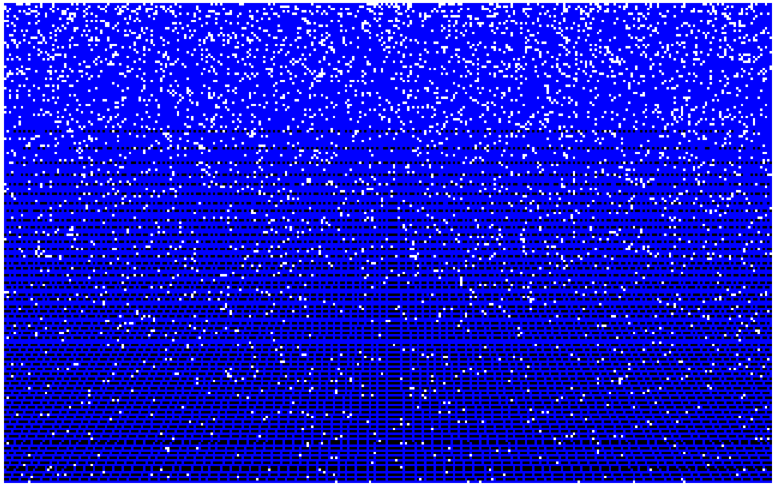
Gravitatie en kosmologie

FEW cursus



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Kromlijnige coördinaten: 28 oktober 2013

Inhoud

- **Inleiding**
  - Overzicht
- **Klassieke mechanica**
  - Galileo, Newton
  - Lagrange formalisme
- **Quantumfenomenen**
  - Neutronensterren
- **Wiskunde I**
  - Tensoren
- **Speciale relativiteitstheorie**
  - Minkowski
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- **Wiskunde II**
  - Algemene coordinaten
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- **Algemene relativiteitstheorie**
  - Einsteinvergelijkingen
  - Newton als limiet
- **Kosmologie**
  - Friedmann
  - Inflatie
- **Gravitatiestraling**
  - Theorie
  - Experiment

Special relativity

- Consider speed of light as invariant in all reference frames

Coordinates of spacetime

Cartesian coordinates

$$x^0 \equiv ct = t$$
$$x^1 \equiv x$$
$$x^2 \equiv y$$
$$x^3 \equiv z.$$

denote as x^μ

superscripts

spacetime indices: greek

space indices: latin

- SR lives in special four dimensional manifold: Minkowski spacetime (Minkowski space)

Coordinates are x^μ

Elements are events

Vectors are always fixed at an event; four vectors V^μ Abstractly V

- Metric on Minkowski space $\eta_{\mu\nu}$ as matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inner product of two vectors (summation convention)

$$A \cdot B \equiv \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

Spacetime interval

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

Often called `the metric'

Signature: +2

Proper time

$$d\tau^2 \equiv -ds^2$$

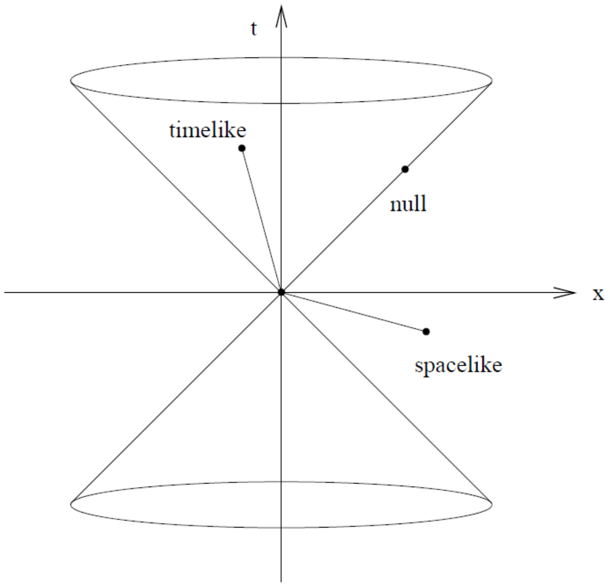
Measured on travelling clock

Special relativity

- Spacetime diagram

  Points are spacelike, timelike or nulllike separated from the origin

  Vector V^μ with negative norm $V \cdot V < 0$ timelike



- Path through spacetime

  Path is parameterized $x^\mu(\lambda)$

  Path is characterized by its tangent vector $dx^\mu/d\lambda$ spacelike, timelike or null

