E | Gamma matrices

This appendix reviews the properties of γ-matrices. In 4 space-time dimensions we have already given an explicit representation of these matrices in chapter 5. The set-up of this appendix is kept more general; motivated by dimensional regularization and by recent discussions of higher-dimensional theories in the context of Kaluza-Klein supergravity and superstrings we summarize the properties of γ-matrices in arbitrary space time dimension D. For this reason we adopt a notation which is different from that used in the main text. In D-dimensional Minkowski space the space components carry indices $1, 2, ...D-1$, and the purely imaginary time component carries index D. Readers who are emotionally attached to 4-dimensional space time can simply insert $D = 4$, or, if they only need a certain 4-dimensional formula, they are advised to consult section E.3 and parts of the later sections where explicit results for $D = 4$ are listed.

E.1. The Clifford algebra

We consider a representation of the D-dimensional Clifford algebra

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}\mathrm{I}\,, \qquad a, b = 1, ...D\,. \tag{E.1}$$

Repeated multiplication of the γ-matrices leads to a set of 2^D matrices Γ^A

$$\Gamma^A\ : \mathrm{I}, \gamma_a\,, \gamma_{ab}\,, \gamma_{abc}\ \ldots\,, \tag{E.2}$$

where

$$\gamma_{ab} = \gamma_a\gamma_b \qquad (a < b)\,, \qquad \gamma_{abc} = \gamma_a\gamma_b\gamma_c \qquad (a < b < c) \qquad ,\text{etc}\,. \tag{E.3}$$

In (E.2) we only include ordered strings of different γ-matrices; products in which the γ-matrices appear in different order or the same γ-matrix appears more than once can be reduced to one of these by using the anticommutation relation (E.1). On account of (E.1) the matrices $\gamma_{a_1\ldots a_n}$ are antisymmetric in the indices $a_1, \ldots, a_n$, so they can also be defined as an antisymmetrized product of γ-matrices

$$\gamma_{a_1\ldots a_n} = \frac{1}{n!}\sum_{\substack{\text{perm}\\ [a_1\ldots a_n]}} (-)^P\gamma_{a_1}\gamma_{a_2}\ldots\gamma_{a_n}\,. \tag{E.4}$$

As there are $\binom{n}{D}$ different ways of selecting n different indices between 1 and D, there are $\binom{n}{D}$ matrices $\gamma_{a_1\ldots a_n}$. Therefore the total number of matrices Γ^A is

$$\sum_{n=0}^{D}\binom{D}{n} = 2^D\,. \tag{E.5}$$

It is useful to define the degree of the matrices Γ^A as the number of γ-matrices contained in the products (E.4). Obviously the degree ranges between 0 and D: the identity matrix has zero degree, and the product in which each γ-matrix appears once has degree D. Because the latter plays a special role in what follows we denote it by $\tilde{\gamma}$; hence

$$\tilde{\gamma} = \gamma_1\gamma_2\gamma_3 \ldots \gamma_D \,. \tag{E.6}$$

The product of two matrices Γ^A and Γ^B can be reduced to a simpler form by using (E.1) to cancel all pairs of equal γ-matrices. One is then left with a string of different γ-matrices, which constitutes a matrix Γ^C. As this operation involves an interchange of γ-matrices there may be a number of sign changes, so we write

$$\Gamma^A\Gamma^B = +\Gamma^C \,. \tag{E.7}$$

Note that the degree of Γ^C is equal to or less than the sum of the degrees of Γ^A and Γ^B. By similar arguments one finds (no summation over B)

$$\Gamma^B\Gamma^A\Gamma^B = \pm\Gamma^A \,. \tag{E.8}$$

We note two special examples of (E.7). First of all, $\Gamma^C = \mathrm{I}$ if and only if $\Gamma^A = \Gamma^B$; the sign is related to the degree of Γ^A,

$$(\Gamma^A)^2 = \alpha_n^2 \mathrm{I} \,, \tag{E.9}$$

where α_n equals 1 or i in accordance with

$$\alpha_n^2 = (-)^{\frac{1}{2}n(n-1)} \,. \tag{E.10}$$

Hence $\alpha_n = 1$ for $n = 0, 1$ modulo 4 (i.e. $n = 4N$ and $n = 1 + 4N$, where N is a positive integer) and $\alpha_n =$i for $n = 2, 3$ modulo 4. The sign factor in (E.10) arises because reordering the indices in $\gamma_{a_1\ldots a_n}$ in opposite order $a_n, a_{n-1}, \ldots a_1$ induces $\sum_{i=1}^{n}(i-1) = \frac{1}{2}n(n-1)$ minus signs, so that

$$\gamma_{a_1a_2\ldots a_n} = \alpha_n^2 \gamma_{a_na_{n-1}\ldots a_1} \,, \tag{E.11}$$

and obviously $\gamma_{a_1a_2\ldots a_n}\gamma_{a_na_{n-1}\ldots a_1} = \mathrm{I}$ (no summation over $a_1 - a_n$).

Secondly $\Gamma^C = \tilde{\gamma}$ if the sum of the degrees of Γ^A and Γ^B equals D, and if Γ^A and Γ^B contain no identical γ-matrices; explicitly

$$\gamma_{a_1\ldots a_n}\gamma_{a_{n+1}\ldots a_D} = \varepsilon_{a_1\ldots a_D}\tilde{\gamma} \,, \tag{E.12}$$

where $\varepsilon_{a_1\ldots a_n}$ is the D-dimensional Levi-Civita symbol (with normalization $\varepsilon_{123\ldots D} = +1$).

