Introduction to Supersymmetry

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1 Relativistic particles

a. Energy and momentum

The energy and momentum of relativistic particles are related by

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4. \tag{1}$$

In covariant notation we define the momentum four-vector

$$p^\mu = (p^0, \mathbf{p}) = \left(\frac{E}{c}, \mathbf{p}\right), \qquad p_\mu = (p_0, \mathbf{p}) = \left(-\frac{E}{c}, \mathbf{p}\right). \tag{2}$$

The energy-momentum relation (1) can be written as

$$p^\mu p_\mu + m^2 c^2 = 0, \tag{3}$$

where we have used the Einstein summation convention, which implies automatic summation over repeated indices like μ.

Particles can have different masses, spins and charges (electric, color, flavor, ...). The differences are reflected in the various types of fields used to describe the quantum states of the particles. To guarantee the correct energy-momentum relation (1), any free field Φ must satisfy the Klein-Gordon equation

$$\left(-\hbar^2 \partial^\mu \partial_\mu + m^2 c^2\right) \Phi = \left(\frac{\hbar^2}{c^2}\, \partial_t^2 - \hbar^2 \boldsymbol{\nabla}^2 + m^2 c^2\right) \Phi = 0. \tag{4}$$

Indeed, a plane wave

$$\Phi = \phi(k)\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{5}$$

satisfies the Klein-Gordon equation if

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}. \tag{6}$$

From now on we will use natural units in which $\hbar = c = 1$. In these units we can write the plane-wave fields (5) as

$$\Phi = \phi(p)\, e^{ip\cdot x} = \phi(p)\, e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}. \tag{7}$$

b. Spin

Spin is the *intrinsic* angular momentum of particles. The word ‘intrinsic’ is to be interpreted somewhat differently for massive and massless particles. For *massive* particles it is the angular momentum in the rest-frame of the particles, whilst for massless particles –for which no rest-frame exists– it is the angular momentum w.r.t. the direction of motion. We illustrate this for case of spin-1/2 particles (fermions).

Massless fermions

A massless spin-1/2 particle, can have its spin polarized parallel or antiparallel to the momentum. In this 2-dimensional space of states the angular momentum in the direction of motion, the *helicity*, and its eigenvalues (in units $\hbar$) are given by

$$\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2|\mathbf{p}|} = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2E} \to \pm\frac{1}{2}. \tag{8}$$

The Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ all have eigenvalues $(+1, -1)$; they have the standard representation

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

An important property of the Pauli matrices is, that they satisfy the angular momentum commutation relations

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\, \varepsilon_{ijk}\sigma_k. \tag{10}$$

In addition they also satisfy the anticommutation relations

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\, \delta_{ij}\, \mathbf{1}. \tag{11}$$

As a result, for $m = 0$ we get

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = \frac{1}{2}\, p_i p_j \left(\sigma_i\sigma_j + \sigma_j\sigma_i\right) = \mathbf{p}^2\, \mathbf{1} = E^2\, \mathbf{1}. \tag{12}$$

The quantum theory of a massless fermion can therefore be formulated in terms of a 2-component field

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{13}$$

satisfying a partial differential equation

$$i\partial_t \Phi = \mp i \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \Phi, \tag{14}$$

known as the massless Dirac equation. It has plane-wave solutions

$$\Phi = \begin{bmatrix} \phi_1(p) \\ \phi_2(p) \end{bmatrix} e^{ip\cdot x}, \tag{15}$$

provided

$$\mathbf{p} \cdot \boldsymbol{\sigma}\, \Phi = \pm E\, \Phi, \tag{16}$$

which can be rewritten in the form

$$\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\, \Phi = \pm\frac{1}{2}\, \Phi. \tag{17}$$

Therefore Φ represents a state of definite helicity $\pm 1/2$; moreover

$$\mathbf{p}^2\, \Phi = (\mathbf{p} \cdot \boldsymbol{\sigma})^2\, \Phi = \pm E\, (\mathbf{p} \cdot \boldsymbol{\sigma})\, \Phi = E^2\, \Phi, \tag{18}$$

which is the correct energy-momentum relation for a massless particle. Both relations (17) and (18) are implied by the massless Dirac equation (14).

Massive fermions
For a massive fermion[1]

$$(E + \mathbf{p} \cdot \boldsymbol{\sigma})\, (E - \mathbf{p} \cdot \boldsymbol{\sigma}) = E^2 - (\mathbf{p} \cdot \boldsymbol{\sigma})^2 = E^2 - \mathbf{p}^2 = m^2 \neq 0. \tag{19}$$

Therefore the field Φ can not satisfy the massless Dirac equation:

$$i\, (\partial_t \pm \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \Phi \neq 0.$$

As a result we can introduce a second 2-component field X defined by

$$\begin{aligned} i\, (\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \Phi &= m\mathrm{X}, \\ i\, (\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \mathrm{X} &= m\Phi. \end{aligned} \tag{20}$$

It follows, that

$$-\, (\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, (\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \Phi = im\, (\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \mathrm{X} = m^2\, \Phi, \tag{21}$$

but also

$$-\, (\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, (\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \Phi = \left(-\partial_t^2 + (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})^2 \right) \Phi = \left(-\partial_t^2 + \boldsymbol{\nabla}^2 \right) \Phi. \tag{22}$$

Combining these results, we reobtain the Klein-Gordon equation

$$\left(-\partial^2 + m^2 \right) \Phi = 0, \qquad \partial^2 \equiv \partial^\mu \partial_\mu = -\partial_t^2 + \boldsymbol{\nabla}^2. \tag{23}$$

In terms of plane-wave fields

$$\Phi = \left[\begin{array}{c} \phi_1(p) \\ \phi_2(p) \end{array} \right] e^{ip\cdot x}, \qquad \mathrm{X} = \left[\begin{array}{c} \chi_1(p) \\ \chi_2(p) \end{array} \right] e^{ip\cdot x}, \tag{24}$$

eqs. (20) take the form

$$(E - \mathbf{p} \cdot \boldsymbol{\sigma})\Phi = m\mathrm{X}, \qquad (E + \mathbf{p} \cdot \boldsymbol{\sigma})\mathrm{X} = m\Phi. \tag{25}$$

These equations imply

$$(E + \mathbf{p} \cdot \boldsymbol{\sigma})(E - \mathbf{p} \cdot \boldsymbol{\sigma})\Phi = m(E + \mathbf{p} \cdot \boldsymbol{\sigma})\mathrm{X} = m^2\, \Phi, \tag{26}$$

[1] In the following we do not write explicitly the unit matrix **1** when multiplication by a scalar is intended.

which reproduces the result (19) as required.

