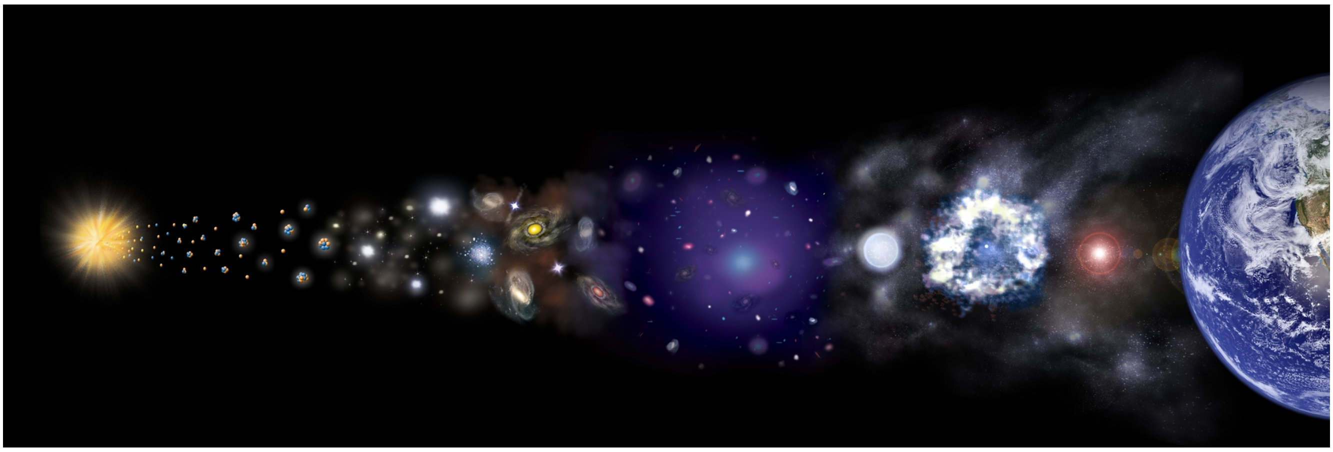
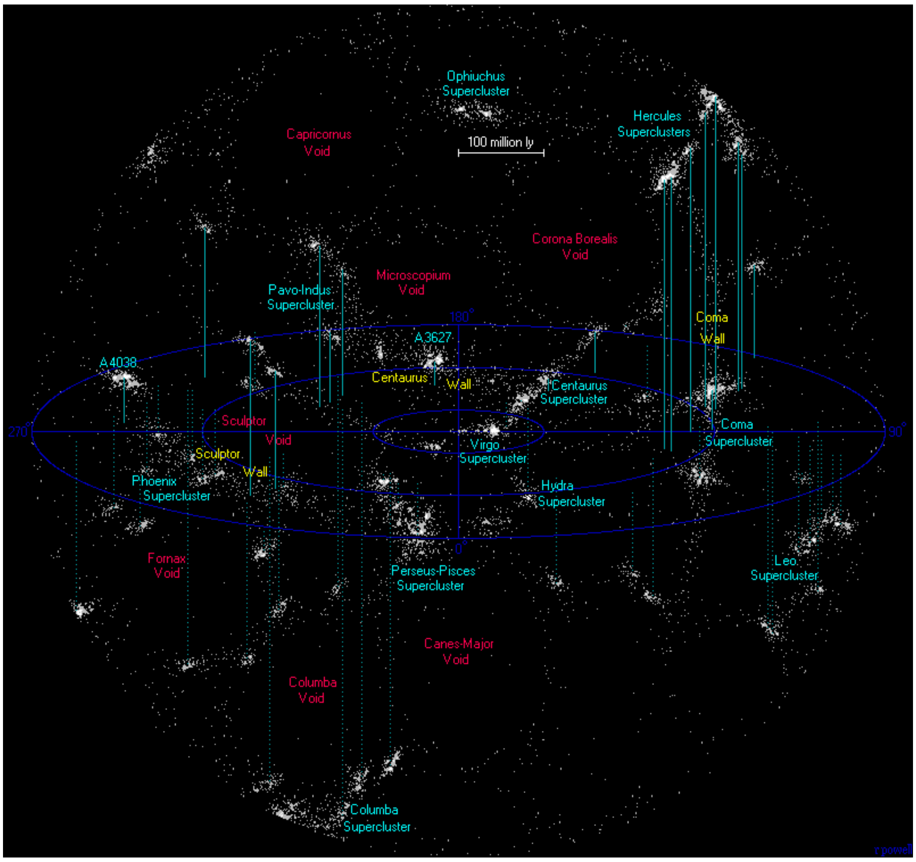
Topical lectures,
20-03-2019