In specific cases it is convenient to write (E.7) in covariant form. Most results follow from repeated use of the products $\gamma_a\gamma_{b_1\ldots b_n}$ and $\gamma_{b_1\ldots b_n}\gamma_a$. If a is different from $b_1 - b_n$, this product equals $\gamma_{ab_1\ldots b_n}$ or $\gamma_{b_1\ldots b_n a}$ which have degree $n+1$; if a is equal to one of the indices $b_1 - b_n$, say b_i then the two equal γ-matrices gives I, so that one is left with $\pm\gamma_{b_1\ldots b_n}$ with the index b_i deleted. Explicitly

$$\gamma_a\gamma_{b_1\ldots b_n} = \gamma_{ab_1\ldots b_n} + \sum_{i=1}^{n}(-)^{i+1}\delta_{ab_i}\gamma_{b_1\ldots\hat{b}_i\ldots b_n} \,. \tag{E.13}$$

where $\hat{b}_i$ indicates that the index b_i has been deleted. Similarly

$$\gamma_{b_1\ldots b_n}\gamma_a = \gamma_{b_1\ldots b_n a} + \sum_{i=1}^{n}(-)^{n-i}\delta_{ab_i}\gamma_{b_1\ldots\hat{b}_i\ldots b_n} \,. \tag{E.14}$$

From (E.13) and (E.14) it follows that

$$[\gamma_a, \gamma_{b_1\ldots b_n}] = 2\gamma_{ab_1\ldots b_n}\,, (n \quad \text{odd}) \tag{E.15}$$

$$\{\gamma_a, \gamma_{b_1\ldots b_n}\}] = 2\gamma_{ab_1\ldots b_n}\,, (n \quad \text{even}) \tag{E.16}$$

A similar relation is

$$\gamma_{a_1\ldots a_n}\tilde{\gamma} = \frac{\alpha_s^2}{(D-n)!}\varepsilon_{a_1\ldots a_n a_{n+1}\ldots a_D}\gamma_{a_{n+1}\ldots a_D} \,. \tag{E.17}$$

where we sum over the indices $a_{n+1} - a_D$ on the right-hand side. This result is derived by noting that the left-hand side contains the matrices $\gamma_{a_1} - \gamma_{a_n}$ twice, once in $\gamma_{a_1\ldots a_n}$ and once in $\tilde{\gamma}$, so that one is left with a product of the $D-n$ $\gamma-$ matrices that are not present in $\gamma_{a_1\ldots a_n}$ We must divide by $(D-n)!$ because there are $(D-n)!$ terms in the summation over $a_{n+1} - a_D$, each giving rise to the same result; the sign can be verified by choosing particular values for the indices $a_1 - a_n$.

Another important result is

$$\gamma_a \Gamma^A \gamma_a = (-)^n (D-2n)\Gamma^A \,, \tag{E.18}$$

which follows from (no summation over a)

$$\gamma_a\gamma_{b_1\ldots b_n}\gamma_a = (-)^{n-1}\gamma_{b_1\ldots b_n} \qquad a \in b_1 \ldots b_n \,, \tag{E.19}$$
$$\gamma_a\gamma_{b_1\ldots b_n}\gamma_a = (-)^{n}\gamma_{b_1\ldots b_n} \qquad a \neq b_1 \ldots b_n \,. \tag{E.20}$$

As there are n index values in $b_1 - b_n$ and $D-n$ index values unequal to $b_1 - b_n$ summation over all index values leads directly to (E.18).

Equation (E.18) may now be used to obtain information about the trace of Γ^A. Using the cyclicity of the trace one derives

$$\mathrm{Tr}(\gamma_a\Gamma^A\gamma_a) = \mathrm{Tr}(\Gamma^A\gamma_a\gamma_a) = D\,\mathrm{Tr}(\Gamma^A)\,,$$

which according to (E.18) must also be equal to $(-)^n(D-2n)\mathrm{Tr}(\Gamma^A)$. Consequently all matrices Γ^A are traceless with the exception of the unit matrix and possibly $\tilde{\gamma}$ (if D is odd), viz.

$$\mathrm{Tr}(\Gamma^A) = 0 \qquad \Gamma^A \neq \mathrm{I}, \tilde{\gamma}\,, \tag{E.21}$$
$$\mathrm{Tr}(\tilde{\gamma}) = 0 \qquad \text{for} \quad D \quad \text{even}\,. \tag{E.22}$$

E.2. A finite group

According to (E.7) the matrices $\pm\Gamma^A$ form a finite group of 2^{D+1} elements. For finite groups there are strong restrictions on the number and type of inequivalent representations which we will exploit to determine a number of important properties of γ-matrices. Let us start by describing the profile of the group. The *order* of the group, defined as the number of elements, is equal to 2^{D+1}. The group elements can be divided in *classes*: two group elements g_1 and g_2 belong to the same class if there is a group element g such that

$$g g_1 g^{-1} = g_2. \tag{E.23}$$

According to (E.8) and (E.9) $+\Gamma^A$ and $-\Gamma^A$ constitute a class in general, unless Γ^A commutes with all group elements, in which case $+\Gamma^A$ and $-\Gamma^A$ are separate classes. To determine the commuting elements one first determines the elements that commute with all the γ-matrices; there are only two, namely the identity element and (if D is odd) the element $\tilde{\gamma}$ (cf. E.15). Elements commuting with the γ-matrices commute with all Γ^A. Therefore $+\,\mathbf{I}$ and -**I** form separate classes and so do $+\tilde{\gamma}$ and $-\tilde{\gamma}$ if D is odd. Consequently the number of classes is 2^D+1 for even D and 2^D+2 for odd D.

