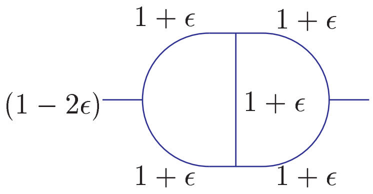
We define:



$$= \sum_{i=0}^{\infty} \epsilon^i C_i$$

We work in $4-2\epsilon$ dimensions. We find:

$$
\begin{aligned}
C_0 &= 6\ \zeta_3 \\
C_1 &= 9\ \zeta_4 \\
C_2 &= 372\ \zeta_5 \\
C_3 &= 915\ \zeta_6 - 864\zeta_3^2 \\
C_4 &= 18450\ \zeta_7 - 2592\ \zeta_3\zeta_4 \\
C_5 &= 52851\ \zeta_8 - 2592\ \zeta_{6,2} - 76680\ \zeta_3\zeta_5 \\
C_6 &= 905368\ \zeta_9 - 130572\ \zeta_4\zeta_5 - 200340\ \zeta_3\zeta_6 + 66384\ \zeta_3^3 \\
C_7 &= 3024018\ \zeta_{10} - 68688\ \zeta_{8,2} - 1777680\ \zeta_5^2 - 3659904\ \zeta_3\zeta_7 \\
&+ 298728\ \zeta_3^2\zeta_4 \\
C_8 &= 49765338\ \zeta_{11} - 134784\ \zeta_{8,2,1} - 2493504\ \zeta_9\zeta_2 \\
&- 9155484\ \zeta_5\zeta_6 - 6579792\ \zeta_4\zeta_7 + 305856\ \zeta_3\zeta_{6,2} \\
&- 10723860\ \zeta_3\zeta_8 + 7961760\ \zeta_3^2\zeta_5 \\
C_9 &= 173709662490/691\ \zeta_{12} - 1990656\ \zeta_{8,2,1,1} + 1754784\ \zeta_{10,2} \\
&+ 4478976\ \zeta_{8,2}\zeta_2 - 229450968\ \zeta_5\zeta_7 + 4478976\ \zeta_5^2\zeta_2 \\
&+ 1656288\ \zeta_4\zeta_{6,2} - 222714072\ \zeta_3\zeta_9 + 6967296\ \zeta_3\zeta_7\zeta_2 \\
&+ 30697056\ \zeta_3\zeta_4\zeta_5 + 21770424\ \zeta_3^2\zeta_6 - 3421440\ \zeta_3^4 \\
C_{10} &= 2695032432\ \zeta_{13} - 2545344\ \zeta_{9,3,1} - 12480480\ \zeta_{10,2,1} \\
&- 140735232\ \zeta_2\zeta_{11} - 456491052\ \zeta_6\zeta_7 + 10285056\ \zeta_5\zeta_{6,2} \\
&- 504765324\ \zeta_5\zeta_8 - 379672164\ \zeta_4\zeta_9 + 8105184\ \zeta_3\zeta_{8,2} \\
&- 573728184\ \zeta_3\zeta_{10} + 338481072\ \zeta_3\zeta_5^2 + 366624144\ \zeta_3^2\zeta_7 \\
&- 21523968\ \zeta_3^3\zeta_4 \\
C_{11} &= 237666053085/14\ \zeta_{14} - 43296768\ \zeta_{10,2,1,1} - 35284608\ \zeta_{10,4} \\
&+ 558916128\ \zeta_{12,2} + 119066112\ \zeta_2\zeta_{10,2} - 6166600704\ \zeta_7^2 \\
&+ 39903840\ \zeta_6\zeta_{6,2} - 12065462640\ \zeta_5\zeta_9 + 281428992\ \zeta_5\zeta_7\zeta_2 \\
&+ 60812208\ \zeta_4\zeta_{8,2} + 682121736\ \zeta_4\zeta_5^2 - 2239488\ \zeta_3\zeta_{8,2,1} \\
&- 11343311568\ \zeta_3\zeta_{11} + 153404928\ \zeta_3\zeta_9\zeta_2 + 1919069712\ \zeta_3\zeta_5\zeta_6 \\
&+ 1339427664\ \zeta_3\zeta_4\zeta_7 - 22581504\ \zeta_3^2\zeta_{6,2} + 1193223096\ \zeta_3^2\zeta_8 \\
&- 552692160\ \zeta_3^3\zeta_5
\end{aligned}
$$

