Gravitational waves

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1 PP-waves

The vacuum Einstein equations admit wave-like solutions propagating with the speed of light. A simple example is provided by the *pp*-waves, which can be described in light-cone co-ordinates[1] with $u = t - z$, $v = t + z$ by the metric

$$-d\tau^2 = -dudv + \Phi(u,x,y)du^2 + dx^2 + dy^2. \tag{1}$$

The metric co-efficient $\Phi(u,x,y)$ parametrizes the deviation from Minkowski space-time; as it depends on z and t only via the light-cone variable u, the profile of Φ at fixed tranverse position (x,y) propagates with the speed of light in the z-direction.

In the space-time (1) all components of the Riemann curvature tensor vanish, except for

$$R_{uiuj} = -\frac{1}{2}\,\Phi_{,ij}, \qquad i,j = (x,y). \tag{2}$$

The only non-trivial component of the Einstein tensor then is

$$G_{uu} = R_{uu} = -\frac{1}{2}\,\Delta_T\,\Phi, \tag{3}$$

where Δ_T is the flat-space laplacian in the transverse (x,y)-plane

$$\Delta_T\,\Phi = \sum_{i=x,y} \Phi_{,ii}. \tag{4}$$

In complex co-ordinates $\zeta = x + iy$, $\bar{\zeta} = x - iy$, the vacuum Einstein equations then reduce to

$$\Phi_{,\bar{\zeta}\zeta} = 0, \tag{5}$$

with the solution

$$\Phi(u,\bar{\zeta},\zeta) = f(u,\zeta) + \bar{f}(u,\bar{\zeta}). \tag{6}$$

We can expand this in a combined power-series/Fourier integral as

$$\Phi(u,\bar{\zeta},\zeta) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(f_n(k)\,e^{-iku}\,\zeta^n + \bar{f}_n(k)\,e^{iku}\,\bar{\zeta}^n \right). \tag{7}$$

Obviously the constant and linear terms in $(\zeta,\bar{\zeta})$ do not contribute to the Riemann curvature. The corresponding metrics represent flat Minkowski space in disguise: they are related to standard cartesian Minkowski co-ordinates by a mere co-ordinate transformation. In contrast, for $n \geq 3$ the space-time has a curvature singularity at transverse infinity $x^2 + y^2 \rightarrow \infty$. The only regular non-trivial solutions are the quadratic ones; in real co-ordinates these read

$$\Phi(u,x,y) = f_2(u)\,\zeta^2 + \bar{f}_2(u)\,\bar{\zeta}^2 \;\equiv\; f_+(u)\,(x^2 - y^2) + 2f_\times(u)\,xy. \tag{8}$$

[1]We use units in which $c = 1$.

The co-efficients $f_+(u)$ and $f_\times(u)$ parametrize the two polarization states of the regular *pp*-waves, as might be expected for a massless field propagating at the speed of light. Note, that a rotation over an angle ϕ in the transverse plane changes the complex co-ordinates by

$$\zeta \rightarrow \zeta' = e^{i\phi}\, \zeta, \qquad \bar{\zeta} \rightarrow \bar{\zeta}' = e^{-i\phi}\, \bar{\zeta}. \tag{9}$$

Thus the modes $\mathcal{O}\left(\zeta^n\right)$ transform under rotations by a phase factor $e^{in\phi}$. The modes $n = 0, 1$ which decouple from gauge-invariant physical quantities (vanishing curvature) can therefore be identified with the scalar and vector (dipole) modes of the *pp*-wave. The singular modes are the octupole and higher 2^n-pole modes. Finally, the regular modes with $n = 2$ are the quadrupole modes with period $\phi \rightarrow \phi + \pi$.

Like electromagnetic waves, *pp*-waves are transversely polarized. This can be seen in two different ways. First, if we consider the geodesics $X^\mu(\tau)$:

$$\frac{d^2 X^\mu}{d\tau^2} + \Gamma_{\lambda\nu}^{\;\;\mu}(X)\, \frac{dX^\lambda}{d\tau} \frac{dX^\nu}{d\tau} = 0, \tag{10}$$

then in the light-cone directions we find

$$\frac{d^2 U}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dU}{d\tau} = \gamma = \text{constant}, \tag{11}$$

whilst from the form of the line element (1) one then infers

$$\gamma\, \frac{dV}{d\tau} + \gamma^2\, \Phi(U, X, Y) - \left(\frac{dX}{d\tau} \right)^2 - \left(\frac{dY}{d\tau} \right)^2 = 1. \tag{12}$$

This equation can be recast in the form

$$\left(\frac{dt}{d\tau} \right)^2 - \left(\frac{d\mathbf{r}}{d\tau} \right)^2 = 1 - \gamma^2\, \Phi \quad \Rightarrow \quad \frac{dt}{d\tau} = \sqrt{\frac{1 - \gamma^2 \Phi}{1 - \mathbf{v}^2}}. \tag{13}$$

It follows, that there are two source of time dilation: a special relativistic effect of kinematical origin (non-zero velocity $\mathbf{v} = d\mathbf{r}/dt$); and a general relativistic effect of gravitational origin (non-zero potential Φ).

Next, the components of the geodesic equation in the transverse direction can be written in the form

$$\left(\begin{array}{c} X''(U) \\ Y''(U) \end{array} \right) = - \left(\begin{array}{cc} f_+(U) & f_\times(U) \\ f_\times(U) & -f_+(U) \end{array} \right) \left(\begin{array}{c} X(U) \\ Y(U) \end{array} \right), \tag{14}$$

where we have used equation (11) to reparametrize the geodesics in terms of the light-cone time U instead of proper time τ. It is clear that these equations describe two parametric oscillators in the transverse plane with real and imaginary frequencies, respectively. In particular, for constant amplitudes $f_{(+,\times)}$ we can diagonalize the symmetric traceless mode matrix to find solutions of the type

$$X(U) = X(0) \cos kU, \qquad Y(U) = Y(0) \cosh kU, \tag{15}$$

where $k^2 = f_+^2 + f_\times^2$.

