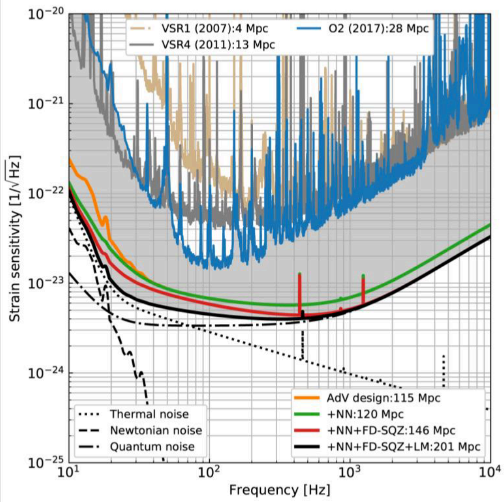
Learned from first lecture

- Idealized Interferometer: Conceptual design and crude analysis
  1. Encoding GW signal in phase shift of light — $\Delta\phi \sim \Delta L \sim h(t)$
  2. Increasing signal strength via bounces in arms — $\Delta\phi \sim B$
  3. Limit on accuracy of phase measurement — $\Delta\phi \geq \frac{1}{\sqrt{N_\gamma}}$
  4. Required laser power; energetic quantum limit

Sensitivity curves

- What is shown?
  - Unit 1/√Hz
- Noise contributions
  - Fundamental limits
  - Ways to improve
  - Behavior of IFO



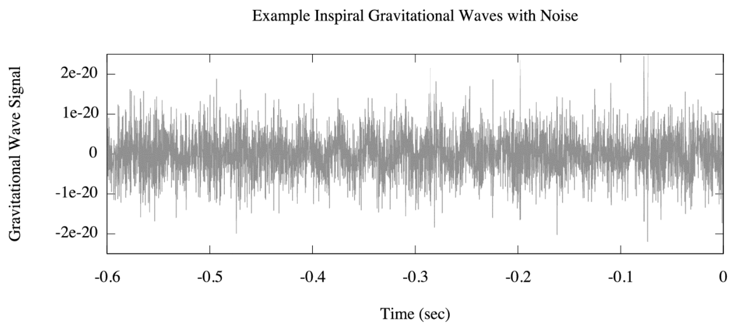
→ Smaller time intervals
Smaller wavelengths

Signal and noise

- Time series y(t)
  - Deterministic
  - Random

Sum of both

- Signal with noise
  - Laser field
  - Current from photodiode
  - Strain

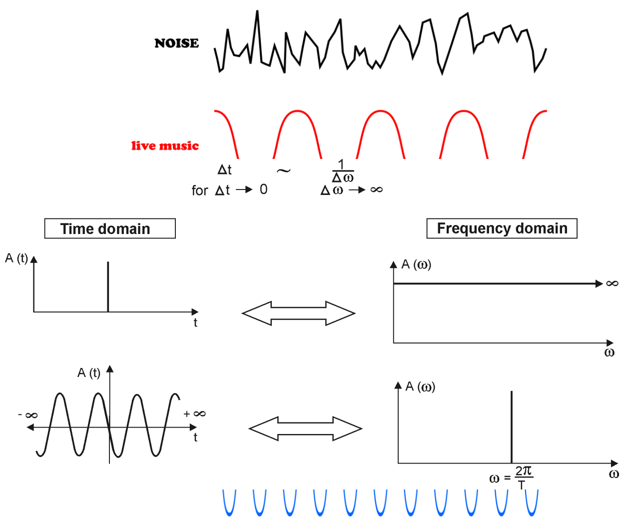


- Mathematical tools to deal with time series
  - Fourier transform
  - Cross-correlation, auto-correlation, convolution

Power spectrum

Fourier transform

$$\boxed{Y(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t) e^{-2\pi f t} dt}$$



- Convert to frequency domain (and back)
  - Show different frequency components
  - Operations in frequency domain easier
  - Use filtering