  For timelike paths: use proper time τ as parameter

  Calculate as

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

Tangent vector $U^\mu = dx^\mu/d\tau$

Four-velocity

Normalized $\eta_{\mu\nu}U^\mu U^\nu = -1$

Momentum four-vector $p^\mu = mU^\mu$

Mass m

Energy is time-component p^0

Particle rest frame $E = mc^2$

Moving frame for particle with three-velocity $v = dx/dt$ along *x*-axis

$$p^\mu = (\gamma m, v\gamma m, 0, 0)$$

Small *v*

$$p^0 = m + \tfrac{1}{2}mv^2$$
$$p^1 = mv$$

Energie-impuls tensor

- Perfecte vloeistof (in rustsysteem)
  - Energiedichtheid ρ
  - Isotrope druk *P*

$T^{\mu\nu}$ diagonaal, met $T^{11} = T^{22} = T^{33}$

- In rustsysteem

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

- In tensorvorm (geldig in elke systeem)

We hadden $T^{\mu\nu}_{\text{stof}} = \rho U^{\mu} U^{\nu}$

Probeer $T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^{\mu} U^{\nu}$

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

We vinden

$$T^{\mu\nu}_{\text{perfecte vloeistof}} = \left(\rho + \frac{P}{c^2}\right) U^{\mu} U^{\nu} + P g^{\mu\nu}$$

Verder geldt

$$\partial_{\mu} T^{\mu\nu} = 0$$

Voor stof: *P = 0*

Tensors – coordinate invariant description of GR

- Linear space – a set *L* is called a linear space when
  - Addition of elements is defined $\vec{a}+\vec{b}$ is element of *L*
  - Multiplication of elements with a real number is defined $\lambda\vec{a}$
  - *L* contains *0* $\vec{a}+0=\vec{a}$
  - General rules from algebra are valid $\vec{a}+\vec{b}=\vec{b}+\vec{a},\ \lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b},\ etc.$
- Linear space *L* is *n*-dimensional when
  - Define vector basis $\vec{e}_1, ..., \vec{e}_n$ Notation: $\{\vec{e}_i\}$
  - Each element (vector) of L can be expressed as $\vec{A}=\sum_{i=1}^{n} A^i\vec{e}_i$ or $\vec{A}=A^i\vec{e}_i$
  - Components are the real numbers A^i
  - Linear independent: none of the $\vec{e}_i$'s can be expressed this way
  - Notation: vector component: upper index; basis vectors lower index
- Change of basis
  - *L* has infinitely many bases
  - If $\vec{e}_i$ is basis in *L*, then $\vec{e}_j{}'$ is also a basis in *L*. One has $\vec{e}_j{}'=\Lambda^i{}_j\vec{e}_i$ and $\vec{e}_i=G^j{}_i\,\vec{e}_j{}'$
  - Matrix *G* is inverse of Λ $G^j{}_i\Lambda^k{}_j=\delta^k{}_i$
  - In other basis, components of vector change to $A^{i\prime}=G^i{}_jA^j$
  - Vector $\vec{A}$ is geometric object and does not change!

contravariant
covariant

1-FORMS AND DUAL SPACES

- 1-form
  - GR works with geometric (basis-independent) objects
  - Vector is an example
  - Other example: real-valued function of vectors $\tilde{p}(\vec{a})$
  - Imagine this as a machine with a single slot to insert vectors: real numbers result
- Dual space
  - Imagine set of all 1-form in *L*
  - This set also obeys all rules for a linear space, dual space. Denote as *L**
  - When *L* is *n*-dimensional, also *L** is *n*-dimensional
  - For 1-form $\tilde{p}$ and vector $\vec{V}$ we have $\tilde{p}(\vec{V}) = \tilde{p}(V^i\vec{e}_i) = \tilde{p}(\vec{e}_i)V^i$
  - Numbers $\{\tilde{p}(\vec{e}_i)\}$ are components p_i of 1-form $\tilde{p}$
- Basis in dual space
  - Given basis $\{\vec{e}_i\}$ in *L*, define 1-form basis $\{\tilde{\omega}^i\}$ in *L** (called dual basis) by $\tilde{\omega}^i(\vec{e}_j) = \delta^i{}_j$
  - Can write 1-form as $\tilde{p} = p_i\tilde{\omega}^i$, with p_i real numbers
  - We now have $\tilde{p}(\vec{V}) = p_iV^i$
  - Mathematically, looks like inner product of two vectors. However, in different spaces
  - Change of basis yields $\tilde{\omega}^{i\prime} = G^i{}_j\tilde{\omega}^j$ and $p_i{}' = \Lambda^j{}_i p_j$ (change covariant!)
  - Index notation by Schouten
  - Dual of dual space: *L*** = *L*