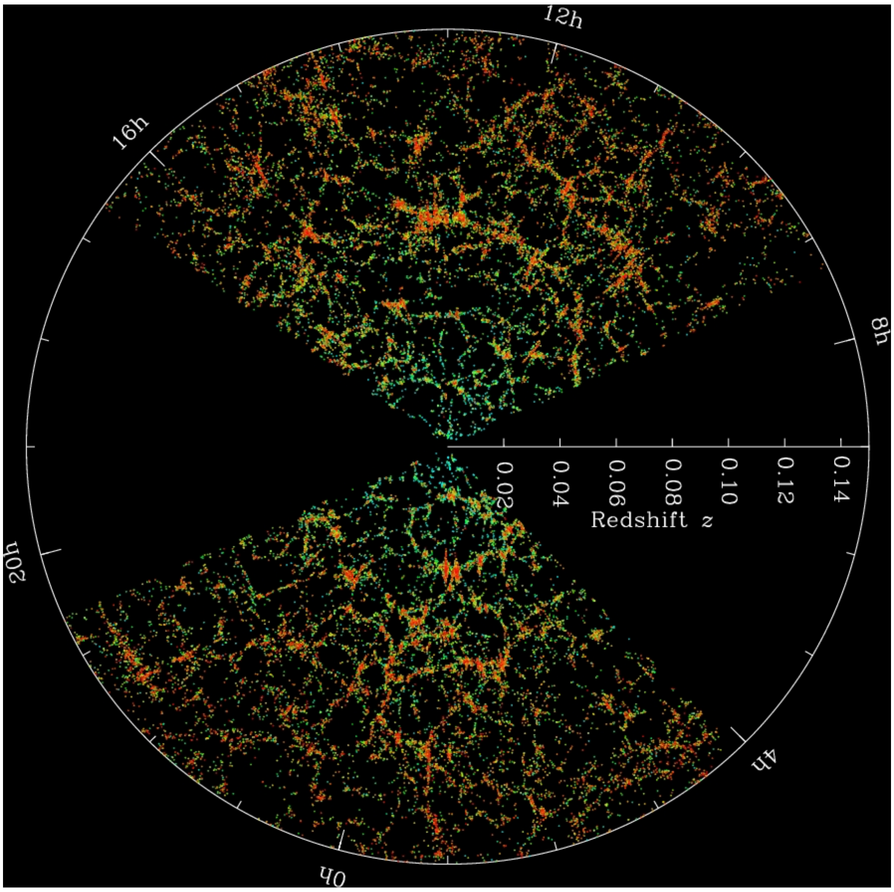
Nikhef

General Relativity and Cosmology

J.W. van Holten

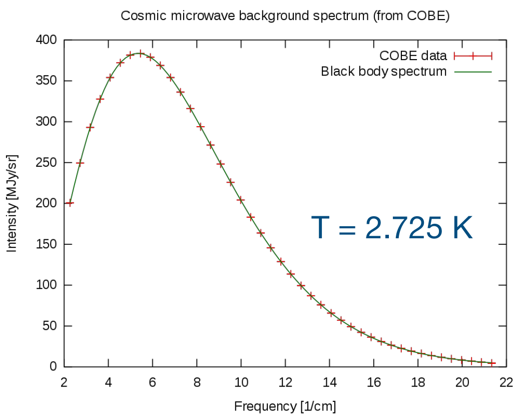






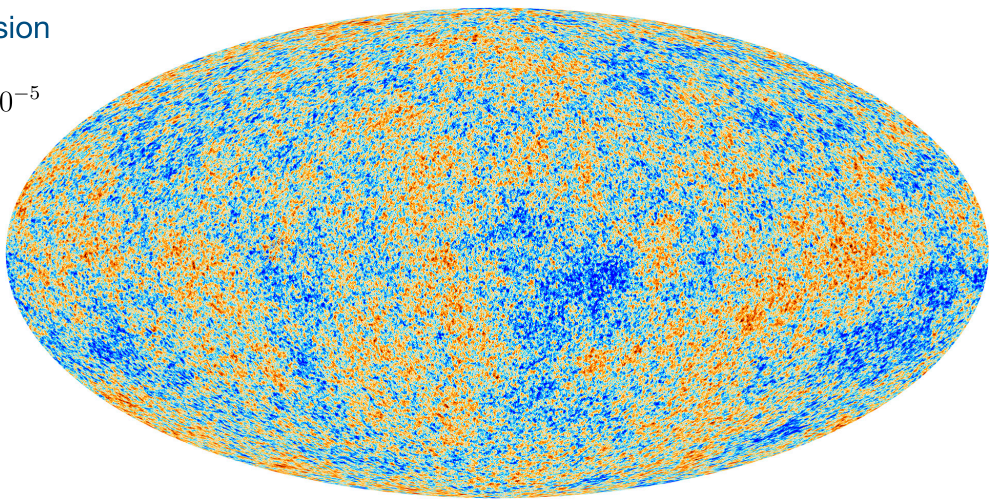
Distribution of galaxies in a wedge of the universe centered on our home galaxy

