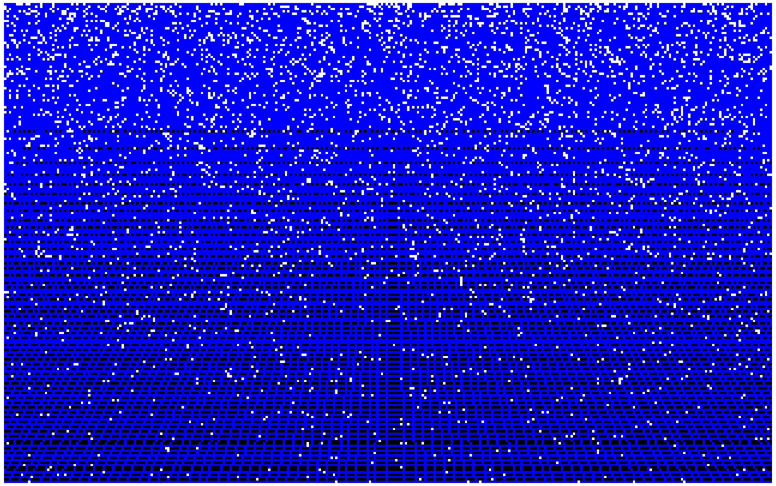
General Relativity

a summary



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Nikhef: April 9, 2010

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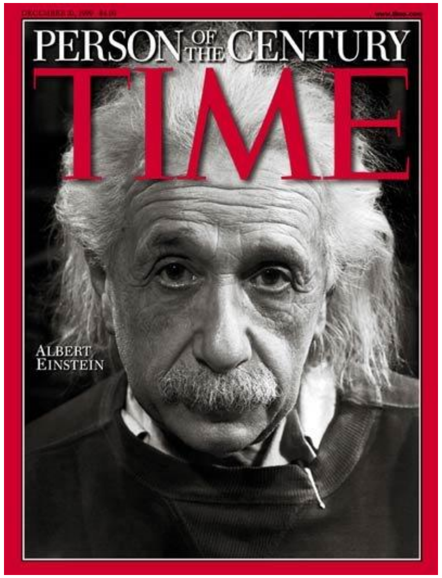
Motivation

Einstein gravity :

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

Gravity as a geometry
Space and time are physical objects

Most beautiful physical theory



- Gravitation

  Least understood interaction

  Large world-wide intellectual activity

  - Theoretical: ART + QM, black holes, cosmology
  - Experimental: Interferometers on Earth and in space, gravimagnetism (Gravity Probe B)

- Gravitational waves

  Dynamical part of gravitation, all space is filled with GW

  Ideal information carrier, almost no scattering or attenuation

  The entire universe has been transparent for GWs, all the way back to the Big Bang

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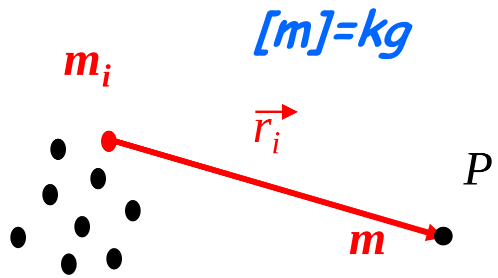
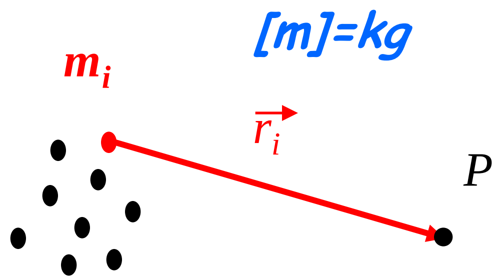
Newtonian gravity

Newton's Law:



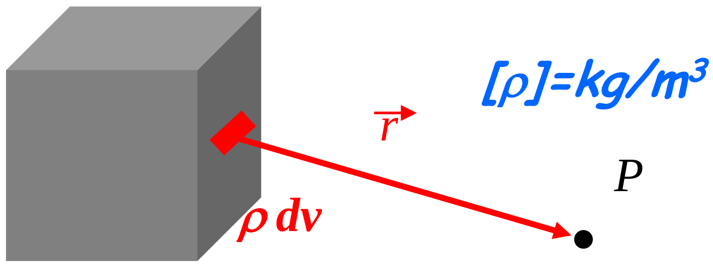
$$\vec{F} = m\vec{g}_P \equiv -G\sum_{i=1}^{N} \frac{m m_i}{r_i^2}\hat{r}_i$$

Discrete:



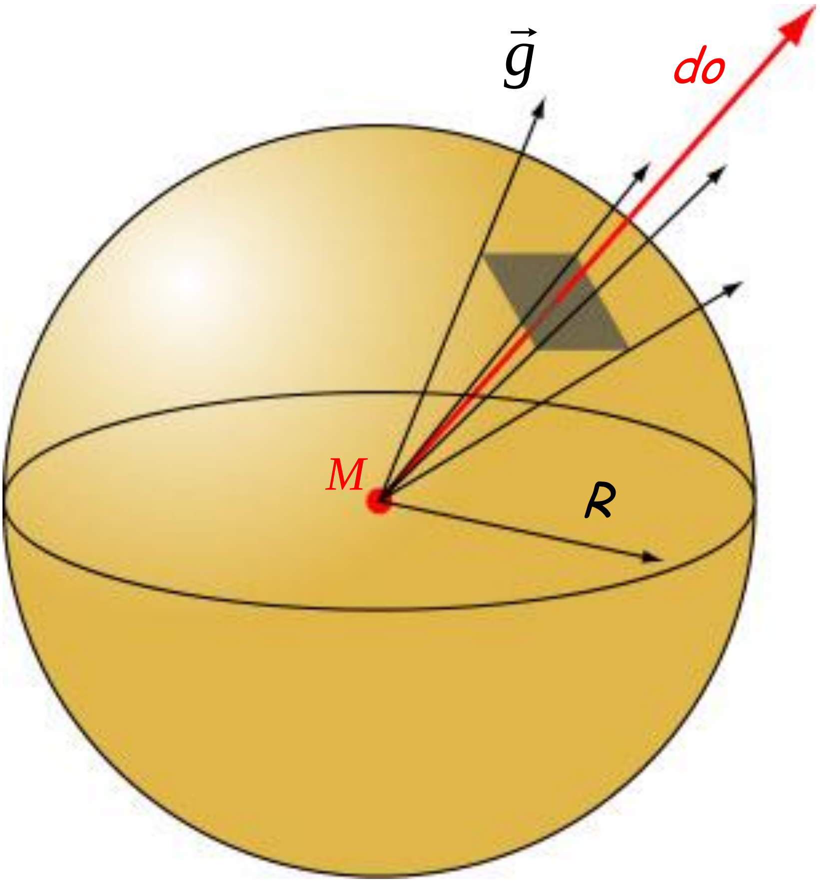
$$\vec{g}_P \equiv -G\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i$$

- **Continuous:**



$$\vec{g}_P \equiv -G \int_{volume} dv \frac{\rho}{r^2}\hat{r}$$

Gravitational flux



Mass M in center of sphere

Flux F_g through surface of sphere:

$$F_g \equiv \oint_{sphere} \vec{g} \cdot d\vec{o} = \oint_{sphere} \frac{-GM}{R^2} do \qquad (\vec{g} // d\vec{o})$$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{-GM}{R^2} R^2 \sin\theta \, d\varphi \, d\theta \qquad (r\theta\varphi)$$

$$= -GM \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\varphi = -GM\,4\pi = -4\pi GM$$

In essense:

- $g \propto 1/r^2$
- surface area $\propto r^2$

$F_g = -4\pi GM$ holds for every closed surface; not only for that of a sphere with M at center!