Moreover, eqs. (25) can be rewritten in the form

$$\begin{aligned} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\, \Phi &= \frac{1}{2} \left(\Phi - \frac{m}{E} \mathrm{X} \right), \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2E}\, \mathrm{X} &= -\frac{1}{2} \left(\mathrm{X} - \frac{m}{E} \Phi \right). \end{aligned} \tag{27}$$

This shows, that in the relativistic limit $E \gg m$ the components Φ represent positive-helicity states, and the components X represent negative-helicity states. Hence both particle helicities are present in the massive Dirac equations (20), and the number of field components is doubled accordingly.

Covariant form of the Dirac equations
The Dirac equations (20) can be written in a more concise form as a 4×4-matrix equation

$$\left(\begin{array}{cc} m & -i(\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) \\ -i(\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) & m \end{array} \right) \left(\begin{array}{c} \Phi \\ \mathrm{X} \end{array} \right) = \left(\begin{array}{c} -i(\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\mathrm{X} + m\Phi \\ -i(\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\Phi + m\mathrm{X} \end{array} \right) = 0. \tag{28}$$

This can be cast in a manifestly covariant form by defining a four-component field, or *spinor*:

$$\Psi = \left(\begin{array}{c} \Phi \\ \mathrm{X} \end{array} \right) = \left(\begin{array}{c} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{array} \right) \equiv \left(\begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \right). \tag{29}$$

The Pauli matrices then are generalized to a set of four 4×4-matrices, the Dirac matrices, defined in terms of 2×2-blocks by

$$\gamma^0 = -\,\gamma_0 = \left(\begin{array}{cc} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{array} \right), \qquad \boldsymbol{\gamma} = \left(\begin{array}{cc} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{array} \right). \tag{30}$$

With these definitions

$$\gamma \cdot \partial \equiv \gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} = \left(\begin{array}{cc} 0 & \partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ \partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & 0 \end{array} \right). \tag{31}$$

the Dirac equations (28) can be summarized in the form

$$\left(-i\gamma \cdot \partial + m \right) \Psi = 0. \tag{32}$$

Observe, that for $m = 0$ the equation reduces to two separate equations for the positive and negative helicity components. We can distinguish these components by introducing another Dirac matrix

$$\gamma_5 = \left(\begin{array}{cc} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{array} \right), \tag{33}$$

such that in the limit $E \gg m$ the positive and negative helicity components of Ψ are characterized by the eigenvalue ± 1 under the action of γ_5:

$$\Psi_+ \equiv \frac{1}{2}(1+\gamma_5)\,\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \quad \Psi_- \equiv \frac{1}{2}(1-\gamma_5)\,\Psi = \begin{pmatrix} 0 \\ \mathrm{X} \end{pmatrix} \tag{34}$$

such that

$$\gamma_5 \Psi_\pm = \pm \Psi_\pm. \tag{35}$$

Charge conjugation and Majorana spinors
The Dirac matrices satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}\,\mathbf{1}, \tag{36}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric. Actually, these relations can be extended to include γ_5:

$$\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0, \qquad \gamma_5^2 = \mathbf{1}. \tag{37}$$

Under hermitean conjugation

$$\gamma_0^\dagger = \gamma_0, \qquad \boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}. \tag{38}$$

These relations can be summarized in a single equation

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0. \tag{39}$$

The hermitean conjugate of the Dirac equation (32) can therefore be rewritten in the form

$$\Psi^\dagger \left(i\gamma^\dagger \cdot \overleftarrow{\partial} + m \right) = \Psi^\dagger\gamma_0 \left(i\gamma\cdot \overleftarrow{\partial} + m \right)\gamma_0 = 0. \tag{40}$$

It is customary to define $\overline{\Psi} = \Psi^\dagger\gamma_0$, in terms of which the conjugate Dirac equation becomes

$$\overline{\Psi} \left(i\gamma\cdot \overleftarrow{\partial} + m \right) = 0. \tag{41}$$

Next, we introduce the charge-conjugation matrix

$$C = C^{-1} = -C^T = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{42}$$

Observing that in the 2-dimensional case of the Pauli matrices there is an identity

$$\sigma_2\sigma_i\sigma_2 = -\sigma_i^T, \tag{43}$$

it follows that

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \qquad C^{-1}\gamma_5 C = \gamma_5^T. \tag{44}$$

This leads us to consider transposition of the conjugate Dirac equation (41):

$$\left(i\gamma^T \cdot \partial + m\right) \overline{\Psi}^T = C^{-1} \left(-i\gamma \cdot \partial + m\right) C\overline{\Psi}^T = 0. \tag{45}$$

Now define the charge conjugate spinor

$$\Psi^c \equiv C\overline{\Psi}^T. \tag{46}$$

Eq. (45) implies, that if Ψ is a solution of the Dirac equation, the charge conjugate spinor is a solution of the Dirac equation as well:

$$\left(-i\gamma \cdot \partial + m\right) \Psi^c = 0. \tag{47}$$

It is therefore possible to restrict the number of independent components of a spinor by requiring that it is self-conjugate:

$$\Psi^c = \Psi. \tag{48}$$

Such a spinor is called a *Majorana spinor*. In terms of the helicity components this becomes

$$\mathrm{X} = \sigma_2 \Phi^* \quad \Leftrightarrow \quad \Phi = -\sigma_2 \mathrm{X}^*. \tag{49}$$

Physically it implies, that the particle described by a Majorana spinor is its own anti-particle: the negative-helicity state is the conjugate of the right-handed helicity state of the same spinor, and not independent.