Finally the *commutator subgroup*, consisting of all elements $g_1 g_2 g_1^{-1} g_2^{-1}$, has only two elements, $+\mathbf{I}$ and -**I** (cf. E.8 and E.9). In what follows we only need the order of the group, the number of classes and the order of the commutator subgroup. These numbers are listed in table E.2. Now we summarize the following results of finite

Table E.1: Properties of the finite group consisting of the matrices Γ^A defined in the text.

Property	D even	D odd
order of the group	2^{D+1}	2^{D+1}
number of classes	$2^{D+1}+1$	$2^{D+1}+2$
order of commutator subgroup	2	2

group theory:
(i) The number of inequivalent (i.e. not related by a similarity transformation $\gamma_a \to S\gamma_a S^{-1}$) irreducible representations equals the number of classes.
(ii) The number of inequivalent one-dimensional representations equals the order of the group divided by the order of the commutator subgroup.
(iii) The sum of the squares of the dimension of the irreducible representations equals the order of the group.
(iv) All representations of the group are equivalent (through a similarity transformation) to a unitary representation.
Using these results one straightforwardly derives that the group in question has 2^D one-dimensional representations (in such representations the Γ^A are represented by numbers). For even D there is only one other representation of dimension $2^{\frac{1}{2}D}$. For

odd D there are two other representations with dimensions d_1 and d_2 satisfying

$$d_1^2 + d_2^2 = 2^D \,. \tag{E.24}$$

As we shall see below the two representations are both $2^{\frac{1}{2}(D-1)}$ dimensional. Furthermore, in all representations the Γ^A can be chosen unitary. Because of (E.9) this implies that the Γ^A are either hermitean or antihermitean, viz.

$$(\Gamma^A)^\dagger = (\Gamma^A)^{-1} = \alpha_n^2 \Gamma^A \,, \tag{E.25}$$

so that the matrices $\alpha_n \Gamma^A$ are always hermitean. In particular one can always choose

$$\gamma^{a\dagger} = \gamma^a \,. \tag{E.26}$$

Although the one-dimensional representations are genuine representations of the finite group, they do not correspond to representations of the Clifford algebra because the Γ^A are just numbers which cannot satisfy the anticommutation relation (E.1).

Hence only the higher-dimensional representations are relevant for our purpose. From this we conclude that the γ-matrices are unique (i.e. up to a similarity transformation) in even dimensions; for odd dimensions there are two inequivalent representations. There are two ways of understanding the odd-dimensional case. The first is to start from the observation that the group contains an element other than the identity which commutes with all group elements, namely $\tilde\gamma$. Because $\tilde\gamma$ is (anti)hermitean (cf. E.25) it can be diagonalized with eigenvalues $\pm\alpha_D$. Correspondingly we may now decompose all matrices according to a subspace where $\tilde\gamma = \alpha_D \mathrm{I}$, and a subspace where $\tilde\gamma = -\alpha_D \mathrm{I}$; because all Γ^A commute with $\tilde\gamma$ there are no matrix elements connecting these two subspaces. Consequently we can restrict $\tilde\gamma$ to

$$\tilde\gamma = \pm\alpha_D \mathrm{I} \,, \tag{E.27}$$

each corresponding to an (inequivalent) representation of the odd-dimensional Clifford algebra. The second approach starts from the even-dimensional algebra, which one extends to the odd-dimensional case by making the identification

$$\gamma_{D+1} = \pm\alpha_D \gamma \quad (\alpha_D^2 = (-)^{\frac{1}{2}D(D-1)} \,, D\,\text{even}) \,.$$

It is easy to verify that the set $\{\gamma_1, \gamma_2 \ldots, \gamma_D, \gamma_{D+1}\}$ now generates an odd-dimensional Clifford algebra, with

$$\gamma_1\gamma_2 \ldots \gamma_D \gamma_{D-1} = \pm\alpha_D \mathrm{I} \,. \tag{E.28}$$

Note that the sign in (E.27) and (E.28) cannot be changed by a similarity transformation so that this condition characterizes truly inequivalent representations. As both representations have the same dimension it follows from (E.21) that the two inequivalent representations have dimension $2^{\frac{1}{2}(D-1)}$ (D odd). The results obtained so far are summarized in table E.2. [a] To show that, in odd dimensions, the matrices $\gamma_{a_1 \ldots a_n}$ with $0 \leqslant n \leqslant D$, are overcomplete, one uses (E.17) and (E.27).

Table E.2: Properties of γ-matrices in even and odd space time dimensions[a].

D even	$\gamma_a = \gamma_a^\dagger (a = I, ..., D)$ are $2^{D/2} \times 2^{D/2}$ matrices, which are unique modulo a similarity transformation
	all Γ^A are linearly independent
	all $2^{D/2} \times 2^{D/2}$ matrices can be decomposed into $\gamma_{a_1,\ldots a_n} (0 \leqslant n \leqslant D)$
D odd	$\gamma_a = \gamma_a^\dagger (a = I, ..., D)$ are $2^{(D-1)/2} \times 2^{(D-1)/2}$ matrices, which are not unique; there are two representations o
	not all Γ^A are linearly independent
	all $2^{(D-1)/2} \times 2^{(D-1)/2}$ matrices can be decomposed into $\gamma_{a_1,\ldots a_n} (0 \leqslant n \leqslant (D-1)/2)$

E.3. Gamma matrices in $D = 4$ dimensions

For $D = 4$ the γ-matrices have already been defined in chapter where a particular representation was written down. That representation had the advantage that γ_4 was diagonal. Another useful representation is the one where γ_5 is diagonal (chiral representation) or the one where all γ-matrices are real (Majorana representation). Since most of the calculations presented in this book are independent of the explicit form of the γ-matrices we refer to other textbooks for explicit representations other than that of chapter 5 (see, for instance, Itzykson and Zuber (1980); their convention differs from ours in that their γ_1, γ_2 and γ_3 contain an extra factor i and their γ_0 is our γ_4; cf. appendix B).

