Classical fields

in relativistic framework

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1. Notation and conventions

Generic space-time co-ordinates are denoted by x^μ, $\mu = (0,1,2,3)$. They can either refer to cartesian co-ordinates:

$$x^\mu \rightarrow (ct, x, y, z),$$

or to spherical co-ordinates

$$x^\mu \rightarrow (ct, r, \theta, \varphi),$$

or to some other convenient co-ordinate system. Using the Einstein summation convention for repeated indices the Lorentz-invariant line element of Minkowski space-time can be written in such co-ordinates as

$$\begin{aligned} \eta_{\mu\nu} dx^\mu dx^\nu &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 && \textit{in cartesian co-ordinates}, \\ &= -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 && \textit{in spherical co-ordinates.} \end{aligned}$$

Therefore the components of the Minkowski metric $\eta_{\mu\nu}$ depend on the co-ordinates used

$$\begin{aligned} \eta_{\mu\nu} &\rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \textit{in cartesian co-ordinates}, \\ &\rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} && \textit{in spherical co-ordinates.} \end{aligned}$$

Note that the metric is always symmetric: $\eta_{\mu\nu} = \eta_{\nu\mu}$.
Problem 1.1
The relations between cartesian and spherical co-ordinates are:

$$ct = ct, \quad x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

a. Take $x^\mu = (ct, x, y, z)$ and $\eta_{\mu\nu}$ to refer to the cartesian co-ordinates and metric components, and $\bar{x}^\mu = (ct, r, \theta, \varphi)$ and $\bar{\eta}_{\mu\nu}$ to refer to the spherical co-ordinates and metric components; show that by the chain rule

$$\eta_{\mu\nu} \, dx^\mu dx^\nu = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\kappa} d\bar{x}^\kappa \frac{\partial x^\nu}{\partial \bar{x}^\lambda} d\bar{x}^\lambda = \bar{\eta}_{\kappa\lambda} \, d\bar{x}^\kappa d\bar{x}^\lambda.$$

b. Show that the symmetry of $\eta_{\mu\nu}$ automatically implies the symmetry of $\bar{\eta}_{\mu\nu}$.

2. Lorentz invariance

The Minkowski line element is invariant under Lorentz transformations. Write a Lorentz transformation as a linear co-ordinate transformation $x^\mu \to \bar{x}^\mu$:

$$\bar{x}^\mu = \Lambda^\mu_{\ \ \nu}\, x^\nu. \tag{1}$$

The *invariance* of the line element is expressed by

$$\eta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = \eta_{\mu\nu} \Lambda^\mu_{\ \ \kappa} dx^\kappa\, \Lambda^\nu_{\ \ \lambda} dx^\lambda = \eta_{\kappa\lambda} dx^\kappa dx^\lambda \tag{2}$$

with the same components of the Minkowski metric on both sides; thus invariance means $\bar{\eta}_{\mu\nu} = \eta_{\mu\nu}$. The metric has an inverse denoted by $\eta^{\mu\nu}$:

$$\eta^{\mu\lambda}\, \eta_{\lambda\nu} = \delta^\mu_\nu. \tag{3}$$

Problem 2.1
a. From the invariance of the line element argue that

$$\eta_{\mu\nu} = \eta_{\kappa\lambda} \Lambda^\kappa_{\ \ \mu} \Lambda^\lambda_{\ \ \nu}. \tag{4}$$

b. Use this to show that

$$(\Lambda^{-1})^\mu_{\ \ \nu} = \eta_{\nu\kappa}\, \Lambda^\kappa_{\ \ \lambda}\, \eta^{\lambda\mu} \equiv \Lambda_\nu^{\ \ \mu}. \tag{5}$$

Here we have introduced the convention that indices can be raised or lowered by the Minkowski metric and its inverse.
c. Prove that $\det \Lambda = \pm 1$.

Problem 2.2
Consider two 4-vectors with components a^μ and b^μ. Define the scalar product

$$a \cdot b = \eta_{\mu\nu}\, a^\mu b^\nu. \tag{6}$$

Prove that the scalar product is invariant under Lorentz transformations:

$$\bar{a}^\mu = \Lambda^\mu_{\ \ \nu}\, a^\nu, \quad \bar{b}^\mu = \Lambda^\mu_{\ \ \nu}\, b^\nu \quad \Rightarrow \quad \bar{a} \cdot \bar{b} = a \cdot b. \tag{7}$$

Note that the convention to raise and lower indices with the Minkoswki metric and its inverse allows us to write equivalently:

$$a \cdot b = \eta_{\mu\nu}\, a^\mu b^\nu = a^\mu b_\mu = a_\mu b^\mu = \eta^{\mu\nu} a_\mu b_\nu. \tag{8}$$

Problem 2.3
Show that Lorentz transformations act on the vector components with lower indices as follows:

$$\bar{a}_\mu = \Lambda_\mu^{\ \ \nu}\, \bar{a}_\nu = a_\nu\, (\Lambda^{-1})^\nu_{\ \ \mu}. \tag{9}$$

Problem 2.4
Consider the transformation

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{cc|cc} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & \multicolumn{2}{c}{0} \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & & \\ \multicolumn{2}{c}{0} & 1 & 0 \\ & & 0 & 1 \end{array} \right), \tag{10}$$

where $\beta = v/c < 1$.
a. Show that it satisfies the properties (4) and that $\det \Lambda = 1$.
b. Compute the inverse as in eq. (5); how can you directly verify the correctness of your result?

3. Proper time

The motion of a point-like particle is described by giving its position as a function of time. Together these points form a curve in space-time, the *world line* of the particle. Introduce a parameter τ which labels the points of the world line in order of increasing time. Then the world line is given by the functions $\xi^{\mu}(\tau)$ which give the space-time co-ordinates of the particle at the point labeled τ. The ξ^{μ} must be distinguished from the general space-time co-ordinates x^{μ} which can label any point in space-time, whether it is on the world line or not.

Now we can choose a specific Lorentz-invariant parameter τ by defining it so as to satisfy

$$\begin{aligned} c^2 d\tau^2 &= -\eta_{\mu\nu}\, d\xi^{\mu} d\xi^{\nu} = c^2 dt^2 - d\boldsymbol{\xi}^2 \\ &= c^2 dt^2 \left(1 - \frac{\mathbf{v}^2}{c^2} \right), \qquad \mathbf{v} = \frac{d\boldsymbol{\xi}}{dt}. \end{aligned} \tag{11}$$

By definition $\mathbf{v}$ is the instantaneous velocity as measured in laboratory co-ordinates, and it follows that $d\tau = dt$ whenever the particle velocity momentarily vanishes: $\mathbf{v}(t) = 0$. Therefore τ measures the passage of time by a clock in an inertial frame with respect to which the particle is momentarily at rest. This parameter τ is called the *proper time*, and by construction it is (inertial) observer independent.

Problem 3.1
The proper 4-velocity of the particle is defined as the tangent vector of the world line at proper time τ:

$$u^{\mu}(\tau) = \frac{d\xi^{\mu}}{d\tau}. \tag{12}$$

a. Show that under a Lorentz transformation (1) the proper 4-velocity transforms as a 4-vector:

$$\bar{u}^{\mu} = \Lambda^{\mu}{}_{\nu}\, u^{\nu}. \tag{13}$$

b. Show that u^{μ} is a time-like vector of fixed magnitude:

$$u^2 \equiv u \cdot u = -c^2. \tag{14}$$

c. The proper 4-acceleration is $a^\mu = du^\mu/d\tau$. Prove that it is orthogonal in the 4-dimensional sense to the 4-velocity:

$$a \cdot u = 0. \tag{15}$$

4. Momentum and force

The 4-momentum of a particle of mass m is

$$p^\mu = m u^\mu. \tag{16}$$

Problem 4.1
a. Derive expressions for the components of p^μ:

$$p^0 = \frac{mc}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{17}$$

b. Show that

$$p^2 = -m^2 c^2. \tag{18}$$

The special relativistic generalization of Newton's second law is

$$\frac{dp^\mu}{d\tau} = f^\mu. \tag{19}$$

Problem 4.2
a. Explain why under Lorentz transformations f^μ must transform as 4-vector:

$$\bar{f}^\mu = \Lambda^\mu{}_\nu f^\nu. \tag{20}$$

b. Show that

$$f \cdot u = 0 \quad \text{and} \quad f^0 = \frac{\mathbf{f} \cdot \mathbf{v}}{c}. \tag{21}$$

c. What is the physical meaning of $\mathbf{f} \cdot \mathbf{v}$? Explain why it implies that we can identify $\mathcal{E} = c\,p^0$ with the energy of the particle.
d. Show that the constraint (21) on relativistic force laws is satisfied automatically if there is an anti-symmetric tensor $f^{\mu\nu}$ such that

$$f^\mu = f^{\mu\nu} u_\nu, \quad f^{\mu\nu} = -f^{\nu\mu}. \tag{22}$$

An example of a relativistic force law is the Lorentz force on a charged particle. In this case

$$f^{\mu\nu} = q F^{\mu\nu}, \tag{23}$$

where q is the electric charge of the particle and $F^{\mu\nu}$ is the electro-magnetic field strength tensor with components

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{24}$$

Problem 4.3
a. Translate the relativistic Lorentz force equation

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{25}$$

to the laboratory frame and derive the equations

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right), \quad \frac{d\mathcal{E}}{dt} = q\mathbf{E}\cdot\mathbf{v}. \tag{26}$$

b. Argue why under Lorentz transformations the field components must transform such that

$$\bar{F}^{\mu\nu}(\bar{x}) = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda}(x). \tag{27}$$

5. Electro-magnetic fields

Define the 4-dimensional permutation symbol by

$$\varepsilon_{\mu\nu\kappa\lambda} = -\varepsilon^{\mu\nu\kappa\lambda} \equiv \begin{cases} +1 & \text{if } (\mu\nu\kappa\lambda) = \text{ even permutation of } (0123); \\ -1 & \text{if } (\mu\nu\kappa\lambda) = \text{ odd permutation of } (0123); \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Problem 5.1
a. Prove that

$$\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\kappa\gamma}\eta_{\lambda\delta}\,\varepsilon^{\alpha\beta\gamma\delta} = -\eta\,\varepsilon_{\mu\nu\kappa\lambda}, \tag{29}$$

where $\eta = \det \eta_{\mu\nu}$. Hence the permutation symbol is strictly not a covariant tensor in the usual sense.
b. Show that a better-behaved tensor is defined by

$$\eta_{\mu\nu\kappa\lambda} = \sqrt{-\eta}\,\varepsilon_{\mu\nu\kappa\lambda}, \qquad \eta^{\mu\nu\kappa\lambda} = \frac{1}{\sqrt{-\eta}}\,\varepsilon^{\mu\nu\kappa\lambda} = \eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\kappa\gamma}\eta^{\lambda\delta}\,\eta_{\alpha\beta\gamma\delta}. \tag{30}$$

What values do the components of these tensors take?

