Quantumelektrodynamica

Elektromagnetische interactie van deeltjes

Inhoud

- Spin-0 deeltjes
  - Klein-Gordon vergelijking
  - Oplossingen voor positieve en negatieve energie
- Het fotonveld
  - Gauge invariance, minimale substitutie
  - Interactie van spin-0 deeltjes met het fotonveld
- Spin-½ deeltjes
  - Dirac vergelijking
- Quantumelektrodynamica
  - Elektron-muon verstrooiing
  - $e^+e^- \to \mu^+\mu^-$, $q\bar{q}$, …

Klein-Gordon vergelijking

Klein-Gordon vergelijking

j^μ kun je als volgt vinden

KG: $0=\left(\partial_\mu\partial^\mu+m^2\right)\phi \Rightarrow 0=\phi^*\left(\partial_\mu\partial^\mu+m^2\right)\phi$

KG^*: $0=\left(\partial_\mu\partial^\mu+m^2\right)\phi^* \Rightarrow 0=\phi\left(\partial_\mu\partial^\mu+m^2\right)\phi^*$

$$\overline{0=\phi^*\partial_\mu\partial^\mu\phi-\phi\partial_\mu\partial^\mu\phi^*}=\partial_\mu\left(\phi^*\partial^\mu\phi-\phi\partial^\mu\phi^*\right)\equiv\partial_\mu j^\mu$$

\`Vertaal' $p^2=m^2$ naar QM ($p^\mu \leftrightarrow i\partial^\mu \equiv (i\partial t, -i\nabla)$ dan $E=p^2/2m \rightarrow$ Schrödinger vergelijking)

$$(\partial_\mu\partial^\mu+m^2)\phi=0\begin{cases}\phi=Ne^{-ip\cdot x}=Ne^{-iEt+i\vec{p}\cdot\vec{x}}\\ j^\mu=(\rho,\vec{j})=i(\phi^*\delta^\mu\phi-\phi\delta^\mu\phi^*)=2p^\mu|N|^2=2|N|^2(E,\vec{p})\end{cases}$$

Problemen
$$\begin{cases}E^2=\vec{p}^{\,2}+m^2 \Rightarrow E=\pm\sqrt{\vec{p}^{\,2}+m^2}\\ \rho=2|N|^2E\begin{cases}\geq 0 & E\geq 0\\ \leq 0 & E\leq 0\end{cases}\end{cases}$$

<u>Historisch:</u>
vergeet Klein-Gordon en gebruik Dirac vergelijking

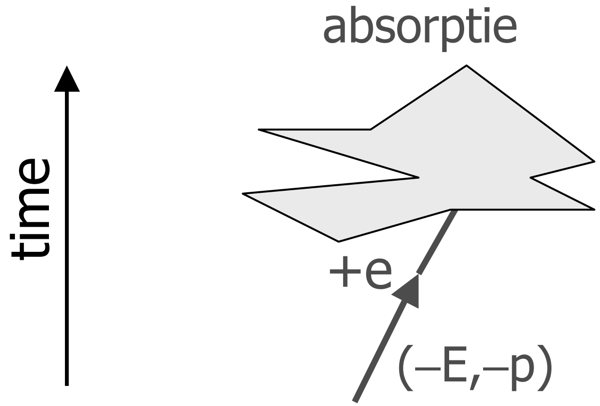
Beter: her-interpretatie van j^μ

$$j^\mu \rightarrow q\times j^\mu=2|N|^2(q\times E, q\times\vec{p})\begin{cases}j^0\geq 0 & \text{deeltje met } q>0\\ j^0<0 & \text{deeltje met } q<0\end{cases}$$

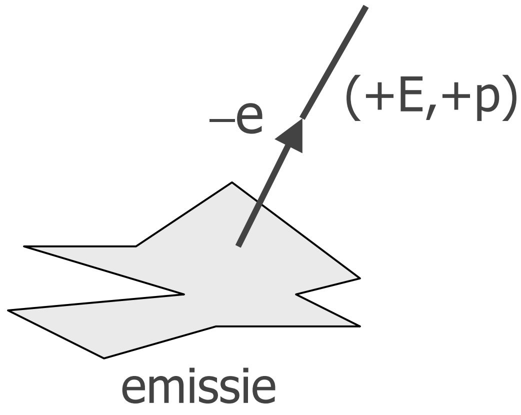
D.w.z. Oplossingen Klein-Gordon vgl.: $\begin{cases}E>0 & \text{deeltje met } q=-|e|\\ E<0 & \text{anti-deeltje met } q=+|e|\end{cases}$

Deze interpretatie is van: Pauli & Weisskopf
Stückelberg & Feynman

Deeltje $\leftrightarrow$ antideeltje



$$\Delta(\text{systeem}) = \begin{cases} -\vec{p} \\ -E \\ +e \end{cases}$$



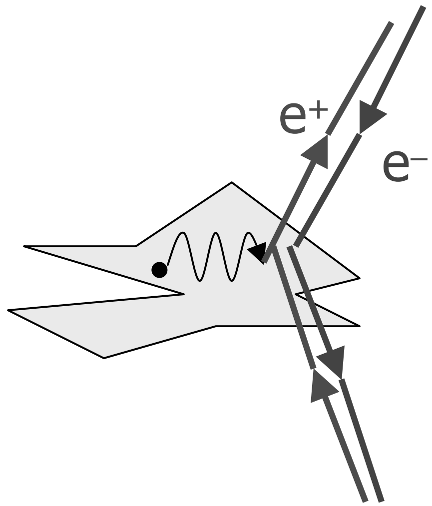
Voor een systeem is er geen verschil tussen:

Emissie: e^- met $p^\mu = (+E, +\vec{p})$
Absorptie: e^+ met $p^\mu = (-E, -\vec{p})$

In termen van de geladen stroom (dichtheid):

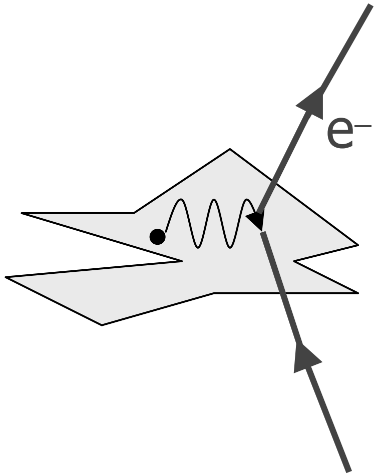
$$\begin{aligned} j^{\mu}_{+(E,\vec{p})}(e^+) &= +e \times |N|^2(+E, +\vec{p}) \\ &= -e \times |N|^2(-E, -\vec{p}) \\ &= j^{\mu}_{-(E,\vec{p})}(e^-) \end{aligned}$$

Ofwel: de volgende scenario's zijn identiek!