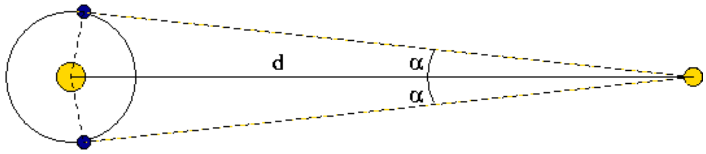
Average spectral intensity (Fyras data)



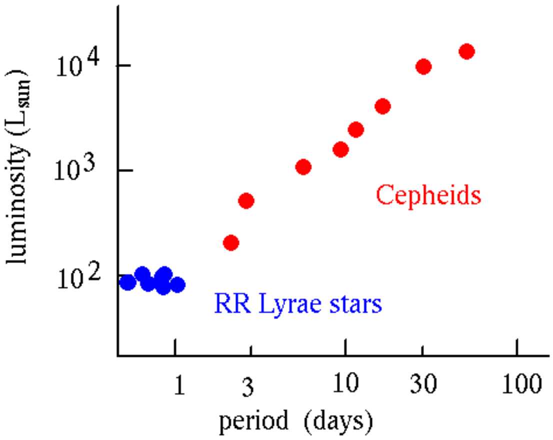
CMB
Planck mission

$$\frac{\Delta T}{T} \sim 10^{-5}$$





distance measurement by parallax

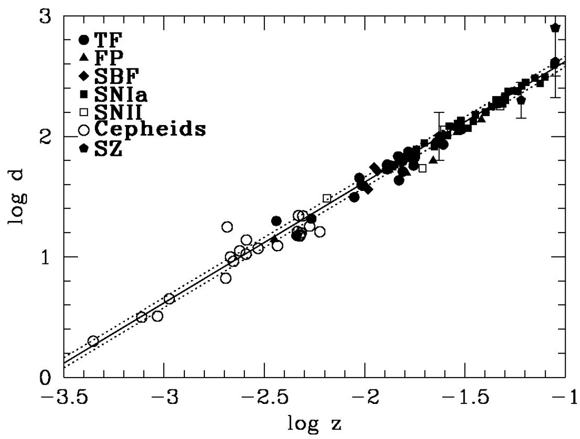


Henrietta Leavitt

Edwin Hubble

Cepheids in Andromeda (1923)

redshift vs distance



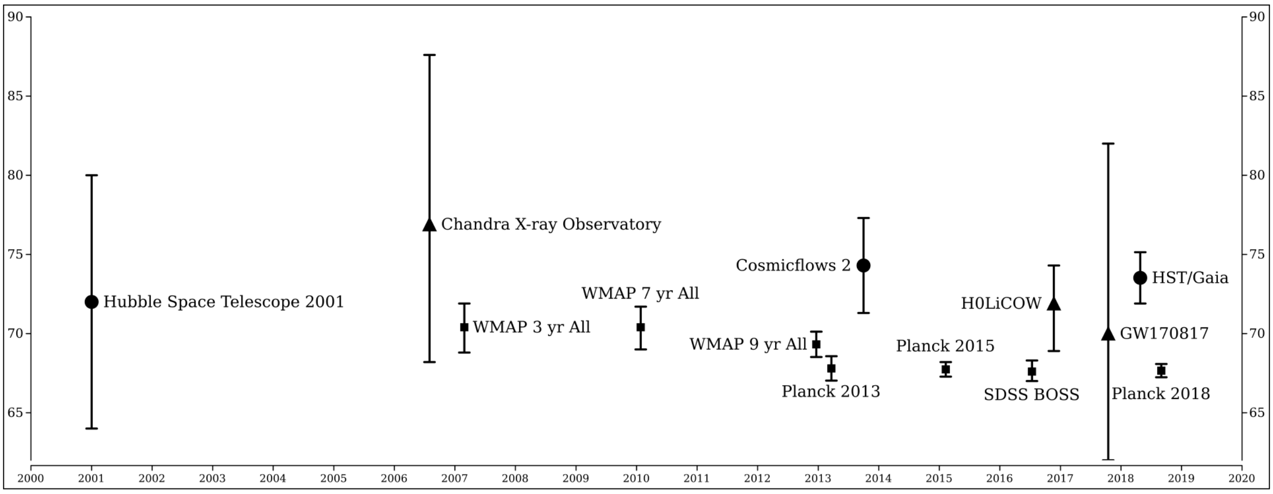
(velocity of source)

Doppler: $$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \simeq \frac{v}{c}$$

Hubble: $$z = \frac{H_o\, d}{c} \quad \Leftrightarrow \quad v = H_o\, d$$

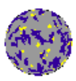
Universe expands!