Maxwell's equations in the standard formulation are

$$\begin{aligned} &\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad c^2\nabla\times\mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial\mathbf{E}}{\partial t}, \\ &\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \quad \nabla\cdot\mathbf{B} = 0, \end{aligned} \tag{31}$$

where ρ is the charge density and $\mathbf{j}$ the 3-dimensional electric current density.

Problem 5.2
a. Show that the equations can be written in covariant relativistic notation as

$$\partial_\mu F^{\mu\nu} = -\frac{j^\mu}{\epsilon_0 c^2}, \qquad \eta_{\mu\nu\kappa\lambda}\,\partial^\nu F^{\kappa\lambda} = 0. \tag{32}$$

Here the electric 4-current density is defined by $j^\mu = (\rho c, \mathbf{j})$.
b. Prove that consistency of the first equation requires an equation of continuity

$$\partial_\mu j^\mu = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}. \tag{33}$$

c. The charge in a spatial volume V is

$$Q = \int_V d^3x\, \rho(\mathbf{x}, t) \tag{34}$$

Show that the any change in Q must be compensated by a flow of current across the boundary surface ∂V:

$$\frac{dQ}{dt} = -\oint_{\partial V} d^2\sigma\, j_n, \tag{35}$$

where $d^2\sigma$ is a 2-dimensional surface element and $j_n = \mathbf{n} \cdot \mathbf{j}$ is the normal component of $\mathbf{j}$ across this surface element, $\mathbf{n}$ being the 3-dimensional outward normal unit vector to $d^2\sigma$. In particular the charge Q is conserved if no net current flows across the total boundary surface.

Problem 5.3
For a charged particle the charge and current densities are described by

$$\rho(x) = q\,\delta^3(\mathbf{x} - \boldsymbol{\xi}(t)), \quad \mathbf{j}(x) = q\mathbf{v}\,\delta^3(\mathbf{x} - \boldsymbol{\xi}(t)). \tag{36}$$

Show that these equations can be combined into one:

$$j^\mu = q \int d\tau\, u^\mu\, \delta^4(x - \xi(\tau)). \tag{37}$$

Units

In the following we will use units of length and of charge such that the numerical values of the speed of light and the electric constant are $c = 1$ and $\epsilon_0 = 1$. The equations (26) for the Lorentz force and (36) for the charge and current densities are not manifestly affected by this choice. The inhomogeneous Maxwell equations simplify to

$$\partial_\mu F^{\mu\nu} = -j^\nu, \tag{38}$$

and the energy-momentum dispersion relation (18) to

$$p^2 = -m^2. \tag{39}$$

6. Energy-momentum tensor

There are many ways to derive dynamical equations: equations of motion for particles, fields or other continuous media. The most common ones are lagrangean and hamiltonian action principles. For relativistic systems there is an alternative procedure based on energy-momentum conservation. Instead of postulating a lagrangean or hamiltonian one postulates a symmetric tensor

$$T^{\mu\nu} = T^{\nu\mu} = 0, \tag{40}$$

with the property that

$$P^{\mu} = \int_V d^3x \, T^{\mu 0} \tag{41}$$

represents the 4-vector of total energy-momentum inside the volume V. For an isolated system this must be conserved, which can be guaranteed by requiring that the energy-momentum tensor is divergence-free:

$$\partial_{\mu} T^{\mu\nu} = 0. \tag{42}$$

Problem 6.1
Equation (42) is an equation of continuity like (33). Show that it implies a conservation law

$$\frac{dP^{\mu}}{dt} = - \oint_{\partial V} d^2\sigma \, T^{\mu}_{\;\;n}, \tag{43}$$

where $T^{\mu}_{\;\;n} = T^{\mu i} \, n_i$ represents a normal component of the tensor for every μ. What is the physical interpretation of the term on the right-hand side?

The procedure can be illustrated with the simple example of a free point particle with world line $\xi^{\mu}(\tau)$, for which

$$T^{\mu\nu}_{part} = m \int d\tau \, u^{\mu} u^{\nu} \delta^4(x - \xi(\tau)). \tag{44}$$

Problem 6.2
a. Show that

$$p^{\mu} = \int d^3x \, T^{\mu 0}_{part} = m u^{\mu}. \tag{45}$$

b. Using that $u^{\mu} = d\xi^{\mu}/d\tau$ prove the result

$$\partial_{\mu} T^{\mu\nu}_{part} = m \int d\tau \, \frac{du^{\nu}}{d\tau} \, \delta^4(x - \xi(\tau)), \tag{46}$$

and explain why

$$\partial_{\mu} T^{\mu\nu}_{part} = 0 \quad \Leftrightarrow \quad \frac{dp^{\mu}}{d\tau} = 0.$$

Problem 6.3
For the electro-magnetic field we define

$$T_{em}^{\mu\nu} = F^{\mu\lambda} F^{\nu}_{\ \lambda} - \frac{1}{4} \eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}.$$

a. Prove that

$$T_{em}^{00} = \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right), \quad T_{em}^{i0} = \left(\mathbf{E} \times \mathbf{B} \right)^i. \tag{47}$$

b. Compute the divergence:

$$\partial_\mu T_{em}^{\mu\nu} = \left(\partial_\mu F^{\mu\lambda} \right) F^{\nu}_{\ \lambda} + \frac{1}{2} \left(\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} \right). \tag{48}$$

Explain that the last cyclic term always vanishes, and that the first term vanishes in the absence of charged matter.
c. Combine the energy-momentum tensors of the electro-magnetic field and the point particle to get

$$\partial_\mu \left(T_{part}^{\mu\nu} + T_{em}^{\mu\nu} \right) = \int d\tau \delta^4 (x - \xi(\tau)) \left[\frac{dp^\nu}{d\tau} - q F^{\nu}_{\ \lambda} u^\lambda \right] + \left[\partial_\mu F^{\mu\lambda} + j_{part}^{\nu} \right] F^{\nu}_{\ \lambda}, \tag{49}$$

where j_{part}^{ν} is given by equation (37). This vanishes if the Lorentz force equation (25) and the inhomogeneous Maxwell equations (38) are satisfied.

7. Ideal fluids

A fluid is a continuous form of matter, matter in circumstances where its particle nature becomes irrelevant like a gas or a liquid. Its motion is described by the current density

$$j^\mu = \rho u^\mu, \tag{50}$$

where ρ is the fluid density (the continuum limit of particle number density), and u^μ is the local 4-velocity of flow. Thus

$$u^2 = -1 \quad \Rightarrow \quad j^2 = -\rho^2. \tag{51}$$

Like the electric current it satisfies an equation of continuity

$$\partial_\mu j^\mu = 0. \tag{52}$$

Problem 7.1
Derive the conservation law for the total amount of fluid in a volume V:

$$N = \int_V d^3x \, j^0.$$

Problem 7.2
Using eq. (11) one can express u^μ in terms of the laboratory velocity $\mathbf{v} = d\boldsymbol{\xi}/dt$. Show that in terms of this the equation of continuity takes the form

$$\frac{\partial}{\partial t} \frac{\rho}{\sqrt{1-\mathbf{v}^2}} + \nabla \cdot \frac{\rho \mathbf{v}}{\sqrt{1-\mathbf{v}^2}} = 0. \tag{53}$$

What is the non-relativistic limit $\mathbf{v}^2 \ll 1$ of this equation?

The dynamics of a fluid is driven by the pressure density p and energy density ε. For an ideal isotropic and non-viscous fluid the energy-momentum tensor is taken to be

$$T_f^{\mu\nu} = p\eta^{\mu\nu} + (\varepsilon + p) u^\mu u^\nu. \tag{54}$$

Problem 7.3
a. Show in cartesian co-ordinates that

$$T_f^{00} = \frac{\varepsilon + p\mathbf{v}^2}{1-\mathbf{v}^2}, \qquad T_f^{i0} = \frac{(\varepsilon + p)v^i}{1-\mathbf{v}^2}, \qquad T^{ij} = p\delta^{ij} + (\varepsilon + p)\, \frac{v^i v^j}{1-\mathbf{v}^2},$$

and that for a *static* fluid ($\mathbf{v} = 0$):

$$T_f^{\mu\nu} = \left(\begin{array}{cc} \varepsilon & 0 \\ 0 & p\,\delta^{ij} \end{array} \right). \tag{55}$$

b. From the equation of continuity for energy-momentum show that

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\varepsilon + p}{1-\mathbf{v}^2} \right) + \nabla \cdot \left(\frac{(\varepsilon + p)\mathbf{v}}{1-\mathbf{v}^2} \right), \\ \nabla p + \frac{\partial p}{\partial t} \mathbf{v} &= -\frac{\varepsilon + p}{1-\mathbf{v}^2} \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right). \end{aligned} \tag{56}$$

This is the relativistic form of the Euler equations.

Problem 7.4
Take the energy-momentum continuity equation

$$\partial_\mu T_f^{\mu\nu} = \partial^\nu p + \partial_\mu \left((\varepsilon + p) u^\mu u^\nu \right) = 0,$$

and combine it with the equation of continuity for the fluid flow

$$\partial_\mu \left(\rho u^\mu \right) = 0,$$

to show that

$$\rho u^\mu \left[p\, \partial_\mu \left(\frac{1}{\rho} \right) + \partial_\mu \left(\frac{\varepsilon}{\rho} \right) \right] = 0. \tag{57}$$

As ρ is the fluid density it follows that $v = 1/\rho$ is the specific volume of the fluid: the volume per unit amount of fluid. Then $u = \varepsilon v$ is the specific energy (energy per unit amount of fluid). Therefore equation (57) is equivalent to

$$u^\mu \left(p\, \partial_\mu v + \partial_\mu u \right) = 0. \tag{58}$$

If the fluid is in thermal equilibrium the specific entropy is related to the specific volume and energy by

$$T ds = du + p dv. \tag{59}$$

Relation (58) then implies that an ideal fluid in thermal equilibrium flows adiabatically:

$$u^\mu \partial_\mu s = 0.$$

Problem 7.5
Check that this is equivalent with

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0. \tag{60}$$

To specify completely the dynamics of a fluid in thermal equilibrium we need to give an equation of state relating the internal energy density ε and the fluid density ρ:

$$\varepsilon = f(\rho). \tag{61}$$

Problem 7.6
a. Show that the adiabaticity of the flow implies that p is the Legendre transform of $f(\rho)$:

$$p = \rho f'(\rho) - f(\rho). \tag{62}$$

b. Work out the relation between energy density and pressure for a power-law equation of state

$$\varepsilon = \rho^\alpha. \tag{63}$$

c. For a very relativistic fluid where the mass density of the fluid is negligeable compared to its energy density the equation of state is

$$\varepsilon = 3p. \tag{64}$$

What are the corresponding relations between ε, p and ρ?
d. Let m be the specific mass of the fluid: the mass of a unit amount of fluid. For a classical fluid the specific energy density in the non-relativistic limit is

$$u = m + \frac{3}{2}\, pv. \tag{65}$$

Derive the results

$$\varepsilon = m\rho + \kappa \rho^{5/3}, \qquad p = \frac{2\kappa}{3}\, \rho^{5/3}. \tag{66}$$