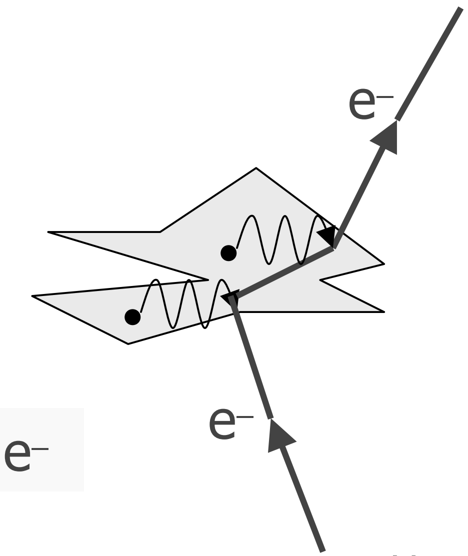


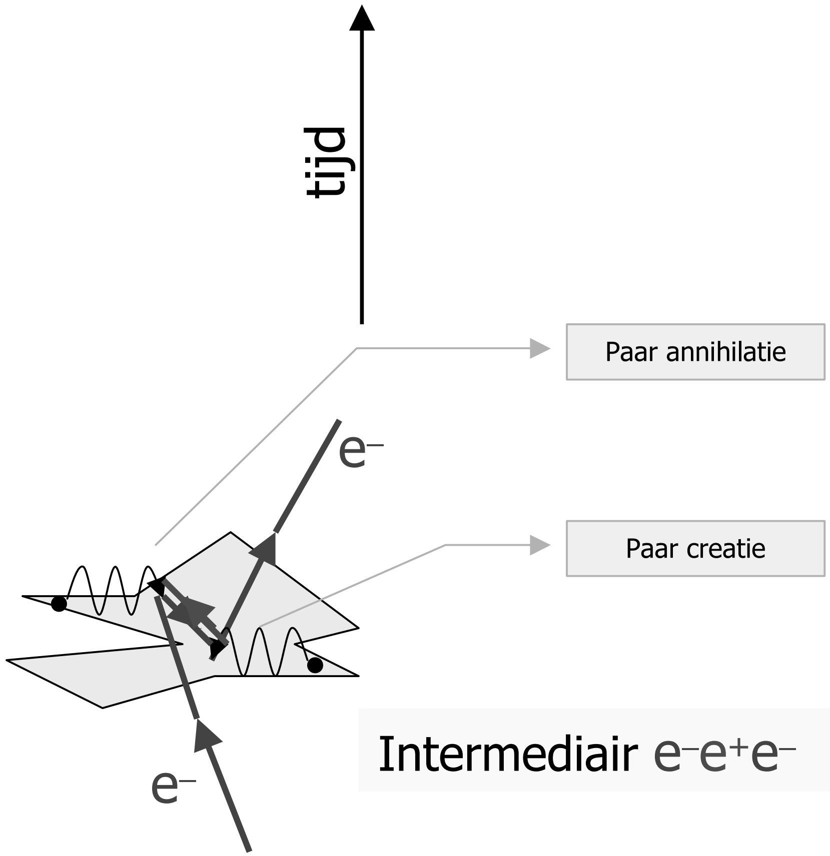
Storingsrekening

1^e orde:



2^e orde:



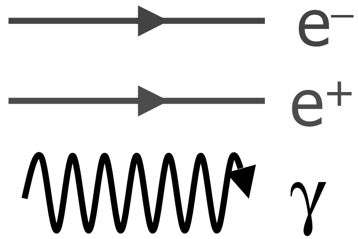


Intermediair e–

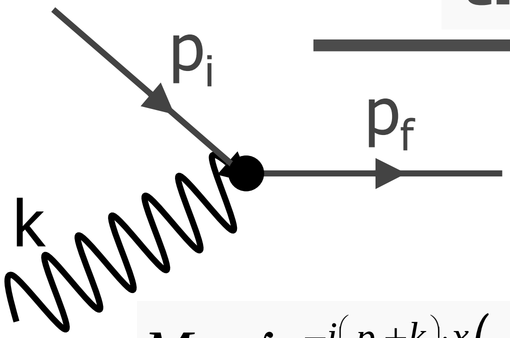
Intermediair e–e+e–

Vanwege de antideeltjes wordt het vacuum een complex systeem:
e^+e^- paren kunnen ontstaan uit het vacuum of erin opgaan.

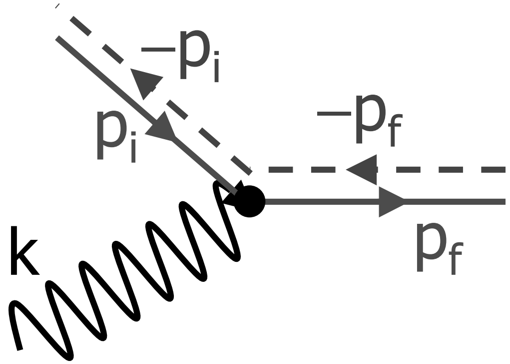
Energie en impulsbehoud



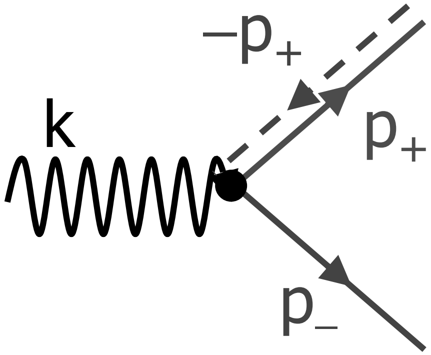
tijd



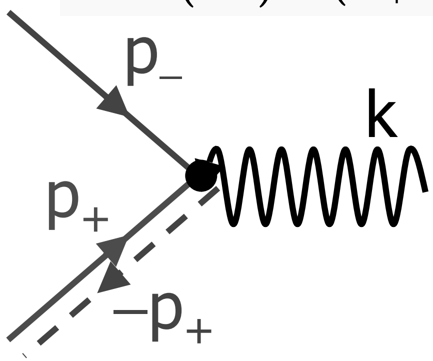
$$M \propto \int e^{-i(-p_f+k)\cdot x}\left(e^{+ip_i\cdot x}\right)^* dx$$
$$=(2\pi)^4\delta\left(E_i+\omega-E_f\right)\times\delta\left(\vec{p}_i+\vec{k}-\vec{p}_f\right)$$



$$M \propto \int e^{-i(p_i+k)\cdot x}\left(e^{-ip_f\cdot x}\right)^* dx$$
$$=(2\pi)^4\delta\left(E_i+\omega-E_f\right)\times\delta\left(\vec{p}_i+\vec{k}-\vec{p}_f\right)$$



$$M \propto \int e^{-ip_-\cdot x}\left(e^{-i(k-p_+)\cdot x}\right)^* dx$$
$$=(2\pi)^4\delta\left(E_++E_--\omega\right)\times\delta\left(\vec{p}_++\vec{p}_--\vec{k}\right)$$



$$M \propto \int e^{-i(k-p_+)\cdot x}\left(e^{-ip_-\cdot x}\right)^* dx$$
$$=(2\pi)^4\delta\left(\omega-E_+-E_-\right)\times\delta\left(\vec{k}-\vec{p}_+-\vec{p}_-\right)$$

Ijkinvariantie

Maxwellvergelijkigen en Lorentz invariantie

Maxwell
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

Stel $h/2\pi=c=1$ en introduceer potentiaal $A^\mu=(V,A)$ en stroom $j^\mu=(\rho,j)$:

$$\begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} \longrightarrow \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V \longrightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{\nabla}\times\vec{A}}{\partial t} = -\frac{\partial\vec{B}}{\partial t} \end{cases}$$

Er geldt $\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = 4\pi j^\nu$

Compacter met $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & -B_z & +B_y \\ +E_y & +B_z & 0 & -B_x \\ +E_z & -B_y & +B_x & 0 \end{pmatrix}$$

$$\boxed{\partial_\mu F^{\mu\nu} = 4\pi j^\nu \quad \begin{cases} \nu = 0: & \vec{\nabla}\cdot\vec{E} = 4\pi j^0 = 4\pi\rho \\ \nu = k: & \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 4\pi\vec{j} \end{cases}}$$

Ijkinvariantie (gauge invariance)

Vrijheid in de keuze van ijkveld A^{μ} (gauge field)

$$\begin{cases} A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\lambda = (V + \frac{\partial\lambda}{\partial t}, -\vec{A} + \vec{\nabla}\lambda) \\ A^{\mu} \rightarrow A'^{\mu} = A^{\mu} + \partial^{\mu}\lambda = (V + \frac{\partial\lambda}{\partial t}, +\vec{A} - \vec{\nabla}\lambda) \end{cases} \qquad \forall\lambda$$

Omdat
$$\begin{cases} \vec{B} \rightarrow \vec{B'} = \vec{\nabla} \times \vec{A'} = \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{\nabla}\lambda = \vec{\nabla} \times \vec{A} = \vec{B} \\ \vec{E} \rightarrow \vec{E'} = -\vec{\nabla}V' - \frac{\partial \vec{A'}}{\partial t} = -\vec{\nabla}(V + \frac{\partial\lambda}{\partial t}) - \frac{\partial}{\partial t}(\vec{A} - \vec{\nabla}\lambda) = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \vec{E} \end{cases}$$