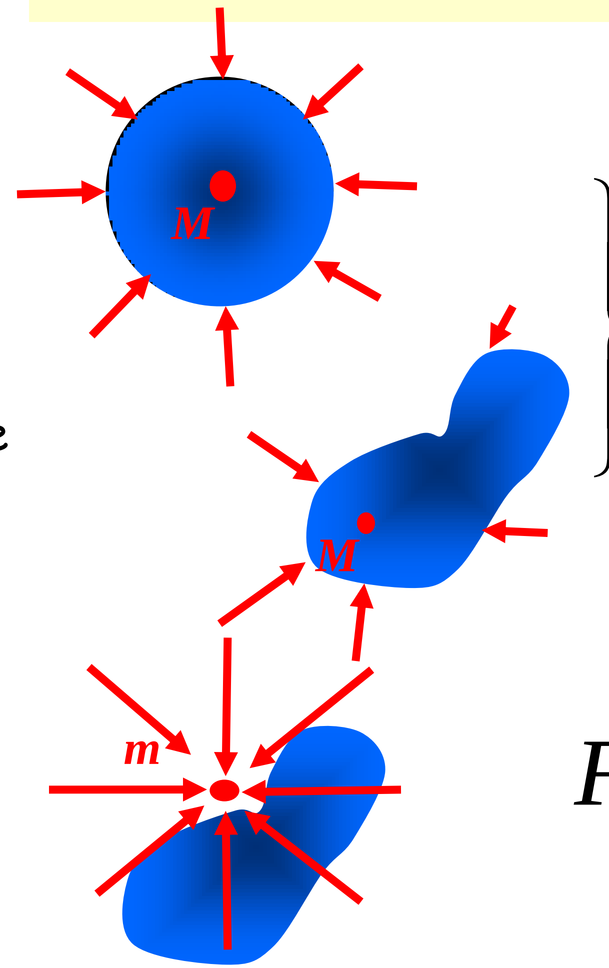
Gauss law

$$F_g \equiv \oint_{area\,O} \vec{g}\cdot d\hat{o} = -4\pi G \sum_{\text{in V}} M_i$$

Mass M enclosed by sphere

Mass M enclosed by arbitrary surface

Mass m outside arbitrary surface

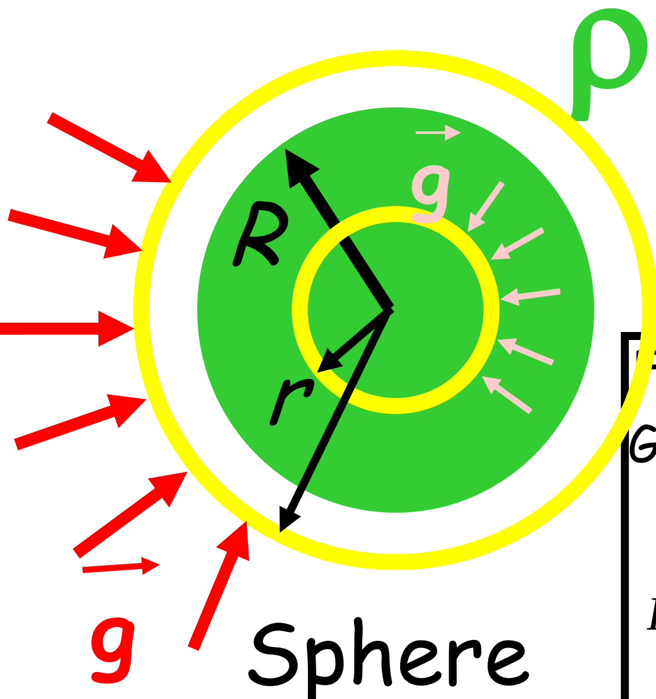


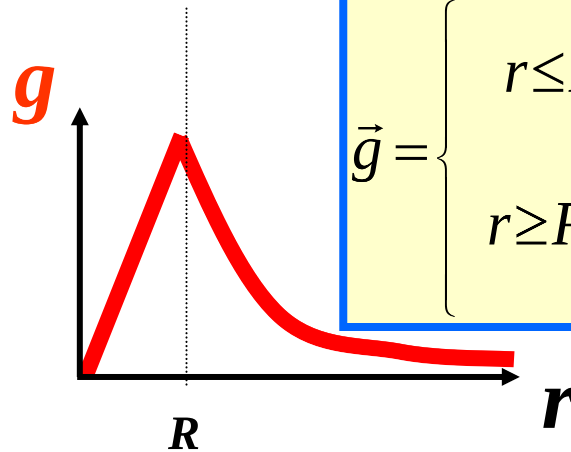
$$F_g = -4\pi G M$$

$$F_g = 0$$

Gauss law – example

Volume sphere:
- Mass distribution: ρ kg/m^3
- symmetry: g ⊥ sphere, g(r)
- "Gauss box": small sphere





$$\vec{g}=\begin{cases} r\le R:\ \vec{g}=-\dfrac{4\pi G\rho\vec{r}}{3} \\ r\ge R:\ \vec{g}=-\dfrac{4\pi G\rho R^3\hat{r}}{3r^2} \end{cases}$$

Flux: $F_g=4\pi r^2 g$

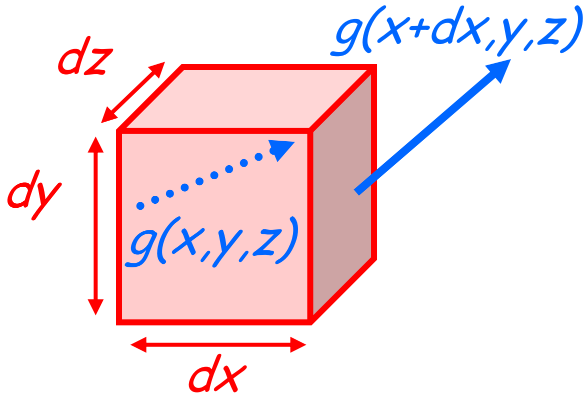
Gauss law:

$$F_g=4\pi G\sum_{enclosed}M \Rightarrow \begin{cases} r<R:\ 4\pi r^2 g=-4\pi G\dfrac{4}{3}\pi r^3\rho \Leftrightarrow g=-\dfrac{4\pi G\rho r}{3} \\ r>R:\ 4\pi r^2 g=-4\pi G\dfrac{4}{3}\pi R^3\rho \Leftrightarrow g=-\dfrac{4\pi G\rho R^3}{3r^2} \end{cases}$$

Gauss law – mathematics

Consider locally (Gauss):

$$\oint_{oppervlak} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volume} \rho dv$$



$$\oint_{area} \vec{g}\cdot d\vec{o} = -dxdy\left(g_z(x,y,z+dz) - g_z(x,y,z)\right) +$$
$$-dzdx \left(g_y(x,y+dy,z) - g_y(x,y,z)\right) +$$
$$-dydz \left(g_x(x+dx,y,z) - g_x(x,y,z)\right)$$
$$= -dxdydz\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}\right)$$

$$4\pi G \int_{volume} \rho dv = 4\pi G\, dxdydz\, \rho(x,y,z)$$

Compact notation: use "divergence":

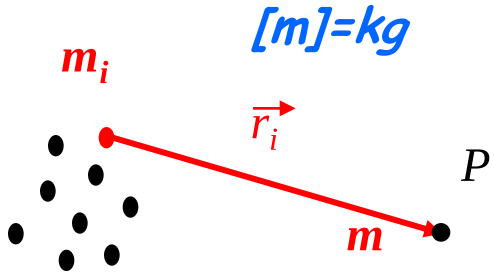
$$\vec{\nabla}\cdot\vec{g} \equiv \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

Thus

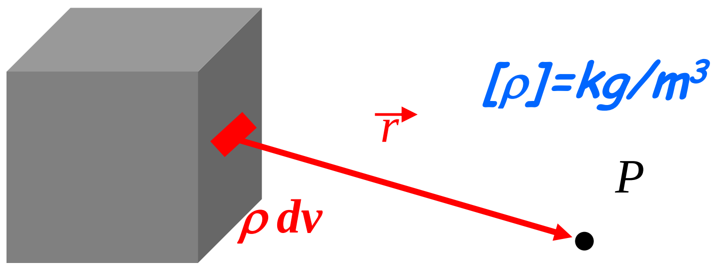
$$\oint_{area} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volume} \rho(\vec{r}) dv \iff \vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G \rho(\vec{r})$$

Gravitational potential – Poisson equation

Law of gravity:



$$\vec{F} = m\vec{g}_P \equiv -G\sum_{i=1}^{N}\frac{m m_i}{r_i^2}\hat{r}_i = -m\nabla\Phi(\vec{r})$$



$$\vec{g}_P \equiv -G \int_{volume} dv \frac{\rho}{r^2}\hat{r} = -\nabla\Phi(\vec{r})$$

$$\vec{g}(\vec{r}) \equiv -\nabla\Phi(\vec{r})$$

$$\int_{volume} \vec{\nabla}\cdot\vec{g}\,dv = \oint_{surface} \vec{g}\cdot d\vec{o} = -4\pi G \int_{volume} \rho\,dv \;\Rightarrow\; \vec{\nabla}\cdot\vec{g} = -4\pi G\rho$$

$$-\nabla\cdot\vec{g}(\vec{r}) = \nabla\cdot\nabla\Phi(\vec{r}) = \nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