2 Actions and symmetries

a. Actions for fields

Spin-0 particles are described by scalar fields, which satisfy the Klein-Gordon equation (4) without any additional constraints on polarization states:

$$(-\partial^2 + m^2)\Phi = 0. \tag{50}$$

Assuming the field Φ is real, this equation can be obtained from an action principle by defining

$$S = \int d^4x \left(-\frac{1}{2}\, \partial^\mu \Phi\, \partial_\mu \Phi - \frac{m^2}{2}\, \Phi^2 \right), \tag{51}$$

and requiring that it is stationary under variations $\Phi \rightarrow \Phi + \delta\Phi$:

$$\delta S \simeq \int d^4x\, \delta\Phi \left(\partial^2 - m^2 \right) \Phi = 0, \tag{52}$$

where the $\simeq$ symbol signifies equality up to partial integration. Clearly for this condition to be satisfied for arbitrary variation $\delta\Phi$, the Klein-Gordon equation (50) must hold.

Similarly we can define an action for spin-1/2 fields which is stationary if the Dirac equation is satisfied. For a Majorana spinor this action is

$$S = \int d^4x \left(-\frac{i}{2}\, \overline{\Psi}\gamma \cdot \partial\Psi + \frac{m}{2}\, \overline{\Psi}\Psi \right). \tag{53}$$

In this action it is necessary to take the components of the spinor Ψ to be anti-commuting quantities:

$$\psi_a \psi_b = -\psi_b \psi_a, \tag{54}$$

otherwise the action reduces zero after partial integration. As for Majorana particles the spinor field Ψ and its conjugate are directly related by $\Psi = \Psi^c = C\overline{\Psi}^T$, the effect of a variation $\Psi \rightarrow \Psi + \delta\Psi$ then is

$$\delta S \simeq \int d^4x\, \delta\overline{\Psi} \left(-i\gamma \cdot \partial + m \right) \Psi = 0, \tag{55}$$

requiring the Dirac equation to hold.

Remark
The mass term in the action (53) for a Majorana spinor can be written up to a factor 2 as

$$m\, \overline{\Psi}\Psi = m\, \overline{\Psi} C \overline{\Psi}^T = m\, \bar{\psi}_a C_{ab} \bar{\psi}_b.$$

Now C is antisymmetric: $C_{ab} = -C_{ba}$; therefore the above mass term vanishes unless the components of Ψ are anti-commuting:

$$\bar{\psi}_a \bar{\psi}_b = -\bar{\psi}_b \bar{\psi}_a.$$

A similar argument can be made for the kinetic term

$$\overline{\Psi}\gamma \cdot \partial\, \Psi = \overline{\Psi}\gamma^{\mu} C \partial_{\mu} \overline{\Psi}^{T},$$

which reduces up to a factor 2 to a total divergence $\partial_{\mu}(\overline{\Psi}\gamma^{\mu} C \overline{\Psi}^{T})$ unless the components of Ψ are anti-commuting.

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b. Symmetries and conservation laws

Consider a system of two scalar fields representing free particles of mass m_1 and m_2; the action for these particles is

$$S = \int d^4x \left(-\frac{1}{2}\, \partial^{\mu}\Phi_1 \partial_{\mu}\Phi_1 - \frac{1}{2}\, \partial^{\mu}\Phi_2 \partial_{\mu}\Phi_2 - \frac{m_1^2}{2}\, \Phi_1^2 - \frac{m_2^2}{2}\, \Phi_2^2 \right). \tag{56}$$

Variation of this action w.r.t. the fields Φ_1, resp. Φ_2 gives the two Klein-Gordon equations

$$\frac{\delta S}{\delta \Phi_1} = \left(\partial^2 - m_1^2 \right) \Phi_1 = 0, \qquad \frac{\delta S}{\delta \Phi_2} = \left(\partial^2 - m_2^2 \right) \Phi_2 = 0. \tag{57}$$

A rotation between the fields can be defined by the transformation

$$\Phi_1' = \cos\theta\, \Phi_1 - \sin\theta\, \Phi_2, \qquad \Phi_2' = \sin\theta\, \Phi_1 + \cos\theta\, \Phi_2. \tag{58}$$

For small θ this reduces to infinitesimal transformations $\delta\Phi_a = \Phi_a' - \Phi_a$ of the form

$$\delta\Phi_1 = -\theta\, \Phi_2, \qquad \delta\Phi_2 = \theta\, \Phi_1. \tag{59}$$

Under such transformations the action changes by

$$\begin{aligned} \delta S &= \theta \int d^4x \, \left(\partial\Phi_2 \cdot \partial\Phi_1 - \partial\Phi_1 \cdot \partial\Phi_2 + m_1^2 \Phi_2 \Phi_1 - m_2^2 \Phi_1 \Phi_2 \right) \\ &= \theta \left(m_1^2 - m_2^2 \right) \int d^4x \, \Phi_1 \Phi_2. \end{aligned} \tag{60}$$

Thus we see, that the action is invariant: $\delta S = 0$, if and only if the masses of the two types of particles are equal: $m_1^2 = m_2^2$. Clearly, when this condition is satisfied the two fields $\Phi_{1,2}$ satisfy the same Klein-Gordon equation (57) and therefore any linear combination of these two fields also satisfies the same equation. This observation can be used to simplify the model (56) by combining the two real fields in a single complex field

$$A = \frac{1}{\sqrt{2}} \left(\Phi_1 + i\Phi_2 \right), \qquad A^* = \frac{1}{\sqrt{2}} \left(\Phi_1 - i\Phi_2 \right). \tag{61}$$

If the masses are equal, the action (56) is equivalent with

$$S = \int d^4x \left(-\partial A^* \cdot \partial A - m^2 A^* A \right), \tag{62}$$

and the rotation (58) takes the form of a phase transformation

$$A' = e^{i\theta} A. \tag{63}$$

It is clear that the action (61) is invariant under these transformations.