The notation in chapter 5 differs from the one used in this appendix so far. One easily verifies the correspondence

$$\gamma_a \to \gamma_\mu \,, \tag{E.29}$$

$$\gamma_{ab} \to \gamma_{\mu\nu} = \mathrm{i}\sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,, \tag{E.30}$$

$$\gamma_{abc} \to \gamma_{\mu\nu\rho} = \tfrac{1}{2}\mathrm{i}(\sigma_{\mu\nu}\gamma_\rho + \gamma_\rho\sigma_{\mu\nu}) = -\varepsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5\,, \tag{E.31}$$

$$\gamma_{abcd} \to \gamma_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}\gamma_5\,, \tag{E.32}$$

$$\tilde{\gamma}_5 \to \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4\,, \tag{E.33}$$

(cf. 5.8-5.9), where we have used the defining expressions for $\gamma_{a_1\ldots a_n}$ and $\tilde{\gamma}$ and relations such as (E.16-E.17). Choosing I$\,, \gamma_\mu\,, \sigma_{\mu\nu}$ and γ_5 as an independent set of (hermitean) 4×4 matrices one derives from section E.l

$$\gamma_\rho\gamma_\rho = 4\,, \qquad \gamma_\rho\gamma_\mu\gamma_\rho = -2\gamma_\mu\,, \qquad \gamma_\rho\sigma_{\mu\nu}\gamma_\rho = 0\,, \tag{E.34}$$

$$\gamma_\rho\sigma_{\mu\nu} = -\mathrm{i}(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu - \varepsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5)\,, \tag{E.35}$$

$$\sigma_{\mu\nu}\gamma_\rho = -\mathrm{i}(\delta_{\mu\rho}\gamma_\nu + \delta_{\nu\rho}\gamma_\mu - \varepsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5)\,, \tag{E.36}$$

$$\sigma_{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\gamma_5\,, \tag{E.37}$$

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = 2\mathrm{i}(\delta_{\mu\rho}\sigma_{\nu\sigma} - \delta_{\nu\rho}\sigma_{\mu\sigma} - \delta_{\mu\sigma}\sigma_{\nu\rho} + \delta_{\nu\sigma}\sigma_{\mu\rho})\,, \tag{E.38}$$

$$\{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} = 2\mathrm{i}(\delta_{\mu\rho}\sigma_{\nu\sigma} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \varepsilon_{\mu\nu\rho\sigma}\gamma_5)\,. \tag{E.39}$$

Contraction of γ-matrices with four-vectors A_μ, B_μ, etc. leads to identities such as

$$A\!\!\!/ B\!\!\!/ + B\!\!\!/ A\!\!\!/ = 2A\cdot B\,, \tag{E.40}$$

$$\gamma_\mu A\!\!\!/ + A\!\!\!/\gamma_\mu = 2A_\mu\,, \qquad \gamma_\mu A\!\!\!/\gamma_\mu = -2A\!\!\!/ \tag{E.41}$$

$$\gamma_\mu A\!\!\!/ B\!\!\!/\gamma_\mu = 4A\cdot B\,, \qquad \gamma_\mu A\!\!\!/ B\!\!\!/ C\!\!\!/\gamma_\mu = -2C\!\!\!/ B\!\!\!/ A\!\!\!/ \tag{E.42}$$

Furthermore there is a hermiticity relation

$$\gamma_4 A\!\!\!/^\dagger\gamma_4 = -\bar{A}\!\!\!/ \tag{E.43}$$

where $\bar{A}\!\!\!/ = \bar{A}_\mu\gamma_\mu$.

E.4. The trace over products of gamma matrices

Motivated by dimensional regularization we first discuss the trace over products of γ-matrices in arbitrary dimension D. From section E.1 the general strategy is clear: one decomposes products of γ-matrices in terms of the $\gamma_{a_1\ldots a_n}$ by means of repeated use of (E.13) and (E.14). The coefficients of the $\gamma_{a_1\ldots a_n}$ do not depend on the value of D as long as one does not consider products of $\tilde{\gamma}$ (the analogue of γ_5 in 4 dimensions). Subsequently one performs the trace, which according to (E.21) picks out the coefficient of the identity matrix and (if D is odd) of the matrix $\tilde{\gamma}$. As the definition of $\tilde{\gamma}$ itself depends on D) the trace for odd D will be dimension-dependent. For instance,

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c) \neq 0, \qquad \text{if} \quad D = 1, 3\,, \tag{E.44}$$

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e) \neq 0, \qquad \text{if} \quad D = 1, 3, 5\,, \qquad \text{etc}\,. \tag{E.45}$$

