I. GRAVITATIONAL WAVES FROM BINARY BLACK HOLES

Two black holes (or two neutron stars, or a neutron star and a black hole) with masses m_1 and m_2 are in a circular orbit around the common center of mass (CM), with angular frequency ω. Assume the distance R between the two is large enough that the black holes can be viewed as point particles and that the effect of orbital energy loss through GW emission is negligible. With respect to an observer at a distance r, the system is oriented such at the line of sight makes an angle ι with the normal to the orbital plane.

(1.1) Pick a coordinate system (x, y, z) such that the direction to the observer is along the z axis, and the CM of the system is at the origin $(0, 0, 0)$. Without loss of generality we may assume that the orbital plane is oriented in such a way that its intersection with the (x, y) plane is in the x axis. Write an expression for the positions $\mathbf{x}_1$, $\mathbf{x}_2$ of the black holes as a function of time.

Solution. One has

$$\begin{aligned} \mathbf{x}_1(t) &= \frac{m_2}{m_1+m_2} R\,\hat{\mathbf{e}}(t) = \frac{\mu}{m_1} R\,\hat{\mathbf{e}}(t) \\ \mathbf{x}_2(t) &= -\frac{m_1}{m_1+m_2} R\,\hat{\mathbf{e}}(t) = -\frac{\mu}{m_2}\,\hat{\mathbf{e}}(t) \end{aligned} \tag{1.1}$$

where

$$\hat{\mathbf{e}}(t) = (\cos(\omega t), \cos(\iota)\,\sin(\omega t), \sin(\iota)\,\sin(\omega t)) \tag{1.2}$$

and in (1.1) we have introduced the *reduced mass* μ, given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.3}$$

(1.2) In the TT gauge, a gravitational wave propagating in the z-direction corresponds to a metric perturbation

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \tag{1.4}$$

and in the quadrupole approximation one has

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}(\ddot{M}_{11} - \ddot{M}_{22}), \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}, \end{aligned} \tag{1.5}$$

where in each case the RHS is evaluated at the retarded time $t - r/c$. Compute the relevant moments M_{ij} and their double time derivatives $\ddot{M}_{ij}$.

Solution. One has

$$\begin{aligned} M^{ij}(t) &= \frac{1}{c^2}\int d^3x\, T^{00}(t, \mathbf{x})\, x^i x^j \\ &= \int d^3x\, \rho(t, \mathbf{x})\, x^i x^j, \end{aligned} \tag{1.6}$$

where in the last line we approximated $T^{00}/c^2 \simeq \rho$, with ρ the density. Since we are effectively dealing with two point particles, this density is given by

$$\rho(t,\mathbf{x}) = m_1\delta^3(\mathbf{x} - \frac{\mu}{m_1}R\hat{\mathbf{e}}(t)) + m_2\delta^3(\mathbf{x} + \frac{\mu}{m_2}R\hat{\mathbf{e}}(t)). \tag{1.7}$$

Substituting this into (1.6) and performing the integral, we get

$$\begin{aligned} M^{ij}(t) &= \left[m_1\frac{\mu^2}{m_1^2}R^2 + m_2\frac{\mu^2}{m_2^2}R^2\right]\hat{e}^i(t)\hat{e}^j(t) \\ &= \mu R^2\hat{e}^i(t)\hat{e}^j(t). \end{aligned} \tag{1.8}$$

In particular, using (1.2),

$$\begin{aligned} M_{11} &= \mu R^2\cos^2(\omega t), \\ M_{22} &= \mu R^2\cos^2(\iota)\,\sin^2(\omega t), \\ M_{12} &= \mu R^2\cos(\iota)\,\cos(\omega t)\,\sin(\omega t). \end{aligned} \tag{1.9}$$

The double time derivatives are (using some trigonometric identities):

$$\begin{aligned} \ddot{M}_{11} &= -2\mu R^2\omega^2\cos(2\omega t), \\ \ddot{M}_{22} &= 2\mu R^2\omega^2\cos^2(\iota)\,\cos(2\omega t), \\ \ddot{M}_{12} &= -2\mu R^2\omega^2\cos(\iota)\,\sin(2\omega t). \end{aligned} \tag{1.10}$$

(1.3) Substitute the results into (1.5) and evaluate at the retarded time $t_{\rm ret}$ to find the gravitational wave polarizations.

Solution. We find

$$\begin{aligned} h_+ &= \frac{4}{r}\frac{G\mu R^2\omega^2}{c^4}\,\frac{1+\cos^2(\iota)}{2}\,\cos(2\omega t_{\rm ret} + 2\phi), \\ h_\times &= \frac{4}{r}\frac{G\mu R^2\omega^2}{c^4}\,\cos(\iota)\,\sin(2\omega t_{\rm ret} + 2\phi). \end{aligned} \tag{1.11}$$

With our choice of time origin, $\phi = \pi/2$ in order to absorb an overall minus sign, but infinitely many choices are possible, all leading to different phase offsets ϕ.

(1.4) Explain why the radiation is at *twice* the orbital frequency.