Returning to the theory defined by the action (56) we next define a four-vector, the *Noether current*, as

$$J_\mu = \Phi_2 \partial_\mu \Phi_1 - \Phi_1 \partial_\mu \Phi_2. \tag{64}$$

Using the Klein-Gordon equations (57), the current satisfies the field equation

$$\begin{aligned} \partial^\mu J_\mu &= \partial\Phi_2 \cdot \partial\Phi_1 + \Phi_2 \partial^2 \Phi_1 - \partial\Phi_1 \cdot \partial\Phi_2 - \Phi_1 \partial^2 \Phi_2 = m_1^2 \Phi_2 \Phi_1 - m_2^2 \Phi_1 \Phi_2 \\ &= \left(m_1^2 - m_2^2\right) \Phi_1 \Phi_2. \end{aligned} \tag{65}$$

Therefore if the masses are equal: $m_1^2 - m_2^2 = 0$, we have a divergence-free current:

$$\partial^\mu J_\mu = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{66}$$

where the space and time components of the current are denoted by $J^\mu = (\rho, \mathbf{j})$. We recognize this as an equation of continuity, like the Euler equation in fluid mechanics. It leads directly to a conservation law for the total charge in a volume V:

$$Q = \int_V d^3x \, J^0 = \int_V d^3x \, \rho. \tag{67}$$

The derivation of the conservation law is straightforward:

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial \rho}{\partial t} = - \int_V d^3x \, \boldsymbol{\nabla} \cdot \mathbf{j} = - \oint_{\Sigma = \partial V} d^2 \sigma j_n, \tag{68}$$

with j_n the normal component of the current across the surface Σ which forms the boundary of the volume V. If there is no net current across this surface, or if we extend the volume over all of space, with fields and currents vanishing at infinity, then we immediately derive

$$\frac{dQ}{dt} = 0. \tag{69}$$

This conservation law is a direct consequence of the invariance of the action under the transformations (59), which forces the equality of the masses $m_1 = m_2$. In contrast, if $m_1 \neq m_2$ the rotation symmetry (58), (59) is said to be broken and the total charge Q is not conserved.

The relation between symmetries and conservation laws is a very general one; after its discoverer it is known as Noether's theorem.

3 Supersymmetry

Supersymmetry is a symmetry implying particles of different spin to have the same mass, charge and other properties. The simplest example is an extension of the previous model of two scalar fields with a single Majorana spinor field with the same mass:

$$S \;=\; \int d^4x \left(-\frac{1}{2}\partial^\mu \Phi_1 \partial_\mu \Phi_1 - \frac{1}{2}\partial^\mu \Phi_2 \partial_\mu \Phi_2 - \frac{m^2}{2}\left(\Phi_1^2 + \Phi_2^2\right) - \frac{1}{2}\,\overline{\Psi}\left(i\gamma \cdot \partial - m\right)\Psi \right), \tag{70}$$

where $\Psi^c = \Psi$ as in (53). We have already seen, that this action is invariant if we transform the scalar fields among themselves by a rotation (58) or (59). However, the action is also invariant under a set of transformations which mix the scalar fields with the spinor field. The infinitesimal form of these transformations is

$$\begin{aligned} &\delta\Phi_1 = \bar{\epsilon}\Psi, \qquad \delta\Phi_2 = i\bar{\epsilon}\gamma_5\Psi, \\ &\delta\overline{\Psi} = \bar{\epsilon}\left(-i\gamma \cdot \partial + m\right)\Phi_1 + i\bar{\epsilon}\gamma_5\left(-i\gamma \cdot \partial + m\right)\Phi_2. \end{aligned} \tag{71}$$

Here ϵ is a parameter which itself is a Majorana spinor[2]. Observe, that the transformation $\delta\Psi$ is directly obtained from $\delta\overline{\Psi}$ by charge conjugation:

$$\delta\Psi = \delta\Psi^c = C\,\delta\overline{\Psi}^T. \tag{72}$$

A somewhat lengthy but straightforward calculation now shows that the action is invariant:

$$\delta S \simeq \int d^4x \left[\delta\Phi_1(\partial^2 - m^2)\Phi_1 + \delta\Phi_2(\partial^2 - m^2)\Phi_2 - \delta\overline{\Psi}\left(i\gamma \cdot \partial - m\right)\Psi\right] = 0. \tag{73}$$

This result holds only because all masses are equal. It is also important to observe, that the model describes two spin-0 particles and one spin-1/2 fermion with two spin polarization states; the total number of bosonic particle states is therefore equal to the total number of fermionic particle states. This is a general condition for a supersymmetry to be possible. The simple model described here is known as the Wess-Zumino model.

The supercurrent
As for the rotation symmetry between the scalar fields, also for supersymmetry there is an associated conserved current, appropriately called the *supercurrent*. For the Wess-Zumino model (70) it takes the form

$$S^\mu = \left[i\gamma \cdot \partial\left(\Phi_1 - i\gamma_5\Phi_2\right) - m\left(\Phi_1 + i\gamma_5\Phi_2\right)\right]\gamma^\mu\Psi. \tag{74}$$

Clearly each component S_μ is itself a spinor. The field equations (52) and (55) imply that this current is divergence free:

$$\partial_\mu S^\mu = 0. \tag{75}$$

[2]Hence ϵ has four components, and $\epsilon = \epsilon^c = C\bar{\epsilon}^T$.