A second way to see the transverse nature of the *pp*-waves is by making a co-ordinate transformation such that the line element (1) takes the form

$$-d\tau^2 = -d\bar{u}d\bar{v} + a^2 d\bar{x}^2 + b^2 d\bar{y}^2 = -d\bar{t}^2 + d\bar{z}^2 + a^2 d\bar{x}^2 + b^2 d\bar{y}^2. \tag{16}$$

For *pp*-waves which have been diagonalized ($f_\times = 0$), this co-ordinate transformation reads explicitly

$$\begin{aligned} \bar{u} &= u, \qquad \bar{v} = v + \Lambda(u, x, y), \\ \bar{x} &= \frac{x}{a(u)}, \quad \bar{y} = \frac{y}{b(u)}, \end{aligned} \tag{17}$$

where

$$\Lambda(u, x, y) = \frac{a'}{a}\, x^2 + \frac{b'}{b} y^2, \tag{18}$$

and $a(u)$ and $b(u)$ are the solutions of the equations

$$-\frac{a''}{a} = \frac{b''}{b} = f_+. \tag{19}$$

For instance, if $f_+ = k^2 =$ constant, then typical solutions are

$$a(u) = a(0) \cos ku, \qquad b(u) = b(0) \cosh ku. \tag{20}$$

The metric (16) is manifestly non-flat only in the transverse plane. Of course, the co-ordinate transformation (17) is non-singular only for $(a, b) \neq 0$, and indeed the metric (16) has a co-ordinate singularity there. By going back to the original line element (1) it is easily observed that there is no curvature singularity at such points.

2 Small-amplitude waves

In this section we consider gravitational fields which can be considered small perturbations of flat Minkowski space-time:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \qquad \|\delta g_{\mu\nu}\| \ll 1. \tag{21}$$

For bookkeeping purposes it is often convenient to introduce the dimensional gravitational coupling constant

$$\kappa = \sqrt{\frac{8\pi G}{c^4}} \approx 0.46 \times 10^{-21}\, \mathrm{kg}^{-\frac{1}{2}}\ \mathrm{m}^{-\frac{1}{2}}\ \mathrm{s}, \tag{22}$$

and write

$$\delta g_{\mu\nu} = 2\kappa h_{\mu\nu}, \qquad \|h_{\mu\nu}\| \ll \frac{1}{2\kappa}. \tag{23}$$

The perturbation field with components $h_{\mu\nu}$ then has the standard dimensions of a bosonic field: squared gradients $(\partial h/\partial x)^2$ have the dimension of an energy density.

Whilst the decomposition of the metric (21) is exact (it defines $h_{\mu\nu}$), the inverse metric requires an infinite power series expansion in terms of $h_{\mu\nu}$:

$$g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa\, \eta^{\mu\kappa} h_{\kappa\lambda} \eta^{\lambda\nu} + 4\kappa^2\, \eta^{\mu\kappa} h_{\kappa\lambda} \eta^{\lambda\rho} h_{\rho\sigma} \eta^{\sigma\nu} + \mathcal{O}(\kappa^3). \tag{24}$$

Similarly, the determinant of the metric is expanded as

$$-g = 1 + 2\kappa\, h + \mathcal{O}(\kappa^2), \tag{25}$$

where

$$h = \eta^{\mu\nu} h_{\mu\nu} \tag{26}$$

is the minkowskian trace of the symmetric tensor field $h_{\mu\nu}$. From now on we use the convention that indices are raised and lowered with the Minkowski metric; hence

$$h_{\mu}^{\;\;\nu} = h_{\mu\lambda} \eta^{\lambda\nu}, \tag{27}$$

etc. To first order in κ the connection and Riemann tensor take the form

$$\begin{aligned} \Gamma_{\mu\nu}^{\;\;\;\lambda} &= \kappa \left(\partial_\mu h_\nu^{\;\;\lambda} + \partial_\nu h_\mu^{\;\;\lambda} - \partial^\lambda h_{\mu\nu} \right) + \mathcal{O}(\kappa^2) \\ R_{\mu\nu\kappa}^{\;\;\;\;\;\lambda} &= \kappa \left(\partial_\mu \partial_\kappa h_\nu^{\;\;\lambda} - \partial_\nu \partial_\kappa h_\mu^{\;\;\lambda} + \partial^\lambda \partial_\nu h_{\mu\kappa} - \partial^\lambda \partial_\mu h_{\nu\kappa} \right) + \mathcal{O}(\kappa^2) \end{aligned} \tag{28}$$

It follows that the Einstein tensor is

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R \\ &= \kappa \left(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial_\lambda h_\nu^{\;\;\lambda} - \partial_\nu \partial_\lambda h_\mu^{\;\;\lambda} - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\kappa h^{\lambda\kappa} \right) + \mathcal{O}(\kappa^2) \end{aligned} \tag{29}$$

It is straightforward to check that the linearized expression for $G_{\mu\nu}$ is invariant under the gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{30}$$

This implies, that 4 of the 10 degrees of freedom in $h_{\mu\nu}$ can be gauged away.

The expression for $G_{\mu\nu}$ can be simplified by switching to a new fluctuation field $\bar{h}_{\mu\nu}$ defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} h \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} \bar{h}, \tag{31}$$

with

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h. \tag{32}$$

Substitution of the field redefinition into the expression (29) gives

$$G_{\mu\nu} = \kappa \left(\Box \bar{h}_{\mu\nu} - \partial_\mu \partial_\lambda \bar{h}_\nu^{\;\;\lambda} - \partial_\nu \partial_\lambda \bar{h}_\mu^{\;\;\lambda} + \eta_{\mu\nu} \partial_\kappa \partial_\lambda \bar{h}^{\kappa\lambda} \right) + \mathcal{O}(\kappa^2), \tag{33}$$

which is invariant under the modified gauge transformations

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi. \tag{34}$$

It follows, that the Einstein equation

$$G_{\mu\nu} = -\kappa^2 T_{\mu\nu},$$

in the linearized approximation reduces to

$$\Box\bar{h}_{\mu\nu} - \partial_\mu\partial_\lambda\bar{h}_\nu^{\ \lambda} - \partial_\nu\partial_\lambda\bar{h}_\mu^{\ \lambda} + \eta_{\mu\nu}\partial_\kappa\partial_\lambda\bar{h}^{\kappa\lambda} = -\kappa\bar{T}_{\mu\nu}. \tag{35}$$

where $\bar{T}_{\mu\nu}$ are the components of the energy-momentum tensor in the Newtonian (flat space-time) approximation. This is confirmed by taking the four-divergence of the left- and right-hand side of eq. (35) which results in the equation of local energy-momentum conservation in flat space-time:

$$\partial^\mu \bar{T}_{\mu\nu} = 0. \tag{36}$$

This constraint on the sources of the fields $\bar{h}_{\mu\nu}$ reflects the gauge invariance (34).