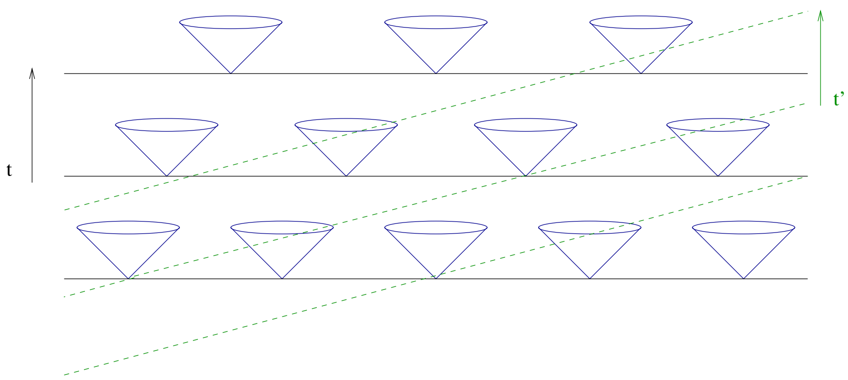
Recent determinations of Hubble constant H_0



H_0 = 68 - 72 km/s/Mpc

Cosmic time





scale factor

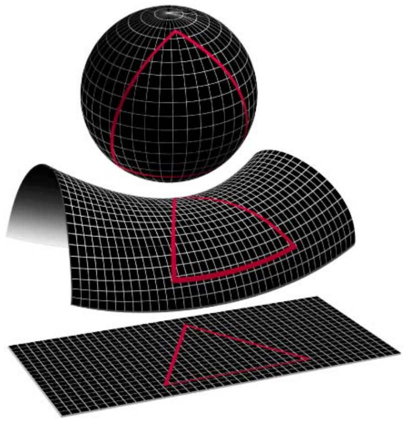
Space-time intervals
in expanding flat space:

$$
\begin{aligned}
ds^2 &= -c^2dt^2 + a^2(t)\,[dx^2 + dy^2 + dz^2] \\
&= -c^2dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]
\end{aligned}
$$

General spaces with constant curvature
(Friedmann-Lemaitre

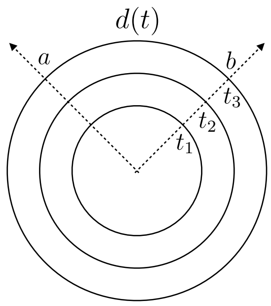
$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

flat:	$k = 0$
spherical:	$k = +1$
hyperbolic:	$k = -1$



redshift - scale factor:

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} \quad \Leftrightarrow \quad 1 + z = \frac{a_0}{a}$$



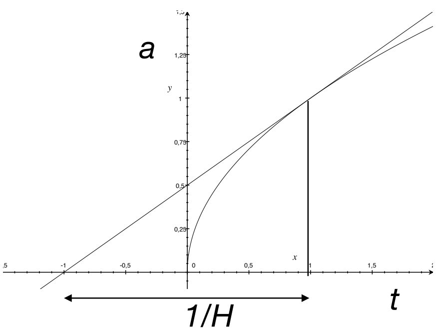
$$d(t) = \left[\int_a^b ds\right]_{t=fixed} = a(t)\left[\int_a^b ds\right]_{a=1}$$

$$\longrightarrow \quad v = \dot{d} = \frac{\dot{a}}{a}\, d = H d$$

with $H(t) = \left.\frac{\dot{a}}{a}\right|_t$

Hubble parameter = relative expansion rate

(at various times throughout the history of the universe)



Cosmic dynamics

Einstein equations (GR):

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\varepsilon}{3c^2} - \frac{kc^2}{a^2}$$ (Friedmann equation)

$$\frac{d(\varepsilon a^3)}{dt} + p\,\frac{da^3}{dt} = 0$$ (local energy conservation)

cf. $\frac{dE}{dt} = -p\,\frac{dV}{dt}$

where ε = energy density, p = pressure — of cosmic constituents

in addition: need equation of state $\varepsilon(p) \longrightarrow (\varepsilon, p, a)$