Gebruik vrijheid om A^{μ} te vereenvoudigen *(covariant)*

Lorentz conditie $\qquad \partial_{\mu}A^{\mu} = 0 \Rightarrow \partial_{\nu}\partial^{\nu}A^{\mu} \equiv \Box A^{\mu} = 4\pi j^{\mu}$

A^{μ} is nog steeds niet uniek; omdat

$$A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\lambda \qquad \Box\lambda \equiv \partial_{\mu}\partial^{\mu}\lambda = 0$$

We kunnen bijvoorbeeld de oplossing kiezen *(niet covariant)*

$A^0 = 0 \quad \text{oftewel:} \quad \vec{\nabla} \cdot \vec{A} = 0$ Coulomb conditie

Foton golffunctie

In vacuüm $j^\mu=0$ en daarom $$0 = \partial_\nu \partial^\nu A^\mu \equiv \Box A^\mu$$

Met als oplossingen (vlakke golven)

$$A^\mu(x) = N\epsilon^\mu e^{-ip\cdot x} \quad \text{met:} \quad p^2 = p_\mu p^\mu = 0 \quad \Rightarrow \quad E = |\vec{p}|$$

Gauge keuze:

$\boxed{\text{Lorentz conditie}}$ $$\partial_\mu A^\mu = 0 \;\Rightarrow\; \epsilon^\mu p_\mu = 0$$

$\boxed{\text{Coulomb conditie}}$ $$\begin{cases} A^0 = 0 \\ \vec{\nabla}\cdot\vec{A} = 0 \end{cases} \Rightarrow \begin{cases} \epsilon^0 = 0 \\ \vec{\epsilon}\cdot\vec{p} = 0 \end{cases}$$

In plaats van 4 vrijheidsgraden (ε^μ) slechts 2

transversaal

$$\begin{cases} \vec{\epsilon}_1 = (1,0,0) \\ \vec{\epsilon}_2 = (0,1,0) \end{cases}$$

circulair

$$\begin{cases} \vec{\epsilon}_+ = \dfrac{-\vec{\epsilon}_1 + i\vec{\epsilon}_2}{\sqrt{2}} = \dfrac{(-1,+i,0)}{\sqrt{2}} \\ \vec{\epsilon}_- = \dfrac{+\vec{\epsilon}_1 - i\vec{\epsilon}_2}{\sqrt{2}} = \dfrac{(+1,-i,0)}{\sqrt{2}} \end{cases}$$

$$\vec{\varepsilon}_+ \to \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} (-\cos\varphi + i\sin\varphi)/\sqrt{2} \\ (-\sin\varphi + i\cos\varphi)/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\varphi}\left(-1/\sqrt{2}\right) \\ e^{-i\varphi}\left(i/\sqrt{2}\right) \\ 0 \end{pmatrix}$$

rotatie φ rond de *z*-axis: $$\vec{\epsilon}_\pm \longrightarrow e^{\mp i\phi} \times \vec{\epsilon}_\pm \quad \Rightarrow \quad \begin{cases} \vec{\epsilon}_+ & \text{Voor } m = +1 \\ \vec{\epsilon}_- & \text{Voor } m = -1 \end{cases}$$

Interactie van spin-0 deeltjes met het fotonveld

$$(\partial_\mu\partial^\mu+m^2)\phi=0\begin{cases}\phi=Ne^{-ip\cdot x}=Ne^{-iEt+i\vec{p}\cdot\vec{x}}\\ j^\mu=(\rho,\vec{j})=i(\phi^*\delta^\mu\phi-\phi\delta^\mu\phi^*)=2p^\mu|N|^2=2|N|^2(E,\vec{p})\end{cases}$$

Interacties: Klein-Gordon veld en A^μ

\`minimale substitutie'
(of via lokale ijkinvariantie)

$$\boxed{p^\mu\rightarrow p^\mu+eA^\mu\ \Rightarrow\ \begin{cases}i\partial^\mu\rightarrow i\partial^\mu+eA^\mu\\ \partial^\mu\rightarrow\partial^\mu-ieA^\mu\end{cases}}$$

Dan
$$\begin{aligned}0&=(\partial_\mu\partial^\mu+m^2)\phi\\ &\rightarrow[(\partial_\mu-ieA_\mu)(\partial^\mu-ieA^\mu)+m^2]\phi\\ &=[\partial_\mu\partial^\mu+m^2-ie(A_\mu\partial^\mu+\partial_\mu A^\mu)-\cancel{e^2A_\mu A^\mu}]\phi\\ &\equiv[\partial_\mu\partial^\mu+m^2]\phi+V\phi\\ V&=-ie(A_\mu\partial^\mu+\partial_\mu A^\mu)-\cancel{e^2A_\mu A^\mu}\\ &\approx-ie(A_\mu\partial^\mu+\partial_\mu A^\mu)\end{aligned}$$

Hogere orde (essentieel voor gauge-invariance!)

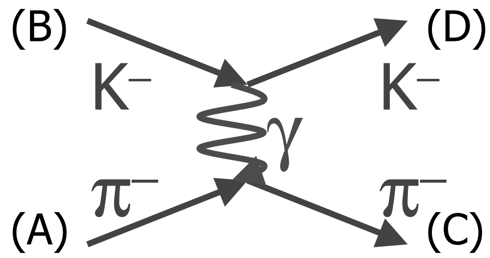
$$\begin{aligned}T_{fi}&=-i\int\phi_f^*V\phi_i d^4x\\ &\approx+i\int\phi_f^*ie(A_\mu\partial^\mu+\partial_\mu A^\mu)\phi_i d^4x\\ &=+i(ie)\int(\phi_f^*A_\mu[\partial^\mu\phi_i]+\phi_f^*[\partial_\mu A^\mu\phi_i])d^4x\\ &=+i(ie)\int(\phi_f^*A_\mu[\partial^\mu\phi_i]-[\partial^\mu\phi_f^*]A_\mu\phi_i)d^4x\\ &\equiv-i\int j^\mu_{fi}A_\mu d^4x\\ j^\mu_{fi}&=-ie(\phi_f^*[\partial^\mu\phi_i]-[\partial^\mu\phi_f^*]\phi_i)\end{aligned}$$

partiële integratie

gebruik: $\phi=Ne^{-ip\cdot x}$

$$\boxed{j^\mu_{fi}=-eN_iN_f^*(p_i^\mu+p_f^\mu)e^{i(p_f-p_i)\cdot x}}$$

$\pi^- K^- \to \pi^- K^-$ verstrooiing



Veronderstel: π^- verstrooit aan A^μ t.g.v. de K^-. Hoe vind je A^μ? Ansatz: Vindt een oplossing van de Maxwell vgl. met als stroom term de `overgangsstroom' die hoort bij K^-!

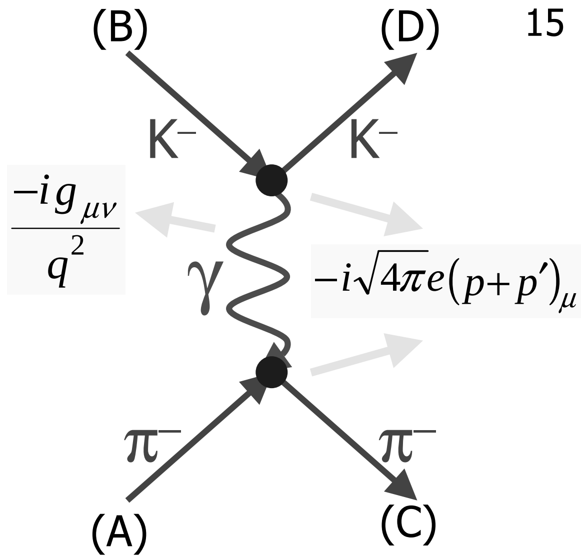
$$\boxed{\Box A^\mu = 4\pi j^\mu_{BD} = -4\pi e N_B N_D^* (p_B^\mu + p_D^\mu) e^{i(p_D - p_B)\cdot x}}$$

$$\Box e^{-iqx} = -q^2 e^{-iqx} \quad \longrightarrow \quad A^\mu = \frac{4\pi}{q^2} e N_B N_D^* (p_B^\mu + p_D^\mu) e^{i(p_D - p_B)\cdot x}$$