An action which is gauge invariant modulo the constraint (36) and which is extremized by fields satisfying equation (35) is

$$S = \int d^4x \left[-\frac{1}{2}\left(\partial_\lambda\bar{h}_{\mu\nu}\right)^2 + \left(\partial_\lambda\bar{h}_\mu^{\ \lambda}\right)^2 + \frac{1}{4}\left(\partial_\lambda\bar{h}\right)^2 + \kappa\bar{h}^{\mu\nu}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right)\right], \tag{37}$$

where $\bar{T} = \eta^{\mu\nu}\bar{T}_{\mu\nu}$. The gauge invariance can be used to impose some restrictions on the form of $\bar{h}_{\mu\nu}$. A standard gauge condition (the de Donder gauge) is to require

$$\partial_\lambda\bar{h}_\mu^{\ \lambda} = 0. \tag{38}$$

In this gauge eq. (35) simplifies to the inhomogenous wave-equation

$$\Box\,\bar{h}_{\mu\nu} = -\kappa\bar{T}_{\mu\nu}. \tag{39}$$

The formal solution of this equation is provided by the retarded Green's function:

$$\bar{h}_{\mu\nu}(\mathbf{r}, t) = -\frac{\kappa}{4\pi}\int d^3\mathbf{r}'\,\frac{\bar{T}_{\mu\nu}\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|\right)}{|\mathbf{r} - \mathbf{r}'|}. \tag{40}$$

3 Momentum representation

In empty space the field eqs. (38), (39) can be solved straightforwardly in terms of plane waves:

$$\bar{h}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^2}\,\varepsilon_{\mu\nu}(k)e^{ik\cdot x}. \tag{41}$$

The field equations then imply

$$k^2\varepsilon_{\mu\nu}(k) = 0, \qquad k^\mu\varepsilon_{\mu\nu}(k) = 0. \tag{42}$$

Therefore the wave vector k_μ is light-like, and the amplitude $\varepsilon_{\mu\nu}(k)$ is space-time orthogonal.

The gauge transformations (34) of $h_{\mu\nu}$ imply an equivalent set of gauge transformations of the amplitude $\varepsilon_{\mu\nu}(k)$:

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu \alpha_\nu + k_\nu \alpha_\mu - \eta_{\mu\nu} k \cdot \alpha, \tag{43}$$

where $\alpha_\mu(k)$ are the Fourier transforms of the gauge parameters $\xi_\mu(x)$. Observe, that on the light cone $k^2 = 0$ the gauge condition (42) is preserved by the transformation (43):

$$k^\mu \varepsilon'_{\mu\nu} = k^\mu \varepsilon_{\mu\nu} + k^2 \alpha_\nu + k_\nu k \cdot \alpha - k_\nu k \cdot \alpha = 0. \tag{44}$$

Hence a physical solution remains a physical solution after a gauge transformations, as expected.

The gauge transformations can be used to bring the amplitude $\varepsilon_{\mu\nu}$ in a particular form. We illustrate this for waves propagating in the z-direction:

$$k_\mu = (k, 0, 0, k) \quad \Rightarrow \quad \varepsilon_{0\nu} = \varepsilon_{3\nu}. \tag{45}$$

Choose the gauge parameters α_μ to satisfy

$$\begin{aligned} & k(\alpha_0 + \alpha_3) = -\varepsilon_{00} = -\varepsilon_{03} = -\varepsilon_{33}, \\ & k\alpha_3 = \frac{1}{4}\left(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33}\right), \\ & k\alpha_i = -\varepsilon_{0i} = -\varepsilon_{3i}, \qquad i = (1,2). \end{aligned} \tag{46}$$

After this transformation the amplitude takes a simple form

$$\varepsilon'_{\mu\nu}(k) = \varepsilon_{11}(k)\, e^+_{\mu\nu} + \varepsilon_{12}(k)\, e^\times_{\mu\nu}, \tag{47}$$

where the polarization tensors $e^{(+,\times)}$ are symmetric, transverse and traceless, both in the 3- and 4-dimensional sense:

$$e^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

As a result, for plane-wave fields defined in this way $h = \bar{h} = 0$, and

$$\bar{h}_{\mu\nu}(x) = h_{\mu\nu}(x). \tag{49}$$

4 Canonical spin-2 field theory

To discuss the dynamics of gravitational radiation fields, it is convenient to work in the hamiltonian formulation. This formulation is conveniently constructed using the ADM formulation of general relativity [1]. In the ADM formulation the space-time metric is parametrized as

$$g_{\mu\nu} = \begin{pmatrix} \kappa^2 \left(-N^2 + \gamma^{mn} N_m N_n\right) & \kappa N_n \\ \kappa N_m & \gamma_{mn} \end{pmatrix}. \tag{50}$$

The functions N and N_m are called the lapse and shift functions, respectively; γ_{mn} is the 3-dimensional space-like metric, and γ^{mn} its 3-dimensional inverse. To adapt the ADM formalism to the linearized theory, we slightly redefine the lapse function N and take

$$\begin{aligned} N &= \frac{1}{\kappa} - h_{00}, \qquad N_m = 2h_{0m} = 2h_{0m}, \\ \gamma_{mn} &= \delta_{mn} + 2\kappa h_{mn}. \end{aligned} \tag{51}$$

Then we find the following expressions for the fields $\bar{h}_{\mu\nu}$:

$$\begin{aligned} \bar{h}_{00} &= \frac{1}{2}\left(h_{kk} - N + \frac{1}{\kappa}\right), \qquad \bar{h}_{0m} = \bar{h}_{m0} = \frac{1}{2}\, N_m, \\ \bar{h}_{mn} &= h_{mn} - \frac{1}{2}\,\delta_{mn}\left(h_{kk} + N - \frac{1}{\kappa}\right). \end{aligned} \tag{52}$$

