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Black holes and gravitational waves I

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Notation and conventions

In these notes we use units in which the speed of light $c = 1$. Greek indices $\mu, \nu, ...$ denote components of 4-vectors and 4-tensors. They take the values $(0, 1, 2, 3)$, which for local *cartesian* co-ordinates correspond to (t, x, y, z), but in general refer to one time and three space dimensions. By upper and lower indices we distinguish between contravariant and covariant components, which differ in transformation properties under general co-ordinates transformations:
– under transformations $x \rightarrow x'$ the components of a contravarant vector a^μ transform as a 1-form:

$$a^\mu \quad \rightarrow \quad a'^\mu = a^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

– and the components of a covariant vector transform as a gradient:

$$a_\mu \quad \rightarrow \quad a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu.$$

We use the summation convention: multiplication of a corresponding pair of covariant and contravariant components implies summation over all such pairs:

$$a^\mu b_\mu = \sum_{\mu=0}^{3} a^\mu b_\mu \equiv a \cdot b.$$

It follows, that the contraction of a covariant and a contravariant vector is invariant:

$$a'^\mu b'_\mu = a^\nu \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\lambda}{\partial x'^\mu} b_\lambda = a^\mu b_\mu.$$

This holds also for the the components of the metric tensor:

$$g'_{\mu\nu}(x') = \frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{\kappa\lambda}(x),$$

and its inverse:

$$g'^{\mu\nu}(x') = g^{\kappa\lambda}(x) \frac{\partial x'^\mu}{\partial x^\kappa} \frac{\partial x'^\nu}{\partial x^\lambda}.$$

Therefore the metric tensor and its inverse can be used to change contravariant into covariant components and vice-versa:

$$a_\mu = g_{\mu\nu} a^\nu \quad \Leftrightarrow \quad a^\mu = g^{\mu\nu} a_\nu.$$

In flat space-time the metric in cartesian components is the Minkowski metric

$$g_{\mu\nu} \quad \rightarrow \quad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1).$$

Hence *in flat space-time*

$$a \cdot b \quad \rightarrow \quad -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3.$$

Latin indices $i, j, k, ... = (1,2,3)$ are used to denote the spatial components of vectors and tensors: $a^\mu = a^0, a^i$.

Covariant derivatives create a rank $(n+1)$-tensor out of a rank-n tensor; its components are constructed with the Riemann-Christoffel connection

$$\Gamma_{\mu\nu}^{\;\;\lambda} = \frac{1}{2}\, g^{\lambda\kappa} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right).$$

The contravariant components of a rank-n tensor t are are

$$(\nabla_\mu t)^{\mu_1 ... \mu_n} = \partial_\mu t^{\mu_1 ... \mu_n} + \Gamma_{\mu\nu}^{\;\;\mu_1}\, t^{\nu\mu_2 .. \mu_n} + ... + \Gamma_{\mu\nu}^{\;\;\mu_n}\, t^{\mu_1 ... \mu_{n-1}\nu}.$$

In the literature and in these notes the notation is often somewhat sloppy by omitting the parentheses on the left-hand side of this equation and writing it as $\nabla_\mu t^{\mu_1 ... \mu_n}$; usually no confusion should arise. Similarly for covariant components:

$$\nabla_\mu t_{\mu_1 ... \mu_n} = (\nabla_\mu t)_{\mu_1 ... \mu_n} = \partial_\mu t_{\mu_1 ... \mu_n} - \Gamma_{\mu\mu_1}^{\;\;\;\nu}\, t_{\nu\mu_2 ... \mu_n} - ... - \Gamma_{\mu\mu_n}^{\;\;\;\nu}\, t_{\mu_1 ... \mu_{n-1}\nu}.$$

The components of the Riemann tensor are given by

$$R_{\mu\nu\kappa}^{\;\;\;\;\lambda} = \partial_\mu \Gamma_{\nu\kappa}^{\;\;\lambda} - \partial_\nu \Gamma_{\mu\kappa}^{\;\;\lambda} - \Gamma_{\mu\kappa}^{\;\;\sigma} \Gamma_{\nu\sigma}^{\;\;\lambda} + \Gamma_{\nu\kappa}^{\;\;\sigma} \Gamma_{\mu\sigma}^{\;\;\lambda}.$$

They satisfy the Bianchi identities

$$R_{\mu\nu\kappa}^{\;\;\;\;\lambda} + R_{\nu\kappa\mu}^{\;\;\;\;\lambda} + R_{\kappa\mu\nu}^{\;\;\;\;\lambda} = 0, \qquad \nabla_\mu R_{\nu\lambda\kappa}^{\;\;\;\;\sigma} + \nabla_\nu R_{\lambda\mu\kappa}^{\;\;\;\;\sigma} + \nabla_\lambda R_{\mu\nu\kappa}^{\;\;\;\;\sigma} = 0.$$

The purely covariant components have the symmetry properties

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} = R_{\kappa\lambda\mu\nu}.$$

The Ricci tensor is the contraction of the Riemann tensor, with components

$$R_{\mu\nu} = R_{\mu\lambda\nu}^{\;\;\;\;\lambda}.$$

a. Geodesic motion

1. From the metric postulate

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma_{\lambda\mu}^{\;\;\;\kappa}\, g_{\kappa\nu} - \Gamma_{\lambda\nu}^{\;\;\;\kappa}\, g_{\mu\kappa} = 0, \tag{1}$$

derive the expression for the components of the Riemann-Christoffel connection $\Gamma_{\mu\nu}^{\;\;\;\lambda}$.

2. a. Define the geodesic hamiltonian (the hamiltonian generating geodesic world-lines for ideal point masses) by

$$H = \frac{1}{2}\, g^{\mu\nu}(x) p_\mu p_\nu. \tag{2}$$

Prove that Hamilton's equations

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}, \tag{3}$$

are equivalent with the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\kappa\nu}^{\;\;\;\mu} \frac{dx^\kappa}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{4}$$

b. Show that for time-like geodesics (world-lines of massive particles) one can take $H = -1/2$ by identifying the world-line co-ordinate λ (the affine parameter) with the proper time. Also show, that for light-like geodesics (massless particles) $H = 0$.