General relativity

$$\nabla^2 \Phi(\vec{r}) = 4\pi G \rho(\vec{r})$$

- Einstein's gravitation

  Spacetime is a curved pseudo-Riemannian manifold with a metric of signature (-,+,+,+)

  The relationship between matter and the curvature of spacetime is given by the Einstein equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

Units: $c = 1$ and often $G = 1$

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Special relativity

- Consider speed of light as invariant in all reference frames

Coordinates of spacetime

Cartesian coordinates

$$x^0 \equiv ct = t$$
$$x^1 \equiv x$$
$$x^2 \equiv y$$
$$x^3 \equiv z.$$

denote as x^μ

superscripts

spacetime indices: greek

space indices: latin

- SR lives in special four dimensional manifold: Minkowski spacetime (Minkowski space)

Coordinates are x^μ

Elements are events

Vectors are always fixed at an event; four vectors V^μ Abstractly V

- Metric on Minkowski space $\eta_{\mu\nu}$ as matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inner product of two vectors (summation convention)

$$A \cdot B \equiv \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

Spacetime interval

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

Often called `the metric'

Signature: +2

Proper time

$$d\tau^2 \equiv -ds^2$$

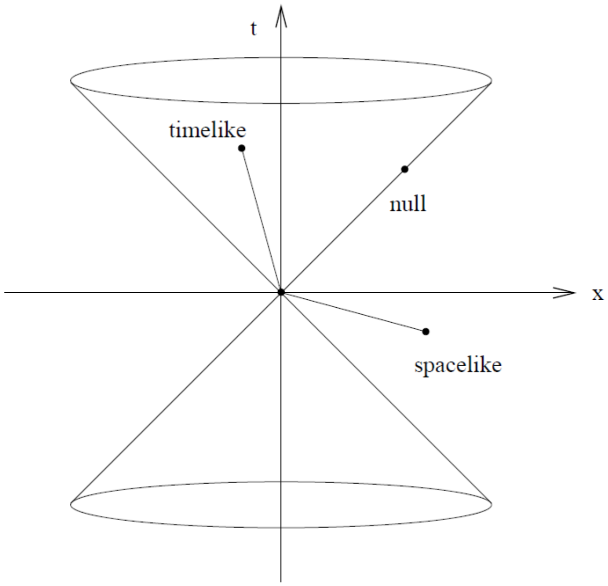
Measured on travelling clock

Special relativity

- Spacetime diagram

Points are spacelike, timelike or nulllike separated from the origin

Vector V^μ with negative norm $V \cdot V < 0$ is timelike



- Path through spacetime

Path is parameterized $x^\mu(\lambda)$

Path is characterized by its tangent vector $dx^\mu/d\lambda$ as spacelike, timelike or null

For timelike paths: use proper time τ as parameter

Calculate as

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

Tangent vector $U^\mu = dx^\mu/d\tau$ — Four-velocity

Normalized $\eta_{\mu\nu}U^\mu U^\nu = -1$

Momentum four-vector $p^\mu = mU^\mu$ — Mass m

Energy is time-component p^0

Particle rest frame $E = mc^2$

Moving frame for particle with three-velocity $v = dx/dt$ along *x*-axis

$$p^\mu = (\gamma m, v\gamma m, 0, 0)$$

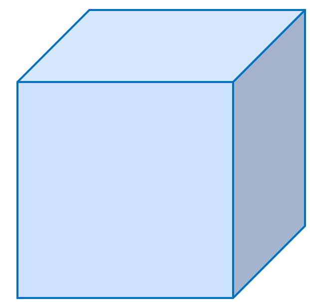
Small *v*

$$p^0 = m + \tfrac{1}{2}mv^2$$
$$p^1 = mv$$

Inertia of pressure

- SRT: when pressure of a gas increases, it is more difficult to accelerate the gas (inertia increases)
- Exert force *F*, accelerate to velocity *v* << *c*

$$\frac{1}{2}mv^2 = \frac{1}{2}\rho V v^2$$



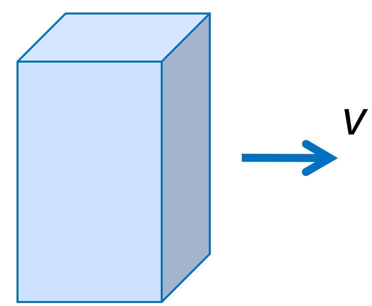
Volume *V*

Density ρ

Pressure *P*

- SRT: Lorentz contraction shortens box

$$\vec{F}\cdot d\vec{s} = -P\Delta V$$



- Energy needed to accelerate gas

$$E = \frac{1}{2}mv^2 - P\Delta V = \frac{1}{2}\rho V v^2 + \frac{1}{2}\frac{v^2}{c^2}PV = \frac{1}{2}\left(\rho + \frac{P}{c^2}\right)v^2 V$$

$$\Delta L = L\sqrt{1-\frac{v^2}{c^2}} \approx -\frac{1}{2}\frac{v^2}{c^2}L$$

additional inertia of gas pressure

Energy – momentum tensor: `dust'

- Energy needed to accelerate gas

  Dependent on reference system

  0 – component of four-momentum

$$E = \frac{1}{2}\left(\rho + \frac{P}{c^2}\right) v^2 V$$

- Consider `dust'

  Collection of particles that are at rest wrt each other

  Constant four-velocity field

$$U^\mu(x)$$

Flux four-vector $N^\mu = nU^\mu$

- Rest system
  - n and m are 0-components of four-vectors

Particle density in rest system

Mass density in rest system $\rho = nm$

Energy density in rest system ρc^2

- Moving system
  - N^0 is particle density
  - N^i particle flux in x^i – direction

$$N^\mu = \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad p^\mu = mU^\mu = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ρc^2 is the $\mu = 0, \nu = 0$ component of tensor $p \otimes N$

$$T^{\mu\nu}_{\text{stof}} = p^\mu N^\nu = mnU^\mu U^\nu = \rho U^\mu U^\nu$$

The gas is pressureless!

Energy – momentum: perfect fluid

- Perfect fluid (in rest system)
  - Energy density ρ
  - Isotropic pressure P
- In rest system
- Tensor expression (valid in all systems)

$T^{\mu\nu}$ diagonal, with $T^{11} = T^{22} = T^{33}$

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

We had $T^{\mu\nu}_{\text{stof}} = \rho U^\mu U^\nu$

Try $T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu$

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

We find

$$T^{\mu\nu}_{\text{fluid}} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu + P g^{\mu\nu}$$

In addition

$$\nabla^\mu T_{\mu\nu} = 0$$

Components of $T_{\mu\nu}$ are the flux of the μ^{th} momentum component in the ν^{th} direction

In GR there is no *global* notion of energy conservation

Einstein's equations extent Newtonian gravity: $\nabla^2\Phi = 4\pi G\rho$ $\quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

Tensors – coordinate invariant description of GR

- Linear space – a set *L* is called a linear space when
  - Addition of elements is defined $\vec{a}+\vec{b}$ is element of *L*
  - Multiplication of elements with a real number is defined $\lambda\vec{a}$
  - *L* contains *0* $\vec{a}+0=\vec{a}$
  - General rules from algebra are valid $\vec{a}+\vec{b}=\vec{b}+\vec{a},\ \lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b},\ etc.$

- Linear space *L* is *n*-dimensional when
  - Define vector basis $\vec{e}_1,\ldots,\vec{e}_n$ Notation: $\{\vec{e}_i\}$
  - Each element (vector) of L can be expressed as $\vec{A}=\sum_{i=1}^{n}A^i\vec{e}_i$ or $\vec{A}=A^i\vec{e}_i$
  - Components are the real numbers A^i
  - Linear independent: none of the $\vec{e}_i$'s can be expressed this way
  - Notation: vector component: upper index; basis vectors lower index

- Change of basis
  - *L* has infinitely many bases
  - If $\vec{e}_i$ is basis in *L*, then $\vec{e}_j{}'$ is also a basis in *L*. One has $\vec{e}_j{}'=\Lambda^i{}_j\vec{e}_i$ and $\vec{e}_i=G^j{}_i\,\vec{e}_j{}'$
  - Matrix *G* is inverse of Λ $G^j{}_i\Lambda^k{}_j=\delta^k{}_i$
  - In other basis, components of vector change to $A^{i\prime}=G^i{}_jA^j$
  - Vector $\vec{A}$ is geometric object and does not change!