Tensors

- Tensors
  - So far, two geometric objects: vectors and 1-forms
  - Tensor: linear function of *n* vectors and *m* 1-forms (picture machine again)
  - Imagine (*n*,*m*) tensor $T = T(x_1, ..., x_n, y_1, ..., y_m)$
  - Where x_i live in *L* and y_j in *L**
  - Expand objects in corresponding spaces: $x_i = x_i^l \vec{e}_l$ and $y_j = y_{jk}\tilde{\omega}^k$
  - Insert into *T* yields $T = T(x_1, ..., x_n, y_1, ..., y_m) = T_{l_1 l_2 ... l_n}^{k_1 k_2 ... k_m} x_1^{l_1} x_2^{l_2} ... x_n^{l_n} y_{1k_1} y_{2k_2} ... y_{mk_m}$
  - with tensor components $T_{l_1 l_2 ... l_n}^{k_1 k_2 ... k_m} = T(\vec{e}_{l_1}, \vec{e}_{l_2}, ..., \vec{e}_{l_n}, \tilde{\omega}^{k_1} \tilde{\omega}^{k_2}, ..., \tilde{\omega}^{k_m})$
  - In a new basis $T_{j_1 j_2 ... j_n}^{i'_1 i'_2 ... i'_n} = G^{i_1}{}_{k_1} G^{i_2}{}_{k_2} ... G^{i_m}{}_{k_m} \Lambda^{l_1}_{j_1} \Lambda^{l_2}_{j_2} ... \Lambda^{l_n}_{j_n} T_{l_1 l_2 ... l_n}^{k_1 k_2 ... k_m}$
  - Mathematics to construct tensors from tensors: tensor product, contraction. This will be discussed when needed

Kromlijnige coördinaten

Cartesische coördinaten

Punt in 2D euclidische ruimte: *x* en *y*

Kromlijnige coördinaten

Punt in 2D euclidische ruimte: ξ en η

Transformatie $\Lambda^{\alpha'}{}_{\beta}$

Voor de afstand tussen 2 punten geldt

Transformatie moet één op één zijn

$$\xi = \xi(x,y), \qquad \Delta\xi = \frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y,$$

$$\eta = \eta(x,y), \qquad \Delta\eta = \frac{\partial\eta}{\partial x}\Delta x + \frac{\partial\eta}{\partial y}\Delta y.$$

$$\det\begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} \neq 0$$

Voorbeeld: poolcoördinaten

$r = \sqrt{x^2 + y^2}$ en $\theta = \arctan\frac{y}{x}$

$x = r\cos\theta$ en $y = r\sin\theta$

$$\Delta r = \frac{x}{r}\Delta x + \frac{y}{r}\Delta y = \cos\theta\Delta x + \sin\theta\Delta y,$$

$$\Delta\theta = -\frac{y}{r^2}\Delta x + \frac{x}{r^2}\Delta y = -\frac{1}{r}\sin\theta\Delta x + \frac{1}{r}\cos\theta\Delta y,$$

Vectoren en 1-vormen

Vector

Transformeert net als verplaatsing

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$\Delta\vec{r}$

Er geldt

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta} \qquad \text{met} \qquad \Lambda^{\alpha'}{}_{\beta} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{pmatrix}$$