De overgangsamplitude wordt dus:

$$\begin{aligned} T_{fi} &= -i \int j^\mu_{AC} A_\mu d^4x \\ &= -ie^2 \int (N_A N_C^* (p_A^\mu + p_C^\mu) e^{i(p_A - p_C)\cdot x}) \frac{4\pi}{(p_D - p_B)^2} (N_B N_D^* (p_\mu^B + p_\mu^D) e^{i(p_D - p_B)\cdot x}) d^4x \\ &= -ie^2 N_A N_C^* N_B N_D^* (p_A^\mu + p_C^\mu) \frac{4\pi}{(p_D - p_B)^2} (p_\mu^B + p_\mu^D)(2\pi)^4 \delta(p_A + p_B - p_C - p_D) \end{aligned}$$

$\pi^- K^- \rightarrow \pi^- K^-$ verstrooiing



De amplitude M wordt ($q \equiv p_D - p_B = p_A - p_C$):

$$\begin{aligned} -i\mathcal{M} &= +ie^2(p_A^\mu + p_C^\mu)\frac{4\pi}{q^2}(p_\mu^B + p_\mu^D) \\ &= [i\sqrt{4\pi}e(p_A^\mu + p_C^\mu)] \times \frac{-i}{q^2} \times [i\sqrt{4\pi}e(p_\mu^B + p_\mu^D)] \\ &= [i\sqrt{4\pi}e(p_A^\mu + p_C^\mu)] \times \frac{-ig_{\mu\nu}}{q^2} \times [i\sqrt{4\pi}e(p^{B\nu} + p^{D\nu})] \end{aligned}$$

Plug in standaard uitdrukking voor $d\sigma_{A+B\rightarrow C+D}/d\Omega$:

$$d\sigma = \frac{(2\pi)^4\delta(p_A + p_B - p_C - p_D)}{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}} \times |\mathcal{M}|^2 \times \frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D} = \frac{1}{64\pi^2 s}\frac{|\vec{p_f}|}{|\vec{p_i}|}|\mathcal{M}|^2 d\Omega$$

Notatie: $\begin{cases} \vec{p}_A = -\vec{p}_B \equiv \vec{p}_i \quad \text{en} \quad \vec{p}_C = -\vec{p}_D \equiv \vec{p}_f \\ s \equiv (p_A + p_B)^2 = (p_C + p_D)^2 = (E_A + E_B)^2 = (E_C + E_D)^2 \end{cases}$

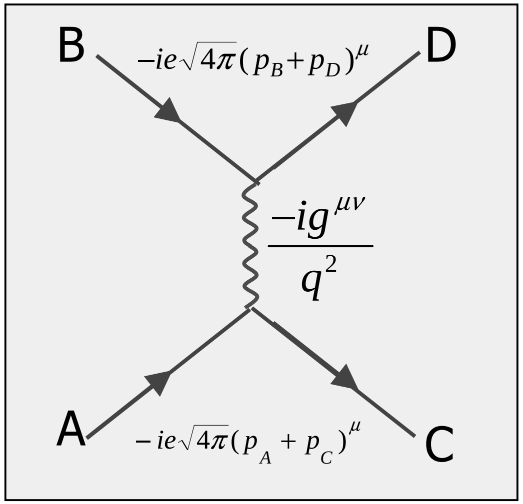
verwaarloos massa's

$$\boxed{\begin{aligned} -i\mathcal{M} &= i4\pi e^2\frac{(p_A + p_C)\cdot(p_B + p_D)}{(p_A - p_C)^2} \approx i4\pi e^2\frac{6E^2 + 2E^2\cos\theta}{-2E^2 + 2E^2\cos\theta} \\ \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s} \times (4\pi e^2)^2 \times \left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2 = \frac{\alpha^2}{4s} \times \left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2 \end{aligned}}$$

‘Feynman regels’

Vertexfactor: voor elke vertex introduceer een factor $-ie\sqrt{(4\pi)}p^{\mu}$.
e : koppeling van deeltje aan het e.m. veld,
p^{μ}: som van 4-impulsen voor en na de vertex

Propagator: voor elke interne lijn introduceer een factor $-ig^{\mu\nu}/q^2$
q: de 4-impuls van het uitgewisselde quantum

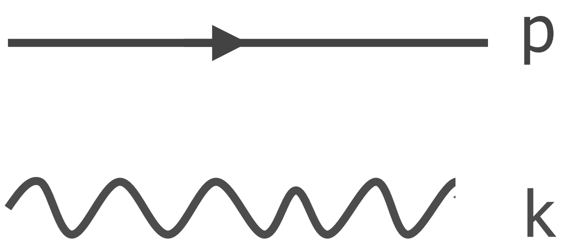


Houd rekening met impulsbehoud in elke vertex (uitdrukking voor q in termen van p_A, p_B, p_C, p_D)