contravariant
covariant

1-FORMS AND DUAL SPACES

- 1-form
  - GR works with geometric (basis-independent) objects
  - Vector is an example
  - Other example: real-valued function of vectors $\tilde{p}(\vec{a})$
  - Imagine this as a machine with a single slot to insert vectors: real numbers result
- Dual space
  - Imagine set of all 1-form in *L*
  - This set also obeys all rules for a linear space, dual space. Denote as *L**
  - When *L* is *n*-dimensional, also *L** is *n*-dimensional
  - For 1-form $\tilde{p}$ and vector $\vec{V}$ we have $\tilde{p}(\vec{V}) = \tilde{p}(V^i \vec{e}_i) = \tilde{p}(\vec{e}_i)V^i$
  - Numbers $\{\tilde{p}(\vec{e}_i)\}$ are components p_i of 1-form $\tilde{p}$
- Basis in dual space
  - Given basis $\{\vec{e}_i\}$ in *L*, define 1-form basis $\{\tilde{\omega}^i\}$ in *L** (called dual basis) by $\tilde{\omega}^i(\vec{e}_j) = \delta^i{}_j$
  - Can write 1-form as $\tilde{p} = p_i \tilde{\omega}^i$, with p_i real numbers
  - We now have $\tilde{p}(\vec{V}) = p_i V^i$
  - Mathematically, looks like inner product of two vectors. However, in different spaces
  - Change of basis yields $\tilde{\omega}^{i'} = G^{i}{}_{j}\tilde{\omega}^j$ and $p_i{}' = \Lambda^j{}_i p_j$ (change covariant!)
  - Index notation by Schouten
  - Dual of dual space: *L*** = *L*

Tensors

- Tensors
  - So far, two geometric objects: vectors and 1-forms
  - Tensor: linear function of *n* vectors and *m* 1-forms (picture machine again)
  - Imagine (*n*,*m*) tensor $T = T(x_1, ..., x_n, y_1, ..., y_m)$
  - Where x_i live in *L* and y_j in *L**
  - Expand objects in corresponding spaces: $x_i = x_i^l \vec{e}_l$ and $y_i = y_{ik}\tilde{\omega}^k$
  - Insert into *T* yields $T = T(x_1, ..., x_n, y_1, ..., y_m) = T^{k_1 k_2 ... k_m}_{l_1 l_2 ... l_n} x_1^{l_1} x_2^{l_2} ... x_n^{l_n} y_{1k_1} y_{2k_2} ... y_{mk_m}$
  - with tensor components $T^{k_1 k_2 ... k_m}_{l_1 l_2 ... l_n} = T(\vec{e}_{l_1}, \vec{e}_{l_2}, ..., \vec{e}_{l_n}, \tilde{\omega}^{k_1} \tilde{\omega}^{k_2}, ..., \tilde{\omega}^{k_m})$
  - In a new basis $T^{i'_1 i'_2 ... i'_n}_{j_1 j_2 ... j_n} = G^{i_1}{}_{k_1} G^{i_2}{}_{k_2} ... G^{i_m}{}_{k_m} \Lambda^{l_1}_{j_1} \Lambda^{l_2}_{j_2} ... \Lambda^{l_n}_{j_n} T^{k_1 k_2 ... k_m}_{l_1 l_2 ... l_n}$
  - Mathematics to construct tensors from tensors: tensor product, contraction. This will be discussed when needed

Curvilinear coordinates

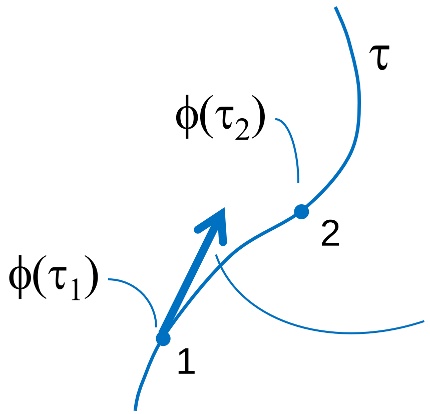
Derivate of scalar field

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau}$$

$$= \frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial x}U^x + \frac{\partial\phi}{\partial y}U^y + \frac{\partial\phi}{\partial z}U^z$$

$$= \tilde{\mathrm{d}}\phi(\vec{U})$$

$$\tilde{\mathrm{d}}\phi \underset{\mathcal{O}}{\rightarrow} \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$



$$\vec{U} = \begin{pmatrix} U^t \\ U^x \\ U^y \\ U^z \end{pmatrix} = \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix}$$

tangent vector

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = < \tilde{\mathrm{d}}\phi, \vec{V} >$$

Magnitude of derivative of ϕ in direction of $\vec{V}$

Derivative of scalar field ϕ along tangent vector $\vec{V}$

$$\nabla_{\vec{V}}\phi = V^\alpha \frac{\partial\phi}{\partial x^\alpha}$$

$$\nabla_{\vec{V}} = V^\alpha \frac{\partial}{\partial x^\alpha} \quad \text{en} \quad \vec{V} = V^\alpha \vec{e}_\alpha$$

⇒

$$\vec{V} = \nabla_{\vec{V}} = \partial_{\vec{V}} = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}$$

$$\vec{e}_\alpha = \frac{\partial\mathcal{P}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}$$

Example

Transformation $x = u + v, \qquad y = u - v, \qquad z = 2uv + w,$

Position vector $\vec{r} = (u+v)\vec{i} + (u-v)\vec{j} + (2uv+w)\vec{k}$

Base vectors $\vec{e}_\alpha = \frac{\partial \mathcal{P}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}$

Natural basis

$$\begin{aligned} \vec{e}_u &= \partial\vec{r}/\partial u = \vec{i} + \vec{j} + 2v\vec{k}, \\ \vec{e}_v &= \partial\vec{r}/\partial v = \vec{i} - \vec{j} + 2u\vec{k}, \\ \vec{e}_w &= \partial\vec{r}/\partial w = \vec{k} \end{aligned}$$

Metric is known

Non orthonormal $\vec{e}_u \cdot \vec{e}_v = 4uv,\ \vec{e}_v \cdot \vec{e}_w = 2u \text{ en } \vec{e}_w \cdot \vec{e}_u = 2v$

Inverse transformation $u = \frac{1}{2}(x+y), \qquad v = \frac{1}{2}(x-y), \qquad w = z - \frac{1}{2}(x^2 - y^2)$

Dual basis

$$\begin{aligned} \vec{e}^u &= \nabla u = \tfrac{1}{2}\vec{i} + \tfrac{1}{2}\vec{j}, \\ \vec{e}^v &= \nabla v = \tfrac{1}{2}\vec{i} - \tfrac{1}{2}\vec{j}, \\ \vec{e}^w &= \nabla w = -x\vec{i} + y\vec{j} + \vec{k} = -(u+v)\vec{i} + (u-v)\vec{j} + \vec{k} \end{aligned}$$

Derivative of a vector

$$\vec{V} = V^\alpha \vec{e}_\alpha$$

α is 0 - 3

$$\rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

Set β to 0

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\mu \Gamma^\alpha{}_{\mu\beta}\vec{e}_\alpha$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}\right)\vec{e}_\alpha$$

Notation

$$\frac{\partial V^\alpha}{\partial x^\beta} = V^\alpha{}_{,\beta}$$

$$V^\alpha{}_{;\beta} \equiv V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}$$