Solution. This is because the components of the quadrupole tensor (1.8) return to the same value after only *half* a period, as they are invariant under $\hat{\mathbf{e}} \to -\hat{\mathbf{e}}$. It is not difficult to see that this property is generic for rigidly rotating systems, by considering the more general expression (1.6). For this reason the frequency $f_{\rm gw} = 2f_{\rm orb} = 2\omega/(2\pi)$ (where $f_{\rm orb}$ is the orbital frequency) is often called the *gravitational wave frequency*. The name is apt if only quadrupole radiation is being studied, but in reality also higher multipole moments will come into play. These introduce *harmonics* with frequencies $nf_{\rm orb}$, $n = 1, 2, 3, \ldots$. The harmonic with $n = 2$ is only the dominant contribution to the gravitational waveform.

(1.5) If the black holes are sufficiently far apart, we can use the Newtonian centripetal force to express the separation R in terms of ω and the component masses m_1, m_2. Define the chirp mass

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}, \tag{1.12}$$

and write h_+, $h_\times$ in terms of it.

Solution. Consider, e.g., the particle with mass m_1. The centripetal force that keeps it on its orbit around the CM with radius $(\mu/m_2)\,R$ is provided by the gravitational force exerted by m_2 over a distance R, so

$$\frac{m_1\left((\mu/m_1)R\right)^2\omega^2}{(\mu/m_1)R} = \frac{G m_1 m_2}{R^2}. \tag{1.13}$$

Solving for R, we find

$$R = (GM)^{1/3}\omega^{-2/3}, \tag{1.14}$$

where $M = m_1 + m_2$ is the total mass of the binary system. Substituting the above expression into (1.11), we find

$$\begin{aligned} h_+ &= \frac{4}{r}\frac{G\mathcal{M}_c^{5/3}\omega^{2/3}}{c^4}\,\frac{1+\cos^2(\iota)}{2}\,\cos(2\omega t_{\rm ret} + 2\phi), \\ h_\times &= \frac{4}{r}\frac{G\mathcal{M}_c^{5/3}\omega^{2/3}}{c^4}\,\cos(\iota)\,\sin(2\omega t_{\rm ret} + 2\phi). \end{aligned} \tag{1.15}$$

In reality, binary black holes won't just keep moving on a circle; gravitational waves carry away orbital energy, causing the orbits to shrink. From Eq. (1.14), this implies an increase in angular frequency ω. A positive power of ω appears in the amplitudes, which will also increase monotonically. Hence both the signal amplitude and frequency increase in a steady "chirp". To leading order, the way the component masses m_1, m_2 affect the chirping is through the chirp mass, $\mathcal{M}_c$.

(1.6) What do the polarizations look like when $\iota = 0$ (i.e., the system is seen face-on) and when $\iota = \pi/2$ (edge-on)?

Solution.

- *Edge-on.* If $\iota = \pi/2$ so that $\cos(\iota) = 0$ then $h_\times$ is identically zero, and we only have the "plus" polarization. If only one of the two polarizations are present then the radiation is said to be *linearly polarized.* In retrospect one could have expected this to be the case here, because the observer only sees the component masses move on straight line segments.

- *Face-on.* If $\iota = 0$ to that $\cos(\iota) = 1$ then the "plus" and "cross" polarizations have *equal amplitudes* but are out of phase by $\pi/2$:

$$\begin{aligned} h_+ &= \mathcal{A}(\omega, r, \mathcal{M}_c)\,\cos(2\omega t_{\rm ret} + 2\phi), \\ h_\times &= \mathcal{A}(\omega, r, \mathcal{M}_c)\,\sin(2\omega t_{\rm ret} + 2\phi). \end{aligned} \tag{1.16}$$

  This is referred to as *circular polarization.*

(1.7) What is the total power emitted in gravitational waves for the Earth-Sun system? What is the total power emitted by a system consisting of two black holes with masses $m_1 = m_2 = 10\, M_\odot$ at radii of $20G(m_1+m_2)/c^2$ (i.e., quite close but still comforably far from the last stable orbit)? (The mass of the Sun is 2×10^{30} kg and that of the Earth, 6×10^{24} kg.)

Solution. In terms of the gravitational wave frequency $f_{\rm gw}$, the power is

$$P_{\rm gw} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G\mathcal{M}_c \pi f_{\rm gw}}{c^3}\right)^{10/3}. \tag{1.17}$$

For the Earth-Sun system, $f_{\rm gw} = 2 \times 1/(365 \times 24 \times 3600\,{\rm s})$ while the chirp mass is $\mathcal{M}_c \simeq 9.7 \times 10^{26}$ kg. Substituting into (1.17), we find $P \simeq 8$ Watts, not even enough to power a light bulb.

For the two black holes, the specified orbital radius leads to a gravitational wave frequency of

$$f_{\rm gw} = \frac{1}{\pi}\left(\frac{GM}{R^3}\right)^{1/2} \simeq 144\,{\rm Hz}, \tag{1.18}$$

and the chirp mass is $\mathcal{M}_c \simeq 4.9 \times 10^{30}$. This leads to a power of $P_{\rm gw} \simeq 2.7 \times 10^{44}$ W.