I. COMPLETE GRAVITATIONAL COLLAPSE: A TOY MODEL

Complete gravitational collapse happens when the pressure is no longer capable of sustaining a star. The simplest model one can think of assumes that there is no pressure at all ($P = 0$), but of course non-zero density. The matter is then pressureless dust, with energy-momentum tensor

$$T_{\mu\nu} = \rho\, u^\mu u^n. \tag{1.1}$$

Let as assume a ball of pressureless dust that is homogeneous and isotropic, which starts out from rest with a finite radius R. Outside there is a spherically symmetric vacuum spacetime, which we assume to be asymptotically flat. By Birkhoff's theorem, the outside geometry must be Schwarzschild.

(1.1) At the surface of the dust ball, dust particles are moving on geodesics of the external Schwarzschild geometry. Using the expressions for geodesics derived in the notes, show that the motion of the surface is determined by

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{R}\frac{R-r}{r}. \tag{1.2}$$

Up to the factor $2M/R$, this is a well-known differential equation, namely that of a cycloid. Consider a point P on a wheel that is rolling in the x-direction. Let η be the angle with the vertical made by the line from the center of the wheel to P. If the radius of the wheel is $\mathcal{R}$, show that the x and y components of P depend on η through

$$\begin{aligned} x &= \mathcal{R}(\eta + \sin\eta), \\ y &= \mathcal{R}(1 + \cos\eta). \end{aligned} \tag{1.3}$$

Also show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{2\mathcal{R} - y}{y}. \tag{1.4}$$

By comparing (1.4) with (1.2), infer a parametric solution to (1.2) similar to (1.3).

Solution. We have seen that the radial part of the orbital motion of a massive particle is governed by

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{\tilde{L}^2}{r^2} + 1\right) = \frac{1}{2}\tilde{E}^2. \tag{1.5}$$

Because the entire setup is spherically symmetric, the particles at the surface of the dust ball will just move radially inwards. In that case $0 = p^\phi = m\tilde{L}^2/r^2$, so $\tilde{L} = 0$. At the start of the collapse, $dr/d\tau = 0$ and $r = R$, so $\tilde{E}^2 = 1 - 2M/r$. Hence the above equation simplifies to (1.2).

Deriving (1.3) and (1.4) is trivial. To arrive at a parametric solution of (1.2), it suffices to identify $y = r$, $x = (2M/r)^{1/2}\tau$, and $\mathcal{R} = R/2$. The equations (1.3) then become

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta + \sin\eta), \tag{1.6}$$

$$r = \frac{R}{2}(1 + \cos\eta). \tag{1.7}$$

(1.2) Explain why the interior geometry of the dust ball must be that of a closed ($k = +1$) FLRW Universe. The Einstein equations then reduce to two coupled differential equations for the scale factor a:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{1}{a^2}, \tag{1.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho, \tag{1.9}$$

where the dot denotes derivation with respect to the proper time of the dust, and we recall that $P = 0$. Derive a parametric solution for a of a comoving observer in terms of the maximum value a_m, and for the proper time, using a parameter η as above. Also derive an expression for the density ρ as a function of η.

Solution. In the absence of a cosmological constant, only the closed Universe can be appropriate, since we have assumed there to be an instant in time when the star was at rest - in the language of FLRW, a moment of maximum expansion. By differentiating Eq. (1.8) and substituting $\ddot{a}$ into Eq. (1.9), we find

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho. \tag{1.10}$$

The solution is

$$\rho = \frac{\rho_0}{a^3}. \tag{1.11}$$

If we define a_m through

$$a_m = \frac{8\pi\rho_0}{3}, \tag{1.12}$$

then Eq. (1.8) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{a_m}{a^3} - \frac{1}{a^2}, \tag{1.13}$$

and we see that a_m is the initial scale factor, when the star is at rest ($da/d\tau = 0$). Moreover,

$$\left(\frac{da}{d\tau}\right)^2 = \frac{a_m - a}{a}. \tag{1.14}$$

Note that this is again of the same form as the cycloid equation (1.4). A parametric solution is obtained by identifying $y = a$, $x = \tau$, and $\mathcal{R} = a_m/2$. The equations (1.3) become

$$\tau = \frac{1}{2}a_m(\eta + \sin\eta), \tag{1.15}$$

$$a = \frac{1}{2}a_m(1 + \cos\eta). \tag{1.16}$$

From (1.11) and (1.12),

$$\rho = \left(\frac{3a_m}{8\pi}\right) a^{-3} = \left(\frac{3}{8\pi a_m^2}\right) \left[\frac{1}{2}(1 + \cos\eta)\right]^{-3}. \tag{1.17}$$