However, for even dimensions only the coefficient of the identity matrix is relevant, so let us concentrate on even values of D. To demonstrate a typical example, take the product of two and four γ-matrices, which we evaluate by using (E.13):

$$\gamma_a\gamma_b = \gamma_{ab} + \delta_{ab}\,, \tag{E.46}$$

$$\gamma_a\gamma_b\gamma_c\gamma_d = \gamma_a\gamma_b(\gamma_{cd} + \delta_{cd}) \tag{E.47}$$

$$= \gamma_a(\gamma_{bcd} + \delta_{bc}\gamma_d - \delta_{bd}\gamma_c + \delta_{cd}\gamma_b) \tag{E.48}$$

$$= \gamma_{abcd} + \delta_{ab}\gamma_{cd} - \delta_{ac}\gamma_{bd} + \delta_{ad}\gamma_{bc} + \delta_{bc}\gamma_{ad} - \delta_{bd}\gamma_{ac} \tag{E.49}$$

$$+ \delta_{cd}\gamma_{ab} + \delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\,. \tag{E.50}$$

Taking the trace and using (E.21) leads to

$$\mathrm{Tr}(\gamma_a\gamma_b) = \delta_{ab}\mathrm{Tr}(\mathrm{I})\,, \tag{E.51}$$

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) = (\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd})\mathrm{Tr}(\mathrm{I})\,. \tag{E.52}$$

Along the same lines one finds

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f) = (\delta_{ab}\delta_{cd}\delta_{ef} - \delta_{ab}\delta_{ce}\delta_{df} - \delta_{ac}\delta_{bf}\delta_{de} + \delta_{ac}\delta_{be}\delta_{df} \tag{E.53}$$

$$+ \delta_{ad}\delta_{bf}\delta_{ce} - \delta_{ad}\delta_{be}\delta_{cf} + \delta_{bc}\delta_{af}\delta_{de} - \delta_{bc}\delta_{ae}\delta_{df} \tag{E.54}$$

$$- \delta_{bd}\delta_{af}\delta_{ce} + \delta_{bd}\delta_{af}\delta_{be} - \delta_{cd}\delta_{af}\delta_{be} + \delta_{cd}\delta_{ae}\delta_{bf} \tag{E.55}$$

$$+ \delta_{ad}\delta_{bc}\delta_{ef} - \delta_{ac}\delta_{bd}\delta_{ef} + \delta_{ab}\delta_{cd}\delta_{ef}\mathrm{Tr}(\mathrm{I})\,. \tag{E.56}$$

The D-dependence now resides entirely in $\mathrm{Tr}(\mathbf{I})$ which equals (for the irreducible representation)

$$\mathrm{Tr}(\mathbf{I}) = 2^{\frac{1}{2}D} \,. \tag{E.57}$$

This number just represents the fact that in different dimensions a spinor field has a different number of components, just as the components of a vector field depend on D. In the context of dimensional regularization the D-dependence of (E.51) and (E.53) is not crucial, as follows from the observation that the trace is always associated with a fermion loop. Changing the number of fermions when moving away from 4 dimensions therefore changes the weight of the diagram, and since we are making an analytic continuation from $D = 4$ we are allowed to change the number of fermions in some continuous fashion, such that the D-dependence of (E.57) is cancelled. Consequently we may use (E.51) and (E.53) for $D = 4$ when applying dimensional regularization.

This is not true if the trace contains the matrix $\tilde{\gamma}$ (or γ_5 in $D = 4$), because $\tilde{\gamma}$ itself depends on D, and just as demonstrated for odd D in (E.44) the trace will sensitively depend on D. Hence we just list some results for $D = 4$, which can be found by using the same procedure as above.

$$\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu) = 0 \,, \tag{E.58}$$

$$\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4\varepsilon_{\mu\nu\rho\sigma} \,, \tag{E.59}$$

$$\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda\gamma_\tau) = 4\delta_{\mu\nu}\,\varepsilon_{\rho\sigma\lambda\tau} - 4\delta_{\mu\rho}\,\varepsilon_{\nu\sigma\lambda\tau} + 4\delta_{\nu\rho}\,\varepsilon_{\mu\sigma\lambda\tau} \tag{E.60}$$

$$+ 4\delta_{\sigma\lambda}\,\varepsilon_{\mu\nu\rho\tau} - 4\delta_{\sigma\tau}\,\varepsilon_{\mu\nu\rho\lambda} + 4\delta_{\lambda\tau}\,\varepsilon_{\mu\nu\rho\sigma} \,, \tag{E.61}$$

where we substituted $\mathrm{Tr}(\mathrm{I}) = 4$. The last equation can be written in a variety of ways by exploiting the Schouten identity (A.17).

E.5. Lorentz transformations and chirality

Lorentz transformations act on spinors as

$$\psi \to \psi' = \exp(\tfrac{1}{4}\theta_{ab}\gamma_{ab})\psi \,, \tag{E.62}$$

where $\theta_{ab} = \bar{\theta}_{ab} = -\theta_{ba}$ are the parameters of the D-dimensional Lorentz group. Note that (E.62) coincides with the four-dimensional result given in (5.11). To show that (E.62) represents the action of the Lorentz group, it suffices to verify that the commutation relations of $-\frac{1}{2}\mathrm{i}\gamma_{ab}$ and $+\frac{1}{2}\mathrm{i}\gamma_{ab}$ coincide with those of the Lorentz group generators M_{ab} and M_{cd} [which take the same form as in $D = 4$; cf. (A.43)]. Just as in $D = 4$ Minkowski space one defines a conjugate spinor

$$\bar{\psi} \equiv \psi^{\star T}\gamma_D \,, \tag{E.63}$$

(where γ_D is the analogue of γ_4 and T denotes that ψ^* is regarded as a row spinor) transforming under Lorentz transformations as