Systeem *(x,y)*

Systeem *(ξ, η)*

1-vorm

Beschouw scalairveld ϕ

Definieer 1-vorm $\tilde{\mathrm{d}}\phi$ met componenten

$$\tilde{\mathrm{d}}\phi \rightarrow \left(\frac{\partial\phi}{\partial\xi}, \frac{\partial\phi}{\partial\eta}\right)$$

Transformatiegedrag volgt uit kettingregel

$$\frac{\partial\phi}{\partial\xi} = \frac{\partial x}{\partial\xi}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial\xi}\frac{\partial\phi}{\partial y} \qquad \frac{\partial\phi}{\partial\eta} =$$

We vinden (transformatie met inverse!)

$$\begin{pmatrix} \partial\phi/\partial\xi \\ \partial\phi/\partial\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{pmatrix} \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix} \qquad \text{met} \qquad \Lambda^{\alpha}{}_{\beta'} = \begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{pmatrix}$$

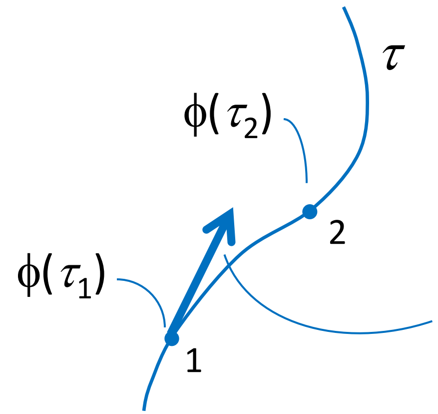
Kromlijnige coördinaten

Afgeleide scalair veld

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau}$$

$$= \frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial x}U^x + \frac{\partial\phi}{\partial y}U^y + \frac{\partial\phi}{\partial z}U^z$$

$$= \tilde{\mathrm{d}}\phi(\vec{U}) \qquad \tilde{\mathrm{d}}\phi \underset{\mathcal{O}}{\rightarrow} \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$



$$\vec{U} = \begin{pmatrix} U^t \\ U^x \\ U^y \\ U^z \end{pmatrix} = \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix}$$

raakvector (tangent vector)

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = <\tilde{\mathrm{d}}\phi, \vec{V}>$$

De waarde van de afgeleide van *f* in de richting $\vec{V}$

Afgeleide van scalair veld ϕ langs raakvector $\vec{V}$

$$\nabla_{\vec{V}}\phi = V^\alpha \frac{\partial\phi}{\partial x^\alpha}$$

$$\nabla_{\vec{V}} = V^\alpha \frac{\partial}{\partial x^\alpha} \quad \text{en} \quad \vec{V} = V^\alpha \vec{e}_\alpha \quad \Rightarrow$$

$$\vec{V} = \nabla_{\vec{V}} = \partial_{\vec{V}} = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}$$

$$\vec{e}_\alpha = \frac{\partial\mathcal{P}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}$$

Voorbeeld 1

Euclidische ruimte

Transformatie $x = u + v, \quad y = u - v, \quad z = 2uv + w,$

Plaatsvector $\vec{r} = (u+v)\vec{i} + (u-v)\vec{j} + (2uv+w)\vec{k}$

Basisvectoren $\vec{e}_\alpha = \frac{\partial \mathcal{P}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}$

Natuurlijke basis

$$\begin{aligned} \vec{e}_u &= \partial\vec{r}/\partial u = \vec{i} + \vec{j} + 2v\vec{k}, \\ \vec{e}_v &= \partial\vec{r}/\partial v = \vec{i} - \vec{j} + 2u\vec{k}, \\ \vec{e}_w &= \partial\vec{r}/\partial w = \vec{k} \end{aligned}$$

Metriek bekend

Niet orthonormaal $\vec{e}_u \cdot \vec{e}_v = 4uv,\ \vec{e}_v \cdot \vec{e}_w = 2u \text{ en } \vec{e}_w \cdot \vec{e}_u = 2v$

Inverse transformatie $u = \frac{1}{2}(x+y), \quad v = \frac{1}{2}(x-y), \quad w = z - \frac{1}{2}(x^2 - y^2)$

Duale basis

$$\begin{aligned} \vec{e}^u &= \nabla u = \tfrac{1}{2}\vec{i} + \tfrac{1}{2}\vec{j}, \\ \vec{e}^v &= \nabla v = \tfrac{1}{2}\vec{i} - \tfrac{1}{2}\vec{j}, \\ \vec{e}^w &= \nabla w = -x\vec{i} + y\vec{j} + \vec{k} = -(u+v)\vec{i} + (u-v)\vec{j} + \vec{k} \end{aligned}$$

Voorbeeld 2

<u>Voorbeeld:</u> basis 1-vormen en vectoren in poolcoördinaten.