3. For any well-behaved functions $A(x,p)$ and $B(x,p)$ of the space-time position and momentum of a point particle, define the Poisson brackets

$$\{A, B\} = \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x^\mu}. \tag{5}$$

a. Verify that Hamilton's equations (3) can be written as

$$\frac{dx^\mu}{d\lambda} = \{x^\mu, H\}, \qquad \frac{dp_\mu}{d\lambda} = \{p_\mu, H\}. \tag{6}$$

Show that as a result

$$\frac{dA}{d\lambda} = \{A, H\}, \tag{7}$$

for any well-behaved function $A(x,p)$.

b. Prove the following properties of Poisson brackets:

$$\{A, B\} = -\{B, A\}, \quad \{\alpha_1 A_1 + \alpha_2 A_2, B\} = \alpha_1 \{A_1, B\} + \alpha_2 \{A_2, B\}, \tag{8}$$

and the Jacobi identity:

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0. \tag{9}$$

b. Symmetries and constants of motion

1. *Isometries.* Line elements near a point $\mathcal{P}$ with co-ordinates x^μ correspond to space-time intervals
$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{10}$$
Consider a set of nearby points $\mathcal{P}'$, with co-ordinates x'^μ, and define
$$\xi^\mu = x'^\mu - x^\mu, \tag{11}$$
such that to any vector $\xi^\mu(x)$ attached to the point $\mathcal{P}$ there corresponds a point $\mathcal{P}'$.
a. Show that to first order in ξ the line element at $\mathcal{P}'$ takes the form
$$ds'^2 = g_{\mu\nu}(x') dx'^\mu dx'^\nu = \left(g_{\mu\nu}(x) + 2\nabla_\mu \xi_\nu(x)\right) dx^\mu dx^\nu. \tag{12}$$
b. Provide the arguments to show that the local geometry at $\mathcal{P}'$ is the same as that at $\mathcal{P}$ provided
$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{13}$$
Any vector field $\xi^\mu(x)$ satisfying this condition is called a Killing vector field; the corresponding motion $\mathcal{P} \rightarrow \mathcal{P}'$ in the space-time manifold which leaves the local geometry invariant is called an isometry.

2. *Constants of motion.* Let $x^\mu(\lambda)$ represent the world line of a point-like particle. A constant of motion is any quantity A which takes the same value on every point of the world line:
$$\frac{dA}{d\lambda} = 0. \tag{14}$$
An obvious example is the electric charge q of a stable point particle.
a. Show that in general a quantity $A(x, p)$ ia a constant of motion if
$$\{A, H\} = 0. \tag{15}$$
b. Prove that if A and B are constants of motion, the quantity
$$C(A, B) = \{A, B\} \tag{16}$$
is also a constant of motion.
Rem.: The inverse is not necessarily true: not every constant of motion can be written as the Poisson bracket of two other constants of motion.
One says that the set of constants of motion is *closed* under the Poisson bracket operation; a set of elements with the bracket relation (16) under which it is closed, is called a Poisson-Lie algebra.

3. *Covariant brackets.* For any function world-line function $A(x,p)$, define the covariant derivative

$$\mathcal{D}_\mu A \equiv \partial_\mu A + \Gamma_{\mu\nu}^{\;\;\lambda}\, p_\lambda \frac{\partial A}{\partial p_\nu}. \tag{17}$$

a. Show that for any A which is an analytical function of the momenta

$$A(x,p) = \sum_n \frac{1}{n!}\, \alpha^{\mu_1 \ldots \mu_n}(x)\, p_{\mu_1} \ldots p_{\mu_n}$$

the covariant derivative takes the form

$$\mathcal{D}_\mu A = \sum_n \frac{1}{n!}\, (\nabla_\mu \alpha)^{\nu_1 \ldots \nu_n}\, p_{\nu_1} \ldots p_{\nu_n}, \tag{18}$$

with as usual

$$(\nabla_\mu \alpha)^{\nu_1 \ldots \nu_n} = \partial_\mu \alpha^{\nu_1 \ldots \nu_n} + \Gamma_{\mu\lambda}^{\;\;\nu_1}\, \alpha^{\lambda \nu_2 \ldots \nu_n} + \ldots + \Gamma_{\mu\lambda}^{\;\;\nu_n}\, \alpha^{\nu_1 \ldots \nu_{n-1} \lambda}.$$

b. Prove, that the Poisson brackets of two functions A and B can be written in the covariant form

$$\{A, B\} = \mathcal{D}_\mu A\, \frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\, \mathcal{D}_\mu B. \tag{19}$$

c. Prove, that a quantity

$$J(x,p) = \xi^\mu(x) p_\mu \tag{20}$$

is a contant of motion if and only if $\xi^\mu(x)$ is a Killing vector:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad \Leftrightarrow \quad \frac{dJ}{d\lambda} = 0. \tag{21}$$

The general correspondence between constants of motion and continuous symmetries is known as Noether's theorem[1].

d. Show, that a quantity

$$K(x,p) = \frac{1}{2}\, K^{\mu\nu}(x) p_\mu p_\nu \tag{22}$$

is a constant of motion if $K_{\mu\nu}$ is a rank-2 Killing tensor, characterized by the property

$$\nabla_{(\lambda} K_{\mu\nu)} = \nabla_\lambda K_{\mu\nu} + \nabla_\mu K_{\nu\lambda} + \nabla_\nu K_{\lambda\mu} = 0. \tag{23}$$

e. Show that the hamiltonian H is a constant of motion, and that the metric $g_{\mu\nu}$ is a Killing tensor.

f. Investigate the transformations of (x,p) generated by K through the Poisson brackets.

[1] After the German-American mathematician Emmy Noether (1882-1935).

4. Let $\{e_i^\mu(x)\}_{i=1}^r$ be a basis for the complete set of Killing vectors in a manifold, such that any Killing vector ξ^μ is a linear combination of the basis vectors:

$$\xi^\mu(x) = \sum_{i=1}^r c^i e_i^\mu(x). \tag{24}$$

a. Show that to these vectors there corresponds a basis for the constants of motion $J(x,p)$:

$$J(x,p) = \sum_{i=1}^r c^i J_i(x,p), \qquad J_i(x,p) = e_i^\mu(x) p_\mu. \tag{25}$$

b. From eq. (16) it follows, that the bracket of two constants of motion J_i can be expanded in this basis as

$$C(J_i, J_j) = \{J_i, J_j\} = \sum_{k=1}^r c_{ij}^{\;\;k} J_k. \tag{26}$$

The numbers $c_{ij}^{\;\;k}$ characterize the Poisson-Lie algebra of the $\{J_i\}$; they are called the structure contants of the algebra. Show that

$$c_{ij}^{\;\;k} = -c_{ji}^{\;\;k}. \tag{27}$$

c. From the Jacobi identity (9) prove that for any $(ijkn)$ the structure constants satisfy the quadratic relation

$$\sum_{m=1}^r \left(c_{ij}^{\;\;m} c_{mk}^{\;\;\;n} + c_{jk}^{\;\;m} c_{mi}^{\;\;\;n} + c_{ki}^{\;\;m} c_{mj}^{\;\;\;n} \right) = 0. \tag{28}$$

d. From the Poisson-Lie algebra structure (26) derive a relation between the basis Killing vectors e_i^μ:

$$e_j^\nu (\nabla_\nu e_i)^\mu - e_i^\nu (\nabla_\nu e_j)^\mu = e_j^\nu \partial_\nu e_i^\mu - e_i^\nu \partial_\nu e_j^\mu = \sum_{k=1}^r c_{ij}^{\;\;k} e_k^\mu. \tag{29}$$