N.B.: from now on use units in which $c = 1$

Important examples

1. Radiation (relativistic particle gas):

$$p = \frac{\varepsilon}{3} \quad \Rightarrow \quad \varepsilon_r a^4 = \text{constant.}$$

2. Non-relativistic matter (dust):

$$p = 0 \quad \Rightarrow \quad \varepsilon_m a^3 = \text{constant.}$$

3. Cosmological constant (vacuum energy):

$$p = -\varepsilon \quad \Rightarrow \quad \varepsilon_v = \text{constant.}$$

more generally: $p = w\varepsilon \quad \Rightarrow \quad \varepsilon a^{3(1+w)} = \text{constant}$

Time evolution of scale factor

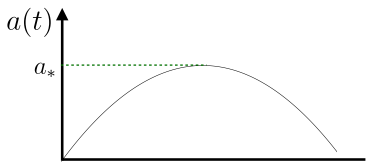
in a flat universe: $k = 0 \longrightarrow a(t) \sim t^{2/[3(1+w)]}$

radiation dominated regime (RD) $a_r(t) \sim \sqrt{t}$

matter dominated regime (MD) $a_m(t) \sim t^{2/3}$

vacuum-energy dominated regime (VD) $a_v(t) \sim e^{Ht}$

Is there an end to expansion?



$$\dot{a}^2 = \frac{8\pi G}{3}\varepsilon - k = \frac{8\pi G\varepsilon_0}{3}\left(\frac{a_0}{a}\right)^{1+3w} - k \overset{a=a_*}{=} 0$$

$$\ddot{a}_* = -\frac{(1+3w)}{2}\frac{k}{a_*} \leq 0$$

Critical density

$$k = 0 \quad \Rightarrow \quad \varepsilon_c = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G}\frac{\dot{a}^2}{a^2}.$$

With present value H_0:

$$\varepsilon_{c0} = 5.2\,\mathrm{GeV/m^3}$$

Friedman eqn.:

$$\varepsilon = \frac{3H^2}{8\pi G} + \frac{3k}{8\pi G a^2} = \varepsilon_c\left(1 + \frac{k}{a^2H^2}\right)$$

$$= \varepsilon_{r0}\left(\frac{a_0}{a}\right)^4 + \varepsilon_{m0}\left(\frac{a_0}{a}\right)^3 + \varepsilon_{v0}.$$

Reparametrization

The present values of the various contribution to the energy density can be represented as fractions of this critical density:

$$\Omega_m \equiv \frac{\varepsilon_{m\,0}}{\varepsilon_{c\,0}}, \qquad \Omega_r \equiv \frac{\varepsilon_{r\,0}}{\varepsilon_{c\,0}}, \qquad \Omega_v \equiv \frac{\varepsilon_{v\,0}}{\varepsilon_{c\,0}}, \qquad \Omega_k \equiv -k/(a_0^2 H_0^2)$$

$\longrightarrow$ the Friedman eqn. takes the form

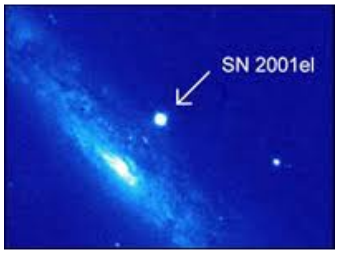
$$\frac{\varepsilon_c}{\varepsilon_{c\,0}} = \frac{H^2}{H_0^2} = \Omega_r \left(\frac{a_0}{a}\right)^4 + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_v + \Omega_k \left(\frac{a_0}{a}\right)^2.$$

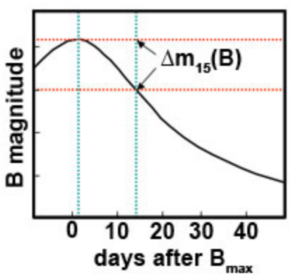
In particular

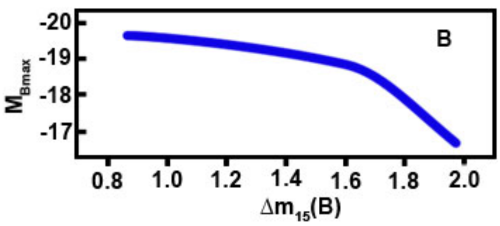
$$\Omega_r + \Omega_m + \Omega_v + \Omega_k = 1.$$