Sound waves
Consider a fluid at rest:

$$\mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} = \nabla \rho = 0. \tag{67}$$

In such a fluid there may be density fluctuations and these can propagate through the fluid in the form of sound waves. We expand the variables in the equilibrium values plus first-order fluctuations:

$$\rho = \rho_0 + \rho_1, \quad \varepsilon = \varepsilon_0 + \varepsilon_1, \quad p = p_0 + p_1, \quad \mathbf{v} = \mathbf{v}_1, \tag{68}$$

where the last equation implements the equilibrium condition $\mathbf{v}_0 = 0$ in the fluid rest frame.
Problem 7.7
a. Derive the first-order expressions

$$\varepsilon_1 = f_0' \, \rho_1, \quad p_1 = f_0'' \, \rho_0 \, \rho_1, \tag{69}$$

where $f_0^n \equiv f^n(\rho_0)$.
b. Show that the fluid equations (56) imply the first-order fluctuation equations

$$\frac{\partial \varepsilon_1}{\partial t} + (\varepsilon_0 + p_0) \nabla \cdot \mathbf{v}_1 = 0, \qquad \nabla p_1 + (\varepsilon_0 + p_0) \frac{\partial \mathbf{v}_1}{\partial t} = 0. \tag{70}$$

c. From these equations derive the wave equations

$$\frac{1}{c_s^2} \frac{\partial^2 \rho_1}{\partial t^2} - \Delta \rho_1 = 0, \quad \frac{1}{c_s^2} \frac{\partial^2 \, \mathbf{v}_1}{\partial t^2} - \Delta \mathbf{v}_1 = 0, \tag{71}$$

where the speed of sound (in units of the speed of light) is

$$c_s^2 = \frac{\rho_0 f_0''}{f_0'} = \left(\frac{\partial p}{\partial \varepsilon} \right)_0 . \tag{72}$$

d. Compute the speed of sound for relativistic and classical non-relativistic fluids as a function of density and as a function of pressure.

8. Vector fields

The homogeneous Maxwell equations

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0,$$

have general solution in terms of a vector field A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{73}$$

Problem 8.1
a. Check this result.
b. Show that the inhomogeneous Maxwell equations become

$$\Box A_\mu - \partial_\mu \partial \cdot A = -j_\mu. \tag{74}$$

c. Verify that these equations are invariant under specific redefinitions of the vector field:

$$A_\mu \quad \rightarrow \quad A'_\mu = A_\mu + \partial_\mu \Lambda, \tag{75}$$

for arbitrary scalar functions $\Lambda(x)$. These are called *gauge transformations*.

Gauge transformations can be used to find a specific solution of the Maxwell equations satisfying the Lorenz condition, or *Lorenz gauge*:

$$\partial \cdot A = 0. \tag{76}$$

This vector field then satisfies the inhomogeneous wave equation

$$-\Box A_\mu = \frac{\partial^2 A_\mu}{\partial t^2} - \Delta A_\mu = j_\mu. \tag{77}$$

Therefore in empty space (where the current density vanishes: $j_\mu = 0$) the fields propagate at the speed of light.

Problem 8.2
Consider a free vector field A_μ satisfying the homogeneous Proca equation:

$$-\Box A_\mu + \partial_\mu \partial \cdot A + m^2 A_\mu = 0. \tag{78}$$

a. Check that this equation is *not* invariant under gauge transformations (75), but show that nevertheless it implies the Lorenz condition (76).
b. Use this result to argue that the solutions of the Proca equation also satisfy the Klein-Gordon equation

$$-\Box A_\mu + m^2 A_\mu = 0. \tag{79}$$

An essential difference between the solutions of the free Maxwell and free Proca equations is the different number of polarization states that propagating fields possess. This is easiest studied in using a plane-wave decomposition of the field:

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^2}\, \varepsilon_\mu(k)\, e^{ik\cdot x} \tag{80}$$

Problem 8.3
a. Derive the Proca equation for the wave amplitudes $\varepsilon_\mu(k)$:

$$k^2 \varepsilon_\mu - k_\mu k \cdot \varepsilon + m^2 \varepsilon_\mu = 0. \tag{81}$$

b. Show that reality of the vector field $A^*_\mu(x) = A_\mu(x)$ implies

$$\varepsilon^*_\mu(k) = \varepsilon_\mu(-k). \tag{82}$$

c. Derive the Lorenz condition for the amplitudes:

$$k \cdot \varepsilon = 0, \tag{83}$$

inferring that

$$(k^2 + m^2)\, \varepsilon_\mu = 0. \tag{84}$$

Note that therefore the amplitude can be non-zero only if the 4-momentum k_μ is on the *mass-shell*:

$$k^2 + m^2 = 0 \quad \Leftrightarrow \quad k_0^2 = \mathbf{k}^2 + m^2. \tag{85}$$

Thus we have to take $\varepsilon_\mu(k)$ of the form

$$\varepsilon_\mu(k) = e_\mu(k)\, \delta(k^2 + m^2), \tag{86}$$

where $k_0 = \pm\,\omega$, $\omega = \sqrt{\mathbf{k}^2 + m^2}$.
d. Prove the identity

$$\delta(k^2 + m^2) = \frac{1}{2k_0} \left[\delta(k_0 - \omega) - \delta(k_0 + \omega) \right]. \tag{87}$$

d. Using this result, derive the following expression for the plane-wave decomposition of the vector field

$$A_\mu(x) = \int \frac{d^3k}{8\pi^2\omega} \left[e_\mu(\mathbf{k}, \omega) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + e_\mu^*(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right], \tag{88}$$

where the on-shell amplitude $e_\mu(\mathbf{k}, \omega)$ is restricted by

$$e_0 = \frac{\mathbf{k} \cdot \mathbf{e}}{\sqrt{\mathbf{k}^2 + m^2}}. \tag{89}$$

Hence for $m^2 > 0$ it follows that in the limit $|\mathbf{k}| \to 0$ also $e_0 \to 0$ and the plane waves have only three *spatial* components. In general the time component is not independent and the wave modes of the vector field are completely determined by the spatial components.

In the massless case $m^2 = 0$, which applies to the free Maxwell field, there are additional complications. In this case equation (81) reduces to

$$k^2 \varepsilon_\mu - k_\mu k \cdot \varepsilon = 0. \tag{90}$$

Problem 8.4
a. Show that this equation is invariant under gauge transformations

$$\varepsilon_\mu(k) \;\rightarrow\; \varepsilon'_\mu(k) = \varepsilon_\mu(k) + k_\mu \alpha(k). \tag{91}$$

What is the relation between $\alpha(k)$ and the gauge function $\Lambda(x)$ in eqn. (75)?
b. Explain that for $k^2 \neq 0$ the only solutions are

$$\varepsilon_\mu = k_\mu \beta, \quad \beta = \frac{k \cdot \varepsilon}{k^2}, \tag{92}$$

and therefore by taking $\alpha = -\beta$

$$\varepsilon'_\mu(k) = 0. \tag{93}$$

c. Conclude therefore that non-trivial solutions of the free Maxwell-type wave equations require

$$\varepsilon_\mu(k) = e_\mu(k)\,\delta(k^2), \tag{94}$$

where $k_0 = \pm\,\omega = \pm\sqrt{\mathbf{k}^2}$, and show that in the plane-wave expansion (88) the amplitudes satisfy

$$\omega e_0 = \mathbf{k}\cdot\mathbf{e}. \tag{95}$$

d. Prove that this condition is still invariant under residual gauge transformations

$$e'_0 = e_0 + \omega a \qquad \mathbf{e}' = \mathbf{e} + \mathbf{k}\,a, \tag{96}$$

for arbitrary $a(\mathbf{k},\omega)$.
d. Show that one can find a such that simultaneously

$$e'_0 = 0 \quad \text{and} \quad \mathbf{k}\cdot\mathbf{e}' = 0. \tag{97}$$

Therefore the plane waves in Maxwell's theory are described not just by amplitudes restricted to have no time component, but these amplitudes are also polarized transversely to the direction of propagation. This carries automatically over to the electric and magnetic field components $(\mathbf{E},\mathbf{B})$, as these do not depend on the choice of the gauge function $a(\mathbf{k},\omega)$. Thus massless Maxwell-type vector wave fields have only *two* polarization states, in contrast to massive Proca-type wave fields which have *three.*

Problem 8.5
a. Show that the electric and magnetic field components $\mathbf{E}(\mathbf{k})$, $\mathbf{B}(\mathbf{k})$ for fixed wave vector $\mathbf{k}$ are orthogonal to the wave vector and to each other.
b. Also show that the electric and magnetic field components have the same magnitude: $|\mathbf{E}(\mathbf{k})| = |\mathbf{B}(\mathbf{k})|$.

9. Dipole radiation

Having solved the free Maxwell's equations without sources, we now turn to the solution in the presence of a charge-current density j_μ. We consider the situation that the charges are localized in a finite volume of space and ask what electromagnetic waves they emit which can be observed by an observer far away at distances many times the typical size of the source region.

First recall the solution of the inhomogeneous Poisson equation:

$$\Delta\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta^3(\mathbf{r}-\mathbf{r}'). \tag{98}$$

Problem 9.1
Show that the inhomogeneous wave equation

$$-\Box f((\mathbf{r},t) = s(\mathbf{r},t) \tag{99}$$

has a formal solution

$$f(\mathbf{r},t) = \frac{1}{4\pi}\int d^3x' \, \frac{s(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{100}$$

This is called the *retarded solution*, as the integrand involves the source function s evaluated at an earlier time than the time at which the wave field f is evaluated; the difference in times is such that the distance between the source point $\mathbf{r}'$ and the wave point $\mathbf{r}$ is precisely covered by a signal travelling at the speed of light. Of course the *general solution* of the inhomogeneous wave equation (99) is a linear combination of the retarded solution (100) and a solution of the homogeneous wave equation. But the retarded solution is often the most relevant one as it implements the idea of *causality.* The general solution just reflects the possibility that an unrelated free wave is superimposed on the wave emitted by the source.