5. *Rotations.* The 2-D spherical surface with radius R is defined by the equation

$$x^2 + y^2 + z^2 = R^2.$$

a. By transforming to spherical co-ordinates

$$x = R \sin\theta \cos\varphi, \quad y = R \sin\theta \sin\varphi, \quad z = R\cos\theta, \tag{30}$$

show that the length of a line-element on the sphere is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2. \tag{31}$$

b. Show that the geodesics on the sphere are great circles.
c. Prove that the geodesic hamiltonian is

$$H = \frac{1}{2R^2} \left(p_\theta^2 + \frac{1}{\sin^2\theta}\, p_\varphi^2 \right). \tag{32}$$

d. Prove the existence of a triplet of Killing vectors on the sphere

$$\xi_1^a = (-\sin\varphi, -\text{cotg}\,\theta \cos\varphi), \quad \xi_2^a = (\cos\varphi, -\text{cotg}\,\theta \sin\varphi), \quad \xi_3^a = (0, 1), \tag{33}$$

with $a = (\theta, \varphi)$, with the corresponding constants of motion

$$J_1 = -\sin\varphi\, p_\theta - \text{cotg}\,\theta \cos\varphi\, p_\varphi, \quad J_2 = \cos\varphi\, p_\theta - \text{cotg}\,\theta \sin\varphi\, p_\varphi, \quad J_3 = p_\varphi. \tag{34}$$

e. Show, that in 3-dimensional cartesian co-ordinates

$$J_1 = y p_z - z p_y, \quad J_2 = z p_x - x p_y, \quad J_3 = x p_y - y p_x. \tag{35}$$

f. Explain why the isometries generated by the Killing vectors (33) correspond to rotations around the x-, y- and z-axis.
g. Prove that the J_i satisfy the Poisson-Lie algebra

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \tag{36}$$

where the 3-dimensional permutation symbol is defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } (ijk) \text{ is an even permutation of (123);} \\ -1, & \text{if } (ijk) \text{ is an odd permutation of (123);} \\ 0, & \text{in all other cases.} \end{cases} . \tag{37}$$

h. Show that the geodesic hamiltonian on the sphere can be expressed as

$$H = \frac{1}{2R^2} \left(J_1^2 + J_2^2 + J_3^2 \right). \tag{38}$$

c. Schwarzschild space-time

The only solution of the Einstein equations in empty space which is static and spherically symmetric was found by K. Schwarzschild (published in 1916). Static means: the same at all times t; spherically symmetric means: invariant under rotations in 3 dimensional space. It is then convenient to adopt co-ordinates (t, r, θ, φ), which are spherical co-ordinates in the sense that equal time surfaces defined by t = constant and r = constant, are closed surfaces, which in Schwarzschild's solution become spheres of constant curvature.

1. a. A static space-time has a time-independent metric: $\partial_t g_{\mu\nu} = 0$. Show that in terms of the geodesic hamiltonian H this can be written as a bracket relation

$$\{p_t, H\} = 0. \tag{39}$$

b. If p_t is a constant of motion,

$$\xi_t^\mu = (1, 0, 0, 0) \tag{40}$$

must be a Killing vector. Show that the covariant components of this Killing vector are

$$\xi_{t\,\mu} = g_{\mu t}. \tag{41}$$

c. Verify that

$$\nabla_\mu \xi_{t\,\nu} + \nabla_\nu \xi_{t\,\mu} = 0 \quad \Leftrightarrow \quad \partial_t g_{\mu\nu} = 0. \tag{42}$$

Hint: use the metric postulate (1).

d. Show by a similar chain of arguments that invariance of the metric under rotations around the z-axis is guaranteed by the existence of a Killing vector

$$\xi_\varphi^\mu = (0, 0, 0, 1) \quad \Rightarrow \quad \partial_\varphi g_{\mu\nu} = 0. \tag{43}$$

e. Full spherical symmetry requires two more Killing vectors, as in eq. (33):

$$\xi_1^\mu = (0, 0, -\sin\varphi, -\text{cotg}\,\theta\cos\varphi), \quad \xi_2^\mu = (0, 0, \cos\varphi, -\text{cotg}\,\theta\sin\varphi). \tag{44}$$

Together with $\xi_3^\mu = \xi_\varphi^\mu$ these define the three components of angular momentum:

$$J_i = \xi_i^\mu\, p_\mu.$$

Show that this construction reproduces the expressions (34):

$$J_1 = -\sin\varphi\, p_\theta - \text{cotg}\,\theta\cos\varphi\, p_\varphi, \quad J_2 = \cos\varphi\, p_\theta - \text{cotg}\,\theta\sin\varphi\, p_\varphi, \quad J_3 = p_\varphi,$$

and explain why spherical symmetry requires

$$\{J_i, H\} = 0, \tag{45}$$

Up to co-ordinate transformations there exists a unique solution for the geodesic hamiltonian satisfying the conditions (39) and (45); this is the solution discovered by Schwarzschild[2].

[2] The uniqueness was proven by G. Birkhoff in 1923.

2. The standard form of the metric used to parametrize the Schwarzschild solution was introduced by J. Droste[3]:

$$ds^2 = -\left(1 - \frac{\kappa}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{\kappa}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2, \tag{46}$$

where κ is a constant.

a. Show that the geodesic hamiltonian is given by

$$2H = -\frac{p_t^2}{1 - \frac{\kappa}{r}} + \left(1 - \frac{\kappa}{r}\right) p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}. \tag{47}$$

b. Prove, that this hamiltonian satisfies the conditions of time-invariance (39) and spherical symmetry (45).