(1.3) For this model to be viable, the circumference of the star's surface should be the same as measured with the internal FLRW metric and the external Schwarzschild metric. In other words, if $\mathcal{C}$ is the equator at an arbitary moment in time, then we must have

$$\int_{\mathcal{C}} |g^{\rm S}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2} = \int_{\mathcal{C}} |g^{\rm F}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2}, \tag{1.18}$$

where $g^{\rm F}$ is the $k = 1$ FLRW metric with line element

$$ds^2 = -d\tau^2 + a^2(\tau)\,\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{1.19}$$

Show that it is indeed possible to have (1.18) for all times. How does it relate M and R to the initial scale factor a_m and the radial coordinate value χ_0 of the outer boundary?

Solution. We first need to ensure that Eqns. (1.6) and (1.15) represent the same, unique proper time. This we can do by identifying the parameter η in both equations, and demanding

$$\frac{1}{2}a_m = \left(\frac{R^3}{8M}\right)^{1/2}. \tag{1.20}$$

Outside the dust ball, the metric is Schwarzschild:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.21}$$

while on the inside,

$$ds^2 = -d\tau^2 + a^2(\tau)\,\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.22}$$

At a given time, the equator corresponds to a given t, the Schwarzschild coordinate r of the equator is given by (1.7), and $\theta = \pi/2$. Moving along the equator, only ϕ varies; on the Schwarzschild side we then have

$$\begin{aligned}\int_{\mathcal{C}} |g^{\rm S}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2} &= \int_0^{2\pi} r(\eta)\, d\phi \\ &= 2\pi r(\eta) \\ &= \pi R\,(1 + \cos\eta).\end{aligned} \tag{1.23}$$

On the FLRW side, we note that the radial coordinate χ is co-moving with the dust, so the boundary of the dust ball must have fixed χ_0. the equator again has $\theta = \pi/2$, and the scale factor is given by (1.16). Hence

$$\begin{aligned}\int_{\mathcal{C}} |g^{\rm F}_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2} &= \int_0^{2\pi} a(\eta)\,\sin\chi_0\, d\phi \\ &= 2\pi a(\eta)\,\sin\chi_0 \\ &= \pi a_m(1 + \cos\eta)\,\sin\chi_0.\end{aligned} \tag{1.24}$$

The expressions (1.23) and (1.24) will agree for all η if

$$R = a_m \sin\chi_0. \tag{1.25}$$

Together with Eq. (1.20), one finds

$$M = \frac{1}{2} a_m \sin^3 \chi_0. \tag{1.26}$$

(1.4) An observer A is riding on the surface of the dust ball. Derive expressions in in terms of M and R for the time it takes, according to A, for the following things to happen:

(a) the formation of an event horizon;
(b) the appearance of a singularity.

Suppose that the initial radius of the star was $R = 7 \times 10^8$ m, and that its mass is $M = 2 \times 10^{30}$ kg. (These are in fact the radius and mass of the Sun.) In that case, what are the times in (a) and (b)? *Note:* don't forget to reinstate the appropriate powers of G and c.

Solution. If $R > 2M$ then there is no horizon to begin with; only the part of spacetime outside the star is Schwarzschild, and the inside FLRW geometry has no horizons. As the star collapses, the part of spacetime it vacates is replaced by Schwarzschild. A horizon appears when the coordinate radius of the star reaches $r = 2M$. According to Eq. (1.7), this happens when

$$\eta = \arccos\left(\frac{4M}{R} - 1\right). \tag{1.27}$$

From Eq. (1.6), the proper time of A is then

$$\tau_{\rm hor} = \left(\frac{R^3}{8M}\right)^{1/2} \left[\arccos\left(\frac{4M}{R} - 1\right) + \sqrt{1 - \left(\frac{4M}{R} - 1\right)^2}\,\right]. \tag{1.28}$$