Step by step repeating the proof of (68), it follows that there exists a conserved spinorial *supercharge*

$$Q = \int_V d^3x\, S^0. \tag{76}$$

Again, the vanishing of the 4-divergence (75) and the resulting conservation law for the supercharge:

$$\frac{dQ}{dt} = 0, \tag{77}$$

require equality of the boson and fermion masses. As the parameter of the supersymmetry transformations is a Majorana spinor, also the supercurrent and supercharge are Majorana spinors:

$$Q = Q^c = C\overline{Q}^T. \tag{78}$$

Massless particles

In the case of massless scalar and spin-1/2 particles, the action (70) can be simplified in two respects. First, as before we can combine the two real scalar fields in a single complex scalar field

$$A = \frac{1}{\sqrt{2}}\left(\Phi_1 + i\Phi_2\right), \qquad A^* = \frac{1}{\sqrt{2}}\left(\Phi_1 - i\Phi_2\right). \tag{79}$$

Second, we can write the 4-component Majorana spinor Ψ in terms of the complex 2-component left-handed spinor X by using eq. (49):

$$\Psi = \left[\begin{array}{c} -\sigma_2 X^* \\ X \end{array}\right], \qquad \overline{\Psi} = \left[-X^\dagger, X^T\sigma_2\right]. \tag{80}$$

In terms of these fields, the action (70) with $m = 0$ becomes

$$S = \int d^4x \left(-\partial^\mu A^* \partial_\mu A - iX^\dagger \left(\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right) X\right). \tag{81}$$

This complex scalar A and complex spinor X now are partners under supersymmetry transformations

$$\begin{aligned} \delta A &= \eta^\dagger X, \quad \delta X = i[\left(\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right) A]\eta, \\ \delta A^* &= X^\dagger \eta, \quad \delta X^\dagger = -i\eta^\dagger \left(\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right) A^*, \end{aligned} \tag{82}$$

where η is a complex 2-component spinor parameter, representing the left-handed components of ϵ in eq. (71). The action (81) is seen to be invariant under separate phase transformations of A and X:

$$A' = e^{i\alpha} A, \qquad X' = e^{i\beta} X. \tag{83}$$

Supersymmetric gauge theories

The existence of supersymmetry in a field theory of particles is not restricted to fermions and spinless scalar particles. It is equally well possible to construct theories of spin-1 vector fields and fermions with supersymmetry. The simplest case is that of a supersymmetric extension of pure Maxwell theory.

Maxwell's theory of the electro-magnetic field can be formulated in terms of the four-dimensional vector potential $A_\mu = (\phi, \mathbf{A})$, where ϕ is the electric potential and $\mathbf{A}$ the magnetic vector potential. The corresponding electric and magnetic field strengths $(\mathbf{E}, \mathbf{B})$ together make up the components a an anti-symmetric four-tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{84}$$

with the Maxwell equations in empty space being reproduced in the form

$$\partial^\mu F_{\mu\nu} = \partial^2 A_\nu - \partial_\nu \, \partial \cdot A = 0. \tag{85}$$

An important property of the field-strength tensor $F_{\mu\nu}$ is, that it is invariant under gauge transformations, changing the vector field A_μ by the gradient of a scalar Λ:

$$A'_\mu = A_\mu + \partial_\mu \Lambda \quad \Rightarrow \quad F'_{\mu\nu} = F_{\mu\nu}. \tag{86}$$

Because of this arbitrariness we can impose an additional constraint on A_μ, which can be convenienty chosen in the form

$$\partial \cdot A = 0. \tag{87}$$

With this choice the free Maxwell equations (85) reduce to

$$\partial^2 A_\nu = 0. \tag{88}$$

Then all components of the vector field satisfy the massless Klein-Gordon equation, showing that we can associate this field with a massless spin-1 particle, the photon.

To make the theory supersymmetric, it suffices to introduce a massless Majorana fermion λ, which satisfies the Dirac equation

$$-i\gamma \cdot \partial \lambda = 0. \tag{89}$$

This hypothetical fermionic partner of the photon is commonly called the *photino*. Observe that the massless photon can exist in two physical polarization states: right-handed, with its spin parallel to its momentum, and left-handed with its spin anti-parallel to its momentum. The same is true for a masless Majorana fermion. Hence the numbers of bosonic and fermionic particle states are equal, whilst both particles have zero mass. Thus all conditions for the existence of supersymmetry in this theory are fulfilled.

Formally, this can be seen by constructing the action for this theory

$$S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda \right). \tag{90}$$

The variation of this action under arbitrary changes in the fields δA_μ and $\delta\lambda$ is

$$\delta S \simeq \int d^4x \left(\delta A^\nu \partial^\mu F_{\mu\nu} - i \delta \bar{\lambda} \gamma \cdot \partial \lambda \right), \tag{91}$$

and this vanishes identically if the field equations (85) and (89) hold. However, even if the field equations do not hold, the result is still $\delta S = 0$, provided the field variations are of the form

$$\delta A_\mu = \bar{\epsilon} \gamma_\mu \lambda, \qquad \delta \bar{\lambda} = \frac{i}{2} \bar{\epsilon} \gamma^\mu \gamma^\nu F_{\mu\nu}. \tag{92}$$

To prove this, it is necessary to use the following identity for Dirac matrices:

$$(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \gamma^\lambda = 2 \left(\gamma^\mu \eta^{\nu\lambda} - \gamma^\nu \eta^{\mu\lambda} \right) + 2 \varepsilon^{\mu\nu\lambda\kappa} \gamma_5 \gamma_\kappa. \tag{93}$$

In addition, there is an identity for the field strength tensor

$$\varepsilon^{\mu\nu\lambda\kappa} \partial_\lambda F_{\mu\nu} = \varepsilon^{\mu\nu\lambda\kappa} \partial_\lambda \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) = 0, \tag{94}$$

because of the complete anti-symmetry of the permutation symbol $\varepsilon^{\mu\nu\lambda\kappa}$ and the symmetry under the interchange of partial derivatives $\partial_\lambda \partial_\mu = \partial_\mu \partial_\lambda$. This result is known as the Bianchi identity, or also as the *homogeneous* Maxwell equations. It follows, that only the first term on the r.h.s. of the identity (93) is relevant, and the variation of the action becomes

$$\delta S \simeq \int d^4x \left(\bar{\varepsilon} \gamma^\nu \lambda \, \partial^\mu F_{\mu\nu} - \frac{1}{2} \bar{\varepsilon} (\gamma^\nu \lambda \, \partial^\mu - \gamma^\mu \lambda \, \partial^\nu) F_{\mu\nu} \right) = 0, \tag{95}$$

which proves the invariance modulo partial integrations of the action under the supersymmetry transformations (92). As in the Wess-Zumino model, there is a supercurrent which is divergence-free upon use of the field equations:

$$S^\mu = \gamma^\lambda \gamma^\nu \gamma^\mu \lambda \, F_{\lambda\nu} \quad \Rightarrow \quad \partial_\mu S^\mu = 0. \tag{96}$$