The retarded solution of the inhomogeneous Maxwell equation (77) computed in the Lorenz gauge $\partial \cdot A = 0$ reads:

$$A_\mu(\mathbf{r},t) = \frac{1}{4\pi}\int d^3x' \, \frac{j_\mu(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}, \tag{101}$$

Problem 9.2
a. Consider a large sphere S of radius $r \gg d$, where d is linear dimension of the source region. Use equation (43) to infer that the total energy $\mathcal{E}$ inside the sphere changes at a rate

$$\frac{d\mathcal{E}}{dt} = -\int_S d^3x' \nabla_i T^{0i} = -r^2 \oint d^2\Omega \, T^{0r}, \tag{102}$$

where $T^{0r} \equiv T^{0i}\hat{r}_i$ with $\hat{\mathbf{r}}$ the radial unit vector, and $d^2\Omega = \sin\theta \, d\theta d\varphi$ an area element on the unit sphere. What is the meaning of the minus sign?
b. Assume that the energy inside the spherical volume changes only by the flux of radiation crossing the surface. Explain why one can write the flux of energy per unit of time and per unit of area (with a little abuse of notation) as

$$\frac{d\mathcal{E}}{r^2 d^2\Omega \, dt} = -\left[(\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{r}}\right]_{\partial S}, \tag{103}$$

where the subscript ∂S denotes evaluation of the expression on the boundary surface of the sphere S.
c. Show that in this expression one may replace the vectors $\mathbf{E}$ and $\mathbf{B}$ on the

boundary ∂S by their tangential components

$$\frac{d\mathcal{E}}{r^2 d^2\Omega\, dt} = -\left[(\mathbf{E}_\perp \times \mathbf{B}_\perp)\cdot\hat{\mathbf{r}}\right]_{\partial S}, \qquad \mathbf{E}_\perp = \mathbf{E} - (\mathbf{E}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}, \quad \mathbf{B}_\perp = \mathbf{B} - (\mathbf{B}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}. \tag{104}$$

d. Explain why the only components of $(\mathbf{E}_\perp, \mathbf{B}_\perp)$ of interest are those for which the expression for the energy flux in the radial direction does not fall off faster than $\sim 1/r^2$.

Problem 9.3

a. Taking the origin of coordinates inside the source region and the radius of the sphere S very large as before, any point inside the source region satisfies $|\mathbf{r}'| \ll |\mathbf{r}|$. In these circumstances show that expression (101) can be expanded as

$$A_\mu(\mathbf{r},t) = \frac{1}{4\pi r}\int d^3x'\, j_\mu(\mathbf{r}', t-r) + \mathcal{O}(1/r^2), \qquad r = |\mathbf{r}|. \tag{105}$$

b. The leading term is a function of r and t only. Denote it by $a_\mu(r,t)$. Define the retarded dipole moment of the charge distribution by

$$\mathbf{D}(t-r) = \int d^3x'\, \mathbf{r}'\rho(\mathbf{r}', t-r). \tag{106}$$

Using charge conservation (33) show that for the spatial components of the vector potential

$$\mathbf{a}(r,t) = -\frac{1}{4\pi r}\dot{\mathbf{D}}(t-r). \tag{107}$$

c. The leading term of the time component A_0 is the retarded Coulomb potential:

$$a_0(r,t) = \frac{1}{4\pi r}\int d^3x'\, \rho(\mathbf{r}', t-r) = \frac{Q}{4\pi r}. \tag{108}$$

Infer that

$$\nabla a_0(r,t) = -\frac{Q}{4\pi r^2}\,\hat{\mathbf{r}} = \mathcal{O}(1/r^2). \tag{109}$$

d. Derive expressions for the leading contributions to the electric and magnetic fields on the spherical surface at distance r:

$$\mathbf{E} = \frac{1}{4\pi r}\ddot{\mathbf{D}}(t-r), \quad \mathbf{B} = \frac{1}{4\pi r}\,\hat{\mathbf{r}}\times\ddot{\mathbf{D}}(t-r). \tag{110}$$

e. By substitution in equation (103) or (104) derive the result

$$\frac{d\mathcal{E}(r,t)}{d^2\Omega dt} = -\frac{1}{16\pi^2}\left[\ddot{\mathbf{D}}^2 - (\hat{\mathbf{r}}\cdot\ddot{\mathbf{D}})^2\right]_{t-r}, \tag{111}$$

and interpret the result. Integrate the result to show that the total energy flux through the spherical surface is

$$\frac{d\mathcal{E}(r,t)}{dt} = -\frac{1}{6\pi}\,\ddot{\mathbf{D}}^2\Big|_{t-r}. \tag{112}$$

10. Free tensor fields

For a free, massless symmetric tensor field $h_{\mu\nu}$ an energy-momentum tensor can be found reading

$$
\begin{aligned}
T^{\mu\nu}[h] = \; & \frac{1}{2}\,\eta^{\mu\nu}\left[-(\partial_\kappa h_{\rho\sigma})^2 + 3(\partial_\sigma h_\rho^{\;\rho})^2 + 2\partial^\rho h^{\sigma\kappa}\partial_\sigma h_{\rho\kappa} - 4\partial_\rho h^{\rho\sigma}\partial_\sigma h_\kappa^{\;\kappa}\right] \\
& + \partial^\sigma h^{\mu\nu}\left(2\,\partial^\rho h_{\rho\sigma} - 3\,\partial_\sigma h_\rho^{\;\rho}\right) + \left(\partial^\mu h^{\nu\sigma} + \partial^\nu h^{\mu\sigma}\right)\partial_\sigma h_\rho^{\;\rho} \\
& -2\left(\partial^\rho h^{\sigma\nu}\partial^\mu h_{\rho\sigma} + \;\partial^\rho h^{\sigma\mu}\partial^\nu h_{\rho\sigma}\right) + 2\left(\partial_\sigma h^{\sigma\nu}\partial^\mu h_\rho^{\;\rho} + \partial_\sigma h^{\sigma\mu}\partial^\nu h_\rho^{\;\rho}\right) \\
& + \partial^\mu h^{\rho\sigma}\partial^\nu h_{\rho\sigma} - 2\,\partial^\mu h_\rho^{\;\rho}\partial^\nu h_\sigma^{\;\sigma} + 2\,\partial^\sigma h^{\rho\mu}\partial_\sigma h_\rho^{\;\nu} - 2\,\partial_\rho h^{\rho\mu}\,\partial_\sigma h^{\sigma\nu}.
\end{aligned}
\tag{113}
$$

Its divergence vanishes provided

$$
\Box h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\lambda\nu} - \partial_\nu\partial^\lambda h_{\lambda\mu} + \partial_\mu\partial_\nu h_\lambda^{\;\lambda} - \eta_{\mu\nu}\left(\Box h_\lambda^{\;\lambda} - \partial^\kappa\partial^\lambda h_{\kappa\lambda}\right) = 0. \tag{114}
$$

Problem 10.1
By taking the covariant trace of eqn. (114), show that it is equivalent to the following two simpler equations

$$
\Box h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\lambda\nu} - \partial_\nu\partial^\lambda h_{\lambda\mu} + \partial_\mu\partial_\nu h_\lambda^{\;\lambda} = 0, \qquad \Box h_\lambda^{\;\lambda} = \partial^\kappa\partial^\lambda h_{\kappa\lambda}. \tag{115}
$$

Use this to prove that $\partial_\mu T^{\mu\nu}[h] = 0$.

Obviously the energy-momentum tensor (113) is a complicated starting point to define the dynamics. It can be derived much simpler from an action principle.

Problem 10.2
Prove that the field equation (114) is equivalent to the condition for stationary points of the action

$$
S[h] = \int d^4x \left[-\frac{1}{2}\,\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu} + \partial^\mu h^{\nu\lambda}\partial_\nu h_{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h_\lambda^{\;\lambda} + \frac{1}{2}\,\partial^\lambda h_\mu^{\;\mu}\partial_\lambda h_\nu^{\;\nu}\right]. \tag{116}
$$

Problem 10.3
Prove that the field equation (114) and the action (116) are invariant under local gauge transformations

$$
h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{117}
$$

where the gauge parameters $\xi_\mu(x)$ define an arbitrary differentiable vector field.

The gauge invariance can be used to impose a gauge condition; a useful generalization of the Lorenz gauge was proposed by De Donder:

$$
\partial^\lambda h_{\lambda\mu} = \frac{1}{2}\,\partial_\mu h_\lambda^{\;\lambda}. \tag{118}
$$

Problem 10.4
a. Show that in the De Donder gauge the field equations (115) reduce to

$$\Box h_{\mu\nu} = 0, \qquad \partial^\mu \partial^\nu h_{\mu\nu} = 0. \tag{119}$$

b. Establish that these equations are still invariant under gauge transformations (117) with gauge parameters restricted by $\Box \xi_\mu = 0$.

Using the gauge invariance massless tensor fields can be made to satisfy the wave equation (118), and moreover the residual gauge transformations can be used to remove the non-transverse polarization states. As for massless vector fields this can be shown most easily in wave-vector space using the plane-wave decomposition

$$h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^2}\, \varepsilon_{\mu\nu}(k) e^{ik\cdot x}, \quad \varepsilon^*_{\mu\nu}(k) = \varepsilon_{\mu\nu}(-k). \tag{120}$$

Problem 10.5
a. Derive the field equations in this representation:

$$k^2 \varepsilon_{\mu\nu} - k_\mu k^\lambda \varepsilon_{\lambda\nu} - k_\nu k^\lambda \varepsilon_{\lambda\mu} + k_\mu k_\nu \varepsilon_\lambda^{\ \lambda} = 0, \quad k^2 \varepsilon_\lambda^{\ \lambda} = k^\mu k^\nu \varepsilon_{\mu\nu}. \tag{121}$$

b. Check that these equations are invariant under gauge transformations

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu \alpha_\nu + k_\nu \alpha_\mu. \tag{122}$$

c. Show that for $k^2 \neq 0$ the gauge transformations can be used to impose

$$k^\lambda \varepsilon'_{\lambda\mu} - \frac{1}{2}\, k_\mu \varepsilon'^{\ \lambda}_{\lambda} = 0, \tag{123}$$

and as a result

$$k^2 \neq 0 \quad \Rightarrow \quad \varepsilon'_{\mu\nu}(k) = 0. \tag{124}$$

d. Argue that physical amplitudes can be taken of the form

$$\varepsilon_{\mu\nu}(k) = e_{\mu\nu}(k) \delta(k^2), \tag{125}$$

and that the plane-wave expansion (120) can then be converted to

$$h_{\mu\nu}(x) = \int \frac{d^3k}{8\pi^2\omega} \left[e_{\mu\nu}(\mathbf{k}, \omega) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + e^*_{\mu\nu}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right], \tag{126}$$

where $\omega = \sqrt{\mathbf{k}^2}$ and the amplitudes are subject to the De Donder conditions on the light cone:

$$\omega e_{0\mu} + k_i e_{i\mu} = \frac{1}{2}\, k_\mu e_\lambda^{\ \lambda}. \tag{127}$$

Check that these conditions take the component form

$$e_{00} + e_{ii} = -\frac{2k_i}{\omega}\, e_{i0}, \quad k_i e_{ij} - k_j e_{ii} = -\omega e_{0j} + \frac{k_j k_i}{\omega}\, e_{i0}. \tag{128}$$

e. Writing

$$\alpha_\mu(k) = a_\mu(k)\delta(k^2),$$

we can still make gauge transformations restricted to the light cone:

$$e'_{00} = e_{00} - 2\omega a_0, \quad e'_{i0} = e_{i0} + k_i a_0 - \omega a_i, \quad e'_{ij} = e_{ij} + k_i a_j + k_j a_i. \tag{129}$$

Show that one can find (a_0, a_i) such that

$$e'_{00} = e'_{i0} = e'_{ii} = 0, \quad k_i e'_{ij} = 0, \tag{130}$$

and check that in co-ordinate space eqn. (126) then guarantees that

$$h_{00} = h_{i0} = h_{ii} = 0, \quad \nabla_i h_{ij} = 0. \tag{131}$$

Thus in this gauge tensor waves have two polarization states which are space-like, transverse and traceless; therefore it is also known as the TT-gauge.
f. Show that for a wave moving in the z-direction: $\mathbf{k} = (0, 0, k)$, the wave amplitude in the TT-gauge takes the generic form

$$e_{\mu\nu}(\omega, \mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+ & e_\times & 0 \\ 0 & e_\times & -e_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{132}$$

and similar results for the space time fields $h_{\mu\nu}(x)$.