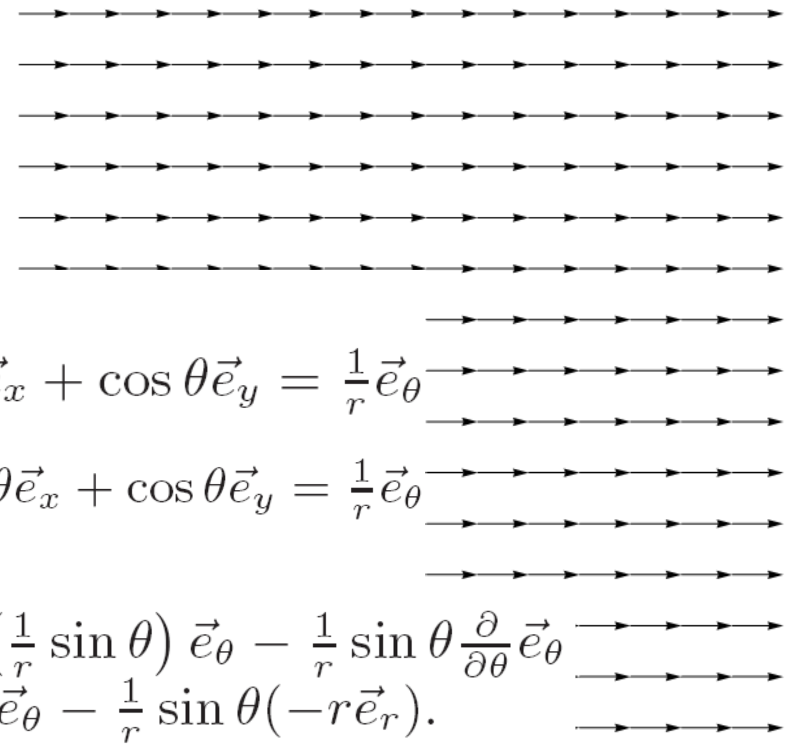
$$\Rightarrow \quad \boxed{\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha{}_{;\beta}\vec{e}_\alpha}$$

Covariant derivative

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensorveld $\nabla \vec{V}$

with components $\quad (\nabla \vec{V})^\alpha{}_\beta = (\nabla_\beta \vec{V})^\alpha = V^\alpha{}_{;\beta}$

Polar coordinates



$$\vec{e}_x \rightarrow (\Lambda^r{}_x, \Lambda^\theta{}_x) = (\cos\theta, -r^{-1}\sin\theta)$$

Calculate $\partial\vec{e}_x/\partial\theta$

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_r &= \frac{\partial}{\partial r}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = 0, \\
\frac{\partial}{\partial\theta}\vec{e}_r &= \frac{\partial}{\partial\theta}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_\theta &= \frac{\partial}{\partial r}(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta \\
\frac{\partial}{\partial\theta}\vec{e}_\theta &= -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y = -r\vec{e}_r.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\vec{e}_x &= \frac{\partial}{\partial\theta}(\cos\theta)\vec{e}_r + \cos\theta\frac{\partial}{\partial\theta}(\vec{e}_r) - \frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\right)\vec{e}_\theta - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\vec{e}_\theta \\
&= -\sin\theta\vec{e}_r + \cos\theta\left(\frac{1}{r}\vec{e}_\theta\right) - \frac{1}{r}\cos\theta\vec{e}_\theta - \frac{1}{r}\sin\theta(-r\vec{e}_r).
\end{aligned}$$

$$\boxed{\partial\vec{e}_x/\partial\theta = 0}$$

Calculate Christoffel symbols

$$\begin{array}{llcll}
(1) & \frac{\partial\vec{e}_r}{\partial r} = 0 & \rightarrow & \Gamma^\mu{}_{rr} = 0 & \text{voor alle } \mu, \\
(2) & \frac{\partial\vec{e}_r}{\partial\theta} = \frac{1}{r}\vec{e}_\theta & \rightarrow & \Gamma^r{}_{r\theta} = 0, & \Gamma^\theta{}_{r\theta} = \frac{1}{r}, \\
(3) & \frac{\partial\vec{e}_\theta}{\partial r} = \frac{1}{r}\vec{e}_\theta & \rightarrow & \Gamma^r{}_{\theta r} = 0, & \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \\
(4) & \frac{\partial\vec{e}_\theta}{\partial\theta} = -r\vec{e}_r & \rightarrow & \Gamma^r{}_{\theta\theta} = -r, & \Gamma^\theta{}_{\theta\theta} = 0.
\end{array}$$

Divergence and Laplace operators

$$V^\alpha{}_{;\alpha} = \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial\theta} + \frac{1}{r}V^r = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^\theta$$

$$\nabla\cdot\nabla\phi \equiv \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

Christoffel symbols and metric

Covariant derivatives
$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}$$
$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^\mu{}_{\alpha\beta}$$

$$\nabla_\beta(p_\alpha V^\alpha) = p_{\alpha;\beta}V^\alpha + p_\alpha V^\alpha{}_{;\beta}$$

In cartesian coordinates and Euclidian space $\nabla_\beta \tilde{V} = g(\nabla_\beta \vec{V}, \ldots)$

This <u>tensor equation</u> is valid for all coordinates $V_{\alpha;\beta} = g_{\alpha\mu} V^\mu{}_{;\beta}$

Take covariant derivative of $V_{\alpha'} = g_{\alpha'\mu'} V^{\mu'}$ ⇒ $V_{\alpha';\beta'} = g_{\alpha'\mu'} V^{\mu'}{}_{;\beta'}$

⇒ $\cancel{V_{\alpha';\beta'}} = g_{\alpha'\mu';\beta'} V^{\mu'} + \cancel{g_{\alpha'\mu'} V^{\mu'}{}_{;\beta'}}$

⇒ $g_{\alpha'\mu';\beta'} = 0$

Directly follows from $g_{\alpha\beta,\mu} = 0$ in cartesian coordinates!

The components of <u>the same</u> tensor ∇g for arbitrary coordinates are $g_{\alpha\beta;\mu}$

<u>Exercise</u>: proof the following

$$\Gamma^\gamma{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

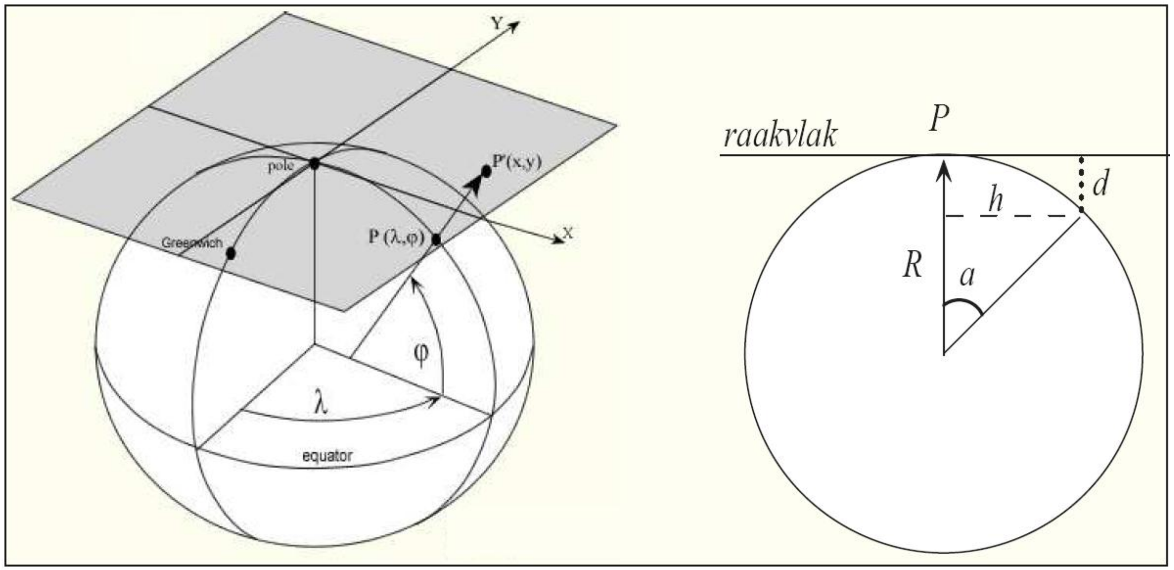
Connection coefficients contain derivatives of the metric

Local Lorentz frame – LLF

Next,we discuss curved spacetime

At each event *P* in spacetime we can choose a LLF:

- we are free-falling (no gravity effects according to *equivalence principle* (EP))
- in LLF one has Minkowski metric



$$\cos x = 1 - \frac{x^2}{2} + \ldots$$

$$g_{jk} = \delta_{jk} + \mathcal{O}\left(\frac{|\vec{x}|^2}{R^2}\right)$$

Locally Euclidian

LLF in curved spacetime

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \quad \text{voor alle } \alpha, \beta;$$

$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(\mathcal{P}) = 0 \quad \text{voor alle } \alpha, \beta, \gamma;$$