Cross-correlation

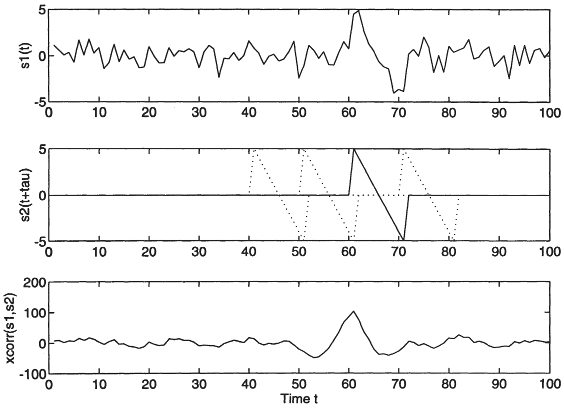
- Cross-correlation

$$x_1 \otimes x_2(\tau) \equiv \int_{-\infty}^{\infty} x_1(t) x_2(t+\tau) dt$$

- Auto-correlation
  - Cross-correlation of time series x(t) with itself

- Convolution

$$x_1 * x_2(\tau) \equiv \int_{-\infty}^{\infty} x_1(t) x_2(\tau - t) dt$$



Cross-correlation

Power spectrum

- Power spectral density

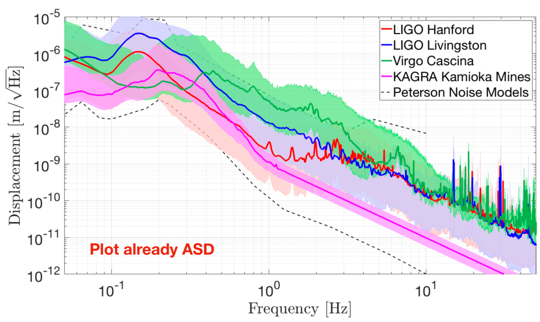
$$\boxed{P_x(f) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \otimes x(\tau) e^{-2\pi f \tau} d\tau}$$

  - First the auto-correlation of a time series, and then take the Fourier transform

- Single-sided power spectrum

$$\boxed{\begin{aligned} S_x(f) &\equiv 2P_x(f), && \text{if } f \geq 0 \\ &\equiv 0, && \text{otherwise} \end{aligned}}$$

- PSD of random process x(t) gives how much noise power there is in a certain frequency band



Handy

$$S_x(f) = \frac{2}{T} |X(f)|^2$$

$$(\Delta y)^2 = \int_0^{\infty} S_y(f) df$$

$$\Delta y|_{\Delta f} = \sqrt{S_y(f) \cdot \Delta f}$$

Power spectrum II

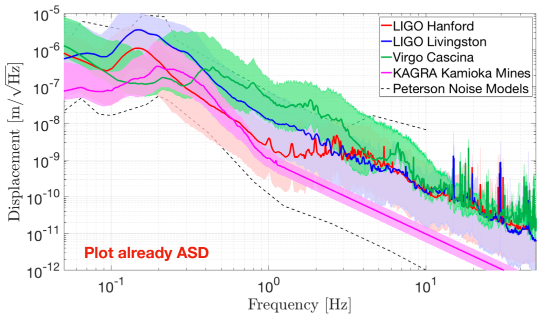
- PSD in [x²/Hz] where x can be in [V], [m], [-],….
- Mean square value

$$\boxed{(\Delta x)^2 = \overline{x^2} = \int_0^\infty S_x(f)df}$$

- Amplitude spectral density (ASD)

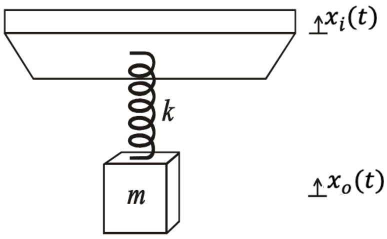
$$\boxed{A_x(f) \equiv \sqrt{S_x(f)}}$$

  - in [x/√Hz]



Linear system

- Linear system
  - ➡ $x_0(t) = \int_{-\infty}^{t} x_i(\tau)g(t-\tau)d\tau$
- Example - Harmonic Oscillator
  - ➡ Filter - x_i and x_o in [m]
  - ➡ Relevant in seismic noise
- Transducer
  - ➡ Input and output have different units
- Impulse respons
  - ➡ Input-output relationship



Equation of motion

$$\boxed{m\ddot{x}_o + k(x_o - x_i) + b\dot{x}_o = 0}$$

Linear system II

- Linear system (convolution)

$$x_0(t) = \int_{-\infty}^{t} x_i(\tau) g(t-\tau) d\tau$$

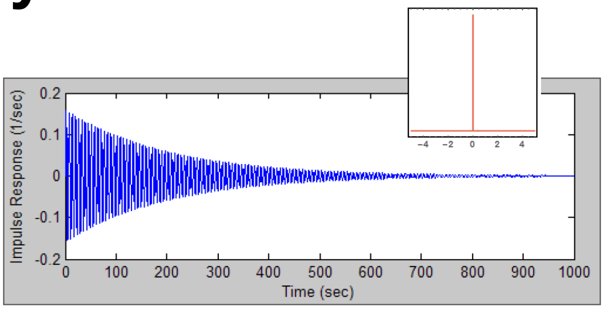
- Hard to calculate in time domain, use FT