11. Tensor fields interacting with external sources

The massless symmetric tensor field in interaction with external sources satisfies an inhomogeneous generalization of equation (114):

$$\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h_\lambda^{\ \lambda} - \eta_{\mu\nu}\left(\Box h_\lambda^{\ \lambda} - \partial^\kappa \partial^\lambda h_{\kappa\lambda}\right) = -t_{\mu\nu}, \tag{133}$$

where $t_{\mu\nu} = t_{\nu\mu}$ represents the material source of the field $h_{\mu\nu}$.
Problem 11.1
Show that consistency of eqn. (133) required the divergence of $t_{\mu\nu}$ to vanish:

$$\partial_\mu t^{\mu\nu} = 0. \tag{134}$$

For all physical systems such a symmetric, divergence-free tensor can be found: the energy-momentum tensor $T_{\mu\nu}[X]$, where X stands for the physical system X.

Therefore the symmetric tensor field can be coupled to any system X by taking $t_{\mu\nu}$ proportional to $T_{\mu\nu}[X]$:

$$\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h_\lambda^{\ \lambda} - \eta_{\mu\nu} \left(\Box h_\lambda^{\ \lambda} - \partial^\kappa \partial^\lambda h_{\kappa\lambda} \right) = -\kappa T_{\mu\nu}[X]. \quad (135)$$

Problem 11.2
Show that the field $h_{\mu\nu}$ and the proportionality constant κ have dimensions

$$[h] = [\kappa^{-1}] = \mathrm{kg}^{1/2}\ \mathrm{m}^{1/2}\ \mathrm{s}^{-1}. \quad (136)$$

In terms of Newton's constant of gravity this is the dimension of $\sqrt{c^4/G}$. It follows that the combination $a_{\mu\nu} \equiv \kappa h_{\mu\nu}$ is dimensionless, and this dimensionless field satisfies an inhomogeneous equation

$$\Box a_{\mu\nu} - \partial_\mu \partial^\lambda a_{\lambda\nu} - \partial_\nu \partial^\lambda a_{\lambda\mu} + \partial_\mu \partial_\nu a_\lambda^{\ \lambda} - \eta_{\mu\nu} \left(\Box a_\lambda^{\ \lambda} - \partial^\kappa \partial^\lambda a_{\kappa\lambda} \right) = -\frac{\alpha G}{c^4}\, T_{\mu\nu}[X], \quad (137)$$

with α a *dimensionless* proportionality constant, which can in principle depend on the type of source X: particles, fluids, fields, etc.

Now there is one interaction in nature that couples to all forms of energy and momentum: gravity. Moreover for gravity the coupling parameter is a *universal* constant, i.e. α has a universal value, the same for all types of sources of energy and momentum[1]. This is reasonable (though not obvious) if one realizes that different forms of energy (mass, motion, heat, radiation) can be converted into each other by a variety of physical processes. As a result the tensor field $h_{\mu\nu}$ can be used to model (weak) gravitational fields in Minkowski space-time, in which case it is customary to take the coupling parameter to have the universal value $\alpha = 8\pi$, and identify

$$\kappa^2 = \frac{8\pi G}{c^4}, \quad (138)$$

where in the following we will again use units in which $c = 1$.

Problem 11.3
Observe that the inhomogeneous equation (135) is still invariant under the gauge transformations (117). Impose the De Donder condition and show it reduces the field equation to an inhomogeneous wave equation:

$$\partial^\lambda h_{\lambda\mu} = \frac{1}{2}\, \partial_\mu h_\lambda^{\ \lambda} \quad \Rightarrow \quad \Box \left(h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\, h_\lambda^{\ \lambda} \right) = -\kappa T_{\mu\nu}[X]. \quad (139)$$

[1] In principle the tensor field then also couples to itself via its own energy-momentum tensor. This makes the tensor theory of gravity in Minkowski space-time highly non-linear. Carrying through a complete analysis ultimately leads to general relativity (GR). The gravitational tensor field we discuss here is therefore a minkowskian approximation of GR valid in a regime in which self-interaction of the gravitational field can be neglected.

Problem 11.4
Consider a massive point particle at rest in the origin of co-ordinates.
a. Using the expression (44) show that for this simple system

$$T_{00} = m\,\delta^3(\mathbf{x}), \quad T_{0i} = T_{ij} = 0. \tag{140}$$

b. Derive the equation for the corresponding static gravitational field as approximated by $h_{\mu\nu}$:

$$\Delta h_{00} = -\frac{\kappa m}{2}\,\delta^3(\mathbf{x}), \quad h_{i0} = 0, \quad h_{ij} = \delta_{ij} h_{00}. \tag{141}$$

c. Derive the solution for the h_{00} component

$$h_{00} = \frac{\kappa m}{8\pi r} \quad \Rightarrow \quad \kappa h_{00} = \frac{Gm}{r}, \tag{142}$$

and conclude that κh_{00} represents the gravitational Newton potential.

12. Quadrupole radiation

After imposing the De Donder condition the action (116) reduces after some partial inegrations to

$$S[h] = \int \left[-\frac{1}{2}\,\partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} + \frac{1}{4}\,\partial^\lambda h_\mu^{\ \mu} \partial_\lambda h_\nu^{\ \nu} \right]. \tag{143}$$

To get the correct field equation for a tensor field interacting with a source X an additional interaction term is needed:

$$S_{int}[h] = \kappa \int d^4x\, h_{\mu\nu} T^{\mu\nu}[X]. \tag{144}$$

Then the full action $S + S_{int}$ is stationary if the field equation (139) holds.

Problem 12.1
a. Show that from this action one can by Legendre transformation obtain a hamiltonian

$$H[h,\pi] = \int d^3x \left[\frac{1}{2}\,\pi_{\mu\nu}^2 - \frac{1}{4}\,\pi^{\lambda\ 2}_{\ \lambda} + \frac{1}{2}\,(\nabla h_{\mu\nu})^2 - \frac{1}{4}\,(\nabla h^\lambda_{\ \lambda})^2 - \kappa h_{\mu\nu} T^{\mu\nu}[X] \right]. \tag{145}$$

b. Evaluate the Hamilton equations

$$\dot{h}_{\mu\nu} = \frac{\delta H}{\delta \pi^{\mu\nu}}, \quad \dot{\pi}_{\mu\nu} = -\frac{\delta H}{\delta h^{\mu\nu}}, \tag{146}$$

and show that they again reproduce the field equation (139).

Problem 12.2
a. The energy of tensor waves inside a spatial volume V is $\mathcal{E} = H|_V$, obtained by restricting the integral (145) to the volume V. By a direct calculation show that

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \int_V d^3x \left[\nabla \cdot (\pi^{\mu\nu} \nabla h_{\mu\nu}) - \kappa h^{\mu\nu} \dot{T}_{\mu\nu}[X]\right] \\ &= \oint_{\partial V} d^2\sigma \, \pi^{\mu\nu} \nabla_n h_{\mu\nu} - \kappa \int_V d^3x \, h^{\mu\nu} \, \dot{T}_{\mu\nu}[X].\end{aligned} \tag{147}$$

The first surface term is the energy carried away by the flux of tensor waves crossing the boundary surface ∂V. This shows that we can interpret the hamiltonian density and the flux density as components of the effective energy-momentum tensor for tensor fields subject to the De Donder condition in Minkowski space-time:

$$\begin{aligned}T^{00}[h] &= \frac{1}{2}\, \pi_{\mu\nu}^2 - \frac{1}{4}\, \pi^{\lambda\,2}_{\ \lambda} + \frac{1}{2}\, (\nabla h_{\mu\nu})^2 - \frac{1}{4}\, (\nabla h^{\lambda}_{\ \lambda})^2, \\ T^{i0}[h] &= -\pi^{\mu\nu} \nabla^i h_{\mu\nu},\end{aligned} \tag{148}$$

with the property that free fields (in regions where there are no sources) satisfy

$$\partial_\mu T^{\mu 0}[h] = 0. \tag{149}$$

The volume term in eqn. (147) represents the change in energy of interaction between the sources X and the tensor field. By total energy conservation this should be compensated by other non-gravitational contributions to the internal energy of the sources which have not been included in $H[h, \pi]$.
b. Make assumptions similar to the case of electro-magnetic radiation: the region where the energy-momentum tensor of the source is non-zero is located in a finite region inside the volume V, which we can again take to be a large sphere S of radius r; and the only contribution to the *total* energy-momentum tensor on the boundary surface of the sphere comes from the tensor waves: the flux of gravitational radiation. Then show that

$$\frac{d\mathcal{E}}{r^2 d^2\Omega \, dt} = \left[(\pi^{\mu\nu} \nabla h_{\mu\nu}) \cdot \hat{\mathbf{r}}\right]_{\partial S} = \left[\frac{\partial h^{\mu\nu}}{\partial t} \frac{\partial h_{\mu\nu}}{\partial r} - \frac{1}{2} \frac{\partial h^{\lambda}_{\ \lambda}}{\partial t} \frac{\partial h^{\mu}_{\ \mu}}{\partial r}\right]_{\partial S}. \tag{150}$$

We therefore need to compute the field components $h_{\mu\nu}$ contributing to this flux.