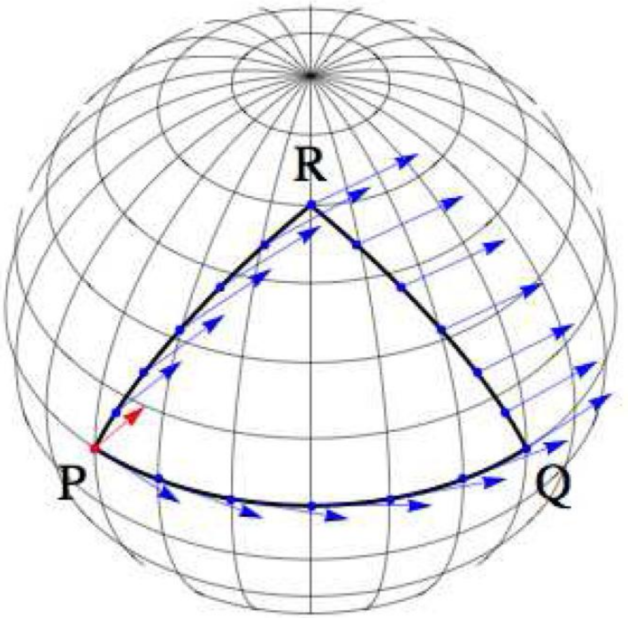
$$\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(\mathcal{P}) \neq 0.$$

At each point $\mathcal{P}$ tangent space is flat

Curvature and parallel transport

Parallel lines can intersect in a curved space
(Euclidian fifth postulate is invalid)

Parallel transport of a vector $\vec{V}$
- project vector after each step to local tangent plane
- rotation depends on curve and size of loop



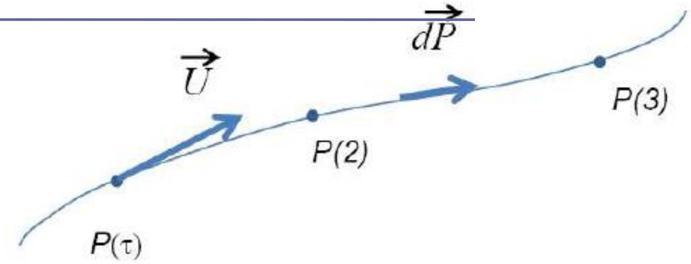
Mathematical description
- interval PQ is curve with parameter λ
- vector field $\vec{V}$ exists on this curve
- vector tangent to the curve $\vec{U} = d\vec{x}/d\lambda$
- we demand that in a LLF its components must be constant

$$\frac{dV^\alpha}{d\lambda} = U^\beta V^\alpha_{\ ,\beta} = U^\beta V^\alpha_{\ ;\beta} = 0 \quad \text{op punt } \mathcal{P}$$

Parallel transport $$U^\beta V^\alpha_{\ ;\beta} = 0 \quad \leftrightarrow \quad \frac{d}{d\lambda}\vec{V} = \nabla_{\vec{U}}\vec{V} = 0$$

Geodesics

Parallel transport $U^\beta V^\alpha_{;\beta} = 0 \quad \leftrightarrow \quad \frac{d}{d\lambda}\vec{V} = \nabla_{\vec{U}}\vec{V} = 0$



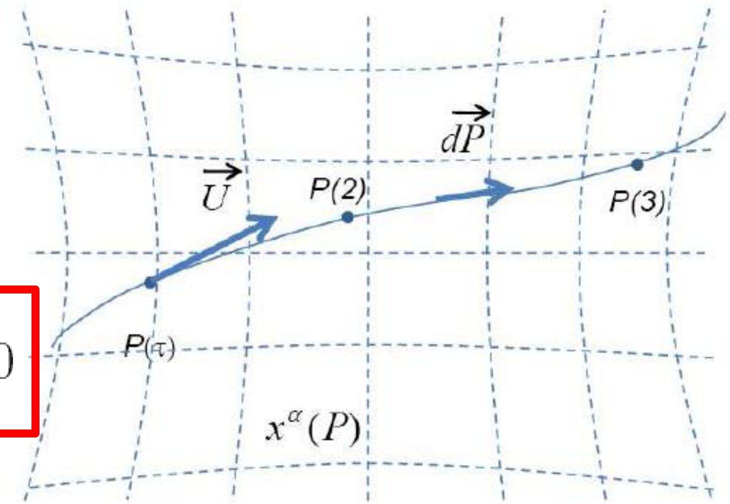
Geodesic: line, as straight as possible $\nabla_{\vec{U}}\vec{U} = 0$

Components of four-velocity $U^\alpha = \frac{dx^\alpha}{d\tau}$

Geodesic equation

$$U^\alpha_{;\mu}U^\mu = 0 \quad \rightarrow \quad \left(U^\alpha_{,\mu} + \Gamma^\alpha_{\mu\nu}U^\nu\right)U^\mu = 0$$

$$\underbrace{\underbrace{U^\alpha_{,\mu}}_{\frac{\partial U^\alpha}{\partial x^\mu}}\underbrace{U^\mu}_{\frac{dx^\mu}{d\tau}}}_{\frac{dU^\alpha}{d\tau}=\frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right)} + \Gamma^\alpha_{\mu\nu}\underbrace{U^\nu}_{\frac{dx^\nu}{d\tau}}\underbrace{U^\mu}_{\frac{dx^\mu}{d\tau}} = 0 \quad \rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$



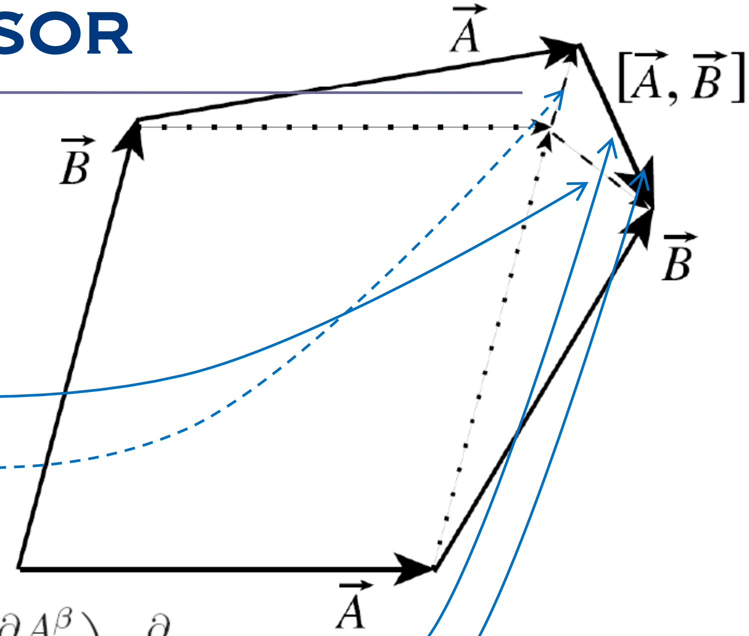
Four ordinary second-order differential equations for the coordinates $x^0(\tau)$, $x^1(\tau)$, $x^2(\tau)$ and $x^3(\tau)$
Coupled through the connection coefficients

Two boundary conditions $x^\alpha(\tau = 0)$

$$\frac{dx^\alpha}{d\tau}(\tau = 0) = U^\alpha(0)$$

Spacetime determines the motion of matter

Riemann tensor



Consider vector fields $\vec{A}$ and $\vec{B}$

Transport $\vec{B}$ along $\vec{A}$

Vector $\vec{B}$ changes by $A^\alpha \partial B^\beta / \partial x^\alpha$

Transport $\vec{A}$ along $\vec{B}$ ⇒ $B^\alpha \partial A^\beta / \partial x^\alpha$

Components of the commutator

$$[\vec{A},\vec{B}] = \Big[\underbrace{A^\alpha \frac{\partial}{\partial x^\alpha}}_{\vec{A}}, \underbrace{B^\beta \frac{\partial}{\partial x^\beta}}_{\vec{B}}\Big] = \underbrace{\left(A^\alpha \frac{\partial B^\beta}{\partial x^\alpha} - B^\alpha \frac{\partial A^\beta}{\partial x^\alpha}\right)}_{[\vec{A},\vec{B}]^\beta} \underbrace{\frac{\partial}{\partial x^\beta}}_{\vec{e}_\beta}$$