Vervolgens berekening van de differentiële werkzame doorsnede

Elektrodynamika (S=0): Feynman regels

Externe lijnen

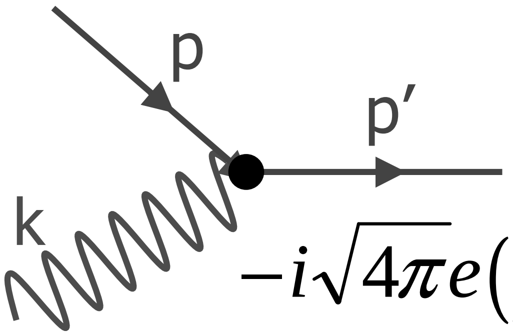


1

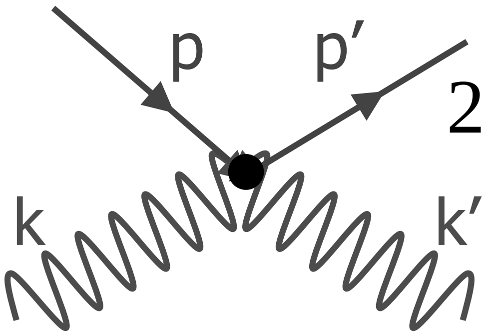
in: ε^{μ}

uit: $\varepsilon^{\mu *}$

Vertices

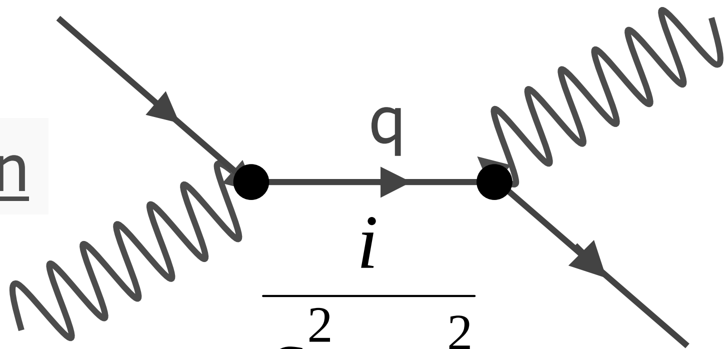


$$-i\sqrt{4\pi}e(p+p')_{\mu}$$

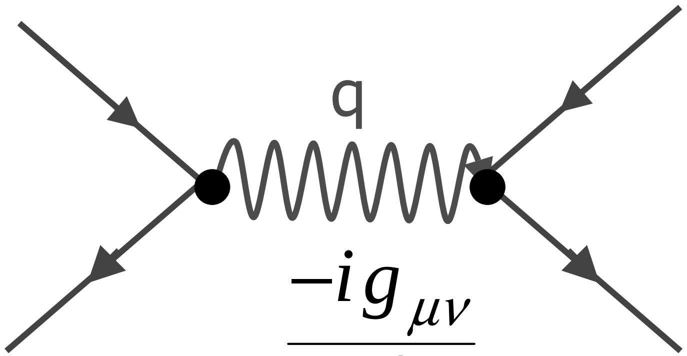


$$2i\,4\pi\,e^2 g_{\mu\nu}$$

Propagatoren

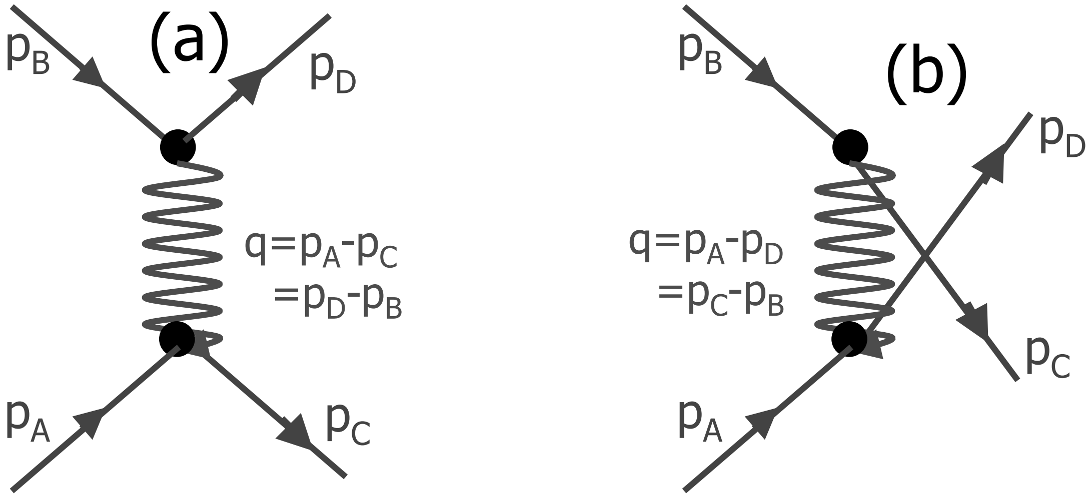


$$\frac{i}{q^2-m^2}$$



$$\frac{-i\,g_{\mu\nu}}{q^2}$$

$\pi^-\pi^- \to \pi^-\pi^-$ verstrooiing



Gewoon regels volgen geeft

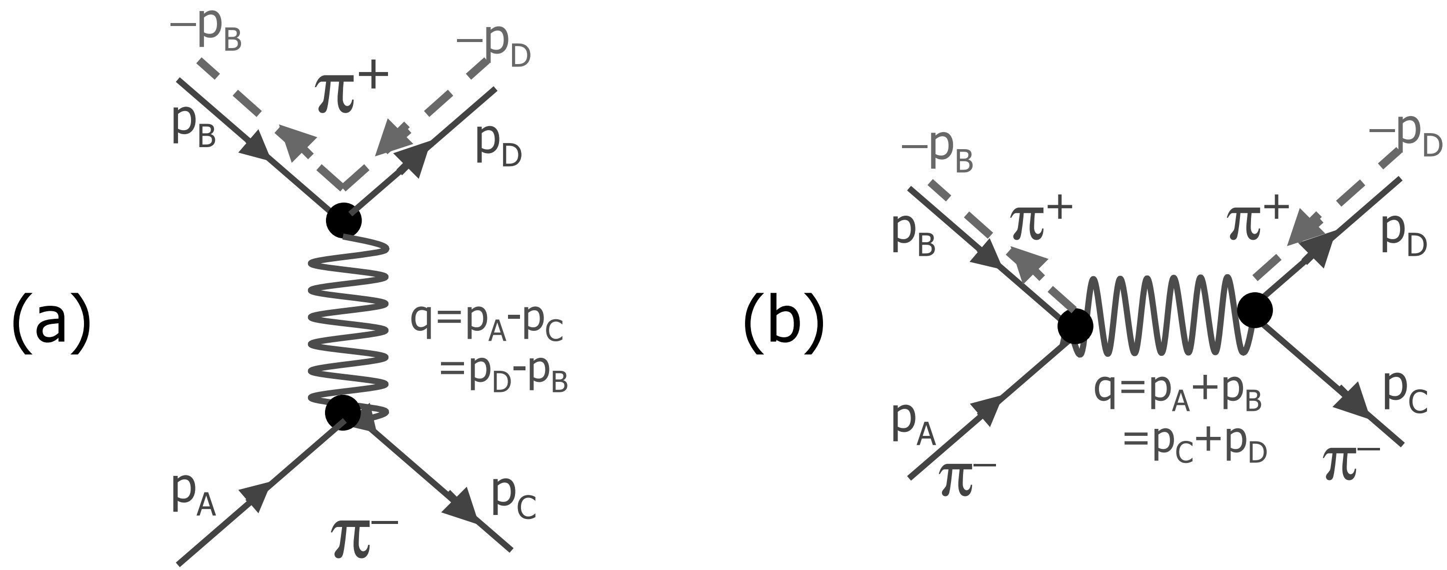
$$
\begin{aligned}
-i\mathcal{M}_a &= i4\pi e^2 \frac{(p_A+p_C)_\mu (p_B+p_D)^\mu}{(p_D-p_B)^2} \\
-i\mathcal{M}_b &= i4\pi e^2 \frac{(p_A+p_D)_\mu (p_B+p_C)^\mu}{(p_C-p_B)^2} \\
-i\mathcal{M} &= -i\mathcal{M}_a - i\mathcal{M}_b \\
&= i4\pi e^2 \times \left(\frac{(p_A+p_C)_\mu (p_B+p_D)^\mu}{(p_D-p_B)^2} + \frac{(p_A+p_D)_\mu (p_B+p_C)^\mu}{(p_C-p_B)^2} \right)
\end{aligned}
$$

En daarmee de werkzame doorsnede

$$
\begin{aligned}
d\sigma &= \frac{1}{128\pi^2 s} \frac{|\vec{p_f}|}{|\vec{p_i}|} |\mathcal{M}|^2 d\Omega \\
&= \ldots\ldots\ldots
\end{aligned}
$$

Let op: faktor ½ identieke deeltjes!

$\pi^-\pi^+ \rightarrow \pi^-\pi^+$ verstrooiing



Gewoon regels volgen geeft

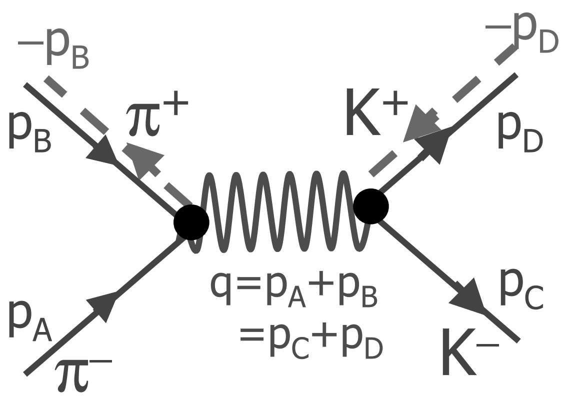
$$\begin{aligned}
-i\mathcal{M}_a &= i4\pi e^2 \frac{(p_A+p_C)_\mu(-p_D-p_B)^\mu}{(p_D-p_B)^2} \\
-i\mathcal{M}_b &= i4\pi e^2 \frac{(p_A-p_B)_\mu(-p_D+p_C)^\mu}{(p_C+p_D)^2} \\
-i\mathcal{M} &= -i\mathcal{M}_a - i\mathcal{M}_b \\
&= i4\pi e^2 \times \left(\frac{(p_A+p_C)_\mu(-p_D-p_B)^\mu}{(p_D-p_B)^2} + \frac{(p_A-p_B)_\mu(-p_D+p_C)^\mu}{(p_C+p_D)^2}\right)
\end{aligned}$$

Etcetera!