c. Compute the components of the Riemann-Christoffel connection

$$\begin{aligned}
&\Gamma_{tt}^{\;\;r} = \frac{\kappa}{2r^2}\left(1 - \frac{\kappa}{r}\right), && \Gamma_{rt}^{\;\;t} = \Gamma_{tr}^{\;\;t} = \frac{\kappa}{2r^2\left(1 - \frac{\kappa}{r}\right)}, \\
&\Gamma_{rr}^{\;\;r} = -\frac{\kappa}{2r^2}\left(1 - \frac{\kappa}{r}\right), && \\
&\Gamma_{\theta\theta}^{\;\;r} = -r\left(1 - \frac{\kappa}{r}\right), && \Gamma_{r\theta}^{\;\;\theta} = \Gamma_{\theta r}^{\;\;\theta} = \frac{1}{r}, \\
&\Gamma_{\varphi\varphi}^{\;\;r} = -r\sin^2\theta\left(1 - \frac{\kappa}{r}\right), && \Gamma_{r\varphi}^{\;\;\varphi} = \Gamma_{\varphi r}^{\;\;\varphi} = \frac{1}{r}, \\
&\Gamma_{\varphi\varphi}^{\;\;\theta} = -\sin\theta\cos\theta, && \Gamma_{\theta\varphi}^{\;\;\varphi} = \Gamma_{\varphi\theta}^{\;\;\varphi} = \mathrm{cotg}\,\theta,
\end{aligned} \tag{48}$$

whilst all others vanish.

d. Derive the components of the Riemann tensor:

$$\begin{aligned}
&R_{trt}^{\;\;\;\;r} = \frac{\kappa}{r^3}\left(1 - \frac{\kappa}{r}\right), && R_{rtr}^{\;\;\;\;t} = -\frac{\kappa}{r^3\left(1 - \frac{\kappa}{r}\right)}, \\
&R_{t\theta t}^{\;\;\;\;\theta} = -\frac{\kappa}{2r^3}\left(1 - \frac{\kappa}{r}\right), && R_{\theta t\theta}^{\;\;\;\;t} = \frac{\kappa}{2r}, \\
&R_{t\varphi t}^{\;\;\;\;\varphi} = -\frac{\kappa}{2r^3}\left(1 - \frac{\kappa}{r}\right), && R_{\varphi t\varphi}^{\;\;\;\;t} = \frac{\kappa}{2r}\sin^2\theta,
\end{aligned} \tag{49}$$

[3]J. Droste was a student of Lorentz, who discovered the static and spherically symmetric solution independently around the same time as Schwarzschild.

and

$$R_{r\theta r}{}^{\theta} = \frac{\kappa}{2r^3\left(1 - \frac{\kappa}{r}\right)}, \qquad R_{\theta r\theta}{}^{r} = \frac{\kappa}{2r},$$

$$R_{r\varphi r}{}^{\varphi} = \frac{\kappa}{2r^3\left(1 - \frac{\kappa}{r}\right)}, \quad R_{\varphi r\varphi}{}^{r} = \frac{\kappa}{2r}\sin^2\theta, \tag{50}$$

$$R_{\theta\varphi\theta}{}^{\varphi} = -\frac{\kappa}{r}, \qquad R_{\varphi\theta\varphi}{}^{\theta} = -\frac{\kappa}{r}\sin^2\theta.$$

All other components not related by symmetry vanish.
e. Prove that this space-time geometry satisfies the Einstein equations in empty space:

$$R_{\mu\nu} = 0. \tag{51}$$

3. a. To interpret the Droste co-ordinate system (46), show that circles centered on the Schwarzschild mass: $r = R =$ constant, $\theta =$ constant, at fixed time $t =$ constant, have circumference $2\pi R$.
   b. Similarly show that spheres of constant $r = R$ at fixed time t have surface area $4\pi R^2$.
   c. What is the fixed-time radial distance between two circles with circumference $2\pi R$ and $2\pi(R + r)$?
   d. To stay at a fixed spatial point, a point mass must be accelerated to counteract the radial geodesic acceleration. What is the 4-velocity $u^\mu = dx^\mu/d\tau$ of a particle at rest at radial distance $r = R$? Compute its proper acceleration

$$a^\mu = \frac{Du^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma_{\lambda\nu}{}^{\mu} u^\lambda u^\nu. \tag{52}$$

   e. Argue that at large radial distances $R \to \infty$ the proper acceleration and the acceleration d^2r/dt^2, measured in Droste co-ordinates become equal. By equating the radial proper acceleration a^r with the newtonian gravitational acceleration of a central mass M, show that $\kappa = 2GM$. The radial distance κ is known as the *Schwarzschild radius.*

4. By construction the Schwarzschild geodesics are characterized by 5 constants of motion: the hamiltonian H, the energy of motion p_t and the 3 components of angular momentum (J_1, J_2, J_3). Conservation of angular momentum implies that both the absolute magnitude and the direction of angular momentum are conserved. As a result, the geodesic motion is confined to a plane, which because of spherical symmetry we may take to be the equatorial plane $\theta = \pi/2$. In this construction only the z-component of angular momentum is non-zero, and every geodesic in the equatorial plane is characterized by two further constants:

$$p_t = -\varepsilon, \qquad J_i = (0, 0, \ell). \tag{53}$$

a. From these equations derive the results

$$p_\theta = 0, \qquad p_\varphi = \ell. \tag{54}$$

b. Show that time-like geodesics in the equatorial plane, parametrized by the proper time τ, satisfy

$$\frac{dt}{d\tau} = \frac{\varepsilon}{1 - \frac{\kappa}{r}}, \quad \frac{d\theta}{d\tau} = 0, \quad \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}. \tag{55}$$

c. By definition the proper time is the affine parameter for which the geodesic hamiltonian $H = -1/2$; derive the relation

$$\left(\frac{dr}{d\tau} \right)^2 + \left(1 - \frac{\kappa}{r} \right) \left(1 + \frac{\ell^2}{r^2} \right) = \varepsilon^2. \tag{56}$$

d. Derive the equation for the radial acceleration

$$\begin{aligned} &\frac{d^2r}{d\tau^2} + \frac{\kappa}{2r^2 \left(1 - \frac{\kappa}{r} \right)} \left[\varepsilon^2 - \left(\frac{dr}{d\tau} \right)^2 \right] - \frac{\ell^2}{r^3} \left(1 - \frac{\kappa}{r} \right) \\ &= \frac{d^2r}{d\tau^2} + \frac{\kappa}{2r^2} - \frac{\ell^2}{r^3} + \frac{3\kappa\ell^2}{2r^4} = 0. \end{aligned} \tag{57}$$

Verify the compatibility with equation (56).