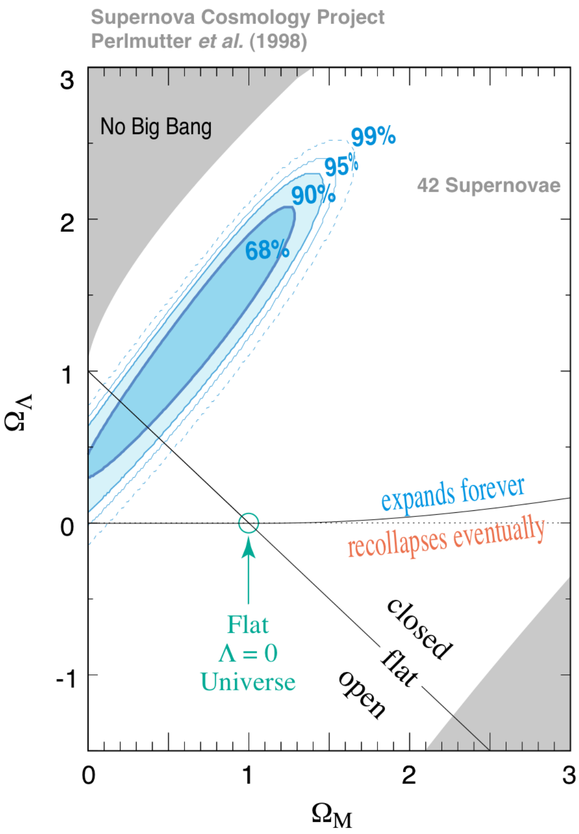
SN Ia: standard candles

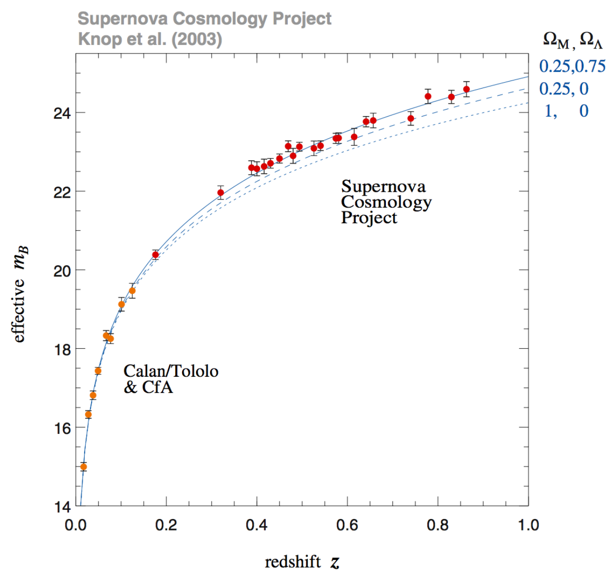
runaway C/O fusion reactions
in a white dwarf star











luminosity m_B too high
compared to red shift z

The age of the universe

$$H = \frac{\dot{a}}{a} = \frac{a_0^2 H_0}{a^2}\sqrt{\Omega_r + \Omega_m \frac{a}{a_0} + \Omega_v \left(\frac{a}{a_0}\right)^4 + \Omega_k \left(\frac{a}{a_0}\right)^2}$$

Then with $x = a/a_0$

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{x dx}{\sqrt{\Omega_r + \Omega_m x + \Omega_v x^4 + \Omega_k x^2}}$$

Standard cosmology (Planck, 2018):

$$\Omega_r = 5 \times 10^{-5} \qquad \Omega_m = 0.308 \qquad \Omega_v = 0.692 \qquad \Omega_k = 0$$

$$H_0 = 70 \text{ km/s/Mpc}$$

$$\longrightarrow \quad t_0 \simeq \frac{1}{H_0}\int_0^1 \frac{\sqrt{x}\, dx}{\sqrt{\Omega_m + \Omega_v x^3}} = \frac{2}{3H_0\sqrt{\Omega_v}} \operatorname{arcsinh}\sqrt{\frac{\Omega_v}{\Omega_m}}$$

$$= 1.38 \times 10^{10}\,\text{yr}$$

Horizon

Light cones ($c = 1$):

$$ds^2 = dt^2 - a^2 dr^2 = 0 \quad \Rightarrow \quad dt = a(t) dr$$

In a time t a photon travels a maximal co-ordinate distance

$$r(t) = \int_0^t \frac{dt'}{a(t')} = \int_0^{a(t)} \frac{da}{a\dot{a}} = \int_0^{a(t)} \frac{da}{a^2 H}$$