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\exp(-\tfrac{1}{4}\theta_{ab}\gamma_{ab}) \,. \tag{E.64}$$

If D is *even* one can define chiral spinors by

$$\psi_\pm = \frac{1 \pm \alpha_D \tilde{\gamma}}{2} \psi \,, \tag{E.65}$$

such that $\psi_\pm$ are eigenspinors of $\alpha_D \tilde{\gamma}$ with eigenvalues ± 1, viz.

$$\alpha_D \tilde{\gamma} \psi_\pm = \frac{\alpha_D \tilde{\gamma} \pm \alpha_D^2 \tilde{\gamma}^2}{2} \psi = \pm \frac{1 \pm \alpha_D \tilde{\gamma}}{2} \psi = \pm \psi_\pm \,. \tag{E.66}$$

As $\tilde{\gamma}$ commutes with γ^{ab}, the chiral spinors transform identically under Lorentz transformations according to (E.62). Furthermore we note that

$$\bar{\psi}_\pm (\alpha_D \tilde{\gamma}) = \mp \bar{\psi}_\pm \,, \tag{E.67}$$

as follows from $\psi^{*T} \gamma_D \alpha_D \tilde{\gamma} = -\psi_\pm^{*T} \alpha_D \tilde{\gamma} \gamma_D = -((\alpha_D \tilde{\gamma})^\dagger \psi_\pm)^{*T} \gamma_D = \mp \bar{\psi}_\pm$. The above properties are true for arbitrary even dimension. Now we concentrate on Lorentz transformations in four dimensions. Using $\sigma_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} \gamma_5$, we write

$$\theta_{\mu\nu} \sigma_{\mu\nu} = \theta_{ij} \sigma_{ij} + 2\theta_{k4} \sigma_{j4} \,, \tag{E.68}$$

$$= (\theta_{ij} - \theta_{k4} \varepsilon_{k4ij} \gamma_5) \sigma_{ij} \,, \tag{E.69}$$

$$= \xi_{ij} \sigma_{ij} \frac{1 + \gamma_5}{2} + \xi_{ij}^* \sigma_{ij} \frac{1 - \gamma_5}{2} \,, \tag{E.70}$$

where $\xi_{ij} = \theta_{ij} - \theta_{k4} \varepsilon_{k4ij}$, or explicitly using $\theta_{k4} = \mathrm{i}\theta_{k0}$ with θ_{k0} real

$$\xi_{12} = \theta_{12} - \mathrm{i}\theta_{30} \qquad \xi_{31} = \theta_{31} - \mathrm{i}\theta_{20} \qquad \xi_{23} = \theta_{23} - \mathrm{i}\theta_{10} \,. \tag{E.71}$$

This shows that chiral spinors transform under Lorentz transformations as

$$\psi_+ \to \psi'_+ = \exp(\tfrac{1}{4} \xi_{ij} \sigma_{ij}) \psi_+ \,, \qquad \psi_- \to \psi'_- = \exp(\tfrac{1}{4} \xi_{ij}^* \sigma_{ij}) \psi_- \,, \tag{E.72}$$

or in other words, as under ordinary *spatial* rotations with *complex* rather than real angles. We leave it to the reader to substitute (5.10) for $\exp(\frac{1}{4} \xi_{ij} \sigma_{ij})$ to see that it decomposes as

$$\exp(\tfrac{1}{4} \xi_{ij} \sigma_{ij}) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \,, \tag{E.73}$$

with U a complex 2×2 matrix with unit determinant. Such matrices generate the group Sl(2, C), so we have established the equivalence of this group with the four-dimensional Lorentz group (the equivalence holds only locally; see appendix C). The above observations form the basis for the 2-component spinor notation.

E.6. Charge conjugation matrix and Majorana spinors

Observing that the matrices $\pm \gamma_a^{\mathrm{T}}$ (where the superscript T denotes the transpose) also satisfy the defining relation (E.1) of the Clifford algebra, one concludes that $\pm \gamma_a^{\mathrm{T}}$ must be related to γ_a by a similarity transformation in view of the uniqueness property of the Clifford algebra (cf. table E.2). Hence matrices $C_\pm$ must exist such that

$$\pm\gamma_a^{\mathrm{T}} = C_\pm \gamma_a C_\pm^{-1} \,. \tag{E.74}$$

The matrices $C_\pm$ are called "charge-conjugation" matrices for reasons mentioned at the end of this section. For even D both C_+ and C_- should exist; however, for odd D there are two inequivalent representations, and one must ensure that the $\pm\gamma_a^{\mathrm{T}}$ do not actually constitute the other representation. To examine this we first show that the matrices Γ^A, defined by (E.4), satisfy

$$(\Gamma^A)^{\mathrm{T}} = (\pm)^n \alpha_n^2 C_\pm \Gamma^A C_\pm^{-1} \,, \tag{E.75}$$

where n is the degree of Γ^A, as follows directly from (E.74) and (E.11). Consequently

$$\tilde\gamma^{\mathrm{T}} = (\pm)^D \alpha_D^2 C_\pm \tilde\gamma C_\pm^{-1} \,,$$

from which one deduces that in odd dimensions, where $\tilde\gamma$ is proportional to the identity (cf. E.27), so that $C_\pm \tilde\gamma C_\pm^{-1} = \tilde\gamma$, *either* C_+ exists (for $\alpha_D^2 = 1$, so $D = 1$ modulo 4), *or* C_- exists (for $\alpha_D^2 = -1$, so $D = 3$, modulo 4).