Voor de basiscoördinaten geldt $\vec{e}_{\alpha'} = \Lambda^{\beta}{}_{\alpha'}\vec{e}_{\beta}$ en dit levert

$$\vec{e}_r = \Lambda^x{}_r\vec{e}_x + \Lambda^y{}_r\vec{e}_y = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y, \qquad (258)$$

en op dezelfde wijze

$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y = -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y. \qquad (259)$$

Merk op dat we gebruiken dat $\Lambda^x{}_r = \frac{\partial x}{\partial r}$. Op dezelfde wijze kunnen we de andere kant op transformeren met $\Lambda^r{}_x = \frac{\partial r}{\partial x}$. De transformatiematrices zijn eenvoudig: we hoeven alleen te kijken naar welke index boven of beneden is en we weten welke afgeleide we dienen te gebruiken.

De basis 1-vormen vinden we op analoge wijze. Er geldt $\tilde{\mathrm{d}}p^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\tilde{\mathrm{d}}p^{\beta}$ en dus

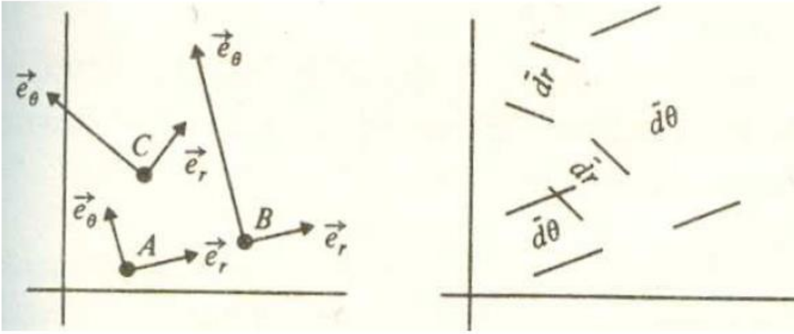
$$\tilde{\mathrm{d}}\theta = \frac{\partial \theta}{\partial x}\tilde{\mathrm{d}}x + \frac{\partial \theta}{\partial y}\tilde{\mathrm{d}}y = -\frac{1}{r}\sin\theta\tilde{\mathrm{d}}x + \frac{1}{r}\cos\theta\tilde{\mathrm{d}}y, \qquad (260)$$

en ook

$$\tilde{\mathrm{d}}r = \cos\theta\tilde{\mathrm{d}}x + \sin\theta\tilde{\mathrm{d}}y.$$

We vinden bijvoorbeeld $|\vec{e}_\theta|^2 = \vec{e}_\theta \cdot \vec{e}_\theta =$

$r^2\sin^2\theta + r^2\cos^2\theta = r^2$, zodat de lengte van $\vec{e}_\theta$ toeneemt met haar afstand tot de oorsprong. We hebben dus ook geen eenheidsbasis meer. Er geldt $|\vec{e}_r| = 1, |\tilde{\mathrm{d}}r| = 1, |\vec{e}_\theta| = r, |\tilde{\mathrm{d}}\theta| = r^{-1}$.



Voorbeeld 2

De inproducten kunnen worden uitgerekend, omdat we de metriek in cartesische coördinaten (x, y) kennen: $\vec{e}_x \cdot \vec{e}_x = \vec{e}_y \cdot \vec{e}_y = 1$ en $\vec{e}_x \cdot \vec{e}_y = 0$. In tensornotatie is dat $g(\vec{e}_\alpha, \vec{e}_\beta) = \delta_{\alpha\beta}$ voor cartesische coördinaten. De metriek $\mathbf{g}$ heeft in poolcoördinaten de componenten $g_{\alpha'\beta'} = g(\vec{e}_{\alpha'}, \vec{e}_{\beta'}) = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'}$ en dit levert $g_{rr} = 1$, $g_{\theta\theta} = r^2$ en $g_{r\theta} = 0$. We kunnen de componenten van $\mathbf{g}$ ook schrijven als

$$(g_{\alpha\beta})_{\text{pool}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{en ook} \quad \mathrm{d}\vec{l} \cdot \mathrm{d}\vec{l} = \mathrm{d}s^2 = |\mathrm{d}r\vec{e}_r + \mathrm{d}\theta\vec{e}_\theta|^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2. \tag{262}$$