$$G(f) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f\tau} d\tau$$

- Convolution becomes

$$X_o(f) = X_i(f) G(f) \qquad \text{or } G(f) = \frac{X_o(f)}{X_i(f)}$$

➡ *Transfer function, gain, filter*



Response due to a single unit impulse at t=0

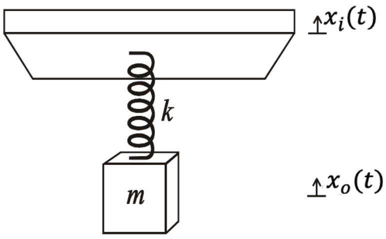
Example

- Sinusoidal input

$$x_i(t) = X_i(f) e^{i2\pi ft}$$

- Linear system ⇒ output also sinusoidal
- Plug into equation of motion
- Frequency response?

$$G(f) = \frac{k}{k + i2\pi fb - m(2\pi f)^2}$$

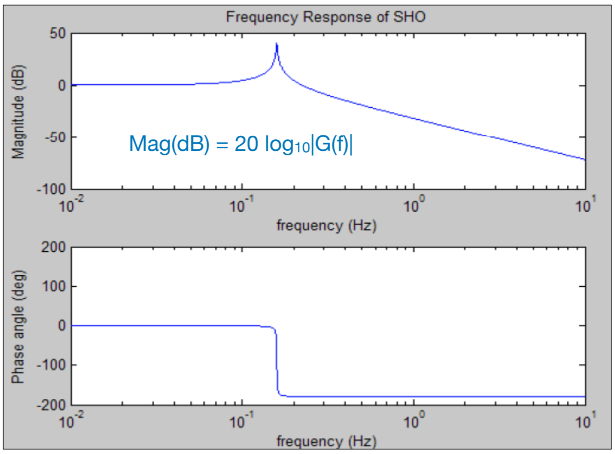


Equation of motion

$$\boxed{m\ddot{x}_o + k(x_o - x_i) + b\dot{x}_o = 0}$$

Bode plot

- For f << f_{res}
  - ➡ 0 dB (or 1) - mass tracks the top
- At resonance
  - ➡ Underdamped - largest response
- For f >> f_{res}
  - ➡ -40 dB/decade (~ $1/f^2$) - mass movement damped



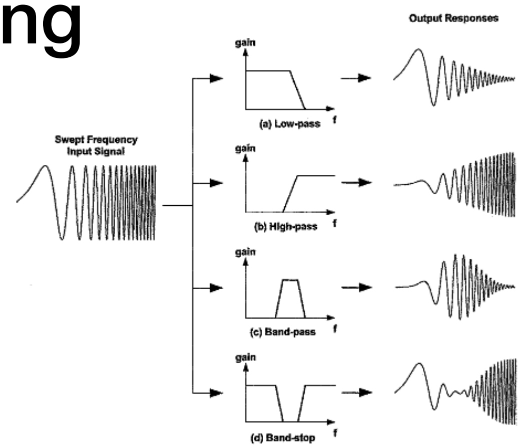
Filtering

- Linear process is a convolution between filter kernel (or transfer function) k and input signal

$$y_{out}(\tau) = \int_{-\infty}^{\infty} k(\tau - t)\, y_{in}(t)\, dt$$

- Effect of filtering on noise spectral density (use *)

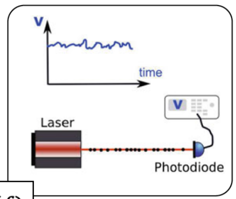
$$S_{y,out}(f) = |K(f)|^2 S_{y,in}(f)$$



$$Y_{out,T}(f) = K(f) Y_{in,T}(f)$$

$$|Y_{out,T}(f)|^2 = |K(f)|^2 |Y_{in,T}(f)|^2$$

Photon shot noise



- Determine shot noise (PSD) of laser beam
  - *Contributes to h(t)*
  - *Via transfer function*

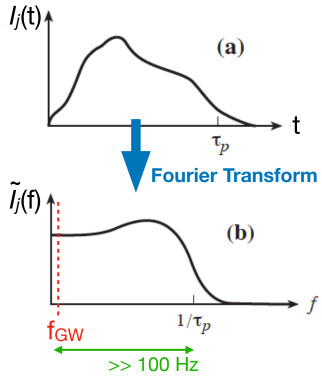
$$\boxed{I(\tau) \sim k(\tau - t)h(t)}$$

$$\boxed{S_I(f) = |K(f)|^2 \cdot S_h(f)}$$

- Sum over pulses (T)
  - *PSD: take square of Fourier transform*
  - *Pulses are uncorrelated and cancel each other*
  - *Use the energy of a photon*
  - *Now write the PSD for the shot noise (not freq. dependent!)*