Commutator is a measure for non-closure

Curvature tensor of Riemann measures the non-closure of double gradients

Consider vector field $\vec{A}$

$$A_{\alpha;\mu\nu} - A_{\alpha;\nu\mu} = [\nabla_\mu, \nabla_\nu] A_\alpha \equiv R^\beta{}_{\alpha\mu\nu} A_\beta$$

$$R^\beta{}_{\alpha\mu\nu} = \left(\frac{\partial \Gamma^\beta_{\alpha\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\beta_{\alpha\mu}}{\partial x^\nu} + \Gamma^\gamma_{\alpha\nu}\Gamma^\beta_{\gamma\mu} - \Gamma^\gamma_{\alpha\mu}\Gamma^\beta_{\gamma\nu}\right)$$

$$A_{\alpha;\mu} = A_{\alpha,\mu} - \Gamma^\beta_{\alpha\mu} A_\beta$$

$$A_{\alpha;\mu\nu} = \frac{\partial}{\partial x^\nu}(A_{\alpha;\mu}) - \Gamma^\beta{}_{\alpha\nu}(A_{\beta;\mu}) - \Gamma^\beta{}_{\mu\nu}(A_{\alpha;\beta})$$

$$\Gamma^\gamma{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

$$R^\alpha{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\right)$$

Riemann tensor – properties

Metric tensor contains all information about intrinsic curvature

Properties Riemann tensor

Antisymmetry $\mathbf{R}(\_,\_,\vec{A},\vec{B}) = -\mathbf{R}(\_,\_,\vec{B},\vec{A})$ of $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$

$\mathbf{R}(\vec{A},\vec{B},\_,\_) = -\mathbf{R}(\vec{B},\vec{A},\_,\_)$ of $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$

Symmetry $\mathbf{R}(\vec{A},\vec{B},\vec{C},\vec{D} = \mathbf{R}(\vec{C},\vec{D},\vec{A},\vec{B})$ of $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\nu\mu}$

Bianchi identities $R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta} = 0$

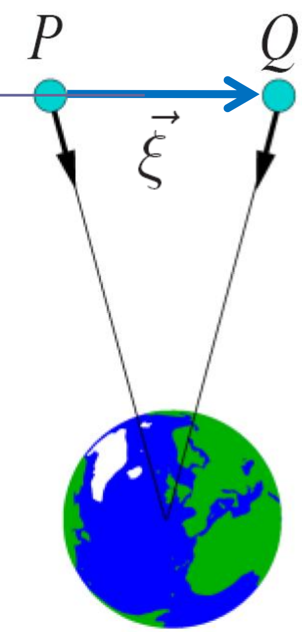
Independent components: 20

Curvature tensor of Ricci $R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}$

Ricci curvature (scalar) $R = R^{\alpha}{}_{\alpha}$

Exercise: demonstrate all this for the description of the surface of a sphere

Tidal forces



Drop a test particle. Observer in LLF: no sign of gravity

Drop two test particles. Observer in LLF: differential gravitational acceleration: tidal force

According to Newton

$$\vec{F}_{\text{grav}} = -m\nabla\Phi(\vec{x}) \qquad \text{en ook} \quad \frac{d^2\vec{x}}{dt^2} = -\nabla\Phi(\vec{x})$$

$$\left(\frac{d^2x_j}{dt^2}\right)_{(P)} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_{(P)} \quad \text{en} \quad \left(\frac{d^2x_j}{dt^2}\right)_{(Q)} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_{(Q)}$$

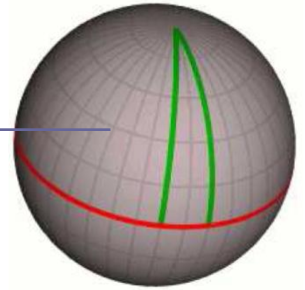
Define $\vec{\xi} = (x_j)_{(P)} - (x_j)_{(Q)}$

$$\Rightarrow \quad \frac{d^2\xi_j}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^j\partial x^k}\right)\xi_k = -\mathcal{E}_{jk}\xi_k \quad \longrightarrow \quad \mathcal{E}_{jk} = \left(\frac{\partial^2\Phi}{\partial x^j\partial x^k}\right)$$

Gravitational tidal tensor

$$\frac{d\vec{\xi}^2}{dt^2} = -\mathbf{E}(\_,\vec{\xi}) \qquad \nabla^2\Phi = 4\pi G\rho$$

Einstein equations

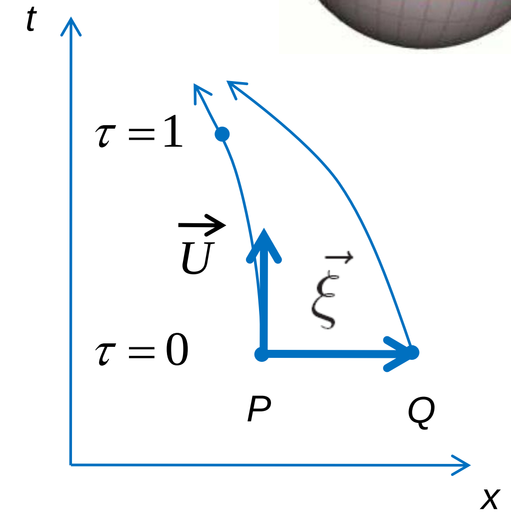


Two test particles move initially parallel

Spacetime curvature causes them to move towards each other

Initially at rest

At $\tau = 0$ one has
$$\left.\begin{aligned}\nabla_{\vec{U}}\vec{\xi} &= 0\\ \vec{U}\cdot\vec{\xi} &= 0\end{aligned}\right\}\quad \text{op punt } \mathcal{P} \text{ voor } \tau = 0$$



Second-order derivative $\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi}$ does not vanish because of curvature

One has $\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi} = -\mathbf{R}(\_,\vec{U},\vec{\xi},\vec{U})$ Follows from $[\nabla_\mu,\nabla_\nu]A_\alpha \equiv R^{\beta}{}_{\alpha\mu\nu}A_\beta$

Describes relative acceleration

In het LLF van deeltje P op tijdstip $\tau = 0$ geldt $U^0 = 1$ en $U^i = 0$

$$(\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi})^j = \frac{\partial^2\vec{\xi}}{\partial t^2} = -R^j_{\alpha\beta\gamma}U^\alpha\xi^\beta U^\gamma = -R^j_{0k0}\xi^k$$

$$\Rightarrow \quad \frac{\partial^2\xi^j}{\partial t^2} = -R^j_{0k0}\xi^k$$

Newton $\dfrac{\partial^2\xi^j}{\partial t^2} = -\mathcal{E}_{jk}\xi^k$

$$R_{j0k0} = \mathcal{E}_{jk} = \frac{\partial^2\Phi}{\partial x^j x^k}$$

$$\nabla^2\Phi = 4\pi G\rho \quad \rightarrow \quad \Phi_{,jk}\delta^{jk} = \mathcal{E}_{jk}\delta^{jk} = \mathcal{E}^j{}_j \quad \Rightarrow \quad R^j{}_{0j0} = 4\pi G\rho \quad ?$$

Einstein equations

Perhaps we expect
However, not a tensor equation (valid in LLF)

$$R^{j}{}_{0j0} = 4\pi G\rho \quad ?$$ (struck through)

$$R_{0000} = 0 \quad R^{0}{}_{000} = 0$$

$$\rightarrow R^{\mu}{}_{0\mu 0} = 4\pi G\rho$$

tensor ↑ scalar ↑

Perhaps one has
Einstein 1912 – wrong

$$R_{\alpha\beta} = 4\pi G T_{\alpha\beta} \quad ?$$ (struck through)

↑ $T_{00} = \rho$

$$R^{\delta}{}_{\alpha\beta\gamma} \approx g_{\alpha\beta,\gamma\delta} + \text{niet lineaire termen}$$

⇒ $$R_{\alpha\beta} \approx g_{\gamma\alpha,\gamma\delta} + \text{niet lineaire termen}$$

Set of 10 p.d.e. for 10 components of $g_{\alpha\beta}$

Problem: We hebben de vrijheid om $x^0(\mathcal{P})$, $x^1(\mathcal{P})$, $x^2(\mathcal{P})$ en $x^3(\mathcal{P})$ te kiezen

⇒ Free choice: 4 van de 10 componenten van $g_{\alpha\beta}$

Einstein tensor $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ Bianchi identities $G^{\alpha\beta}{}_{;\beta} = 0$

Energy – momentum tensor

$$T^{\alpha\beta}{}_{;\beta} = 0$$

⇒ $$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

Einstein equations

Matter tells spacetime how to curve

Weak gravitational fields

GR becomes SRT in a LLF

Without gravitation one has Minkowski metric η

For weak gravitational fields one has $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ met $|h_{\mu\nu}| \ll 1$