After substitution of these field redefinitions into the action (37) and performing a number of partial integrations we then find the result

$$S = S_0 + S_1 = \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_1\right), \tag{53}$$

with S_0 the free action

$$\begin{aligned} S_0 \;=\; & \int d^4x \left(\frac{1}{2}\left(\dot{h}_{mn}\right)^2 - \frac{1}{2}\left(\dot{h}_{kk}\right)^2 - \frac{1}{2}\left(\nabla_k h_{mn}\right)^2 \right. \\ & + \frac{1}{4}\left(\nabla_m h_{kk}\right)^2 + \left(\nabla_k h_{km} - \frac{1}{2}\nabla_m h_{kk}\right)^2 - \nabla_m N \left(\nabla_k h_{km} - \nabla_m h_{kk}\right) \\ & \left. - \nabla_m N_n \left(\dot{h}_{mn} - \delta_{mn}\dot{h}_{kk}\right) + \frac{1}{8}\left(\nabla_m N_n - \nabla_n N_m\right)^2 \right). \end{aligned} \tag{54}$$

Here an overdot denotes a time derivative. In addition, S_1 represents the coupling to the non-gravitational energy-momentum:

$$S_1 = \int d^4x \left((1 - \kappa N)\bar{T}_{00} - \kappa N_k \bar{T}_{k0} + \kappa h_{mn} \bar{T}_{mn}\right). \tag{55}$$

From the expression (54) we observe, that the action S_0 contains no time derivatives of the fields (N, N_m). Thus these fields represent auxiliary degrees of freedom, acting as a kind of

Lagrange multipliers to impose constraints on the remaining degrees if freedom h_{mn}. This is most transparent in the hamilton formulation for the dynamical fields. As a first step we define the field momenta

$$\pi_{mn} = \frac{\partial \mathcal{L}_0}{\partial \dot{h}_{mn}} = \dot{h}_{mn} - \delta_{mn}\dot{h}_{kk} - \frac{1}{2}\left(\nabla_m N_n + \nabla_n N_m\right) + \delta_{mn}\nabla \cdot N. \tag{56}$$

Next we perform a Legendre transformation w.r.t. $\dot{h}_{mn}$ to obtain

$$\begin{aligned} \mathcal{H}_0 &= \pi_{mn}\dot{h}_{mn} - \mathcal{L}_0 \\ &\simeq \frac{1}{2}\pi_{mn}^2 - \frac{1}{4}\pi_{nn}^2 - N_m \nabla_n \pi_{nm} + \frac{1}{2}\left(\nabla_k h_{mn}\right)^2 - \frac{1}{4}(\nabla_m h_{nn})^2 \\ &\quad - \left(\nabla_n h_{nm} - \frac{1}{2}\nabla_m h_{nn}\right)^2 - N\left(\nabla_m \nabla_n h_{mn} - \Delta h_{nn}\right). \end{aligned} \tag{57}$$

Writing $\mathcal{L}_1 = -\mathcal{H}_1$, the action now takes the form

$$S = \int d^4x \left(\pi_{mn}\dot{h}_{mn} - \mathcal{H}_0 - \mathcal{H}_1\right) = \int d^4x \left(\pi_{mn}\dot{h}_{mn} - \mathcal{H}\right). \tag{58}$$

The hamiltonian field equations are obtained by varying this action. First, the dynamical equations read

$$\begin{aligned} \dot{h}_{mn} &= \frac{\partial \mathcal{H}}{\partial \pi_{mn}} = \pi_{mn} - \frac{1}{2}\delta_{mn}\pi_{kk} + \frac{1}{2}\left(\nabla_m N_n + \nabla_n N_m\right), \\ \dot{\pi}_{mn} &= -\frac{\partial \mathcal{H}}{\partial h_{mn}} = \Delta h_{mn} - \nabla_m \nabla_k h_{kn} - \nabla_n \nabla_k h_{km} + \nabla_m \nabla_n h_{kk} \\ &\quad + \delta_{mn}\left(\nabla_k \nabla_l h_{kl} - \Delta h_{kk}\right) + \nabla_m \nabla_n N - \delta_{mn}\Delta N + \kappa \bar{T}_{mn}. \end{aligned} \tag{59}$$

The first equation reproduces the definition of π_{mn}, eq. (56); the second equation describes the dynamics of the fields h_{mn}. In addition variation of the action (58) w.r.t. the lapse and shift functions (N, N_m) gives a set of constraints

$$\begin{aligned} \nabla_m \nabla_n h_{mn} - \Delta h_{nn} &= \kappa \bar{T}_{00}, \\ \nabla_n \pi_{nm} &= \kappa \bar{T}_{m0}. \end{aligned} \tag{60}$$

It is important to note, that these constraints are imposed at all times; this is possible because applying the dynamical equations (59) shows, that the constraints are preserved by the time evolution owing to the local conservation of energy and momentum (36)

$$\dot{\bar{T}}_{00} = \nabla_m \bar{T}_{m0}, \qquad \dot{\bar{T}}_{m0} = \nabla_n \bar{T}_{nm}. \tag{61}$$

These same equations also can be seen to imply that the hamiltonian action, as well as the canonical momenta π_{mn}, are invariant under the gauge transformations

$$h'_{mn} = h_{mn} + \nabla_m a_n + \nabla_n a_m, \qquad N'_m = N_m + 2\dot{a}_m, \qquad N' = N. \tag{62}$$

Also the hamiltonian action and the fields h_{mn} are invariant (modulo boundary terms and conservation of the energy-momentum tensor) under the transformations

$$\pi'_{mn} = \pi_{mn} + \nabla_m \nabla_n a - \delta_{mn} \Delta a, \qquad N'_m = N_m - \nabla_m a, \qquad N' = N + \dot{a}. \tag{63}$$

These transformations can be used to impose gauge conditions on the lapse and shift functions (N, N_m). The gauge conditions can be chosen such that

$$h_{nn} = 0. \tag{64}$$

This is achieved by taking

$$N_m = 0, \tag{65}$$

and

$$N = \frac{1}{\kappa} + \Phi \quad \text{s.t.} \quad \Delta N = \Delta \Phi = \frac{\kappa}{2} \left(\bar{T}_{00} + \bar{T}_{nn} \right), \tag{66}$$

With this choice of gauge $h_{00} = \Phi$ and $h_{m0} = 0$; in particular, in empty space $\Phi = 0$.