De laatste formule in bovenstaande uitdrukking[64] geeft de lengte van een willekeurige en oneindig kleine verplaatsing $\mathrm{d}\vec{l}$, hetgeen een handige manier is om de componenten van de metrische tensor en tegelijkertijd de coördinaten van het lijnelement $\mathrm{d}\vec{l}$ te tonen.

De metriek heeft een inverse

$$(g^{\alpha\beta})_{\text{pool}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}, \tag{263}$$

waarmee geldt dat $g^{rr} = 1$, $g^{r\theta} = 0$ en $g^{\theta\theta} = 1/r^2$. We kunnen dit gebruiken om een afbeelding te maken tussen vectoren en 1-vormen. Stel bijvoorbeeld dat ϕ een scalairveld is en $\tilde{\mathrm{d}}\phi$ haar gradiënt, dan heeft de vector $\vec{\mathrm{d}}\phi$ de componenten

$$(\vec{\mathrm{d}}\phi)^\alpha = g^{\alpha\beta}\phi_{,\beta} \quad \text{en dus} \quad \left\{ \begin{array}{lll} (\vec{\mathrm{d}}\phi)^r & = & g^{r\beta}\phi_{,\beta} = g^{rr}\phi_{,r} + g^{r\theta}\phi_{,\theta} = \frac{\partial\phi}{\partial r} \\ (\vec{\mathrm{d}}\phi)^\theta & = & g^{\theta\beta}\phi_{,\beta} = g^{\theta r}\phi_{,r} + g^{\theta\theta}\phi_{,\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} \end{array} \right. \tag{264}$$

Dus terwijl $(\phi_{,r}, \phi_{,\theta})$ componenten van een 1-vorm zijn, heeft de vector gradiënt componenten $(\phi_{,r}, \phi_{,\theta}/r^2)$. Ondanks dat we in de euclidische ruimte zijn, zien we dat vectoren in het algemeen componenten hebben die verschillen van die van de geassocieerde 1-vormen. Cartesische coördinaten zijn de enige coördinaten waarvoor de componenten hetzelfde zijn (want dan geldt dat $g_{\alpha\beta} = \mathrm{diag}(1, 1)$).

Tensorcalculus

Afgeleide van een vector

$$\vec{V} = V^\alpha \vec{e}_\alpha$$

α is 0 - 3

$$\rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

stel β is 0

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\mu \Gamma^\alpha{}_{\mu\beta}\vec{e}_\alpha$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}\right)\vec{e}_\alpha$$

Notatie

$$\frac{\partial V^\alpha}{\partial x^\beta} = V^\alpha{}_{,\beta}$$

$$V^\alpha{}_{;\beta} \equiv V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha{}_{;\beta}\vec{e}_\alpha$$

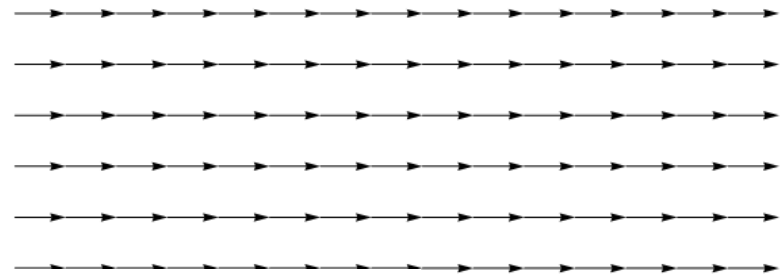
Covariante afgeleide

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensorveld $\nabla \vec{V}$

met componenten $(\nabla \vec{V})^\alpha{}_\beta = (\nabla_\beta \vec{V})^\alpha = V^\alpha{}_{;\beta}$