4 Symmetry breaking

In sect. 2, eq. (60), we found that the rotation symmetry of the two scalar fields was an exact symmetry only if the masses were equal. Equivalently, as follows from eq. (65), the conservation of the Noether charge was seen to hold only up to terms proportional to the mass difference. Actually, this shows that the gradient terms (the kinetic terms) of the fields in the action are always invariant, only the terms involving the masses actually break the symmetry if $m_1^2 \neq m_2^2$.

The same is true for supersymmetry: the action (70) of the Wess-Zumino model is invariant under the supersymmetry transformations (71) provided the boson and fermion masses are all equal. Also the conservation of the supercharge, as follows from the vanishing divergence of the supercurrent (74), requires the equality of these masses.

If the masses are not equal, we say that the symmetry involved is broken *explicitly* by the mass terms. A more subtle situation arises if the symmetry of the action is exact, but the minimal solution of the field equations itself is not invariant under the symmetry. In this case the symmetry is said to be broken *spontaneously.* This situation arises if the effective masses of the fields are not those suggested by the action, but actually determined by the interactions of the fields.

a. Scalar fields

A simple example of this situation arises when the two scalar fields of section 2.b obey a slightly more generalized set of field equations:

$$\partial^2 \Phi_1 - \frac{\partial V}{\partial \Phi_1} = 0, \qquad \partial^2 \Phi_2 - \frac{\partial V}{\partial \Phi_2} = 0, \tag{97}$$

following from variation of the action

$$S = \int d^4x \left(-\frac{1}{2} \left(\partial^\mu \Phi_1 \partial_\mu \Phi_1 + \partial^\mu \Phi_2 \partial_\mu \Phi_2 \right) - V[\Phi_1, \Phi_2] \right). \tag{98}$$

Now take a potential V of the form

$$V[\Phi_1, \Phi_2] = \frac{1}{2g^2} \left(\mu^2 - \frac{g^2}{2} \left(\Phi_1^2 + \Phi_2^2 \right) \right)^2; \tag{99}$$

then the field equations become

$$\left(\partial^2 + \mu^2 - g^2 \left(\Phi_1^2 + \Phi_2^2 \right) \right) \Phi_1 = 0, \qquad \left(\partial^2 + \mu^2 - g^2 \left(\Phi_1^2 + \Phi_2^2 \right) \right) \Phi_1 = 0. \tag{100}$$

These equations have a set of simple constant solutions

$$\Phi_1 = \frac{\mu}{g} \cos \alpha, \qquad \Phi_2 = \frac{\mu}{g} \sin \alpha \quad \Rightarrow \quad \Phi_1^2 + \Phi_2^2 = \frac{\mu^2}{g^2}, \tag{101}$$

for arbitrary constant α; in particular we may choose $\alpha = 0$, such that $\Phi_1 = \mu/g$, and $\Phi_2 = 0$. This solution can always be achieved by a rotation of the fields (Φ_1, Φ_2) such that the constant non-zero solution is pointing in the direction of Φ_1.

Next we look for solutions which are not constant, but consist of a constant term plus a non-constant propagating[3] piece:

$$\Phi_1(x) = \frac{\mu}{g} + \phi_1(x), \qquad \Phi_2(x) = \phi_2(x). \tag{102}$$

Then substitution into the field equations (100) leads to the results

$$\left(\partial^2 - 2\mu^2\right)\phi_1 = \mathcal{O}[\phi_i^2], \qquad \partial^2\phi_2 = \mathcal{O}[\phi_i^2]. \tag{103}$$

In the limit of very small ϕ_1 and ϕ_2, we can disregard terms of order $\phi_i^2 \sim (\phi_1^2, \phi_1\phi_2, \phi_2^2)$, as they are very small compared to (ϕ_1, ϕ_2) themselves. Hence to first approximation the right-hand side of the equations (103) may be taken to vanish, and the fields (ϕ_1, ϕ_2) are solutions of the Klein-Gordon equation with masses $m_1^2 = 2\mu^2$ and $m_2^2 = 0$. Clearly for any $\mu^2 \neq 0$ these masses are different and the rotation symmetry of the starting point, defined by the action (98), is not realized in the solutions for ϕ_1 and ϕ_2. This is due to the constant background field $\langle\Phi_1\rangle = \mu/g$. The massless particle represented by the field ϕ_2 is generically called a Goldstone boson, whilst the massive particle represented by ϕ_1 is called a Higgs boson.

b. Spinor fields

Consider again the action (53) for a (Majorana) spinor field Ψ, but with the mass terms replaced by an interaction with a real scalar field Φ:

$$S = \frac{1}{2}\int d^4x \left(-i\overline{\Psi}\gamma\cdot\partial\Psi + g\Phi\,\overline{\Psi}\Psi\right). \tag{104}$$

The interaction term between the scalar field and the spinor fields is represented graphically by the Feynman diagram of fig. 4.1; depending on the direction of time it can be read either as the emission or absorption of a scalar boson by a fermion, or as pair creation of fermions by a scalar boson, or as fermion annihilation producing a single boson.