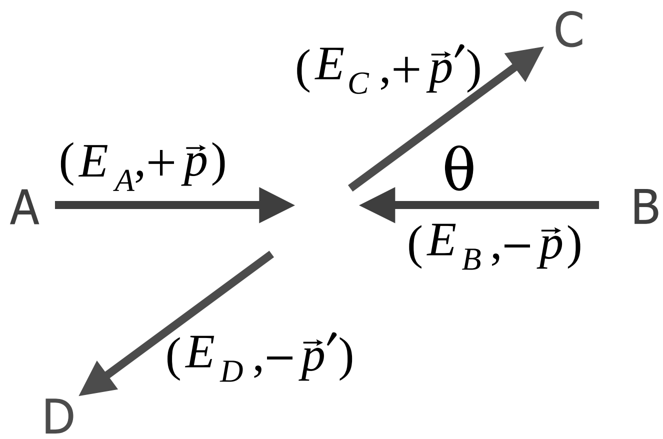
$\pi^+\pi^- \to K^+K^-$ verstrooiing

Volg onderstaande regels

1. Feynman diagram(men)



2. antideeltjes $\to$ tijdomkering van deeltjes



3. amplitude $-iM = i4\pi e^2 \dfrac{(p_A - p_B)_\mu (p_C - p_D)^\mu}{(p_A + p_B)^2}$

4. standaard $d\sigma/d\Omega$ uitdrukking $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{64\pi^2}\dfrac{1}{s}\dfrac{|\vec{p}_f|}{|\vec{p}_i|} \times |M|^2$

5. kies frame & maak 4-vectoren expliciet

$$p_A \to (E_A, +\vec{p} \equiv +\vec{p}_i) \qquad p_B \to (E_B, -\vec{p} \equiv -\vec{p}_i) \quad \text{en} \quad p_C \to (E_C, +\vec{p}\,' \equiv +\vec{p}_f) \qquad p_D \to (E_D, -\vec{p}\,' \equiv -\vec{p}_f)$$

6. relativistische limiet *(verwaarloos massa's)*

$$\frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{4s}\cos^2\theta \quad \Rightarrow \quad \sigma_{tot} = \frac{\pi\alpha^2}{3s}$$

Dirac vergelijking

Golfvergelijkingen

<u>Quantummechanica:</u>
De golffunctie wordt verkregen m.b.v. de transcriptie

$$\begin{cases} E \to i\hbar\partial_t \\ p \to -i\hbar\nabla \end{cases}$$

<u>Schrödinger vergelijking</u>

Klassiek, $E=p^2/2m$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

<u>Klein Gordon vergelijking</u>

Relativistisch, $E^2=p^2c^2+m^2c^4$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\hbar^2\nabla^2\psi + m^2c^2\psi$$

Vergelijking met 2e afgeleide naar de tijd $\Rightarrow$ er bestaan oplossingen met negative energie

<u>Dirac vergelijking</u>

Relativistisch, lineair in $\partial\psi/\partial t$

$$\begin{aligned} i\frac{\hbar}{c}\frac{\partial\psi}{\partial t} &= -i\hbar\vec{\alpha}\cdot\nabla\psi + \beta mc\psi \\ &= (\vec{\alpha}\cdot\vec{p} + \beta mc)\psi \end{aligned}$$

Om aan relativistische uitdrukking $E^2=p^2c^2+m^2c^4$ te voldoen: α en β matrices met:

$$\begin{aligned} (\vec{\alpha}\cdot\vec{p}+\beta mc)^2 &= (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc) \\ &= \beta^2 m^2 c^2 + \sum_i \left[\alpha_i^2 p_i^2 + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0} p_i \beta mc\right] + \\ &\quad \underbrace{\sum_{i>j}\left[(\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j\right]}_{0} \end{aligned}$$

1

Dirac algebra

De matrices α_i en β moeten voldoen aan

$$\boxed{\alpha_i^\dagger = \alpha_i \quad \text{en} \quad \beta^\dagger = \beta}$$

$$\boxed{\alpha_i^2 = \beta^2 = 1}$$

$$\boxed{\{\alpha_i, \alpha_j\} \equiv \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}}$$

$$\boxed{\{\alpha_i, \beta\} \equiv \alpha_i\beta + \beta\alpha_i = 0}$$

Eigenwaarden α_i en β zijn ± 1 en dimensie (*d*) is even

$$i \neq j: \ |\alpha_i\alpha_j| = |-\alpha_j\alpha_i| = (-1)^d|\alpha_j\alpha_i| = \begin{cases} -|\alpha_i\alpha_j|, & d \ \text{oneven} \\ +|\alpha_i\alpha_j|, & d \ \text{even} \end{cases}$$

Voor *d=2* zijn er maximaal 3 anti-commuterende matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Voor *d=4* zijn er inderdaad 4 anti-commuterende matrices
(laagste dimensie waarin Dirac algebra in kan worden gerepresenteerd)

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

Dirac algebra

Expliciet Lorentz-invariante notatie (x β):

$$mc\psi = \frac{i\hbar}{c}\beta\frac{\partial\psi}{\partial t} - \beta\vec{\alpha}\cdot\vec{p}\psi \equiv i\hbar\gamma^{\mu}\partial_{\mu}\psi$$

met:

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k \equiv \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

Dirac algebra:

$$\boxed{\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}}$$

definitie:

$$\begin{cases} \gamma_0 = +\gamma^0 \\ \gamma_k = -\gamma^k \end{cases}$$

Je kunt laten zien dat

$$\begin{aligned} \gamma^{0\dagger} &= +\gamma^0 \\ \gamma^{k\dagger} &= -\gamma^k \\ (\gamma^0)^2 &= +1 \\ (\gamma^k)^2 &= -1 \\ \gamma^{\mu\dagger} &= \gamma^0\gamma^{\mu}\gamma^0 \end{aligned}$$

Let op: γ^{μ} is geen 4-vector, (zie later voor rechtvaardiging gebruik Lorentz index γ)

Dirac vergelijking $\boxed{i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0}$

$$\begin{array}{ccl}
0 & = & -i\hbar\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - mc\psi^\dagger \\
 & = & -i\hbar\partial_0\psi^\dagger\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k - mc\psi^\dagger \\
(\times\gamma^0) & \longrightarrow & -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k\gamma^0 - mc\psi^\dagger\gamma^0 \\
 & = & -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 - i\hbar\partial_k\psi^\dagger\gamma^0\gamma^k - mc\psi^\dagger\gamma^0 \\
 & = & -i\hbar\partial_\mu\psi^\dagger\gamma^0\gamma^\mu - mc\psi^\dagger\gamma^0 \\
(\bar\psi \equiv \psi^\dagger\gamma^0) & \longrightarrow & -i\hbar\partial_\mu\bar\psi\gamma^\mu - mc\bar\psi
\end{array}$$

Het *stroombehoud* wordt verkregen via geconjugeerde Dirac vergelijking:

$$\overline{\psi} \equiv \psi^{+}\gamma^{0}$$

Dirac vergelijkingen voor $\bar\psi$ en ψ

$$\boxed{i\hbar(\partial_\mu\bar\psi)\gamma^\mu + mc\bar\psi = 0}$$

$$\boxed{i\hbar\gamma^\mu(\partial_\mu\psi) - mc\psi = 0}$$

Optellen na multiplicatie met ψ en $\bar\psi$

$$\boxed{0 = i\hbar(\partial_\mu\bar\psi)\gamma^\mu\psi + i\hbar\bar\psi\gamma^\mu(\partial_\mu\psi) = i\hbar\partial_\mu\left[\bar\psi\gamma^\mu\psi\right]}$$

Dus stroom j^μ:

$$j^\mu = \bar\psi\gamma^\mu\psi \left\{ \begin{array}{l} j^0 = \bar\psi\gamma^0\psi = |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \geq 0 \\ j^k = \bar\psi\gamma^k\psi \end{array} \right.$$

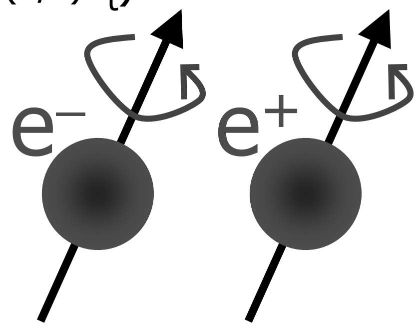
Vgl. Klein-Gordon: $(\partial_\mu\partial^\mu + m^2)\phi = 0 \left\{ \begin{array}{l} \phi = Ne^{-ip\cdot x} = Ne^{-iEt + i\vec p\cdot\vec x} \\ j^\mu = (\rho, \vec j) = i(\phi^*\delta^\mu\phi - \phi\delta^\mu\phi^*) = 2p^\mu|N|^2 = 2|N|^2(E, \vec p) \end{array} \right.$