To prove (64), we first note that the gauge conditions (65), (66) are preserved by residual gauge transformations of the type (62), (63) with

$$\dot{a} = 0, \qquad \dot{a}_m = \frac{1}{2} \nabla_m a. \tag{67}$$

Note that for these residual transformations $\ddot{a}_m = 0$. Under such gauge transformations

$$\begin{aligned} h'_{nn} &= h_{nn} + 2\nabla \cdot a, & \pi'_{nn} &= \pi_{nn} - 2\Delta a, \\ N' &= N, & N'_m &= N_m. \end{aligned} \tag{68}$$

Now at the fixed time $t = 0$ we can choose a_m such that the right-hand side of the first equation vanishes:

$$\nabla \cdot a = -\frac{1}{2} h_{nn} \quad \Rightarrow \quad h'_{nn} = 0. \tag{69}$$

Then at this time

$$\dot{h}_{nn} = -2\nabla \cdot \dot{a} = -\Delta a, \tag{70}$$

and as a result

$$\pi'_{nn} = \pi_{nn} + 2\dot{h}_{nn} = 0 \quad \Rightarrow \quad \dot{h}'_{nn} = 0. \tag{71}$$

Hence by this gauge transformation we have achieved tracelessness $h'_{nn} = \dot{h}'_{nn} = \pi'_{nn} = 0$ as an initial condition. Next observe, that according to the field equation (59) and the gauge choice (65)

$$\dot{\pi}'_{nn} = -2\ddot{h}'_{nn} = \nabla_m \nabla_n h_{mn} - \Delta h_{mn} - 2\Delta N + \kappa \bar{T}_{nn} = 0.$$

The last equality follows by the first constraint (60) and the gauge choice (66), and holds at *all* times. Therefore all higher time derivatives of h'_{nn} vanish at the initial time $t = 0$. Combined with the results (69) and (71) it follows that $h'_{nn} = \pi'_{nn} = 0$ at all times.

With this gauge choice the field equations (59) are

$$
\begin{aligned}
&\pi_{mn} = \dot{h}_{mn}, \\
&\dot{\pi}_{mn} = \Delta h_{mn} - \nabla_m \nabla_k h_{kn} - \nabla_n \nabla_k h_{km} + \delta_{mn} \nabla_k \nabla_l h_{kl} + \nabla_m \nabla_n N - \delta_{mn} \Delta N + \kappa \bar{T}_{mn},
\end{aligned}
\tag{72}
$$

whilst the constraints (60) become

$$
\nabla_m \nabla_n h_{mn} = \kappa \bar{T}_{00}, \qquad \nabla_n \pi_{nm} = \partial_t (\nabla_n h_{nm}) = \kappa \bar{T}_{0m}. \tag{73}
$$

Therefore in empty space the fields h_{mn} are both traceless and transverse:

$$
h_{nn} = 0, \qquad \nabla_n h_{nm} = 0, \tag{74}
$$

and the physical solutions in $D = 4$ space-time dimensions have only 2 dynamical degrees of freedom. Physical free fields then satisfy the wave equation

$$
\Box h_{mn} = \Delta h_{mn} - \ddot{h}_{mn} = 0. \tag{75}
$$

The upshot is, that free fields h_{mn} describe gravitational waves moving at the speed of light. In the language of quantum field theory, this implies that the graviton is a massless particle with two spin states of helicity ± 2.

5 Energy and momentum of the field

In the hamiltonian formulation the action (58) depends only linearly on the lapse and shift functions; in this formalism they are indeed just lagrange multipliers implementing the constraints (60). Once the contraints are taken into account we can freely assign values to (N, N_m), as in eq. (66). The consistency of the contraints with the dynamics is guaranteed by gauge invariance, and indeed the assignment (65), (66) is just a choice of gauge.

The theory under discussion is that of a spin-2 field in flat Minkoswski space, the invariance under time- and space translations imply conservation of energy and momentum. Using the gauge choice $N_m = h_{nn} = 0$ and the constraints (73) of sect. 4, the energy and momentum are given by

$$
E = \int d^3x \, \mathcal{H}, \qquad P_k = \int d^3x \, \Pi_k, \tag{76}
$$

where

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2} \pi_{mn}^2 + \frac{1}{2} (\nabla_k h_{mn})^2 - (\nabla_n h_{nm})^2 - \kappa \, h_{mn} \bar{T}_{mn}, \\
\Pi_k &= \pi_{mn} \nabla_k h_{mn} - 2 \pi_{kn} \nabla_m h_{mn}.
\end{aligned}
\tag{77}
$$

It is easy to verify that upon using the field equations and constraints

$$\frac{\partial \mathcal{H}}{\partial t} = \nabla \cdot \Pi + \kappa \, h_{mn} \dot{\bar{T}}_{mn}. \tag{78}$$

In the absence of sources this equation states the local conservation of energy and momentum of free gravitational radiation in flat space-time. Indeed, from the equation of continuity (78) with $\bar{T}_{mn} = 0$ it follows, that the change in radiation energy in a certain volume Ω with closed boundery $\partial \Omega$ equals the flux through the boundary:

$$\frac{dE_{rad}}{dt} = \int_{\Omega} d^3x \, \nabla \cdot \Pi = \int_{\partial\Omega} d^2\Sigma \, \Pi_n, \tag{79}$$

where Π_n is the normal component of the radiation momentum density on the surface element $d^2\Sigma$. As a result, it is possible to interpret the expressions (77) as the time-components of the energy momentum tensor of the spin-2 field h_{mn}. We emphasize, that this is possible only because we consider fluctuations in a flat space-time background.

6 Frequency representation

A discussion of gravitational radiation is most conveniently formulated in terms of the frequency modes, defined by the Fourier transforms

$$h_{\mu\nu}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, e^{-i\omega t} \, \tilde{h}_{\mu\nu}(\omega, \mathbf{r}), \qquad T_{\mu\nu}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, e^{-i\omega t} \, \tilde{T}_{\mu\nu}(\mathbf{r}, \omega). \tag{80}$$

As the fields $h(\mathbf{r}, t)$ are real, the Fourier modes satisfy

$$\tilde{h}^*_{\mu\nu}(\mathbf{r}, \omega) = \tilde{h}_{\mu\nu}(\mathbf{r}, -\omega). \tag{81}$$

In terms of the frequency modes, the field equations and constraints read

$$\boxed{\begin{aligned} & -(\Delta + \omega^2)\, \tilde{h}_{mn} = \kappa \, \tilde{\theta}_{mn} \\ & \qquad = \nabla_m \nabla_m \tilde{N} + \kappa \left(\tilde{T}_{mn} - \frac{i}{\omega} \left(\nabla_m \tilde{T}_{n0} + \nabla_n \tilde{T}_{m0} \right) + \frac{1}{2}\, \delta_{mn} \left(\tilde{T}_{00} - \tilde{T}_{kk} \right) \right), \\ & -i\omega \nabla_n \tilde{h}_{nm} = \kappa \, \tilde{T}_{m0}, \\ & \nabla_m \nabla_n \tilde{h}_{mn} = \kappa \, \tilde{T}_{00}. \end{aligned}} \tag{82}$$