5. *Radial geodesics*
a. Show that radial geodesics with $\ell = 0$ satisfy

$$\left(\frac{dr}{d\tau} \right)^2 = \varepsilon^2 - 1 + \frac{\kappa}{r}. \tag{58}$$

b. Consider a test particle at rest very far from the origin, i.e. in the limit $r \to \infty$; show that in this limit proper time equals local time, and

$$\frac{dr}{d\tau} = 0, \qquad \ell = 0, \qquad \frac{dt}{d\tau} = \varepsilon = 1. \tag{59}$$

c. Fixing the start of proper time such that an infalling particle reaches $r = \kappa$ at $\tau = 0$, prove that

$$\frac{r}{\kappa} = \left(1 - \frac{3\tau}{2\kappa} \right)^{2/3}. \tag{60}$$

6. *Circular geodesics*
Circular geodesics in the equatorial plane have $r = R =$ constant, and $\theta = \pi/2$.
a. Show that for time-like circular geodesics

$$\varepsilon^2 = \left(1 - \frac{\kappa}{R} \right) \left(1 + \frac{\ell^2}{R^2} \right), \tag{61}$$

b. From the absence of radial acceleration, derive the result

$$\frac{\ell^2}{R^2} = \frac{\kappa}{2R} \frac{1}{\left(1 - \frac{3\kappa}{2R} \right)}, \tag{62}$$

and show that solutions exist only for

$$\ell^2 \geq 3\kappa^2. \tag{63}$$

c. Prove that the innermost stable circular orbit (ISCO) has radial co-ordinate $R = 3\kappa$.
d. Derive the time-dilation factor in a circular orbit:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{3\kappa}{3R}}}. \tag{64}$$

e. Defining the orbital frequency in the laboratory frame (Droste co-ordinates) by $\Omega = d\varphi/dt$, derive Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM} R^3, \tag{65}$$

where $T = 2\pi/\Omega$ is the orbital period as measured by an observer at large distance from the mass M.

7. *Generic bound orbits*
a. Show that for a generic time-like bound geodesic

$$\left(\frac{dr}{d\tau} \right)^2 + V_{eff}(r; \ell) = \varepsilon^2, \tag{66}$$

where

$$V_{eff}(r; \ell) = \left(1 - \frac{\kappa}{r} \right) \left(1 + \frac{\ell^2}{r^2} \right). \tag{67}$$

b. Compute the stationary points of V_{eff}; show that there are three solution for values $\ell^2 > 3\kappa^2$, and one $\ell^2 < 3\kappa^2$. Draw the effective potential V_{eff} in the domain $r \geq \kappa$ for each of these cases and explain the nature of the

orbits near the stationary points.
c. Show that the form of a bound geodesic satisfies

$$\left(\frac{d}{d\varphi}\frac{\ell}{r}\right)^2 = \frac{\ell^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \varepsilon^2 - 1 + \frac{\kappa}{r} - \frac{\ell^2}{r^2} + \frac{\kappa\ell^2}{r^3}. \tag{68}$$

d. Parametrize the solutions by

$$r = \frac{R}{1 + e\cos y(\varphi)}, \tag{69}$$

and explain that if such solutions exist, the points of closest approach (the *periastron*) and the most distant point (the *apastron*) are at

$$r_{peri} = \frac{R}{1+e}, \qquad r_{ap} = \frac{R}{1-e}. \tag{70}$$

Which values does y take at these points?
e. By studying these special points, establish the relations

$$\begin{aligned} \varepsilon^2 &= \left(1 - \frac{\kappa}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) + \frac{e\ell^2}{R^2}\left(1 - \frac{3\kappa}{R}\right), \\ \ell^2 &= \frac{\kappa R^2}{2R - (3+e^2)\kappa}. \end{aligned} \tag{71}$$

f. Prove that solutions (69) exist for $y(\varphi)$ satisfying

$$\left(\frac{dy}{d\varphi}\right)^2 = 1 - \frac{3\kappa}{R}\left(1 + \frac{e}{3}\cos y\right). \tag{72}$$

g. Show that after one turn the periastron has moved by an angular distance

$$\Delta\varphi = \frac{3\pi\kappa}{R} + \mathcal{O}(\kappa^2/R^2). \tag{73}$$

h. The Schwarzschild radius of the sun is $\kappa_{sun} = 2.95$ km and the orbit of Mercury has an average radius $R = 57.9 \times 10^6$ km. Calculate the number of turns of Mercury around the sun per century, and the cumulative shift of the periastron (perihelion).

8. *Light-like geodesics*
Light-like geodesics are geodesics for which the geodesic hamiltonian vanishes: $H = 0$. The energy of motion p_t and the components of angular momentum J_i are still constants of motion; hence light-like geodesics are also confined to a plane, which may be taken to be the equatorial plane

$\theta = \pi/2$. This will be assumed below.
a. Prove that every element of a light-like geodesic is part of a lightcone:

$$-\left(1-\frac{\kappa}{r}\right)dt^2 + \frac{dr^2}{(1-\frac{\kappa}{r})} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 = 0. \quad (74)$$

b. Show that a non-radial ($\ell/\varepsilon \neq 0$) light-like geodesic satisfies

$$\left(\frac{d}{d\varphi}\frac{1}{r}\right)^2 = \left(\frac{\varepsilon}{\ell}\right)^2 - \frac{1}{r^2}\left(1-\frac{\kappa}{r}\right), \quad (75)$$

and that the distance of closest approach is

$$r_m = \frac{\ell}{\varepsilon} - \frac{\kappa}{2} + \mathcal{O}\left[\kappa^2\varepsilon/\ell\right]. \quad (76)$$

c. Define a function $y[r(\varphi)]$ by

$$\cos^2 y = \frac{\ell^2}{\varepsilon^2 r^2}\left(1-\frac{\kappa}{r}\right); \quad (77)$$

derive the results

$$\frac{1}{r} = \frac{\varepsilon}{\ell}\cos y + \frac{\varepsilon^2\kappa}{2\ell^2}\cos^2 y + \mathcal{O}\left[\cos^3 y\right], \quad (78)$$

and

$$\frac{d}{d\varphi}\left(\frac{\ell}{\varepsilon r}\right) = -\sin y\left(1 + \frac{\varepsilon\kappa}{\ell}\cos y + ...\right)\frac{dy}{d\varphi} = -\sin y. \quad (79)$$

In the last equation a choice for the sign of y has been made implicitly.
d. When a lightray comes from large distance r, it is scattered and moves back to large r in a different direction; this is the bending of light by a mass M. Explain why the change in direction is given by:

$$\Delta\varphi = \int_{-\pi/2}^{\pi/2} dy\,\frac{d\varphi}{dy}, \quad (80)$$

and compute $\Delta\varphi$ to first order in M.