Oplossing deeltje in rust: $\vec{p}=\vec{0}$

Dirac vgl. voor $\vec{p}=\vec{0}$ eenvoudig: $$\boxed{i\hbar\gamma^0\partial_0\psi - mc\psi = 0}$$

Splits ψ in twee componenten $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$, dan volgt ($\partial_0 \equiv (1/c)\partial_t$)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A/\partial_t \\ \partial\psi_B/\partial_t \end{pmatrix} = -\frac{imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$



Oplossingen

$$\begin{cases} \psi_A(t) \propto e^{-\frac{imc^2}{\hbar}t}\psi_A(0) \\ \psi_B(t) \propto e^{+\frac{imc^2}{\hbar}t}\psi_B(0) \end{cases}$$

De 4 onafhankelijke oplossingen volledig uitgeschreven:

$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Oplossing bewegend deeltje: $\vec{p} \neq \vec{0}$

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

probeer `Ansatz'

$$\psi = u(p)e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{x})} = u(p)e^{-\frac{i}{\hbar}p\cdot x}$$

$\Longrightarrow$ voor *spinor* u(p)

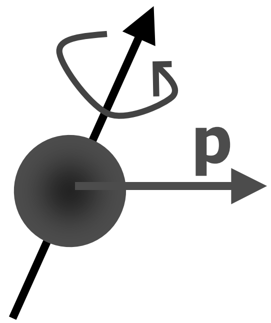
$$(\gamma^\mu p_\mu - mc)\, u(p) = 0$$

splits spinor u(p) in twee componenten

$$u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix}$$

Invullen

$$\begin{aligned} 0 &= (\gamma^\mu p_\mu - mc)\, u(p) = \left(\gamma^0 p^0 - \gamma^k p^k - mc\right) u(p) \\ &= \begin{pmatrix} E/c - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c - mc \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} \\ &= \begin{pmatrix} (E/c - mc)u_A(p) - \vec{p}\cdot\vec{\sigma}u_B(p) \\ \vec{p}\cdot\vec{\sigma}u_A(p) - (E/c + mc)u_B(p) \end{pmatrix} \end{aligned}$$



$$\Longrightarrow \left\{ \begin{array}{lcl} u_A(p) & = & \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p) \\ u_B(p) & = & \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p) \end{array} \right.$$

B.v. voor $u_A(p)$

$$u_A(p) = \frac{c^2}{E^2 - m^2c^4}(\vec{p}\cdot\vec{\sigma})^2 u_A(p)$$

Wat levert $\vec{p}\cdot\vec{\sigma}$?

$$(\vec{p}\cdot\vec{\sigma}) = p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & \dots \\ \dots & \dots \end{pmatrix} = \vec{p}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

En hiermee kan expliciet aangetoond worden dat aan de relativistische vergelijking voor de energie voldaan wordt

$$u_A(p) = \frac{\vec{p}^2 c^2}{E^2 - m^2 c^4} u_A(p) \quad \text{oftewel} \quad E^2 - m^2 c^4 = \vec{p}^2 c^2$$

(niet erg verbazend: had Dirac er in gestopt!)

Volledige oplossingen Dirac vgl.

De 4 onafhankelijke oplossingen worden m.b.v. de Dirac spinoren

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

$$\psi^{(3)} \propto e^{+\frac{i}{\hbar}p\cdot x}\begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{+\frac{i}{\hbar}p\cdot x}\begin{pmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

Normalisatie

Oplossingen E>0 (1 en 2)

$$\begin{aligned} \psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E+mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E+mc^2)^2} \\ &= 1+\frac{E-mc^2}{E+mc^2} = \frac{2E}{E+mc^2} = \frac{2|E|}{|E|+mc^2} \rightarrow N = \sqrt{|E|+mc^2} \end{aligned}$$

Oplossingen E<0 (3 en 4)

$$\begin{aligned} \psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E-mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E-mc^2)^2} \\ &= 1+\frac{E+mc^2}{E-mc^2} = \frac{2E}{E-mc^2} = \frac{2|E|}{|E|+mc^2} \rightarrow N = \sqrt{|E|+mc^2} \end{aligned}$$

Dirac stroom j^μ

De stroom j^μ voor $p_i=0$
(met standaard normalisatie)

$$\begin{cases} j^0 = \bar{\psi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi = \psi^\dagger\psi = 2mc^2 \rightarrow 2|E|\geq 0 \\[2ex] j^k = \bar{\psi}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\psi = \psi^\dagger\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\psi = \vec{0} \end{cases}$$

De stroom j^μ voor $p_i\neq 0$
(met standaard normalisatie)

$$\begin{cases} j^0 = \bar{\psi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi = \psi^\dagger\psi = \begin{cases} +2E & E>0 \\ -2E & E<0 \end{cases} \rightarrow 2|E|\geq 0 \\[2ex] j^k = \bar{\psi}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\psi = \psi^\dagger\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\psi = \begin{cases} +2\vec{p} & E>0 \\ -2\vec{p} & E<0 \end{cases} \end{cases}$$

Waarom? Expliciet uitschrijven, b.v. voor de x-component

$$\begin{aligned} j_x &= |N|^2\left(1, 0, \tfrac{cp_z}{E+mc^2}, \tfrac{c(p_x-ip_y)}{E+mc^2}\right)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \\ &= |N|^2\left(1, 0, \tfrac{cp_z}{E+mc^2}, \tfrac{c(p_x-ip_y)}{E+mc^2}\right)\begin{pmatrix} \frac{c(p_x+ip_y)}{E+mc^2} \\ \frac{cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \\ &= |N|^2\left(\tfrac{c(p_x+ip_y)}{E+mc^2} + \tfrac{c(p_x-ip_y)}{E+mc^2}\right) \\ &= |N|^2\tfrac{2cp_x}{E+mc^2} \quad \rightarrow \quad 2p_x \end{aligned}$$

De waarschijnlijkheidsdichtheid j^0 is altijd positief. Dit resulaat was uitgangspunt van Dirac's werk!

Instrinsieke spin

De E>0 en de E<0 oplossingen van de Dirac vergelijking zijn tweevoudig ontaard. Welke observabele kan deze toestanden onderscheiden?

Gebruik de Hamiltoniaan voor de Dirac vgl.: $H = \vec{\alpha} \cdot \vec{p} + \beta mc$

$\vec{L} \equiv \vec{r} \times \vec{p}$

baanimpulsmoment

$$[H,\vec{L}] = [\vec{\alpha}\cdot\vec{p} + \beta mc, \vec{r}\times\vec{p}] = \alpha_l [p_l, \vec{r}\times\vec{p}] = \alpha_l p_l (\vec{r}\times\vec{p}) - \alpha_l (\vec{r}\times\vec{p}) p_l$$

$$= \alpha_l p_l \varepsilon_{ijk} r_j p_k - \alpha_l \varepsilon_{ijk} r_j p_k p_l \qquad \text{Gebruik: } p_l r_j = r_j p_l + \tfrac{\hbar}{i}\delta_{lj}$$

$$= \alpha_l \tfrac{\hbar}{i} \delta_{lj} \varepsilon_{ijk} p_k = \alpha_l \tfrac{\hbar}{i} \varepsilon_{ilk} p_k = \frac{\hbar}{i} \vec{\alpha}\times\vec{p} = -i\hbar\vec{\alpha}\times\vec{p} \neq \vec{0}$$

$[H,\vec{L}] \neq 0$

$\vec{L}$ niet behouden!