The physical distance covered by this photon is

$$\begin{aligned} d(t) &= a(t)r(t) = a(t) \int_0^{a(t)} \frac{da}{a^2 H} \\ &= \frac{x(t)}{H_0} \int_0^{x(t)} \frac{dx}{\sqrt{\Omega_r + \Omega_m x + \Omega_v x^4}} \end{aligned}$$

at present time t_0: $\quad d_H(t_0) \simeq \dfrac{3.3}{H_0} \simeq 4.6 \times 10^{10}\,\mathrm{yr}$

Cross-over in the energy density

The energy density of matter and vacuum energy was equal:
$\varepsilon_m = \varepsilon_v$, at scale factor when

$$\varepsilon_m = \varepsilon_v \quad \Leftrightarrow \quad \Omega_m \left(\frac{a_0}{a}\right)^3 = \Omega_v$$

$$x = \frac{a}{a_0} = \left(\frac{\Omega_m}{\Omega_v}\right)^{1/3} = 0.76 \longrightarrow \quad t = 1.1 \times 10^{10}\,\mathrm{yr}$$

or $z = 0.3$, at which time $\quad \varepsilon_r = 2 \times 10^{-4}\, \varepsilon_m$

Cross-over matter and radiation:

$$\varepsilon_r = \varepsilon_m \quad \Leftrightarrow \quad \Omega_r = \Omega_m \frac{a}{a_0}$$

or $\quad x = \dfrac{a}{a_0} = \dfrac{\Omega_r}{\Omega_m} \simeq 2 \times 10^{-4} \longleftrightarrow \quad z \sim 0.5 \times 10^4$

$\quad t \sim 4 \times 10^4\,\mathrm{yr}$

Thermal evolution of the universe

radiation / relativistic gas

$$p = \frac{\varepsilon}{3} \longrightarrow \varepsilon a^4 = \text{constant}$$

Stefan-Boltzmann law: *in equilibrium* $\varepsilon = \alpha T^4$

$\longrightarrow$ in RD regime $a \propto \sqrt{t} \propto \frac{1}{T}$

$$\left[\begin{array}{l} \frac{dT}{da} \propto -\frac{1}{a^2} \\ \frac{dT}{dt} \propto -\frac{1}{t^{3/2}} \end{array} \right.$$

non-relativistic gas:

$$pa^3 = nk_BT \quad \text{and} \quad \varepsilon a^3 = nm + \frac{3}{2}nk_BT$$

$\longrightarrow$ in MD regime $a \propto t^{2/3} \propto \frac{1}{\sqrt{T}}$

$$\left[\begin{array}{l} \frac{dT}{da} \propto -\frac{1}{a^3} \\ \frac{dT}{dt} \propto -\frac{1}{t^{7/3}} \end{array} \right.$$

MD regime cools faster than RD regime

Decoupling

- Radiation/matter establishes thermal equilibrium by interactions
- interactions can *cease* if expansion of the universe makes free streaming length larger than horizon distance $d_H(t_0)$

$\longrightarrow$ *freeze-out*

photons: *at time of freeze-out thermal Planck distribution*

$$n_T(\nu)\, d\nu = \frac{8\pi\nu^2 d\nu}{e^{h\nu/k_BT} - 1}$$

after freeze-out

$$a \to a' \longrightarrow \nu \to \nu' = \frac{\nu a}{a'}$$

$$n'(\nu')d\nu' = \left(\frac{a'}{a}\right)^3 \frac{8\pi\nu'^2 d\nu'}{e^{h\nu'/k_BT'} - 1} \quad \text{with} \quad T' = \frac{aT}{a'}$$

$$= n_{T'}(\nu')\, d\nu'$$

quasi-thermal equilibrium

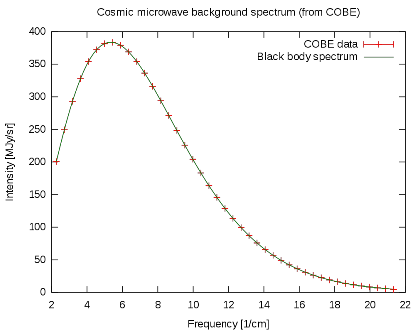
CMB

present temperature: $T = 2.725\,K$

photon density $n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \longrightarrow 0.41 \times 10^9\,\mathrm{m}^{-3}$

energy density $\varepsilon_\gamma = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} \longrightarrow 0.260\,\mathrm{MeV}\ \mathrm{m}^{-3}$

$\Omega_\gamma = 0.46 \times 10^{-4}$



decoupling:

$k_B T_{dec} = 0.27\ \mathrm{eV} \longrightarrow T_{dec} \simeq 3100\,\mathrm{K}$

$$x_{dec} = \frac{a_{dec}}{a_0} = \frac{T_0}{T_{dec}} \simeq 0.87 \times 10^{-3}$$

$$\frac{t_{dec}}{t_0} \simeq \left(\frac{a_{dec}}{a_0}\right)^{3/2} \simeq 360\,000\,\mathrm{yr}$$

Neutrinos

At energies $k_B T > 0.8$ MeV neutrinos interacted with electrons and baryons via pair creation or elastic scattering, e.g.:

$$\nu + \bar{\nu} \leftrightarrow e^- + e^+ \leftrightarrow \gamma + \gamma, \qquad n + \nu \leftrightarrow p^+ + e^-$$

$\longrightarrow$ thermal equilibrium with baryons, electrons and photons.