The conservation laws for energy-momentum take the form

$$-i\omega \tilde{T}_{00} = \nabla_n \tilde{T}_{n0}, \qquad -i\omega \tilde{T}_{0m} = \nabla_n \tilde{T}_{nm}. \tag{83}$$

It is easy to write down a special solution of the first equation (82) in integral form:

$$\tilde{h}_{mn}(\mathbf{r},\omega) = \frac{\kappa}{4\pi} \int d^3\mathbf{r}'\, \frac{e^{i\omega|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\, \tilde{\theta}_{mn}(\mathbf{r}',\omega). \tag{84}$$

Also note, that $\tilde{\theta}_{mn}$ is traceless:

$$\begin{aligned} \tilde{\theta}_{nn} &= \frac{1}{\kappa}\,\Delta N + \left(\tilde{T}_{nn} - \frac{2i}{\omega}\,\nabla_n \tilde{T}_{n0} + \frac{3}{2}\left(\tilde{T}_{00} - \tilde{T}_{nn}\right)\right) \\ &= \frac{1}{\kappa}\,\Delta N - \frac{1}{2}\left(\tilde{T}_{00} + \tilde{T}_{nn}\right) = 0, \end{aligned} \tag{85}$$

whilst the divergence is consistent with that of $\tilde{h}_{mn}$:

$$\nabla_n \tilde{\theta}_{nm} = -\frac{i}{2}\left(\Delta + \omega^2\right) \tilde{T}_{0m} = \frac{1}{2\omega}\left(\Delta + \omega^2\right) \nabla_n \tilde{T}_{nm}. \tag{86}$$

7 Emission of radiation

Consider now the situation where the source of the fields $h_{\mu\nu}$ is localized in a finite volume, which radiates away part of its energy in the form of gravitational radiation. We choose the origin of our co-ordinates in the center of the source, and consider a large sphere of radius R and corresponding volume V; we wish to compute the flux of radiative energy through the surface of the sphere ∂V. The change in energy because of this flux is

$$\frac{dE_{rad}}{dt} = \int_V d^3\mathbf{r}\, \nabla \cdot \Pi = \int_{\partial V} d^2\Sigma\, \Pi^{(n)}.$$

Here $\Pi^{(n)} = \hat{R} \cdot \Pi$, with $\hat{R}$ the radial unit vector normal to the spherical surface element with area $d^2\Sigma = R^2 d\Omega = R^2 \sin\theta d\theta d\varphi$.

As we are interested in the spectral distribution of the energy, we study the Fourier transform of the momentum

$$\tilde{\Pi}_k(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t}\, \Pi_k(\mathbf{r},t) = \frac{i}{\sqrt{2\pi}} \int d\omega'\, \omega'\, \tilde{h}^*_{mn}(\omega') \nabla_k \tilde{h}_{mn}(\omega'+\omega). \tag{87}$$

To obtain the last expression we have used that at the surface of the sphere (where the expression for Π is to be evaluated) we are far away from all material sources and therefore the tensor field is transverse: $\nabla_m \tilde{h}_{mn}(\omega) = 0$ (for $\omega \neq 0$).

We can now compute the *total* amount of energy radiated into the cone with opening angle $d\Omega$ as

$$\begin{aligned} \frac{dE_{rad}}{d\Omega} &= -R^2 \int_{-\infty}^{\infty} dt\, \Pi^{(n)} = \sqrt{2\pi}\, R^2 \tilde{\Pi}^{(n)}(0) \\ &= -\frac{iR^2}{2} \int_{-\infty}^{\infty} d\omega\, \omega\, \tilde{h}^*_{mn}(\omega)\, \overleftrightarrow{\nabla}_R\, \tilde{h}_{mn}(\omega). \end{aligned} \tag{88}$$

Here $\nabla_R = \hat{R} \cdot \nabla$ is the gradient in the radial direction, and we have written the integral in manifestly real form by antisymmetrizing the derivative. The minus sign is included because the energy radiated represents a decrease in energy inside the volume V.

To make further progress, we write

$$\tilde{h}_{mn}(\mathbf{r}, \omega) = \frac{\kappa}{4\pi} \frac{e^{i\omega R}}{R} \tilde{t}_{mn}(\mathbf{r}, \omega). \tag{89}$$

Comparison with the expression (84) shows, that

$$\begin{aligned} \tilde{t}_{mn}(\mathbf{r}, \omega) &= \int d^3\mathbf{r}' \, \frac{R}{|\mathbf{r}' - \mathbf{r}|} \, e^{i\omega(|\mathbf{r}'-\mathbf{r}|-R)} \, \tilde{\theta}_{mn}(\mathbf{r}', \omega) \\ &= \int d^3\mathbf{r}' \, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \, \tilde{\theta}_{mn}(\mathbf{r}', \omega) \left(1 + \frac{\mathbf{r}' \cdot \hat{R}}{R} + \frac{i\omega}{2R} \left(\mathbf{r}'^2 - (\mathbf{r}' \cdot \hat{R})^2 \right) + ... \right), \end{aligned} \tag{90}$$

where we have used $\mathbf{r}^2 = R^2$ and therefore

$$|\mathbf{r}' - \mathbf{r}| = \sqrt{(\mathbf{r}' - \mathbf{r})^2} = R - \mathbf{r}' \cdot \hat{R} + \frac{1}{2R} \left(\mathbf{r}'^2 - (\mathbf{r}' \cdot \hat{R})^2 \right) + ... \tag{91}$$

Using

$$\nabla_R = \frac{d}{dr}, \qquad \mathbf{r}^2 = R^2, \tag{92}$$

the expression for the energy radiated becomes

$$\frac{dE_{rad}}{d\Omega} = \frac{\kappa^2}{16\pi^2} \int_{-\infty}^{\infty} d\omega \left(\omega^2 \, |\tilde{t}_{mn}|^2 + \mathcal{O}(1/R) \right). \tag{93}$$