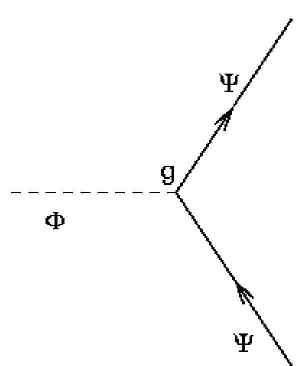

Fig. 4.1: Yukawa coupling of a scalar boson with fermions.

[3]A piece which behaves, to first approximation, like a plane wave with well-defined energy $E = \hbar\omega$ and momentum $\mathbf{p} = \hbar\mathbf{k}$.

Now clearly, if the scalar field has a constant background value, a *vacuum expectation value* $\langle \Phi \rangle = m/g$, then effectively it provides a mass to the fermion, and the field equation for Ψ becomes

$$\Phi(x) = \frac{m}{g} + \phi(x) \quad \Rightarrow \quad (-i\gamma \cdot \partial + m + g\phi)\, \Psi = 0. \tag{105}$$

In the limit of small ϕ the Yukawa interaction with the propagating scalar boson represented by ϕ can be neglected, and the equation reduces to the free Dirac equation with standard plane-wave solutions obeying the free-particle energy-momentum relation

$$E^2 = \mathbf{p}^2 + m^2.$$

c. Vector fields

An important difference between massless and massive vector bosons is the different number of polarization states they possess: a massless vector boson has two helicity states, with $+1$ or -1 unit of spin in the direction of motion, whilst a massive vector boson has three spin states in any direction, quantized in units $(+1, 0, -1)$.

Therefore the dynamical generation of mass for vector bosons is possible only if there is an additional degree of freedom that can serve as the third polarization state. Such a degree of freedom can be provided by a scalar field. As an example we consider the theory of a charged spin-0 particle interacting with the electromagnetic field. The charged bosons are represented by a complex scalar field Φ, coupled to the vector field A_μ through covariant derivatives

$$D_\mu \Phi = (\partial_\mu - ieA_\mu)\, \Phi. \tag{106}$$

Such a derivative transforms in a simple way under combined phase transformations and gauge transformations (86):

$$\Phi \rightarrow \Phi' = e^{i\alpha}\Phi, \qquad A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha, \tag{107}$$

which lead to the transformation rule

$$D_\mu \Phi \rightarrow (D_\mu \Phi)' = e^{i\alpha} D_\mu \Phi. \tag{108}$$

Now consider the modified Klein-Gordon equation

$$\left(-D^2 + M\right) \Phi = 0, \tag{109}$$

where

$$M[|\Phi|^2] = M_0 + M_1 |\Phi|^2 + ... \tag{110}$$

is an arbitrary (possibly field dependent) real quantity. This modified KG-equation is unchanged by the transformations (107) up to an irrelevant multiplicative factor $e^{i\alpha}$.

Next, the Maxwell equations are modified to include a current term

$$\partial^\mu F_{\mu\nu} = j_\nu, \qquad j_\nu = ie\left(\Phi^* D_\nu \Phi - \Phi D_\nu \Phi^*\right). \tag{111}$$

It follows, that

$$\partial^\mu j_\mu = 0, \tag{112}$$

as required by the conservation of electric charge; indeed, from (109) and its conjugate

$$\partial^\mu j_\mu = ie \left(\Phi^* D^2 \Phi - \Phi D^2 \Phi^* \right) = ieM \left(\Phi^* \Phi - \Phi \Phi^* \right) = 0. \tag{113}$$

Observe, that the vanishing four-divergence of the current (112) is also required for consistency by the anti-symmetry of $F_{\mu\nu}$:

$$\partial^\mu \partial^\nu F_{\mu\nu} = \partial^\nu j_\nu = 0. \tag{114}$$

Now consider what happens if we assume that the scalar field Φ has a constant background value:

$$\langle \Phi \rangle = \frac{m}{\sqrt{2e}}. \tag{115}$$

Then, as $\partial_\mu \Phi = 0$:

$$j_\nu = 2e \Phi^* \Phi A_\nu = m^2 A_\nu, \tag{116}$$

and the inhomogeneous Maxwell equation (111) takes the form

$$\partial^\mu F_{\mu\nu} = \partial^2 A_\nu - \partial_\nu \partial \cdot A = m^2 A_\nu \tag{117}$$

Now from this result and eq. (114) we immediately infer that

$$m^2 \, \partial \cdot A = 0, \tag{118}$$

and therefore the equation (117) finally reduces to the Klein-Gordon form

$$\left(-\partial^2 + m^2 \right) A_\nu = 0. \tag{119}$$

Obviously this equation is no longer manifestly gauge invariant; indeed the non-zero vacuum expectation value of Φ assumed in (115) explicitly breaks the gauge invariance of the theory. In addition to the massive gauge field A_μ, we also still have a massive Higgs boson in the theory. Indeed, the vacuum expectation value (115), is a solution of the modified KG-equation (109) only if

$$\langle M \rangle = M_0 + M_1 \frac{m^2}{2e} + ... = 0 \quad \Rightarrow \quad M_0 = -\frac{m^2}{2e} M_1 + ... \tag{120}$$

Now parametrize the full scalar field as

$$\Phi = e^{i\alpha} \left(\frac{m}{\sqrt{2e}} + h \right), \tag{121}$$

and take

$$A_\mu = \frac{1}{e} \partial_\mu \alpha \quad \Rightarrow \quad F_{\mu\nu} = 0. \tag{122}$$

Then the modified Klein-Gordon equation (109) becomes

$$\left(-\partial^2 + m_H^2\right) h = \mathcal{O}[h^2], \qquad m_H^2 = \frac{m^2}{e}\, M_1. \tag{123}$$

In the limit of small amplitude h the right-hand side of the first equation vanishes effectively, and we get a free-field equation for a massive scalar boson h, the Higgs boson.