Problem 12.3
a. In view of equation (139) it is convenient to define

$$\underline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} \, h_{\lambda}^{\ \lambda}. \tag{151}$$

Show that

$$\frac{d\mathcal{E}}{r^2 d^2\Omega\, dt} = \left[\frac{\partial \underline{h}^{\mu\nu}}{\partial t} \frac{\partial \underline{h}_{\mu\nu}}{\partial r} - \frac{1}{2} \frac{\partial \underline{h}^{\lambda}_{\;\lambda}}{\partial t} \frac{\partial \underline{h}^{\mu}_{\;\mu}}{\partial r} \right]_{\partial S}. \tag{152}$$

b. From equations (100) and (139) derive the expression for the fields generated by the sources:

$$\underline{h}_{\mu\nu}(\mathbf{r}, t) = \frac{\kappa}{4\pi} \int_S d^3x' \, \frac{T_{\mu\nu}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}, \tag{153}$$

where for notational convenience explicit reference to the type of source X has been omitted.

c. Following the argument leading to equation (105) we can simplify this expression by retaining only terms that fall off not faster than $1/r$:

$$\underline{h}_{\mu\nu} = \frac{\kappa}{4\pi r} \int_S d^3x' \, T_{\mu\nu}(\mathbf{r}', t - r) + \mathcal{O}(1/r^2). \tag{154}$$

Show that if the energy-momentum tensor of the sources vanishes on the boundary surface ∂S, then to leading order in $1/r$

$$\frac{\partial \underline{h}_{0\mu}}{\partial t} = 0. \tag{155}$$

Argue that the time-components $\underline{h}_{0\mu}$ do not contribute to the energy flux of gravitational waves across the boundary: they represent the newtonian potential of the source as in equation (142).

d. In the absence of source terms on or near the boundary the space-components of the tensor wave fields contributing to the energy flux there must behave like free waves: transverse and traceless. Show that this implies that the relevant components are

$$\underline{h}_{ij} = \frac{\kappa}{4\pi r} \, (\delta_{ik} - \hat{r}_i \hat{r}_k)(\delta_{jl} - \hat{r}_j \hat{r}_l) \left(I_{kl} + \frac{1}{2} \, \delta_{kl} \, \hat{r} \cdot I \cdot \hat{r} \right), \tag{156}$$

where $\hat{r}$ is the outward radial unit vector and

$$I_{ij}(t - r) = \int_S d^3x' \left(T_{ij} - \frac{1}{3} \, \delta_{ij} T_{kk} \right) (\mathbf{r}', t - r). \tag{157}$$

e. From the equation of continuity (42) for $T_{\mu\nu}$ show that

$$\int_S d^3x' T_{ij}(\mathbf{r}', t - r) = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int_S d^3x' \, r'_i r'_j \, T_{00}(\mathbf{r}', t - r). \tag{158}$$

f. For non-relativistic sources T_{00} is dominated by the mass density ρ over the kinetic and potential energy density; then I_{ij} can be replaced by the second derivative of the retarded mass quadrupole moment:

$$I_{ij} = \frac{1}{2} \frac{\partial^2 Q_{ij}}{\partial t^2}, \qquad Q_{ij}(t - r) = \int_S d^3x' \left(r'_i r'_j - \frac{1}{3} \, \delta_{ij} \, \mathbf{r}'^{\,2} \right) \rho(\mathbf{r}', t - r). \tag{159}$$

Check that

$$\underline{h}_{ij} = \frac{\kappa}{8\pi r} \left(\delta_{ik} - \hat{r}_i \hat{r}_k\right) \left(\delta_{jl} - \hat{r}_j \hat{r}_l\right) \frac{\partial^2}{\partial t^2} \left(Q_{kl} + \frac{1}{2}\, \delta_{kl}\, \hat{r} \cdot Q \cdot \hat{r} \right), \tag{160}$$

and explain that

$$\frac{\partial \underline{h}_{ij}}{\partial r} = - \frac{\partial \underline{h}_{ij}}{\partial t} + \mathcal{O}(1/r^2). \tag{161}$$

g. Prove that for large r and after restoring powers of c:

$$\frac{d\mathcal{E}}{d^2\Omega\, dt} = - \frac{G}{8\pi c^5} \left[\mathrm{Tr}\, \dddot{Q}^2 - 2\hat{r} \cdot \dddot{Q}^{\,2} \cdot \hat{r} + \frac{1}{2} (\hat{r} \cdot \dddot{Q} \cdot \hat{r})^2 \right]. \tag{162}$$

and by integrating over all directions

$$\frac{d\mathcal{E}}{dt} = - \frac{G}{5c^5}\, \dddot{Q}_{ij}^{\,2}. \tag{163}$$

13. Newtonian binaries

A non-relativistic type of source of considerable interest is the newtonian binary: two massive bodies (stars) in bound orbit with velocities much less than that of light. The masses nd positions of the two bodies are $(m_1, \mathbf{r}_1)$ and $(m_2, \mathbf{r}_2)$, respectively. Take the center of mass to be the origin of our co-ordinate system:

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0. \tag{164}$$

The relative distance vector is

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1. \tag{165}$$

Problem 13.1
With $M = m_1 + m_2$, show that

$$\mathbf{r}_1 = - \frac{m_2}{M}\, \mathbf{R}, \quad \mathbf{r}_2 = \frac{m_1}{M}\, \mathbf{R} \tag{166}$$

and that Newton's law of gravity implies

$$\ddot{\mathbf{R}} = - \frac{GM}{R^2}\, \hat{\mathbf{R}}, \tag{167}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$ is th unit vector in the direction of $\mathbf{R}$.

The solutions of equation (167) are the well-known Kepler orbits. The bound states are closed planar orbits: ellipses, in the simplest case circles.

Problem 13.2
a. Check that angular momentum is conserved:

$$\mathbf{L} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 = \mu \mathbf{R} \times \dot{\mathbf{R}} = \text{constant}, \tag{168}$$

where the reduced mass is

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}.$$

Explain why this implies that the motion is planar.
b. Take a circular orbit in the x-y-plane:

$$\mathbf{R} = R(\cos\omega t, \sin\omega t, 0). \tag{169}$$

Show that the angular velocity is given by

$$\omega^2 = \frac{GM}{R^3}. \tag{170}$$

c. Compute the quadrupole moment:

$$\begin{aligned} Q_{ij} &= m_1 \left(r_{1i} r_{1j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_1^2 \right) + m_2 \left(r_{2i} r_{2j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_2^2 \right) \\ &= \frac{\mu R^2}{2} \left(\begin{array}{ccc} \cos 2\omega t + \frac{1}{3} & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{array} \right). \end{aligned} \tag{171}$$

d. Derive the power emitted by the binary system as a function of direction:

$$\frac{d\mathcal{E}}{d^2\Omega dt} = -\frac{4G^4 m_1^2 m_2^2 M}{\pi c^5 R^5} \left(\cos^2\theta + \frac{1}{4}\,\sin^4\theta \sin^2 2(\varphi - \omega t) \right), \tag{172}$$

and after integrating over all angles:

$$\frac{d\mathcal{E}}{dt} = -\frac{32 G^4}{5c^5} \frac{m_1^2 m_2^2 M}{R^5}. \tag{173}$$

14. Dynamics in a gravitational field

We wish to derive the motion of a system of point masses in a gravitational field. Using expression (144) for the interaction of a point mass with the tensor field $h_{\mu\nu}$ and the expression (44) for the energy-momentum tensor, the action functional for an interacting point mass with world-line $\xi^\mu(\tau)$ takes the form

$$S_{part}[\xi] = \frac{m}{2} \int d\tau \left(\eta_{\mu\nu} + 2\kappa h_{\mu\nu}(\xi) \right) \dot{\xi}^\mu \dot{\xi}^\nu \tag{174}$$

We can combine the two terms by defining

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(x) \quad \Rightarrow \quad S_{part}[\xi] = \frac{m}{2} \int d\tau\, g_{\mu\nu}(\xi) \dot{\xi}^\mu \dot{\xi}^\nu, \tag{175}$$

and interpret this combination as the metric of a curved space-time in which the particle moves. The actual world-line of the particle then is given by the space-time curve for which this action is stationary under small deviations $\delta\xi^\mu(\tau)$.

Problem 14.1
a. Using the results of problem 11.4, show that to linearized order the metric of a stationary point mass can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2Gm}{r} \right) dt^2 + \left(1 + \frac{2Gm}{r} \right) d\mathbf{r}^2, \tag{176}$$

where t and $\mathbf{r} = (x, y, z)$ are cartesian co-ordinates in the asymptotic Minkowski space-time.
b. By using expression (176) in the action (174), show that for a particle moving in the field of a stationary point mass *to first order* in G

$$\frac{dt}{d\tau} = \varepsilon \left(1 + \frac{2Gm}{r} \right) + ..., \quad \frac{1}{2} \left(\frac{d\mathbf{r}}{d\tau} \right)^2 - \frac{Gm}{r} + ... = \frac{1}{2} \left(\varepsilon^2 - 1 \right), \tag{177}$$

where ε is a constant of motion and the ellipses denote terms of order G^2.

Problem 14.2
a. Show that the world-line is a solution of the geodesic equation

$$\ddot{\xi}^\mu + \Gamma_{\lambda\nu}^{\;\;\mu}(\xi)\, \dot{\xi}^\lambda \dot{\xi}^\nu = 0, \tag{178}$$

where Γ is the Riemann-Christoffel connection for the metric $g_{\mu\nu}$:

$$\Gamma_{\lambda\nu}^{\;\;\mu} = \frac{1}{2}\, g^{\mu\kappa} \left(\partial_\lambda g_{\kappa\nu} + \partial_\nu g_{\kappa\lambda} - \partial_\kappa g_{\lambda\nu} \right), \tag{179}$$

with $g^{\mu\nu}$ the inverse of the metric:

$$g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + 2\kappa^2 h^{\mu\lambda} h_\lambda^{\;\;\nu} + ... \tag{180}$$

c. Show that to first order in κ:

$$\Gamma_{\lambda\nu}^{\;\;\mu} = \kappa \left(\partial_\lambda h_\nu^{\;\;\mu} + \partial_\nu h_\lambda^{\;\;\mu} - \partial^\mu h_{\lambda\nu} \right) + \mathcal{O}(\kappa^2). \tag{181}$$