Subsequently by applying (E.74) twice one proves

$$(C^{-1} C^{\mathrm{T}})\gamma_a = \gamma_a (C^{-1} C^{\mathrm{T}}) \,. \tag{E.76}$$

However, matrices commuting with γ_a commute with all the matrices Γ^A, so they must be proportional to the unit matrix; therefore $C^{\mathrm{T}} = \lambda C$. Substituting this result back into (E.74) shows that $\lambda^2 = 1$, so that C must be symmetric or antisymmetric

$$C^{\mathrm{T}} = \lambda C, \qquad \lambda = \pm 1 \,. \tag{E.77}$$

By similar arguments one shows that

$$(C^\dagger C)\gamma_a = \gamma_a (C^\dagger C) \tag{E.78}$$

in representations where the γ_a are hermitean, from which one concludes that $C^\dagger C$ is proportional to the unit matrix. Again the square of the proportionality constant equals 1, and because $C^\dagger C$ is positive we must have

$$C^\dagger = C^{-1} \,. \tag{E.79}$$

Using (E.74) and (E.75) it follows that also the matrices $C\gamma^A$ must be symmetric or antisymmetric

$$(C_\pm \gamma^A)^{\mathrm{T}} = (\pm)^n \alpha_n^2 \lambda (C_\pm \Gamma^A) \,, \tag{E.80}$$

where n is the degree of Γ^A. This implies that the matrices $C_+\Gamma^A$ with degree $n = 0$ or 1 modulo 4 have the same symmetry as C_+, while the others have opposite symmetry; likewise the matrices $C_-\Gamma^A$ with degree $n = 0$ or 3 modulo 4 have the same symmetry as C_-, while the others have opposite symmetry.

The above arguments suffice to determine the value of λ. One first observes that the complete (sub)set of matrices γ^A (i.e. all Γ^A for even D and all Γ^A with degree

$n \leqslant \frac{1}{2}D - 1$ for odd D; see table E.2) leads to a corresponding independent set $C\Gamma^A$. Knowing the dimension of the matrices (i.e. $2^{\frac{1}{2}D}$ or $2^{\frac{1}{2}(D-1)}$) one knows the number of independent symmetric and antisymmetric matrices which can be compared to the total number of symmetric or antisymmetric matrices defined in terms of the $C\Gamma^A$. Only for one value of λ will these numbers match. Rather than demonstrate how this is done we present the results in table E.3. So far we have been describing

Table E.3: Symmetry properties of the charge conjugation matrices C_+ and C_- in various dimensions. An S indicates that the matrix is symmetric, an A that it is antisymmetric, corresponding to $A = +1$ and $\lambda = -1$ in (E.77) respectively. Entries repeat themselves every eight columns (i.e., the result for $D = 2$ coincides with that for $D = 10$, etc.).

D	2	3	4	5	6	7	8	9	10	11	12
C_+	S	-	A	A	A	-	S	S	S	-	A
C_-	A	A	A	-	S	S	S	-	A	A	A

abstract properties of the Clifford algebra. Let us now consider spinors ψ and define

$$\psi^{\mathrm{c}} \equiv C_{\pm}^{-1}\bar{\psi}^{\mathrm{T}} , \tag{E.81}$$

(sometimes called the Majorana conjugate) where the notation $\bar{\psi}^T$ implies that we write the conjugate field $\bar{\psi}$ as a row spinor. Under Lorentz transformations ψ^c transforms just as the original field ψ, i.e.

$$\psi^{\mathrm{c}} \to (\psi^{\mathrm{c}})' = \exp(\tfrac{1}{4}\theta_{ab}\gamma_{ab})\psi^{\mathrm{c}} , \tag{E.82}$$

as follows from $C_{\pm}^{-1}\gamma_{ab}^{\mathrm{T}} = -\gamma_{ab}C_{\pm}^{-1}$ (cf. E.75). For even D one may consider chiral projections of ψ. In that case ψ^{c} and ψ have equal chirality whenever $\alpha_D^2 = -1$, i.e. for $D = 2$ modulo 4.

For eigenspinors $u(\mathbf{P})$ and $v(\mathbf{P})$ of the Dirac equation one can define corresponding Majorana conjugates $u^{\mathrm{c}}(\mathbf{P})$ and $v^{\mathrm{c}}(\mathbf{P})$. Using (E.74) it is easy to show that

$$\mathrm{i}P\!\!\!/ u(\mathbf{P}) = -mu(\mathbf{P}) , \qquad \mathrm{i}P\!\!\!/ v(\mathbf{P}) = mv(\mathbf{P}) ,$$

implies

$$\mathrm{i}P\!\!\!/ u^{\mathrm{c}}(\mathbf{P}) = \pm mu^{\mathrm{c}}(\mathbf{P}) , \qquad \mathrm{i}P\!\!\!/ v^{\mathrm{c}}(\mathbf{P}) = \pm mv^{\mathrm{c}}(\mathbf{P}) , \tag{E.83}$$

where the upper (lower) sign refers to a Majorana conjugate defined with respect to $C_+(C_-)$. If we use C_- then (E.83) shows that the Majorana conjugate spinors $u^{\mathrm{c}}(\mathbf{P})$ and $v^{\mathrm{c}}(\mathbf{P})$ are linearly related to $v(\mathbf{P})$ and $u(\mathbf{P})$, respectively (note that this relationship does not exist in dimensions $D = 5$ modulo 4, as C_- cannot be defined). For $D = 4$ this was shown explicitly in chapter 5, and C_- was defined in (5.54). Because C relates a spinor field to its complex conjugate, which for electrically charged fermions is associated with particles of opposite charge, it is conventionally called the charge conjugation matrix.