d. Supersymmetry

Next we consider an extension of the massless Wess-Zumino model with a dimensionless coupling of the scalar and spinor fields of strength g:

$$\begin{aligned} S &= \int d^4x \left[-\partial^\mu A^* \partial_\mu A - i X^\dagger \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X - \frac{g^2}{4} \left(A^* A \right)^2 \right. \\ & \qquad \left. - \frac{g}{2} \left(A\, X^T \sigma_2 X + A^*\, X^\dagger \sigma_2 X^* \right) \right]. \end{aligned} \tag{124}$$

This action is invariant under the extended supersymmetry transformations

$$\begin{aligned} &\delta A = \eta^\dagger X, \quad \delta X = i \left[(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) A \right] \eta + F^*\, \sigma_2 \eta^*, \\ &\delta A^* = X^\dagger \eta, \quad \delta X^\dagger = -i \eta^\dagger (\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) A^* + F\, \eta^T \sigma_2, \end{aligned} \tag{125}$$

with

$$F(A) = \frac{g}{2}\, A^2, \tag{126}$$

as discussed in the appendix. The field equations are

$$\left(-\partial^2 + \frac{g^2}{2}\, |A|^2 \right) A = -\frac{g}{2}\, X^\dagger \sigma_2 X^*, \qquad -i \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X = g A^* \sigma_2 X^*. \tag{127}$$

The only consistent solution in terms of constant fields is $A = X = 0$. Therefore in this model there is no spontaneous mass generation, at least at the level of classical field equations.

Consider however, what would happen if we had constant backgrounds $\langle gA \rangle = m\sqrt{2}$, $X = 0$. Then the non-constant part of the fields would to first order satisfy the equations

$$\left(-\partial^2 + m^2 \right) A = 0, \qquad i \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X = \sqrt{2}\, m \Phi, \tag{128}$$

with $\Phi = -\sigma_2 X^*$ as in eq. (49). Clearly, in this situation the bosons and fermions have different masses, and supersymmetry is broken. At the same time we find a constant vacuum expectation value for the quantity F in eq. (126):

$$\langle F(A) \rangle = \frac{m^2}{g}, \tag{129}$$

and there is a constant background enenergy

$$\langle H \rangle = \langle |F(A)|^2 \rangle = \frac{m^4}{g^2} > 0. \tag{130}$$

This is the hallmark of spontaneously broken supersymmetry.

A The superpotential

The simple massless version of the Wess-Zumino model defined by eq. (81) is formulated in terms of a single complex scalar field A and a complex 2-component spinor field X. The most general interaction terms that can be included in the model is defined by a single analytic function of the complex scalar field A, the *superpotential* $W(A)$, or its derivatives $F(A) = W'(A)$ and $F'(A) = W''(A)$. In terms of this function the action then reads

$$\begin{aligned} S &= \int d^4x \left[-\partial^\mu A^* \partial_\mu A - iX^\dagger \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X - F^*(A^*) F(A) \right. \\ & \quad \left. - \frac{1}{2} \left(F'(A) X^T \sigma_2 X + F^{*\prime}(A^*) X^\dagger \sigma_2 X^* \right) \right]. \end{aligned} \tag{131}$$

Here the superscript T denotes transposition (replacing a column vector by a row vector), whilst $*$ denotes ordinary complex conjugation, which for spinors equals hermitean conjugation minus transposition. The action (131) is invariant under the extended supersymmetry transformations

$$\begin{aligned} &\delta A = \eta^\dagger X, \quad \delta X = i \left[(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) A \right] \eta + F^*(A^*) \sigma_2 \eta^*, \\ &\delta A^* = X^\dagger \eta, \quad \delta X^\dagger = -i \eta^\dagger (\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}) A^* + F(A) \eta^T \sigma_2. \end{aligned} \tag{132}$$

For renormalizable theories the superpotential is at most a cubic polynomial. For example, for a cubic monomial

$$W(A) = \frac{g}{3!} A^3 \quad \Rightarrow \quad F(A) = \frac{g}{2} A^2, \quad F'(A) = gA, \tag{133}$$

and therefore in this specific case the action becomes

$$\begin{aligned} S &= \int d^4x \left[-\partial^\mu A^* \partial_\mu A - iX^\dagger \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X - \frac{g^2}{4} (A^* A)^2 \right. \\ & \quad \left. - \frac{g}{2} \left(A\, X^T \sigma_2 X + A^*\, X^\dagger \sigma_2 X^* \right) \right]. \end{aligned} \tag{134}$$

Denoting the right-handed spinor components by $\Phi = -\sigma_2 X^*$, as in eq. (49), this can be written equivalently as

$$\begin{aligned} S &= \int d^4x \left[-\partial^\mu A^* \partial_\mu A - \frac{i}{2} X^\dagger \left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) X - \frac{i}{2} \Phi^\dagger \left(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \right) \Phi \right. \\ & \quad \left. - \frac{g^2}{4} (A^* A)^2 + \frac{g}{2} \left(A\, \Phi^\dagger X + A^*\, X^\dagger \Phi \right) \right]. \end{aligned} \tag{135}$$

In this model all particles are massless, and there is only one dimensionless coupling constant g.

A slightly more general model starts from the superpotential

$$W = \frac{m}{2}\,A^2 + \frac{g}{3!}\,A^3, \tag{136}$$

which is equivalent to starting from

$$W = -\frac{m^2}{2g}A + \frac{g}{3!}A^3, \tag{137}$$

and making the shift

$$A \;\rightarrow\; \frac{m}{g} + A. \tag{138}$$

The upshot is the massive interacting Wess-Zumino model

$$\begin{aligned} S \;\;=\;\; \int d^4x \Big[& -\partial^\mu A^* \partial_\mu A - \frac{i}{2}\,X^\dagger\,(\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,X - \frac{i}{2}\,\Phi^\dagger\,(\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\Phi \\ & - \left| m + \frac{g}{2}\,A \right|^2 A^* A + \frac{1}{2}\,(m + gA)\,\Phi^\dagger X + \frac{1}{2}\,(m + gA^*)\,X^\dagger \Phi \Big]\,. \end{aligned} \tag{139}$$

In this model all particles (bosons and fermions) have mass m, and supersymmetry in manifest.