Problem 14.3
Consider Maxwell theory in the presence of a gravitational field; by a similar procedure the starting point is the action

$$S_{em}[A] \;\; = \;\; -\frac{1}{4} \int d^4x\, F_{\mu\nu} F^{\mu\nu} + \kappa \int d^4x\, h_{\mu\nu} T_{em}^{\mu\nu} \tag{182}$$

a. From this expression, show that in the absence of currents and charges the action can be written as:

$$\begin{aligned} S_{em}[A] &= \int d^4x \left[-\frac{1}{4} \left(1 + \kappa h_\lambda^{\ \lambda}\right) F_{\mu\nu} F^{\mu\nu} + \kappa h_{\mu\nu} F^{\mu\lambda} F^{\nu}_{\ \lambda} \right] \\ &\simeq -\frac{1}{4} \int d^4x \left(1 + \kappa h_\lambda^{\ \lambda}\right) \left(\eta^{\mu\kappa} - 2\kappa h^{\mu\kappa}\right) \left(\eta^{\nu\lambda} - 2\kappa h^{\nu\lambda}\right) F_{\mu\nu} F_{\kappa\lambda} \\ &\simeq -\frac{1}{4} \int d^4x \sqrt{-\det g}\, g^{\mu\kappa} g^{\nu\lambda} F_{\mu\nu} F_{\kappa\lambda}, \end{aligned} \tag{183}$$

where the $\simeq$ sign signifies equality up to terms of order $\mathcal{O}(\kappa^2)$.
b. Derive Maxwell's equations in the presence of a gravitational field:

$$g^{\mu\lambda} \left(\partial_\lambda F_{\mu\nu} - \Gamma_{\lambda\mu}^{\ \ \kappa} F_{\kappa\nu} - \Gamma_{\lambda\nu}^{\ \ \kappa} F_{\mu\kappa} \right) = 0. \tag{184}$$

15. Scalar fields

Real free scalar fields $\varphi(x)$ satisfy the Klein-Gordon equation

$$\left(-\Box + m^2\right) \varphi = 0. \tag{185}$$

Problem 15.1
a. Derive this equation by considering the divergence of the energy-momentum tensor

$$T_{f\,\mu\nu}[\varphi] = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\, \eta_{\mu\nu} \left[(\partial\varphi)^2 + m^2 \varphi^2 \right]. \tag{186}$$

b. Derive it as well from the action

$$S_f[\varphi] = -\frac{1}{2} \int d^4x \left(\partial^\mu \varphi \partial_\mu \varphi + m^2 \varphi^2 \right). \tag{187}$$

c. Show that the coupling of the scalar field to gravity is described by the action

$$\begin{aligned} S[\varphi] &= -\frac{1}{2} \int d^4x \left[\partial^\mu \varphi \partial_\mu \varphi + m^2 \varphi^2 \right] + \kappa \int d^4x\, h^{\mu\nu} T_{\mu\nu}[\varphi] \\ &= \int d^4x \left[-\frac{1}{2} \left(1 + \kappa h_\lambda^{\ \lambda}\right) \left((\partial\varphi)^2 + m^2 \varphi^2 \right) - \kappa h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right] \\ &\simeq -\frac{1}{2} \int d^4x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2 \right). \end{aligned} \tag{188}$$

The plane-wave decomposition

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^2}\, \alpha(k) e^{ik\cdot x} \tag{189}$$

is a solution of the Klein-Gordon equation if the wave vectors are constrained by

$$k^2 + m^2 = 0 \quad \Leftrightarrow \quad k_0^2 = \mathbf{k}^2 + m^2, \tag{190}$$

and therefore we must have

$$\alpha(k) = a(k)\, \delta(k^2 + m^2) \quad \text{such that} \quad k_0 = \pm\omega = \pm\sqrt{\mathbf{k}^2 + m^2}. \tag{191}$$

Problem 15.2
Following the procedure of problem 8.3, derive the plane-wave decomposition of the free scalar field in the form

$$\varphi(x) = \int \frac{d^3k}{8\pi^2\omega} \left[a(\mathbf{k},\omega) e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + a^*(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \right]. \tag{192}$$

The coupling of scalar fields to electro-magnetic fields via the vector potential A_μ is possible only for complex scalar fields, which can be decomposed into two real scalar fields with equal mass:

$$\phi(x) = \varphi_1(x) + i\varphi_2(x), \quad \phi^*(x) = \varphi_1(x) - i\varphi_2(x),$$

such that the free fields (in vacuum) obey the Klein-Gordon equation

$$\left(-\Box + m^2\right) \phi = 0, \quad \left(-\Box + m^2\right) \phi^* = 0. \tag{193}$$

The reason is that it gives rise to a divergence-free four-vector current

$$j_\mu^{(0)} = -iq\phi^* \partial_\mu \phi + iq\phi\, \partial_\mu \phi^* \equiv -iq\phi^* \overleftrightarrow{\partial_\mu}\, \phi, \tag{194}$$

where q is a free parameter.

Problem 15.3
Using the free Klein-Gordon equations (192) prove that

$$\partial^\mu j_\mu^{(0)} = 0. \tag{195}$$

This gives an *Ansatz* for a scalar-field source term in the inhomogeneous Maxwell equations (74)

$$\partial^\mu F_{\mu\nu} = -j_\nu^{(0)}. \tag{196}$$

However this is not the full story, as this equation contradicts the conservation of energy and momentum. Indeed, the divergence of the electro-magnetic energy-momentum tensor (48) vanishes only if $\partial^\mu F_{\mu\nu} = 0$. We can try to mend this by adding terms $\Delta T^{\mu\nu}[A, \varphi]$ involving the vector and scalar fields to $T_{em}^{\mu\nu}$. But then conservation of the full energy-momentum tensor

$$T^{\mu\nu} = T_{em}^{\mu\nu} + T_f^{\mu\nu} + \Delta T_{\mu\nu} : \quad \partial_\mu T^{\mu\nu} = 0 \tag{197}$$

requires the field equation for the scalar field to change as well. The required changes are the following:
a. the free Klein-Gordon equation is modified to become

$$\begin{aligned}\left[-\left(\partial_\mu - iqA_\mu\right)\left(\partial^\mu - iqA^\mu\right) + m^2\right]\phi = 0,\\ \left[-\left(\partial_\mu + iqA_\mu\right)\left(\partial^\mu + iqA^\mu\right) + m^2\right]\phi^* = 0;\end{aligned} \tag{198}$$

b. the current is modified to read

$$\begin{aligned} j_\mu &= -iq\phi^*\left(\partial_\mu - iqA_\mu\right)\phi + iq\phi\left(\partial_\mu + iqA_\mu\right)\phi^* \\ &= -iq\phi^* \overleftrightarrow{\partial}_\mu \phi - 2q^2 A_\mu \left|\phi\right|^2; \end{aligned} \tag{199}$$

c. the energy-momentum tensor is modified to

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{em} + \left(\partial^\mu + iqA^\mu\right)\phi^*\left(\partial^\nu - iqA^\nu\right)\phi + \left(\partial^\nu + iqA^\nu\right)\phi^*\left(\partial^\mu - iqA^\mu\right)\phi \\ &\quad - \eta^{\mu\nu}\left[\left(\partial + iqA\right)\phi^*\cdot\left(\partial - iqA\right)\phi + m^2\left|\phi\right|^2\right]. \end{aligned} \tag{200}$$

Observe that all changes amount to the replacement

$$\partial_\mu\phi \to D_\mu\phi = \left(\partial_\mu - iqA_\mu\right)\phi, \quad \partial_\mu\phi^* \to \left(D_\mu\phi\right)^* = \left(\partial_\mu + iqA_\mu\right)\phi^*, \tag{201}$$

in the corresponding expressions for the free scalar fields.

Problem 15.4
a. Using the modified Klein-Gordon equations (198) prove that

$$\partial^\mu j_\mu = 0. \tag{202}$$

b. Derive the *Ricci identities*

$$\left(D_\mu D_\nu - D_\nu D_\mu\right)\phi = -iqF_{\mu\nu}\,\phi, \quad \left(D_\mu D_\nu - D_\nu D_\mu\right)\phi^* = iqF_{\mu\nu}\,\phi^*. \tag{203}$$

c. Using the modified Klein-Gordon equations and the Ricci identities prove that

$$\partial_\mu T^{\mu\nu} = \left(\partial_\mu F^{\mu\lambda} + j^\lambda\right) F^\nu_{\ \lambda}, \tag{204}$$

and therefore the total energy-momentum is conserved iff the *inhomogeneous* Maxwell equations hold:

$$\partial_\mu F^{\mu\lambda} = -j^\lambda.$$

16. Gauge invariance

The modification of the Klein-Gordon equation allowing consistent coupling of a complex scalar field to the electro-magnetic vector field A_μ consists of the replacement (201) of ordinary partial derivatives by the combination $D = \partial - iqA$. Now the vector field is only defined up to a gauge transformation (75); this is important, as gauge invariance was shown to be crucial to guarantee that electromagnetic waves have only two transversal polarization states, and no third longitudinal one like massive vector fields.

The question then arises how this gauge invariance is preserved in the interaction with the complex scalar field. The solution of this problem is to transform the complex scalar field by changing its phase.

Problem 16.1
Consider a *local* change of phase of the complex field ϕ:

$$\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x). \tag{205}$$

a. Check that

$$D_\mu \phi' = e^{i\alpha} \left(\partial_\mu + i\partial_\mu \alpha - iqA_\mu \right) \phi. \tag{206}$$

b. Show that the term with $\partial_\mu \alpha$ can be compensated by a gauge transformation of the vector field A_μ with gauge parameter $\Lambda = \alpha/q$, and therefore under simultaneous transformation $\phi \to \phi'$ and $A_\mu \to A'_\mu$ one gets

$$D\phi \to (D\phi)' = e^{i\alpha} D\phi. \tag{207}$$

Therefore the modified derivative $D\phi$ transforms under this combined transformation in the same way as the field ϕ itself by a phase factor. For this reason the operator D_μ is called a *covariant derivative*.
c. Prove that the modified Klein-Gordon equation (188) transforms as

$$\left[\left(-D^2 + m^2 \right) \phi \right]' = e^{i\alpha} \left[\left(-D^2 + m^2 \right) \phi \right] = 0. \tag{208}$$

Explicitly:

$$\left[- \left(\partial_\mu - iqA'_\mu \right) \left(\partial^\mu - iqA'^\mu \right) + m^2 \right) \phi' = 0,$$

and similarly for the complex conjugate equation.
d. Show that the scalar-field current j_μ defined by (199) is invariant under the combined gauge transformations:

$$j'_\mu = j_\mu, \tag{209}$$

and that also the inhomogeneous Maxwell equations stay the same in terms of fields A'_μ and ϕ'.
e. Argue that also the energy-momentum tensor (200) is *invariant* under these field redefinitions.

The upshot of this analysis is that
- given a solution (A_μ, ϕ, ϕ^*) of the combined Maxwell-Klein-Gordon equations we can construct a 1-parameter family of solutions by combined gauge and phase transformations

$$A'_\mu = A_\mu + \partial_\mu \Lambda, \quad \phi' = e^{iq\Lambda}\phi, \quad \phi^{*\prime} = e^{-iq\Lambda}\phi^*. \tag{210}$$

- all these solutions are physically equivalent: they possess identically the same energy and momentum as the original solutions.