Decoupling happens when the interaction rate of neutrinos is smaller than the expansion rate of the universe

$$\Gamma_\nu = \sigma_\nu |\mathbf{v}| n < H = \sqrt{\frac{8\pi G \alpha_{eff}}{3}}\,T^2$$

Condition satisfied at $k_B T = 0.8$ MeV when

$$x_{\nu\,dec} = \frac{a_{\nu\,dec}}{a_0} = 0.2 \times 10^{-9}, \qquad t_{\nu\,dec} \sim 1\ \mathrm{sec.}$$

Cosmic neutrino background

- Cosmic ν-background unobserved;
- After decoupling a thermal background of relativistic neutrinos is expected with an effective temperature at late times

$$T_\nu = 0.71\, T_\gamma.$$

If ν's still relativistic now ($m_\nu \leq 0.17$ meV), this temperature is $T_{\nu\,0} = 1.95$ K. For 3 stable relativistic ν-species

$$n_\nu = 3 \times \frac{3}{4} \times \left(\frac{T_\nu}{T_\gamma}\right)^3 \times n_\gamma \quad \rightarrow \quad 0.33 \times 10^9 \text{ m}^{-3},$$

$$\varepsilon_\nu = 3 \times \frac{7}{8} \times \left(\frac{T_\nu}{T_\gamma}\right)^4 \times \varepsilon_\gamma \quad \rightarrow \quad 0.17 \text{ MeV m}^{-3},$$

$$\rightarrow \quad \Omega_{\nu(massless)} \approx 0.30 \times 10^{-4}.$$

Massive neutrinos

Neutrinos are relativistic up to temperatures $k_B T_\nu \sim m_\nu c^2$
At this time the photon temperature is

$$T_\gamma = 1.40\, T_\nu = 1.40\, \frac{m_\nu c^2}{k_B}.$$

The corresponding scale factor is

$$\frac{a}{a_0} = \frac{T_{\gamma 0}}{T_\gamma} = 0.71\, \frac{k_B T_{\gamma 0}}{m_\nu c^2}$$

After that neutrinos are non-relativistic and

$$k_B T_{\nu 0} \;=\; \left(\frac{a}{a_0}\right)^2 k_B T_\nu = 0.5 \left(\frac{k_B T_{\gamma 0}}{m_\nu c^2}\right)^2 m_\nu c^2$$

$$= \frac{0.029\,\mathrm{meV}^2}{m_\nu c^2} = \left[\frac{1\,\mathrm{meV}}{m_\nu c^2}\right] \times 0.34\,\mathrm{K}$$

Primordial nucleosynthesis

With the known baryon-to-photon ratio η_B spontaneous synthesis of light nuclei: D, ^{3}He, ^{4}He, from free nucleons was possible at photon and baryon temperatures between 100 and 10 keV.

^{4}He is by far the most stable of the light nuclei
$\Rightarrow$ practically all neutrons ($> 99\%$) end up in α-particles.

Then the mass of ^{4}He as a fraction of total baryon mass is

$$Y(^4\text{He}) = \frac{4n_\alpha}{n_n + n_p} = \frac{2n_n}{n_n + n_p} = \frac{2(n_n/n_p)}{1 + (n_n/n_p)}.$$

Observed: $Y(^4\text{He}) = 0.24 \quad \Rightarrow \quad n_n/n_p = 0.13$.

Initial condition determined by thermal equilibrium

$$\left(\frac{n_n}{n_p}\right)_0 = e^{-\Delta mc^2/k_B T_0} = 0.2$$

at $k_B T_0 = 0.8$ MeV.

Neutron life time $\tau = 886$ sec

$$\rightarrow \left(\frac{n_n}{n_p}\right)_1 = \frac{n_{n0}\, e^{-\Delta t/\tau}}{n_{p0} + n_{n0}(1 - e^{-\Delta t/\tau})}$$

$= 0.13$ for $\Delta t = 328$ sec.

$\rightarrow k_B T_1 \approx 0.09$ MeV.

Very sensitive to η_B.