Assume a stationary metric $\partial_0 g_{\mu\nu} = 0$

Assume a slow moving particle $dx^i/dt \ll c \; (i = 1,2,3) \quad x^0 = ct$

Worldline of free-falling particle $\Rightarrow \dfrac{dx^i}{d\tau} \ll \dfrac{dx^0}{d\tau}$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad \Rightarrow \quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}c^2\left(\frac{dt}{d\tau}\right)^2$$

Christoffel symbol $\Gamma^\mu{}_{00} = \frac{1}{2}g^{\kappa\mu}(\partial_0 g_{0\kappa} + \partial_0 g_{0\kappa} - \partial_\kappa g_{00}) = -\frac{1}{2}g^{\kappa\mu}\partial_\kappa g_{00} = -\frac{1}{2}\eta^{\kappa\mu}\partial_\kappa h_{00}$

Stationary metric $\Gamma^0{}_{00} = 0$ en $\Gamma^i{}_{00} = \frac{1}{2}\delta^{ij}\partial_j h_{00}$ met $i = 1,2,3$

$$\Rightarrow \quad \frac{d^2t}{d\tau^2} = 0 \quad \text{en} \quad \frac{d^2\vec{x}}{d\tau^2} = -\frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00} \quad \Rightarrow \quad \frac{d^2\vec{x}}{dt^2} = -\frac{1}{2}c^2\nabla h_{00}$$

$dt/d\tau = \text{constant}$

Newton $\dfrac{d^2\vec{x}}{dt^2} = -\nabla\Phi(\vec{x})$

Newtonian limit of GR

$$\Rightarrow \quad h_{00} = 2\Phi/c^2 \quad \Rightarrow \quad \boxed{g_{00} = 1 + h_{00} = \left(1 + \frac{2\Phi}{c^2}\right)}$$

$\dfrac{\Phi}{c^2} = -\dfrac{GM}{c^2 r}$

Earth	10^{-9}
Sun	10^{-6}
White dwarf	10^{-4}

Curvature of time

Spacetime curvature involves curvature of time

$$g_{00} = 1 + h_{00} = \left(1 + \frac{2\Phi}{c^2}\right)$$

Clock at rest $dx^i/dt = 0$

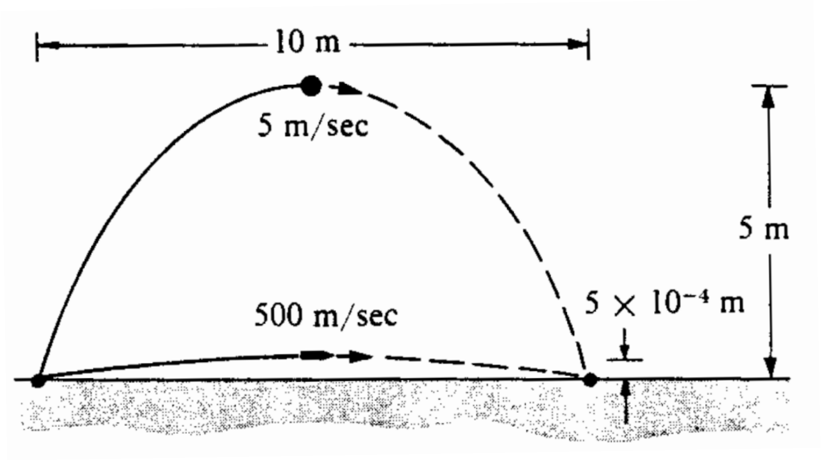
Time interval between two ticks $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2$

$$\Rightarrow \quad d\tau = \left(1 + \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} dt$$

Spacetime interval $$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 + dx^2 + dy^2 + dz^2$$

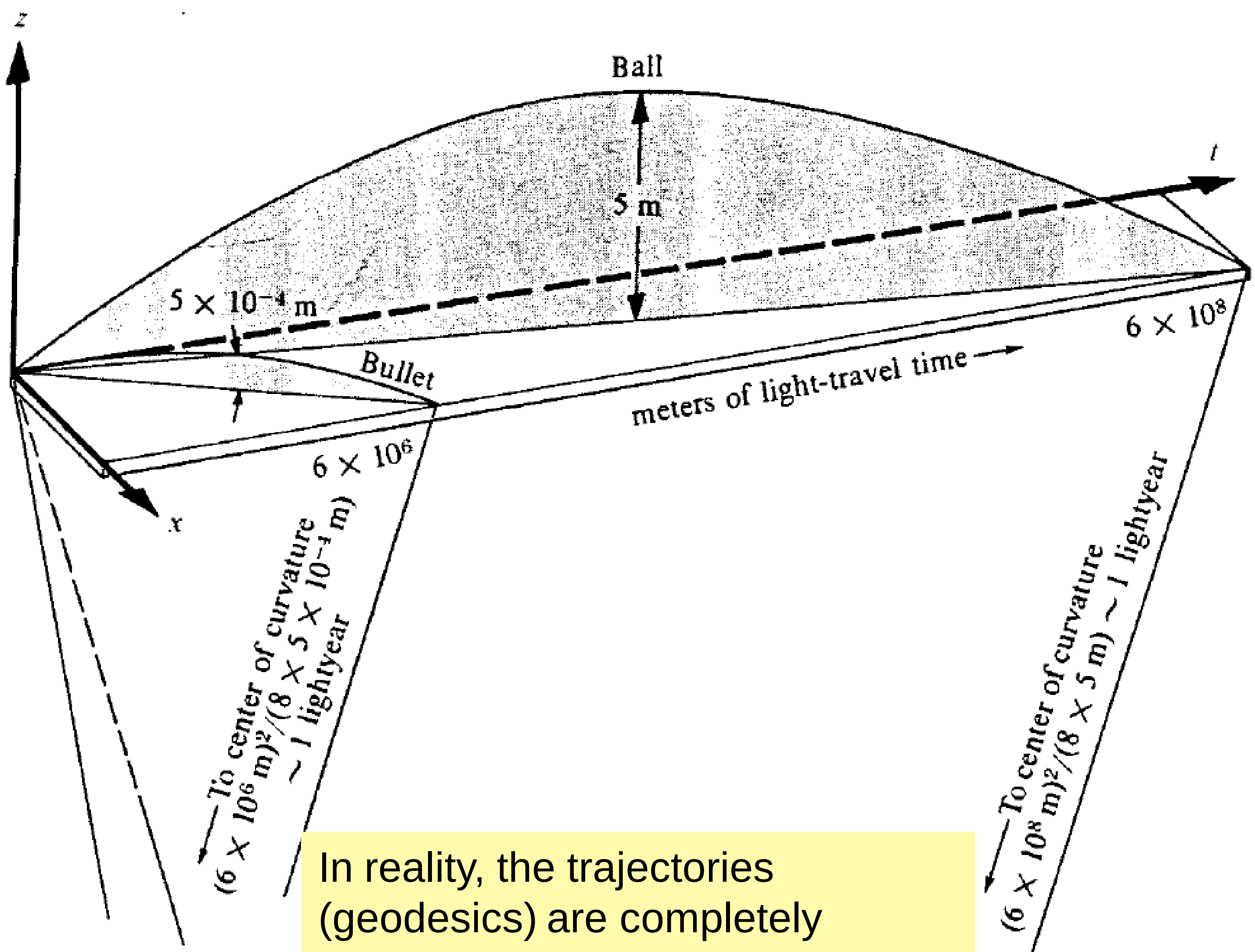
Describes trajectories of particles in spacetime

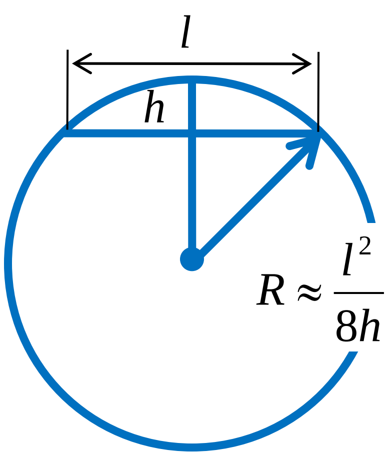
Trajectories of ball and bullet



Spatial curvature is very different

Curvature in spacetime





In reality, the trajectories (geodesics) are completely straight, but spacetime is curved