Voorbeeld: poolcoördinaten

$$\vec{e}_x \longrightarrow (\Lambda^r{}_x, \Lambda^\theta{}_x) = (\cos\theta, -r^{-1}\sin\theta)$$



Bereken $\partial\vec{e}_x/\partial\theta$

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_r &= \frac{\partial}{\partial r}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = 0,\\
\frac{\partial}{\partial\theta}\vec{e}_r &= \frac{\partial}{\partial\theta}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_\theta &= \frac{\partial}{\partial r}(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta\\
\frac{\partial}{\partial\theta}\vec{e}_\theta &= -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y = -r\vec{e}_r.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\vec{e}_x &= \frac{\partial}{\partial\theta}(\cos\theta)\vec{e}_r + \cos\theta\frac{\partial}{\partial\theta}(\vec{e}_r) - \frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\right)\vec{e}_\theta - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\vec{e}_\theta\\
&= -\sin\theta\vec{e}_r + \cos\theta\left(\frac{1}{r}\vec{e}_\theta\right) - \frac{1}{r}\cos\theta\vec{e}_\theta - \frac{1}{r}\sin\theta(-r\vec{e}_r).
\end{aligned}$$

$$\boxed{\partial\vec{e}_x/\partial\theta = 0}$$

Bereken christoffelsymbolen $\dfrac{\partial\vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu$

$$\begin{array}{llclcl}
(1) & \frac{\partial\vec{e}_r}{\partial r} & = & 0 & \rightarrow & \Gamma^\mu{}_{rr} = 0 \quad \text{voor alle } \mu,\\
(2) & \frac{\partial\vec{e}_r}{\partial\theta} & = & \frac{1}{r}\vec{e}_\theta & \rightarrow & \Gamma^r{}_{r\theta} = 0, \quad \Gamma^\theta{}_{r\theta} = \frac{1}{r},\\
(3) & \frac{\partial\vec{e}_\theta}{\partial r} & = & \frac{1}{r}\vec{e}_\theta & \rightarrow & \Gamma^r{}_{\theta r} = 0, \quad \Gamma^\theta{}_{\theta r} = \frac{1}{r},\\
(4) & \frac{\partial\vec{e}_\theta}{\partial\theta} & = & -r\vec{e}_r & \rightarrow & \Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{\theta\theta} = 0.
\end{array}$$

Divergentie en Laplace operatoren

$$V^\alpha{}_{;\alpha} = \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^\theta$$

$$\nabla\cdot\nabla\phi \equiv \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

Christoffelsymbolen en metriek

Covariante afgeleiden

$$V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + V^{\mu}\Gamma^{\alpha}{}_{\mu\beta}$$
$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_{\mu}\Gamma^{\mu}{}_{\alpha\beta}$$

$$\nabla_{\beta}(p_{\alpha}V^{\alpha}) = p_{\alpha;\beta}V^{\alpha} + p_{\alpha}V^{\alpha}{}_{;\beta}$$

In cartesische coördinaten en euclidische ruimte $\nabla_{\beta}\tilde{V} = g(\nabla_{\beta}\vec{V}, \ldots)$

Deze tensorvergelijking geldt in alle coördinaten $V_{\alpha;\beta} = g_{\alpha\mu}V^{\mu}{}_{;\beta}$

Neem covariante afgeleide van $V_{\alpha'} = g_{\alpha'\mu'}V^{\mu'}$ ⇒ $V_{\alpha';\beta'} = g_{\alpha'\mu'}V^{\mu'}{}_{;\beta'}$

⇒ $V_{\alpha';\beta'} = g_{\alpha'\mu';\beta'}V^{\mu'} + g_{\alpha'\mu'}V^{\mu'}{}_{;\beta'}$

⇒ $g_{\alpha'\mu';\beta'} = 0$

Direct gevolg van $g_{\alpha\beta,\mu} = 0$ in cartesische coördinaten!

De componenten van dezelfde tensor ∇g voor willekeurige coördinaten zijn $g_{\alpha\beta;\mu}$

Opgave: bewijs dat geldt

$$\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

Connectiecoëfficiënten bevatten afgeleiden naar de metriek