$$
\begin{aligned}
C_{12} &= 350730486838\ \zeta_{15} - 103514112\ \zeta_{10,2,1,1,1} - 905167872\ \zeta_{11,3,1} \\
&- 4939512192\ \zeta_{12,2,1} - 1200318912\ \zeta_7\zeta_{6,2} - 28378212300\ \zeta_7\zeta_8 \\
&- 37376819964\ \zeta_6\zeta_9 - 814344192\ \zeta_5\zeta_{8,2} - 99946794888/5\ \zeta_5\zeta_{10} \\
&+ 5401221408\ \zeta_5^3 - 58771435032\ \zeta_4\zeta_{11} + 216857088\ \zeta_4\zeta_{8,2,1} \\
&- 17848005787746/691\ \zeta_3\zeta_{12} + 234897408\ \zeta_3\zeta_{8,2,1,1} \\
&- 776392128\ \zeta_3\zeta_{10,2} + 40734425664\ \zeta_3\zeta_5\zeta_7 - 181833984\ \zeta_3\zeta_4\zeta_{6,2} \\
&+ 21548913984\ \zeta_3^2\zeta_9 - 3339239904\ \zeta_3^2\zeta_4\zeta_5 - 1591101792\ \zeta_3^3\zeta_6 \\
&+ 651820608/5\ \zeta_3^5 - 106261906752\ \zeta_2\zeta_{13} + 51757056\ \zeta_2\zeta_{9,3,1} \\
&+ 517570560\ \zeta_2\zeta_{10,2,1} - 528519168\ \zeta_2\zeta_3\zeta_{8,2} - 554397696\ \zeta_2\zeta_3\zeta_5^2 \\
&- 848019456\ \zeta_2\zeta_3^2\zeta_7 \\
C_{13} &= 8259271616688738/3617\ \zeta_{16} + 558544896\ \zeta_{10,4,1,1} \\
&- 24219648\ \zeta_{11,3,1,1} - 2069950464\ \zeta_{12,2,1,1} - 8040083760\ \zeta_{12,4} \\
&+ 198638792640\ \zeta_{14,2} - 1240170912\ \zeta_8\zeta_{6,2} - 1446123470472\ \zeta_7\zeta_9 \\
&- 2413526544\ \zeta_6\zeta_{8,2} - 1247483841048\ \zeta_5\zeta_{11} - 590105088\ \zeta_5\zeta_{8,2,1} \\
&+ 48094412472\ \zeta_5^2\zeta_6 + 507617280\ \zeta_4\zeta_{8,2,1,1} - 3926265696\ \zeta_4\zeta_{10,2} \\
&+ 101859210720\ \zeta_4\zeta_5\zeta_7 - 1006897282056\ \zeta_3\zeta_{13} - 343761408\ \zeta_3\zeta_{9,3,1} \\
&- 703966464\ \zeta_3\zeta_{10,2,1} + 96859067760\ \zeta_3\zeta_6\zeta_7 - 1776100608\ \zeta_3\zeta_5\zeta_{6,2} \\
&+ 116664010128\ \zeta_3\zeta_5\zeta_8 + 99095292864\ \zeta_3\zeta_4\zeta_9 - 598409856\ \zeta_3^2\ \zeta_{8,2} \\
&+ 69270397776\ \zeta_3^2\ \zeta_{10} - 35418446208\ \zeta_3^2\ \zeta_5^2 - 26183070720\ \zeta_3^3\ \zeta_7 \\
&+ 1095179616\ \zeta_3^4\ \zeta_4 + 1001673216\ \zeta_2\zeta_{10,4} - 28565270784\ \zeta_2\zeta_{12,2} \\
&+ 83159301888\ \zeta_2\zeta_7^2 + 132538768128\ \zeta_2\zeta_5\zeta_9 + 105673351680\ \zeta_2\zeta_3\zeta_{11}
\end{aligned}
$$

The multiple zeta values are defined as

$$
\begin{aligned}
\zeta_m &= \sum_{i=1}^{\infty} \frac{1}{i^m} \\
\zeta_{m_1,\cdots,m_p} &= \sum_{m_1 \geq \cdots \geq m_p} \frac{1}{i_1^{m_1}} \cdots \frac{1}{i_p^{m_p}}
\end{aligned}
$$

which is a definition that is different from what is usually encountered in the litterature, but it connects far better with the finite sums.

The choice of independent variables was such that only sums with positive indices were used and in a lexicografic ordering the largest possibility is selected.

To ϵ^9 these values agree with the values given by Broadhurst[1] when converted to the same set of variables.

[1] http://physics.open.ac.uk/~dbroadhu/tetrex12.ps

The method of calculation was based on Bierenbaum and Weinzierl[2] and the implementation was in FORM, using the summer library[3] with one extra routine for the infinite sums. The multiple zeta relations were also obtained with a FORM program using the same method as in the paper on finite sums.

Below we show the computer resources needed by the program that computes the integral.

Power	Weight	time(sec)	Memory(Mb)
1	4	0.36	0.040
2	5	1.22	0.099
3	6	4.01	0.264
4	7	13.06	0.805
5	8	42.04	2.69
6	9	132.6	9.13
7	10	401	29.1
8	11	1251	81.3
9	12	3943	219
10	13	11990	609
11	14	37852	1684
12	15	111821	4725
13	16	341020	12782

Execution time and largest expression size on a P2800 computer running Linux RedHat 9 for the FORM program that computed the integral. FORM needs a disk space that is close to twice the largest expression size.

In the following table we show the computer resources needed by the program that determined a minimal set of multiple zeta values.

[2] hep-ph/0308311

[3] J. Vermaseren, Int.J.Mod.Phys.A14:2037-2076,1999; hep-ph/9806280

Weight	time(sec)	Memory(Mb)	New Constants
1			S_∞
2			ζ_2
3			ζ_3
4			
5	0.02		ζ_5
6	0.09		
7	0.25		ζ_7
8	0.87		$\zeta_6\zeta_2$
9	3.04		ζ_9
10	10.95	0.137	$\zeta_{8,2}$
11	40.09	0.330	$\zeta_{11}, \zeta_{8,2,1}$
12	151.28	0.878	$\zeta_{10,2}, \zeta_{8,2,1,1}$
13	571.71	2.29	$\zeta_{13}, \zeta_{10,2,1}, \zeta_{9,3,1}$
14	2224	6.57	$\zeta_{12,2}, \zeta_{10,4}, \zeta_{10,2,1,1}$
15	9040	19.5	$\zeta_{15}, \zeta_{12,2,1}, \zeta_{11,3,1}, \zeta_{10,2,1,1,1}$
16	36865	59.8	$\zeta_{14,2}, \zeta_{12,4}, \zeta_{12,2,1,1}, \zeta_{11,3,1,1}, \zeta_{10,4,1,1}$

Execution time and largest expression size on a P2800 computer running Linux RedHat 9 for the FORM program that derived the independent multiple zeta values In order to derive this infinite set, one needs to consider the infinite sums as well. This is all explained in the paper that underlies the summer library.