9. *Radial light-like geodesics*
Radial geodesics have fixed angular co-ordinates φ and θ.
a. With $H = 0$, derive the equation

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{\kappa}{r}\right)^2. \quad (81)$$

b. Define the Eddington-Finkelstein co-ordinate

$$r_* = r + \frac{\kappa}{2}\ln\left(\frac{r}{\kappa}-1\right)^2. \quad (82)$$

Show that

$$\frac{dr_*}{dt} = \pm 1. \tag{83}$$

Explain which sign applies to infalling light-rays, and which one to outgoing light-rays.
c. Let

$$v = t + r_*. \tag{84}$$

Calculate dv/dt for ingoing and outgoing light-rays.
d. Show that in terms of the original r-co-ordinate

$$\frac{dv}{dr} = \frac{1}{1 - \frac{\kappa}{r}}. \tag{85}$$

e. In a v-r-diagram, draw light-cones containing ingoing light-rays at $r > \kappa$ and $r < \kappa$; what happens at $r = \kappa$?
f. Perform the same exercises for outgoing light-trays, using

$$u = t - r_* \tag{86}$$

as a light-cone co-ordinate instead of v.

d. Schwarzschild black holes

1. a. Using the results of exercise c.9, explain why the surface $r = \kappa$ is called a *horizon*.
   b. Where is the horizon located on the Eddington-Finkelstein co-ordinate axis r_*?
   c. Compute the radial acceleration a^r, eq. (52), at the horizon.
   d. Show that in terms of the light-cone co-ordinates (u, v), the line element in Schwarzschild space-time can be written in a hybrid way as

$$ds^2 = -\left(1 - \frac{\kappa}{r}\right) du dv + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2, \tag{87}$$

   and explain why the horizon is *not* a singular point of the geometry.
   e. In contrast, in view of the Riemann curvature components (49, 50), $r = 0$ is a real singularity. Explain how to establish this.
   f. Use the result (60) to compute the proper time a test particle needs to move from the horizon $r = \kappa$ to the singularity $r = 0$.
   g. Show that for $r > \kappa$ the equal time surface represented by the equatorial plane can be embedded in a 3-dimensional euclidean space with axial co-ordinates (z, ρ, ψ):

$$ds^2 = dz^2 + d\rho^2 + \rho^2 d\psi^2 = \frac{dr^2}{1 - \frac{\kappa}{r}} + r^2 d\varphi^2, \tag{88}$$

by taking

$$z = 2\kappa\sqrt{\frac{r}{\kappa} - 1}, \qquad \rho = r, \qquad \psi = \varphi. \tag{89}$$

2. *Kruskal-Szekeres co-ordinates*
Co-ordinates which cover both the interior and the exterior region of the Schwarzschild black hole can be defined by taking

$$\begin{aligned} T &= \sqrt{\frac{r}{\kappa} - 1}\, e^{r/\kappa} \sinh \frac{t}{2\kappa}, \\ R &= \sqrt{\frac{r}{\kappa} - 1}\, e^{r/\kappa} \cosh \frac{t}{2\kappa}, \end{aligned} \tag{90}$$

for $r > \kappa$; and

$$\begin{aligned} T &= \sqrt{1 - \frac{r}{\kappa}}\, e^{r/\kappa} \cosh \frac{t}{2\kappa}, \\ R &= \sqrt{1 - \frac{r}{\kappa}}\, e^{r/\kappa} \sinh \frac{t}{2\kappa}, \end{aligned} \tag{91}$$

for $r < \kappa$.
a. Show that

$$R^2 - T^2 = \left(\frac{r}{\kappa} - 1\right) e^{r/\kappa}, \qquad \frac{T}{R} = \begin{cases} \tanh \dfrac{t}{2\kappa}, & r > \kappa; \\ \\ \operatorname{cotanh} \dfrac{t}{2\kappa}, & r < \kappa. \end{cases} \tag{92}$$

b. Show that the line element (46) takes the form

$$ds^2 = \frac{4\kappa^3}{r} e^{-r/\kappa} \left(-dT^2 + dR^2\right) + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{93}$$

where $r(R,T)$ is a function of R and T defined implicitly by the first equation (92).
c. Show that for radial light rays

$$\frac{dR}{dT} = \pm 1. \tag{94}$$

d. Draw a diagram of the space time projected on the R-T plane, with R on the horizontal axis and T on the vertical axis. Draw the line(s) representing the horizon: $r = \kappa$, and the curves representing the singularity $r = 0$.
e. Draw curves of constant t, of constant r in the domains $r > \kappa$ and $0 < r < \kappa$. Indicate the trajectory of a radially ingoing time-like geodesic (the world line of a radially infalling massive particle).

e. Kerr space-time

The Kerr solutions of the Einstein equations describe empty, stationary rotating and axially symmetric space-time. Stationary rotation around the z-axis implies, that the metric components are independent of t and φ, and hence geodesic energy and the z-component of angular momentum are still geodesic constants of motion; However, the metric does depend explicitly on θ, therefore the other components of angular momentum are not conserved on geodesics. The stationary rotation manifests itself by the coupling of time and angular φ-co-ordinate in the metric.

The standard form of the Kerr metric is parametrized by the spherical Boyer-Lindquist co-ordinate system (t, r, θ, φ) and takes the form

$$\begin{aligned} ds^2 &= \frac{-\Delta^2}{r^2 + a^2 \cos^2\theta} \left(dt - a \sin^2\theta \, d\varphi \right)^2 + \frac{r^2 + a^2 \cos^2\theta}{\Delta^2} \left(dr^2 + \Delta^2 \, d\theta \right) \\ &\quad + \frac{(r^2 + a^2)^2 \sin^2\theta}{r^2 + a^2 \cos^2\theta} \left(d\varphi - \frac{adt}{r^2 + a^2} \right)^2 . \end{aligned} \tag{95}$$

Here a is a constant and the quantity Δ is defined as

$$\Delta^2 = r^2 - 2Mr + a^2. \tag{96}$$

It will become clear that M is the central mass and $J = Ma$ is the central angular momentum of the object (star or black hole) described by the Kerr solution.