As ψ and ψ^{c} transform identically under Lorentz transformations it is relevant to investigate if one can impose a reality condition

$$\psi^{\mathrm{c}} = \beta\psi\,, \qquad \text{or} \qquad \bar{\psi}^{\mathrm{T}} = \beta C\psi\,. \tag{E.84}$$

Fields that satisfy (E.84) are called *Majorana spinors*. Multiplying the second equation in (E.84) with the transpose of γ_D and taking the complex conjugate yields

$$\psi = (\beta\gamma_D^{\mathrm{T}}C)^*\psi^* \tag{E.85}$$

$$= \beta^*\gamma_D^\dagger C^*\gamma_D^{\mathrm{T}}\bar{\psi}^{\mathrm{T}} \tag{E.86}$$

$$= \beta^*\gamma^\dagger C^*\gamma_D^{\mathrm{T}}\beta C\psi\,, \tag{E.87}$$

where we again used (E.84). Consequently the following restriction must be satisfied in order that Majorana spinors exist.

$$|\beta|^2\gamma_D^\dagger C_\pm^*\gamma_D^{\mathrm{T}}C_\pm = \mathrm{I} \tag{E.88}$$

or, using (E.26), (E.74), (E.77) and (E.78),

$$|\beta|^2 = \pm\lambda\,. \tag{E.89}$$

Because the left-hand side of this equation is positive, Majorana spinors exist only for those dimensions where C_+ is symmetric or C_- is antisymmetric. Those cases can be read off directly from table E.3 (note the analogy of (E.88) with the reality condition for scalar fields, $\psi^* = \beta\phi$, which requires $|\beta|^2 = 1$). For $D = 2$ modulo 4 it is possible to restrict Majorana spinors to be chiral (see comments following (E.82). Such spinors are called Majorana-Weyl spinors.

E.7. Fierz reordering

In section E.2 we found that the 2^D matrices Γ^A form a complete set of $2^{\frac{1}{2}D}\times 2^{\frac{1}{2}D}$ matrices for even D. For odd D the 2^{D-1} matrices Γ^A of degree less than or equal to $2(D-I)$ are also a complete set of $2^{\frac{1}{2}(D-1)}\times 2^{\frac{1}{2}(D-1)}$ matrices (cf. table E.2). Consequently any matrix of the corresponding dimensionality can be decomposed in terms of the Γ^A:

$$M_{\alpha\beta} = 2^{-\frac{1}{2}D}\sum_{n=0}^{D}\frac{1}{n!}\mathrm{Tr}(M\gamma_{a_n\ldots a_1})(\gamma_{a_1\ldots a_n})_{\alpha\beta}\,, \quad D \quad \text{even}\,, \tag{E.90}$$

$$M_{\alpha\beta} = 2^{-\frac{1}{2}(D-1)}\sum_{n=0}^{\frac{1}{2}(D-1)}\frac{1}{n!}\mathrm{Tr}(M\gamma_{a_n\ldots a_1})(\gamma_{a_1\ldots a_n})_{\alpha\beta}\,, \quad D \quad \text{odd}\,. \tag{E.91}$$

The right-hand side is divided by factors $2^{\frac{1}{2}D}$ and $^{\frac{1}{2}(D-1)}$, which represent the trace of the unit matrix for even and odd D, respectively; the factor $1/n!$ is included to avoid summing $n!$ times over the same matrix $\gamma_{a_1\ldots a_n}$. Observe also that the indices $a_1\ldots a_n$ appear twice but in opposite order to avoid extra minus signs (cf. E.11).

The completeness relation (E.90) can now be used to reorder spinors in expressions such as $(\bar\psi\Gamma^A\chi)(\bar\xi\Gamma^B\zeta)$ For instance in even D one derives directly

$$(\bar\psi\Gamma^A\chi)(\bar\xi\Gamma^B\zeta) = 2^{-\frac{1}{2}D}\sum_{n=0}^{D}\frac{1}{n!}(\bar\psi\Gamma^A\gamma_{a_n\ldots a_1}\Gamma^B\zeta)(\bar\xi\gamma_{a_1\ldots a_n}\chi) \tag{E.92}$$

for *commuting* spinors (for anticommuting spinors there is an extra factor -1.) An example of (E.92)in D = 4 is

$$(\bar\psi\gamma_\mu\chi)(\bar\xi\gamma_\mu\zeta) = \tfrac{1}{4}(\bar\psi\zeta)(\bar\xi\chi) - (\bar\psi\gamma_5\zeta)(\bar\xi\gamma_5\chi) - \tfrac{1}{2}(\bar\psi\gamma_\mu\zeta)(\bar\xi\gamma_\mu\chi) \tag{E.93}$$

$$- \tfrac{1}{2}(\bar\psi\gamma_\mu\gamma_5\zeta)(\bar\xi\gamma_\mu\gamma_5\chi)\,, \tag{E.94}$$

where we used the notation of section E.3. This result simplifies if two of the fields are chiral. For instance replacing χ and ζ by $(1+\gamma_5)\chi$ and $(1+\gamma_5)\zeta$ gives

$$(\bar\psi\gamma_\mu(1+\gamma_5)\chi)(\bar\xi\gamma_\mu(1+\gamma_5)\zeta) = -(\bar\psi\gamma_\mu(1+\gamma_5)\zeta)(\bar\xi\gamma_\mu(1+\gamma_5)\chi) \tag{E.95}$$

for commuting spinors.