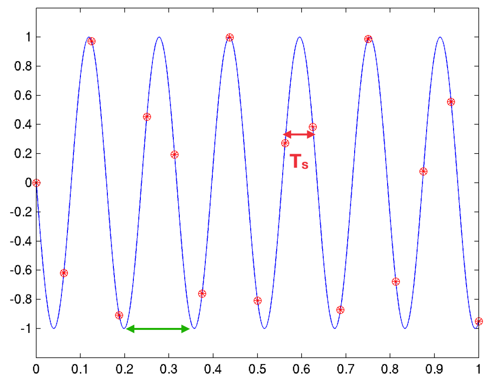
$$I_T(t) = \sum_j I_j(t - t_j)$$

$$S_I(f) = 2I \cdot (\bar{h}\omega_e)$$



What we measure?

- Photo diode
  - $\Delta I_{PD} \rightarrow \Delta\phi \rightarrow \Delta L$
- Time series
  - Virgo sample at 20 kHz
  - Nyquist $f_s/2$ → 10 kHz
- Example 'aliasing'
  - Signal frequency 9.7 Hz
  - Sample with only 16 Hz → ~6 Hz



Signal frequency 9.7 Hz ⇒ f_s=16 Hz

$T_s=1/f_s$=0.063 s

Calibration of IFO

- Linear system

➡ $y(t) = \int_{-\infty}^{t} x(\tau) g(t-\tau) d\tau$

- In frequency domain

➡ $Y(f) = G(f) \cdot X(f)$

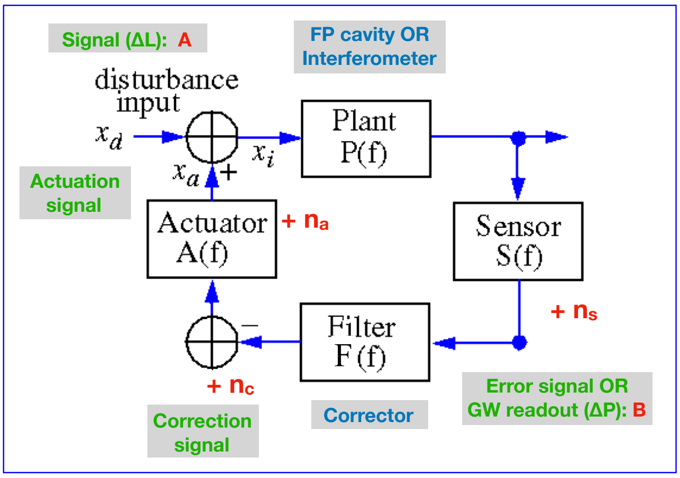
- Feedback control system

➡ $H_{cl}(f) = \frac{H(f)}{1+G(f)}$

- Between 'A' and 'B'

➡ $H_{cl}(f) = \frac{P(f)S(f)}{1+G(f)} = \frac{\Delta P}{\Delta L}$

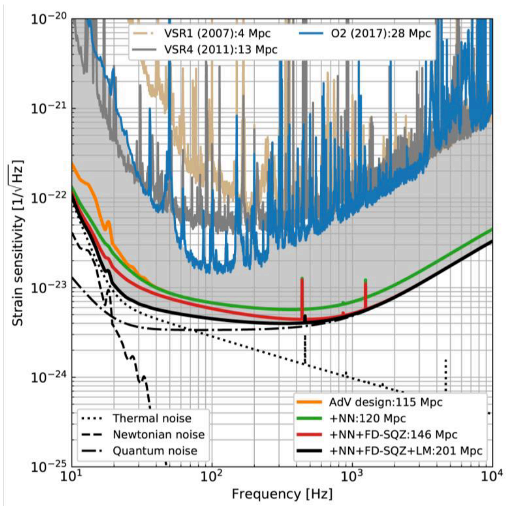
$G(f) = P(f)S(f)F(f)A(f)$



$$h = (\text{GW READOUT}) \times \frac{1+G(f)}{P(f)\,S(f)}$$

Sensitivity curves

- What is shown?
  - Unit 1/√Hz
- Noise contributions
  - Fundamental limits
  - Ways to improve
  - Quantum noise frequency dependent?



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