1. a. Check that in the limit $a \rightarrow 0$ the line element (95) reduces to the Schwarzschild metric.
   b. Prove that the geodesic hamiltonian is

   $$H = \frac{1}{2\rho^2} \left[\Delta^2 p_r^2 + p_\theta^2 + \left(a \sin\theta p_t + \frac{p_\varphi}{\sin\theta} \right)^2 - \frac{1}{\Delta^2} \left((r^2 + a^2) p_t + a p_\varphi \right)^2 \right], \tag{97}$$

   with

   $$\rho^2 = r^2 + a^2 \cos^2\theta. \tag{98}$$

   c. Prove, that p_t and p_φ are constants of motion. As for the Schwarzschild solution, we denote these constants by $p_t = -\varepsilon$, $p_\varphi = \ell$.
   d. Prove that the Kerr geometry possesses a rank-2 Killing tensor

   $$\begin{aligned} K &= \frac{1}{2} K^{\mu\nu} p_\mu p_\nu \\ &= \frac{1}{2\rho^2} \left[-\Delta^2 a^2 \cos^2\theta \, p_r^2 + r^2 p_\theta^2 + r^2 \sin^2\theta \left(a p_t + \frac{p_\varphi}{\sin^2\theta} \right)^2 \right. \\ &\quad \left. + \frac{a^2 \cos^2\theta}{\Delta^2} \left((r^2 + a^2) p_t + a p_\varphi \right)^2 \right]. \end{aligned} \tag{99}$$

2. a. Derive the expressions for the non-vanishing components of the connection:

$$
\begin{aligned}
&\Gamma_{rt}{}^{t} = \frac{M(r^2+a^2)(r^2-a^2\cos^2\theta)}{\Delta^2\rho^4}, \qquad \Gamma_{\theta t}{}^{t} = -\frac{2Mr}{\rho^4}\,a^2\sin\theta\cos\theta \\
&\Gamma_{r\varphi}{}^{t} = -\frac{Ma\sin^2\theta}{\Delta^2\rho^4}\left[2r^2\rho^2+(r^2+a^2)(r^2-a^2\cos^2\theta)\right], \\
&\Gamma_{\theta\varphi}{}^{t} = \frac{2Mr}{\rho^4}\,a^3\sin^3\theta\cos\theta, \\
&\Gamma_{tt}{}^{r} = \frac{M\Delta^2}{\rho^6}\,(r^2-a^2\cos^2\theta), \qquad \Gamma_{t\varphi}{}^{r} = -\frac{Ma\Delta^2\sin^2\theta}{\rho^6}\left(r^2-a^2\cos^2\theta\right), \\
&\Gamma_{rr}{}^{r} = \frac{-M\,(r^2-a^2\cos^2\theta)+ra^2\sin^2\theta}{\Delta^2\rho^2}, \qquad \Gamma_{\theta r}{}^{r} = -\frac{a^2}{\rho^2}\,\sin\theta\cos\theta, \\
&\Gamma_{\theta\theta}{}^{r} = -\frac{r\Delta^2}{\rho^2}, \qquad \Gamma_{\varphi\varphi}{}^{r} = \frac{\Delta^2\sin^2\theta}{\rho^6}\left[Ma^2\sin^2\theta\left(r^2-a^2\cos^2\theta\right)-r\rho^4\right], \\
&\Gamma_{tt}{}^{\theta} = -\frac{2Mra^2}{\rho^6}\,\sin\theta\cos\theta, \qquad \Gamma_{t\varphi}{}^{\theta} = \frac{2Mra}{\rho^6}\,(r^2+a^2)\sin\theta\cos\theta, \\
&\Gamma_{rr}{}^{\theta} = \frac{a^2}{\Delta^2\rho^2}\,\sin\theta\cos\theta, \qquad \Gamma_{r\theta}{}^{\theta} = \frac{r}{\rho^2}, \\
&\Gamma_{\theta\theta}{}^{\theta} = -\frac{a^2}{\rho^2}\,\sin\theta\cos\theta, \qquad \Gamma_{\varphi\varphi}{}^{\theta} = \frac{1}{\rho^6}\,\sin\theta\cos\theta\left[-2Mr(r^2+a^2)-\Delta^2\rho^4\right], \\
&\Gamma_{tr}{}^{\varphi} = \frac{Ma}{\Delta^2\rho^4}\left(r^2-a^2\cos^2\theta\right), \qquad \Gamma_{t\theta}{}^{\varphi} = -\frac{2Mar}{\rho^4}\,\frac{\cos\theta}{\sin\theta}, \\
&\Gamma_{r\varphi}{}^{\varphi} = \frac{1}{\Delta^2\rho^4}\left[\rho^4(r-M)-M(r^2+a^2)(r^2-a^2\cos^2\theta)\right], \\
&\Gamma;_{\theta\varphi}{}^{\varphi} = \frac{\cos\theta}{\sin\theta}+\frac{2Mra^2}{\rho^4}\sin\theta\cos\theta.
\end{aligned}
\tag{100}
$$

b. Derive the components of the Riemann tensor:

$$
\begin{aligned}
R_{trtr} &= \frac{2Mr}{\Delta^2\rho^6}\left(3(r^2+a^2)-4Mr-\rho^2\right)\left(r^2-3a^2\cos^2\theta\right),\\
R_{trt\theta} &= -\frac{6Ma^2}{\rho^6}\sin\theta\cos\theta\left(3r^2-a^2\cos^2\theta\right),\\
R_{t\theta t\theta} &= -\frac{2Mr}{\rho^6}\left(3(r^2+a^2)-2\rho^2-2Mr\right)\left(r^2-3a^2\cos^2\theta\right),\\
R_{t\varphi t\varphi} &= -\frac{2Mr}{\rho^6}\sin^2\theta\left(r^2+a^2-2Mr\right)\left(r^2-3a^2\cos^2\theta\right),\\
R_{trr\varphi} &= \frac{2Mra}{\Delta^2\rho^6}\sin^2\theta\left(3(r^2+a^2)-4Mr\right)\left(r^2-3a^2\cos^\theta\right),\\
R_{tr\theta\varphi} &= -\frac{2Ma}{\rho^6}\sin\theta\cos\theta\left(3(r^2+a^2)-\rho^2\right)\left(3r^2-a^2\cos^\theta\right),\\
R_{t\theta r\varphi} &= -\frac{2Ma}{\rho^6}\sin\theta\cos\theta\left(3(r^2+a^2)-2\rho^2\right)\left(3r^2-a^2\cos^2\theta\right),\\
R_{t\varphi r\theta} &= \frac{2Ma}{\rho^4}\sin\theta\cos\theta\left(3r^2-a^2\cos^2\theta\right),\\
R_{t\theta\theta\varphi} &= -\frac{2Mar}{\rho^6}\sin^2\theta\left(3(r^2+a^2)-2Mr\right)\left(r^2-3a^2\cos^2\theta\right),\\
R_{r\theta r\theta} &= \frac{Mr}{\Delta^2\rho^2}\left(r^2-3a^2\cos^2\theta\right),\\
R_{r\varphi\theta\varphi} &= -\frac{6Ma^2}{\rho^6}\sin^3\theta\cos\theta(r^2+a^2)\left(3r^2-a^2\cos^2\theta\right),\\
R_{r\varphi r\varphi} &= \frac{2Mr}{\Delta^2\rho^6}\sin^2\theta\left(2(r^2+a^2)^2+\Delta^2(r^2+a^2-2\rho^2)\right)\left(r^2-3a^2\cos^2\theta\right),\\
R_{\theta\varphi\theta\varphi} &= -\frac{2Mr}{\rho^6}\sin^2\theta\left(a^2\Delta^2+2(r^2+a^2)^2\right)\left(r^2-3a^2\cos^2\theta\right).
\end{aligned}
\tag{101}
$$