17. The anharmonic charged scalar field

It is possible to make further modifications of the Klein-Gordon equation for a charged scalar field by adding non-linear terms. The simplest non-linear extension is to take

$$-D^2\phi + \kappa\phi + \lambda\left|\phi\right|^2\phi = 0, \quad -D^2\phi^* + \kappa\phi^* + \lambda\left|\phi\right|^2\phi^* = 0, \tag{211}$$

where κ and λ are real parameters. These non-linear equations define the anharmonic scalar field model. Observe, that we have replaced the mass parameter m^2 by the coefficient κ, as these quantities do not necessarily describe the same observable. This will become clear shortly.

Problem 17.1
a. Check that the anharmonic equations (211) still respect the local phase and gauge transformations (210).
b. Show that these equations are derive from a gauge-invariant energy-momentum tensor

$$T^{\mu\nu}_{anh} = T^{\mu\nu}_{em} + D^\mu\phi^* D^\nu\phi + D^\nu\phi^* D^\mu\phi - \eta^{\mu\nu}\left[D\phi^*\cdot D\phi + \kappa\left|\phi\right|^2 + \frac{\lambda}{2}\left|\phi\right|^4\right]. \tag{212}$$

c. From this obtain the energy density:

$$T^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \left|D_t\phi\right|^2 + \left|\mathbf{D}\phi\right|^2 + \kappa\left|\phi\right|^2 + \frac{\lambda}{2}\left|\phi\right|^4. \tag{213}$$

Now consider the energy of fields in empty or partially empty space-time. In the absence of electromagnetic fields: $\mathbf{E} = \mathbf{B} = 0$ the scalr field equation is

$$-\Box\phi + \kappa\phi + \lambda\left|\phi\right|^2\phi = 0, \tag{214}$$

and the scalar field contribution to the energy density is

$$\mathcal{E}_0(\phi) = \left|\dot{\phi}\right|^2 + \left|\nabla\phi\right|^2 + \kappa\left|\phi\right|^2 + \frac{\lambda}{2}\left|\phi\right|^4. \tag{215}$$

Now we first consider the case $\kappa > 0$ and $\lambda > 0$. Then for small field amplitudes:

$$|\phi|^2 \ll \frac{\kappa}{\lambda} \tag{216}$$

the non-linear terms can be neglected and we have a free wave equation with plane-wave solutions similar to (192) such that

$$\omega^2 = \mathbf{k}^2 + \kappa \geq \kappa.$$

Thus κ represents the square of the threshold frequency: $\kappa = m^2$, the frequency in the limit in which the wave length becomes infinite. There are no propagating fields with frequency below this threshold. In fact for fields constant in space and time the only solution of the wave equation (214) is $\phi = 0$.

Next we consider the case $\kappa < 0$ and $\lambda > 0$. Writing $\kappa = -\mu^2$ the dispersion relation for plane waves becomes

$$\mathbf{k}^2 = \omega^2 + \mu^2 \geq \mu^2. \tag{217}$$

This suggest there can be non-oscillating small-amplitude plane waves: $\omega = 0$, but still finite wave length. Such instantaneous waves would imply action at a distance and violate causality. It turns out that the crux of this problem is the assumption of small amplitude fields.

Problem 17.2
a. Consider the energy density of *constant* field configurations: $\dot{\phi} = \nabla\phi = 0$:

$$\mathcal{E}_0 = -\mu^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 . \tag{218}$$

a. Draw the energy density as a function of amplitude $|\phi|$.
b. Show that the absolute minimum is reached for

$$|\phi|^2 = \frac{\mu^2}{\lambda}. \tag{219}$$

Hence the lowest-energy configuration has non-zero amplitude but zero frequency and wave-number. This suggests that the true definition of a small-amplitude field (one which has arbitrarily small energy above the minimum) is

$$\phi(x) = \left(\frac{\mu}{\sqrt{\lambda}} + \sigma(x) \right) e^{i\alpha(x)}, \tag{220}$$

where $\sigma(x)$ and $\alpha(x)$ are real fields representing the amplitude and phase of the scalar field fluctuations.

Problem 17.3
a. Explain how the phase α can be made to vanish: $\alpha = 0$.
b. Show that the scalar field equation (211) in terms of the fluctuations σ reads

$$-D^2\sigma + 2\mu^2\sigma + 3\mu\sqrt{\lambda}\sigma^2 + \lambda\sigma^3 = 0. \tag{221}$$

c. Give a criterion for small-amplitude fluctuations and show that in the absence of electromagnetic fields the satisfy in good approximation the wave equation

$$-\Box\sigma + 2\mu^2\sigma = 0, \tag{222}$$

which is the Klein-Gordon equation with $m_\sigma^2 = 2\mu^2$.
d. Evaluate the electromagnetic current (199) and derive the inhomogeneous form of the Maxwell equation:

$$\partial_\mu F^{\mu\nu} = 2q^2 \left(\frac{\mu^2}{\lambda} + \frac{2\mu}{\sqrt{\lambda}}\,\sigma + \sigma^2 \right) A^\nu. \tag{223}$$

e. Compare with eq. (78) to conclude that in the absence of fluctuating scalar fields ($\sigma = 0$) the vector potential A_μ now satisfies the Proca equation with

$$m_A^2 = \frac{2q^2\mu^2}{\lambda} = \frac{q^2 m_\sigma^2}{\lambda}. \tag{224}$$

f. What is the quantummechanical interpretation of the threshold frequency?
g. Discuss how it is possible for the vector field to have acquired an extra polarization state.

18. Magnetic monopoles

The homogeneous Maxwell equation

$$\nabla \cdot \mathbf{B} = 0$$

is usually interpreted to imply that there exist no magnetic charges, or *magnetic monopoles*. Indeed Gauss's theorem implies that the total magnetic flux through a closed surface ∂V surrounding an volume V vanishes:

$$\int_V d^3x\, \nabla \cdot B = \oint_{\partial V} d^2\sigma\, B_n = 0. \tag{225}$$

However this argument is based on the assumption that the electric and magnetic fields are the fundamental physical objects, and the vector potential $A_\mu = (A_0, \mathbf{A})$ a mathematical construct to parametrize the Maxwell fields $\mathbf{B}$ and $\mathbf{E}$:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = \nabla A_0 - \partial_0 \mathbf{A}.$$

But it is conceivable that the vector field A_μ represents the fundamental physical degrees of freedom and that the properties if the Maxwell fields $F_{\mu\nu}$ are derived from those of A_μ by equation (73):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Then Gauss's theorem should not be applied to the normal component of the magnetic field B_n, but to the normal component of the curl of the vector potential $(\nabla \times \mathbf{A})_n$. If the vector potential is not single valued, Gauss's theorem can't be used in the straightforward manner of (225). Now actually the vector potential is only defined modulo a gauge transformation; therefore it is not necessarily single valued in space: it may be defined differently in different domains provided in the region of overlap the two definitions differ only by a gauge transformation such that the field strengths **B** and **E** are single valued. This offers an escape to the argument for the absence of magnetic monopoles, as was discovered originally by Dirac[2].

Problem 18.1

a. Consider a vector potential defined everywhere except on the negative z-axis in polar co-ordinates by

$$A^+_\varphi = -\kappa(\cos\theta + 1), \qquad 0 < \theta \leq \pi, \tag{226}$$

with κ a constant and all other vectro components vanishing: $A_0 = A_r = A_\theta = 0$. Show that there is a radial magnetic field $\mathbf{B} = (B_r, 0, 0)$

$$B_r = (\nabla \times \mathbf{A})_r = \frac{\kappa}{r^2}. \tag{227}$$

b. Next consider a vector potential defined everywhere except on the positive z-axis by

$$A^-_\varphi = -\kappa(\cos\theta - 1), \qquad 0 \leq \theta < \pi. \tag{228}$$

Show that again there is a radial magnetic field of the same strength (227). Thus we have a single-valued spherically symmetric radial magnetic monopole field in all directions including the full z-axis, parametrized by two different vector potentials in different domains space.

c. Taking a spherical volume of radius r, show that

$$\int_V d^3x \, \nabla \cdot \mathbf{B} = 4\pi\kappa, \tag{229}$$

independent of r. Therefore the parameter κ represents the magnetic charge.

d. Show that the two vector potentials differ by a local gauge transformation

$$A^+_\mu - A^-_\mu = \partial_\mu \Lambda, \tag{230}$$

explaining why the **B**-field is single valued.

There is a more physical to think of this mathematical construction of a magnetic monopole. A spherical surface is a surface of contant magnetic flux B_r; however, if we consider only the vector potential A^+ it is defined only on this

[2] P.A.M. Dirac, Proc. Roy. Soc. A133 (1931), 60

sphere minut the south pole $\theta = 0$. Therefore it is a sphere with a *puncture*; this is the reason Gauss's theorem is modified. Then the magnetic monopole may actually be one pole of a dipole, with all the return flux of size $-4\pi\kappa$ leaving the sphere through this puncture along the negative z-axis. The other pole may be moved on this half-axis to $z = -\infty$. This string of magnetic flux on along the negative z-axis is called the *Dirac string*. Our construction shows, that one might as well have taken a sphere with a puncture at the north pole $\theta = \pi$, with the Dirac string along the positive z-axis, or anywhere else on the sphere with a corresponding arbitrary direction of the Dirac string. As follows from the argument of problem 18.1 these constructions are all equivalent modulo a gauge transformation and should therefore be indistinguishable.

Problem 18.2
Consider the charged scalar field satisfying the modified Klein-Gordon equation

$$\left(-D^2 + m^2\right)\phi = 0, \qquad D_\mu = \partial_\mu - iqA_\mu,$$

or its non-linear extension. Define

$$\phi[C;x] = e^{iq\int_{C(x)} A_\mu dx^\mu}\psi(x), \tag{231}$$

where $C(x)$ is a curve, a path of integration, connecting the origin and the space-time point with co-ordinates x.
a. Prove

$$D_\mu\phi = e^{iq\int_{C(x)} A_\mu dx^\mu}\partial_\mu\psi, \qquad \left(-\Box + m^2\right)\psi = 0. \tag{232}$$

b. In general ψ will be independent of A_μ, but the expression the (231) for ϕ will depend both on A_μ and on the path C. To verify this, take two paths C_1 and C_2 and show that

$$\phi[C_2;x] = e^{iq\int_{C_2(x)} A_\mu dx^\mu}\psi(x) = e^{iq\oint_\gamma A_\mu dx^\mu}\phi[C_1;x], \tag{233}$$

where γ is the closed path formed by going out along C_2 and returning to the origin by C_1.
c. Show that in particular for a purely magnetic vector potential $A_\mu = (0, \mathbf{A})$ and a closed spatial path γ

$$\oint_\gamma \mathbf{A}\cdot d\mathbf{x} = \int_\sigma B_n\, d^2\sigma \equiv \Phi_\gamma, \tag{234}$$

the magnetic flux throug any spatial surface σ bounded by the closed curve γ.
d. For the Dirac monopole, explain why the Dirac string is *unobservable* provided

$$2\kappa q = n, \tag{235}$$

where n is an integer.