Here the minus sign indicates that the energy flows out of the sphere. We also recall, that

$$\frac{\kappa^2}{16\pi^2} = \frac{G}{2\pi}. \tag{94}$$

The only terms in the integrand of (93) which survive for large R are the leading terms of $|\tilde{t}_{mn}|^2$; in this limit eq. (91) reduces to

$$\begin{aligned} \tilde{t}_{mn} &\simeq \int d^3\mathbf{r}' \, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \, \tilde{\theta}_{mn}(\mathbf{r}', \omega) + \mathcal{O}(1/R) \\ &= \int d^3\mathbf{r}' \, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \left[\frac{1}{\kappa} \nabla'_m \nabla'_n \tilde{N} + \left(\tilde{T}_{mn} - \frac{i}{\omega} \left(\nabla'_m \tilde{T}_{n0} + \nabla'_n \tilde{T}_{m0} \right) \right.\right. \\ &\qquad \left.\left. + \frac{1}{2} \delta_{mn} \left(\tilde{T}_{00} - \tilde{T}_{kk} \right) \right) \right]. \end{aligned} \tag{95}$$

Now by partial integration every gradient operator ∇'_m brings down a factor $i\omega\hat{R}_m$; together with the gauge condition (66) this results in the expression

$$\begin{aligned} \tilde{t}_{mn} &\simeq \int d^3\mathbf{r}'\, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \left[\tilde{T}_{mn} - \frac{1}{2}\,\delta_{mn}\tilde{T}_{kk} - \hat{R}_m\hat{R}_k\tilde{T}_{kn} - \hat{R}_n\hat{R}_k\tilde{T}_{km} \right. \\ & \qquad \left. + \frac{1}{2}\hat{R}_m\hat{R}_n\tilde{T}_{kk} + \frac{1}{2}\,\delta_{mn}\hat{R}_k\hat{R}_l\tilde{T}_{kl} + \frac{1}{2}\hat{R}_m\hat{R}_n\hat{R}_k\hat{R}_l\tilde{T}_{kl} \right] \\ &= \int d^3\mathbf{r}'\, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \left(\delta_{mk} - \hat{R}_m\hat{R}_k \right) \left[\tilde{T}_{mn} - \frac{1}{2}\,\delta_{mn} \left(\tilde{T}_{qq} - \hat{R}_p\hat{R}_q\tilde{T}_{pq} \right) \right] \left(\delta_{ln} - \hat{R}_l\hat{R}_n \right). \end{aligned} \tag{96}$$

It follows, that in this approximation $\tilde{t}_{mn}$ is both traceless and transverse in the sense that

$$\hat{R}_m\tilde{t}_{mn} = \tilde{t}_{nm}\hat{R}_m = 0. \tag{97}$$

Furthermore note, that the quantity in the middle

$$\frac{\omega^2}{2}\,\tilde{\Delta}_{mn} = \int d^3\mathbf{r}'\, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \left[\tilde{T}_{mn} - \frac{1}{2}\,\delta_{mn} \left(\tilde{T}_{kk} - \hat{R}_k\hat{R}_l\tilde{T}_{kl} \right) \right] \tag{98}$$

can be expressed in terms of the traceless part of $\tilde{T}_{mn}$. Define

$$\frac{\omega^2}{2}\,\tilde{I}_{mn} = \int d^3\mathbf{r}'\, e^{-i\omega\mathbf{r}'\cdot\hat{R}} \left(\tilde{T}_{mn} - \frac{1}{3}\,\delta_{mn}\tilde{T}_{kk} \right). \tag{99}$$

Then

$$\tilde{t}_{mn} \simeq \frac{\omega^2}{2} \left(\delta_{mk} - \hat{R}_m\hat{R}_k \right) \left(\tilde{I}_{kl} + \frac{1}{2}\,\delta_{kl}\,\hat{R}\cdot\tilde{I}\cdot\hat{R} \right) \left(\delta_{ln} - \hat{R}_l\hat{R}_n \right). \tag{100}$$

Substitution of this result and eq. (94) into (93) then gives the result

$$\frac{dE_{rad}}{d\Omega} = \frac{G}{8\pi} \int_{\infty}^{\infty} d\omega\,\omega^6 \left(|\tilde{I}_{mn}|^2 - 2\hat{R}\cdot\tilde{I}^*\cdot\tilde{I}\cdot\hat{R} + \frac{1}{2}\,|\hat{R}\cdot\tilde{I}\cdot\hat{R}|^2 \right). \tag{101}$$

Now averaging over all directions leads to replacing

$$\langle\hat{R}_m\hat{R}_n\rangle = \frac{1}{3}\,\delta_{mn}, \qquad \langle\hat{R}_k\hat{R}_l\hat{R}_m\hat{R}_n\rangle = \frac{1}{15}\left(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\right), \tag{102}$$

with the result

$$\langle\frac{dE_{rad}}{d\Omega}\rangle = \frac{G}{20\pi} \int_{-\infty}^{\infty} d\omega\,\omega^6\,|\tilde{I}_{mn}|^2. \tag{103}$$

For a source emitting isotropic radiation it follows, that the total energy radiated in all directions is 4π times this quantity:

$$E_{rad} = \frac{G}{5} \int_{-\infty}^{\infty} d\omega\,\omega^6\,|\tilde{I}_{mn}|^2. \tag{104}$$

In the time domain we have

$$I_{mn}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, e^{i\omega t} \, \tilde{I}_{mn}, \tag{105}$$

and

$$E_{rad} = \frac{G}{5} \int_{-\infty}^{\infty} dt \left| \frac{d^3 I_{mn}}{dt^3} \right|^2 . \tag{106}$$

For a continuous source the total energy radiated would formally diverge (although of course in practice there exist no sources which can radiate an infinite amount of energy). In this case one better replaces the total energy by the average energy per unit of time:

$$W_{rad} = \frac{G}{5} \overline{\left| \frac{d^3 I_{mn}}{dt^3} \right|^2}, \tag{107}$$

where the overline denotes a time average

$$\overline{A} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, A(t). \tag{108}$$

For a purely periodic source this can be replaced by the average over a single period.