When $\eta = \pi$, Eq. (1.17) tells us that the density diverges; indeed, according to Eqs. (1.7) and (1.16), $r \to 0$ and $a \to 0$ as $\eta \to \pi$. Again using Eq. (1.6), this occurs at

$$\tau_{\rm sing} = \pi \left(\frac{R^3}{8M}\right)^{1/2}. \tag{1.29}$$

In both cases, reinstating G and c in Eqs. (1.28) and (1.29) implies the replacement $M \to GM/c^2$ and multiplying the expressions by an overall prefactor $1/c$. With the given radius and mass, a horizon appears almost an hour after the start of collapse, at $\eta = 3.139\ldots$ (i.e., already quite close to π). After that, it only takes an additional 1.65 milliseconds for the singularity to occur.

(1.5) An observer B is witnessing the collapse from a large distance. We have seen that the proper time t of observer B is related to the same parameter η as above by

$$\begin{aligned} t \;=\; & 2M \ln\left|\frac{(R/2M-1)^{1/2} + \tan(\eta/2)}{(R/2M-1)^{1/2} - \tan(\eta/2)}\right| \\ & + 2M(R/2M-1)^{1/2}\left[\eta + (R/4M)(\eta + \sin\eta)\right]. \end{aligned} \tag{1.30}$$

The above expression diverges as a certain value of η is approached. What is r at that value?

Solution. The Schwarzschild time t diverges when η approaches a value determined by

$$\tan\left(\frac{\eta}{2}\right) = \left(\frac{R}{2M} - 1\right)^{1/2}. \qquad (1.31)$$

In that case $\cos(\eta/2) = (2M/R)^{1/2}$, hence

$$\cos(\eta) = \frac{4M}{R} - 1. \qquad (1.32)$$

But as we have just seen (Eq. (1.27)), this is when $r = 2M$. Hence, as $r \to 2M$, $t \to \infty$ asymptotically. Observer B never quite sees a horizon appear; rather, he sees the surface of the star asymptotically approach its location. Consequently, he also never sees observer A fall into the black hole.

II. THE SCHWARZSCHILD SPACETIME IN KRUSKAL-SZEKERES COORDINATES

In Schwarzschild coordinates, the black hole metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (2.1)$$

The metric appears to be singular at $r = 2M/r$, even though we know that observers can move through it, in finite proper time. The horizon is just a coordinate singularity. One set of coordinates in which the horizon is manifestly non-singular are the *Kruskal-Szekeres* coordinates.

(2.1) Define coordinates u and v by:

$$\begin{aligned} u &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\frac{t}{4M}, \\ v &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\frac{t}{4M}, \end{aligned}$$

for $r > 2M$, and

$$\begin{aligned} u &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\frac{t}{4M}, \\ v &= \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\frac{t}{4M}, \end{aligned}$$

for $r < 2M$. Show that the metric in these coordinates is

$$ds^2 = -\frac{32M^3}{r}\, e^{-r/2M}(dv^2 - du^2) + r^2(u,v)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \qquad (2.2)$$

where $r(u, v)$ is now not a coordinate but a function of u and v determined implicitly by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2. \tag{2.3}$$

(2.2) Draw a diagram in which the horizontal axis is the u axis, the vertical axis is the v axis, and θ and ϕ are suppressed. What is the meaning of a point on such a diagram? Show that radial null geodesics (i.e., lines with $ds = 0$ and also $d\theta = d\phi = 0$) are at 45 degrees to the u and v axes.

(2.3) On the basis of Eq. (2.3), draw a few representative curves of constant r, for $r < 2M$ as well as for $r = 2M$. Also draw the curves $r = 0$ (the singularity) and $r = 2M$ (the horizon).

(2.4) Characterize the curves of constant t. Draw a few representative ones. Draw $t = -\infty$ and $t = +\infty$.

(2.5) From your conclusion about null geodesics in (2.3), draw a representative timelike geodesic (the spacetime path of a massive particle) which starts outside the horizon and falls into the black hole. Explain why any timelike geodesic which enters the horizon must end at the singularity. What do timelike geodesics look like which do *not* enter the horizon?