c. Prove that

$$R_{\mu\nu}=0. \tag{102}$$

3. a. Show that for $a^2 \leq m^2$:

$$\Delta^2 = 0 \quad \Leftrightarrow \quad r = r_\pm \equiv m \pm \sqrt{m^2 - a^2}. \tag{103}$$

b. Derive the expressions for p_r and p_θ:

$$\begin{aligned} p_r^2 &= \frac{2}{\Delta^2}\left(r^2 H - K\right) + \frac{1}{\Delta^4}\left((r^2+a^2)\varepsilon - a\ell\right)^2, \\ p_\theta^2 &= 3a^2\cos^2\theta\, H + 2K - \left(a\varepsilon\sin\theta - \frac{\ell}{\sin\theta}\right)^2. \end{aligned} \tag{104}$$

c. By using Hamilton's equations show that

$$\frac{dr}{d\lambda} = \frac{\Delta^2}{\rho^2}\, p_r, \qquad \frac{d\theta}{d\lambda} = \frac{p_\theta}{\rho^2}. \tag{105}$$

d. Combine these results for light-like geodesics ($H = 0$) to get

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 &= \frac{1}{\rho^4}\left[\left((r^2+a^2)\varepsilon - a\ell\right)^2 - 2\Delta^2 K\right], \\ \left(\frac{d\theta}{d\lambda}\right)^2 &= \frac{1}{\rho^4}\left[2K - \left(a\varepsilon\sin\theta - \frac{\ell}{\sin\theta}\right)^2\right]. \end{aligned} \tag{106}$$

e. Show that the geodesic flow in the (t, φ)-directions is given by

$$\begin{pmatrix} \frac{dt}{d\lambda} \\ \frac{d\varphi}{d\lambda} \end{pmatrix} = \begin{pmatrix} g^{tt} & g^{t\varphi} \\ g^{\varphi t} & g^{\varphi\varphi} \end{pmatrix} \begin{pmatrix} -\varepsilon \\ \ell \end{pmatrix} = \frac{-1}{\Delta^2 \sin^2\theta} \begin{pmatrix} g_{\varphi\varphi} & -g_{\varphi t} \\ -g_{t\varphi} & g_{tt} \end{pmatrix} \begin{pmatrix} -\varepsilon \\ \ell \end{pmatrix}. \tag{107}$$

f. Derive an expression for the angular velocity:

$$\frac{d\varphi}{dt} = -\frac{g_{t\varphi}\varepsilon + g_{tt}\ell}{g_{\varphi\varphi}\varepsilon + g_{t\varphi}\ell}, \tag{108}$$

and show that for $\ell = 0$ the angular velocity does not vanish:

$$\Omega \equiv \left.\frac{d\varphi}{dt}\right|_{\ell=0} = \frac{a}{(r^2+a^2)}\frac{1}{(1 + \Delta^2\rho^2/2Mr(r^2+a^2))}. \tag{109}$$

In particular on the horizon r_+:

$$\Omega_+^2 = \frac{a}{r_+^2 + a^2}. \tag{110}$$

It follows that geodesics are dragged along by the rotation of space-time.
f. Prove that

$$\Omega^2 = \frac{g_{tt}}{g_{\varphi\varphi}} + \frac{\Delta^2}{g_{\varphi\varphi}^2} \sin^2\theta \geq \frac{g_{tt}}{g_{\varphi\varphi}} \quad \text{for } r \geq r_+. \tag{111}$$

g. Show that

$$\frac{dt}{d\lambda} = \frac{g_{\varphi\varphi}}{\Delta^2 \sin^2\theta} \left(\varepsilon - \Omega\ell\right), \tag{112}$$

and explain that on time-like geodesics

$$\frac{dt}{d\tau} > 0 \quad \Rightarrow \quad \varepsilon - \Omega\ell > 0. \tag{113}$$

h. Finally, rewrite eq. (108) as

$$\left(\varepsilon - \Omega\ell\right) \frac{d\varphi}{dt} = \Omega\varepsilon - \frac{g_{tt}}{g_{\varphi\varphi}}\, \ell, \tag{114}$$

and prove that

$$\begin{aligned} \ell > 0 \quad &\Rightarrow \quad \frac{d\varphi}{dt} > \Omega, \\ \ell < 0 \quad &\Rightarrow \quad \frac{d\varphi}{dt} < \Omega. \end{aligned} \tag{115}$$

i. Compute g_{tt} and show that

$$\begin{aligned} &g_{tt} < 0, \quad \text{if } r > M + \sqrt{M^2 - a^2\cos^2\theta}, \\ &g_{tt} > 0, \quad \text{if } M - \sqrt{M^2 - a^2\cos^2\theta} < r < M + \sqrt{M^2 - a^2\cos^2\theta}. \end{aligned} \tag{116}$$

The region outside the horizon $r > r_+$, where $\Delta^2 > 0$, but where also $g_{tt} > 0$, is called the *ergosphere*.
j. Let ξ_t^μ and ξ_φ^μ be the Killing vectors associated with the constants of motion ε and ℓ; i.e.,

$$\xi_t^\mu\, p_\mu = p_t = -\varepsilon, \qquad \xi_\varphi^\mu\, p_\mu = p_\varphi = \ell. \tag{117}$$

Show that

$$\xi_t^\mu = \delta_t^\mu = (1, 0, 0, 0), \qquad \xi_\varphi^\mu = \delta_\varphi^\mu = (0, 0, 0, 1). \tag{118}$$

k. Compute the norms of these vectors:

$$\xi_t^2 = g_{\mu\nu}\, \xi_t^\mu \xi_t^\nu, \qquad \xi_\varphi^2 = g_{\mu\nu}\, \xi_\varphi^\mu \xi_\varphi^\nu, \tag{119}$$

and the inner product

$$\xi_t \cdot \xi_\varphi = g_{\mu\nu}\, \xi_t^\mu \xi_\varphi^\nu. \tag{120}$$

Show that $\xi_t^2 < 0$ outside the ergosphere, and $\xi_t^2 > 0$ inside the ergosphere, whilst $\xi_\varphi^2 > 0$ everywhere. Prove that $\Omega^2 > 0$ inside the ergosphere, and compute its minimal value there.