8 The quadrupole moment

In the non-relativistic limit the traceless tensor $\tilde{I}_{mn}$ is the Fourier transform of the quadruopole moment:

$$\tilde{I}_{mn}(\mathbf{r}, \omega) = \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \, \tilde{\rho}(\mathbf{r}', \omega) \left(r'_m r'_n - \frac{1}{3} \, \delta_{mn} \, \mathbf{r}'^2 \right), \tag{109}$$

where $\tilde{\rho}(\omega)$ are the frequency components of the mass density. We can rewrite Fourier intergrals of the type (99) as follows

$$\begin{aligned} \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \, \tilde{T}_{mn} &= \frac{1}{2} \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \left(\tilde{T}_{mk} \nabla_k r'_n + \tilde{T}_{nk} \nabla_k r'_m \right) \\ = \frac{i\omega}{2} & \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \left(r'_n \left(\tilde{T}_{m0} + \hat{R}_k \tilde{T}_{km} \right) + r'_m \left(\tilde{T}_{n0} + \hat{R}_k \tilde{T}_{kn} \right) \right), \end{aligned} \tag{110}$$

after partial integration. We can perform the same trick once again to obtain

$$\int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \, \tilde{T}_{mn} = -\frac{\omega^2}{2} \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \, r'_m r'_n \left(\tilde{T}_{00} + 2\hat{R}_k \tilde{T}_{k0} + \hat{R}_k \tilde{T}_{kl} \hat{R}_l \right). \tag{111}$$

Therefore

$$\tilde{I}_{mn} = \int d^3\mathbf{r}' \, e^{-i\omega \mathbf{r}' \cdot \hat{R}} \, \tilde{I} \left(r'_m r'_n - \frac{1}{3} \delta_{mn} \, \mathbf{r}'^2 \right), \tag{112}$$

where

$$\tilde{I} = -\left(\tilde{T}_{00} + 2\hat{R}_k\tilde{T}_{k0} + \hat{R}_k\tilde{T}_{kl}\hat{R}_l\right). \tag{113}$$

Finally, the non-relativistic limit is obtained by making the approximations

$$\tilde{T}_{00} \approx -\tilde{\rho}, \qquad \left(\|\tilde{T}_{0m}\|, \|\tilde{T}_{mn}\|\right) \ll \|\tilde{T}_{00}\|. \tag{114}$$

In this approximation we obtain the quadrupole formula (109).

9 Example: binary stars

To illustrate the above results, we compute the radiation emitted by two masses circulating about a common center of mass, like a binary star system in circular orbit. We compute the orbit in the (non-relativistic) Newtonian approximation.

Consider two objects of mass m_1 and m_2 and positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The total mass is $M = m_1 + m_2$, and the reduced mass is $\mu = m_1 m_2/M$. Choose the origin of co-ordinates in the center of mass:

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0. \tag{115}$$

By conservation of momentum the center of mass remains fixed. The relative motion of the masses is governed by Newton's law of gravity, which takes the form

$$\mu\ddot{\mathbf{r}} = -\frac{G\mu M}{r^2}\,\hat{\mathbf{r}}, \tag{116}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The simplest solutions correspond to circular motion. In spherical co-ordinates they are characterized by

$$r = \text{constant}, \qquad \theta = \frac{\pi}{2} \qquad \varphi = \omega t, \tag{117}$$

where the angular frequency is given by

$$\omega^2 = \frac{GM}{r^3}. \tag{118}$$

The positions of the two masses at time t are then given by

$$\mathbf{r}_1 = \frac{m_2 r}{M}\left(\cos\omega t, \sin\omega t, 0\right), \qquad \mathbf{r}_2 = -\frac{m_1 r}{M}\left(\cos\omega t, \sin\omega t, 0\right). \tag{119}$$

As a result, the traceless part of the moment of inertia normalized as in eq. (109) becomes

$$\begin{aligned} I_{mn}(t) &= m_1\left(r_{1m}r_{1n} - \frac{1}{3}\delta_{mn}\,\mathbf{r}_1^2\right) + m_2\left(r_{2m}r_{2n} - \frac{1}{3}\delta_{mn}\,\mathbf{r}_2^2\right) \\ &= \frac{\mu r^2}{2}\begin{pmatrix} \frac{1}{3} + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \frac{1}{3} - \cos 2\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \end{aligned} \tag{120}$$

and its third time derivative reads

$$\frac{d^3 I_{mn}}{dt^3} = -4\mu r^2\omega^3 \begin{pmatrix} -\sin 2\omega t & \cos 2\omega t & 0 \\ \cos 2\omega t & \sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{121}$$

Taking the square and using the result (118) we find upon inserting the correct factors of c from dimensional analysis

$$W_{rad} = \frac{G}{5c^5}\overline{\left|\frac{d^3 I_{mn}}{dt^3}\right|^2} = \frac{32}{5}\frac{G^4\mu^2 M^3}{c^5 r^5}. \tag{122}$$

We can express this numerically by relating it to the radiation of two solar masses at a distance of 1 A.U. = 1.50×10^{11} m. The result is

$$W_{rad} = 0.43 \times 10^{14} \frac{\left(\frac{m_1}{M_\odot}\right)^2 \left(\frac{m_2}{M_\odot}\right)^2 \left(\frac{m_1+m_2}{2M_\odot}\right)}{\left(\frac{r}{1\text{A.U.}}\right)^5} \quad \text{J s}^{-1}. \tag{123}$$

In the same units, the frequency is given by

$$\nu = \frac{\omega}{2\pi} = 0.4 \times 10^{-7} \frac{\left(\frac{m_1+m_2}{2M_\odot}\right)^{1/2}}{\left(\frac{r}{1\text{A.U.}}\right)^{3/2}} \quad \text{Hz}. \tag{124}$$

Finally, the average flux per square meter measured at a distance R expressed in parsecs (1 pc = 3.1×10^{16} m), is

$$\begin{aligned} \Phi &= \frac{W_{rad}}{4\pi R^2} \\ &= 0.3 \times 10^{-20} \frac{\left(\frac{m_1}{M_\odot}\right)^2 \left(\frac{m_2}{M_\odot}\right)^2 \left(\frac{m_1+m_2}{2M_\odot}\right)}{\left(\frac{R}{1\text{pc}}\right)^2 \left(\frac{r}{1\text{A.U.}}\right)^5} \quad \text{J m}^{-2}\,\text{s}^{-1}. \end{aligned} \tag{125}$$

Of course, the real flux depends on the orientation of the line of sight with respect to the plane of the orbit. Nevertheless, for ordinary binary stars of solar size and at galactic distances both the frequency and the intensity of the radiation are clearly very low.

References

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