Def. $\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \Longrightarrow$

$$[H,\vec{\Sigma}] = [\vec{\alpha}\cdot\vec{p} + \beta mc, \vec{\Sigma}] = p_k [\alpha_k, \Sigma_l] = p_k \begin{pmatrix} 0 & [\sigma_k, \sigma_l] \\ -[\sigma_k, \sigma_l] & 0 \end{pmatrix}$$

$$= p_k \, 2i\varepsilon_{klm} \begin{pmatrix} 0 & \sigma_m \\ -\sigma_m & 0 \end{pmatrix} = 2i \, p_k \, \varepsilon_{klm} \, \alpha_m \equiv 2i\vec{\alpha}\times\vec{p} \neq \vec{0}$$

$[H,\vec{\Sigma}] \neq 0$

$\vec{\Sigma}$ niet behouden!

Slim combineren: $\left[H, \vec{L} + \tfrac{1}{2}\hbar\vec{\Sigma}\right] = 0$ $\quad \Rightarrow \quad \vec{J} \equiv \vec{L} + \tfrac{1}{2}\hbar\vec{\Sigma}$ behouden!

Wat levert $\frac{1}{2}\hbar\vec{\Sigma}$ op ψ?

voor *p=0*

$$\left(\frac{1}{2}\vec{\Sigma}\right)^2\psi = \frac{3}{4}\psi \sim s(s+1)\psi \quad\longrightarrow\quad s=\frac{1}{2}$$

deeltjes met spin=1/2

$\frac{1}{4}(1+1+1) = \frac{3}{4}$

z-component van de spin: ½ Σ_3

$$\frac{1}{2}\Sigma_3\psi = \begin{cases} \psi^{(1)}: & +\frac{1}{2}\times\psi^{(1)} \\ \psi^{(2)}: & -\frac{1}{2}\times\psi^{(2)} \\ \psi^{(3)}: & +\frac{1}{2}\times\psi^{(3)} \\ \psi^{(4)}: & -\frac{1}{2}\times\psi^{(4)} \end{cases}$$

voor *p≠0*

$$\left(\frac{1}{2}\vec{\Sigma}\right)^2\psi = \frac{3}{4}\psi \sim s(s+1)\psi \quad\longrightarrow\quad s=\frac{1}{2}$$

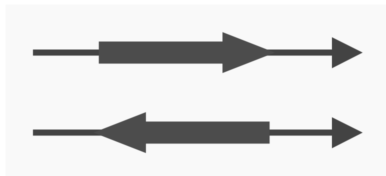
deeltjes met spin=1/2

$[H,\Sigma^2]=0$ maar $[H,\Sigma]\neq 0$ en dus:
ψ i.h.a. geen eigenvector van Σ_3

Spin component // p kan altijd gebruikt worden: $1/2\vec{\Sigma}\cdot\hat{p}$

+½: positieve heliciteit
–½: negatieve heliciteit

Want: $$\left[H,\vec{\Sigma}\cdot\hat{\vec{p}}\right] = \hat{\vec{p}}\cdot\left[H,\vec{\Sigma}\right] = \hat{\vec{p}}\cdot 2i\vec{\alpha}\times\vec{p} = 0$$



De operator $1/2\vec{\Sigma}\cdot\hat{p}$ heet heliciteit; eigenwaarden zijn ±1/2

Spinoren met heliciteit ±1/2

Deeltje spinoren

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

Welke lineaire combinatie zijn ook eigentoestanden van de heliciteit: $1/2\vec{\Sigma}\cdot\hat{p}$

Los op $$\frac{1}{2}\vec{\Sigma}\cdot\hat{p}\times\left(\alpha u^{(1)}(p)+\beta u^{(2)}(p)\right) = \pm\frac{1}{2}\times\left(\alpha u^{(1)}(p)+\beta u^{(2)}(p)\right)$$ voor α en β

$$\vec{\Sigma}\cdot\hat{p} = \frac{1}{|\vec{p}|}\times\begin{pmatrix} +p_z & p_x-ip_y & 0 & 0 \\ p_x+ip_y & -p_z & 0 & 0 \\ 0 & 0 & +p_z & p_x-ip_y \\ 0 & 0 & p_x+ip_y & -p_z \end{pmatrix} \Rightarrow \alpha\begin{pmatrix} +p_z \\ p_x+ip_y \\ \cdots \\ \cdots \end{pmatrix} + \beta\begin{pmatrix} p_x-ip_y \\ -p_z \\ \cdots \\ \cdots \end{pmatrix} = \pm\alpha\begin{pmatrix} |\vec{p}| \\ 0 \\ \cdots \\ \cdots \end{pmatrix} \pm\beta\begin{pmatrix} 0 \\ |\vec{p}| \\ \cdots \\ \cdots \end{pmatrix}$$

Daarom
$$\begin{cases} \alpha p_z+\beta(p_x-ip_y)=\pm\alpha|\vec{p}| \\ \alpha(p_x+ip_y)-\beta p_z=\pm\beta|\vec{p}| \\ \cdots \\ \cdots \end{cases} \Rightarrow \begin{cases} \beta=p_x+ip_y \\ \alpha=p_z\pm|\vec{p}| \end{cases}$$

$$u^{(+)}(p) \propto \begin{pmatrix} p_z+|\vec{p}| \\ p_x+ip_y \\ \cdots \\ \cdots \end{pmatrix} \qquad u^{(-)}(p) \propto \begin{pmatrix} p_z-|\vec{p}| \\ p_x+ip_y \\ \cdots \\ \cdots \end{pmatrix}$$

Lorentz transformaties

Let op: dit geldt voor alle Lorentz transformaties!

Waarnemer S en S′ ($x'^{\mu}=a^{\mu}{}_{\nu}x^{\nu}$) gebruiken dezelfde Dirac vgl.

$$\begin{cases} S: & i\hbar\gamma^{\mu}\partial_{\mu}\psi(x) - mc\psi(x) = 0 \\ S': & i\hbar\gamma^{\mu}\partial'_{\mu}\psi'(x') - mc\psi'(x') = 0 \end{cases}$$

Lineaire relatie tussen ψ en ψ′ $\Longrightarrow$

$$\begin{cases} \psi'(x') = S(a)\psi(x) \quad \text{en} \\ \psi(x) = S^{-1}(a)\psi'(x') = S(a^{-1})\psi'(x') \end{cases}$$

$p \cdot x = p' \cdot x'$

Uit $\psi(x) = u(p)e^{-\frac{i}{\hbar}p\cdot x} \sim u(p) \times \text{Lorentz invariante uitdrukking}$ volgt

S(a) hangt niet expliciet af van x: S(a) $\longrightarrow$ S_L

Dus

$$\begin{aligned} 0 &= i\hbar\gamma^{\mu}\partial'_{\mu}\psi'(x') - mc\psi'(x') \\ &= i\hbar\gamma^{\mu}a_{\mu}^{\ \nu}\partial_{\nu}S_L\psi(x) - mcS_L\psi(x) \Leftrightarrow \\ 0 &= i\hbar S_L^{-1}\gamma^{\mu}a_{\mu}^{\ \nu}S_L\partial_{\nu}\psi(x) - mc\psi(x) \Rightarrow \\ &\sim i\hbar\gamma^{\nu}\partial_{\nu}\psi(x) - mc\psi(x) \Rightarrow \qquad (\times a^{\lambda}{}_{\nu}) \\ \gamma^{\nu} &= S_L^{-1}\gamma^{\mu}a_{\mu}^{\ \nu}S_L \Rightarrow \end{aligned}$$

$$a^{\lambda}{}_{\nu}\gamma^{\nu} = S_L^{-1}\gamma^{\lambda}S_L$$

$$\text{gebruikt}: \begin{cases} \partial'_{\mu} = a_{\mu}{}^{\nu}\partial_{\nu} \\ a_{\kappa}{}^{\mu}a^{\lambda}{}_{\mu} = \delta^{\lambda}_{\kappa} \end{cases}$$

Infinitesimale transformaties: $\Delta^{\mu\nu} = -\Delta^{\nu\mu}$

(gewoon verifiëren; sorry)

$$a^\mu{}_\nu = \delta^\mu{}_\nu + \Delta^\mu{}_\nu \quad \Rightarrow \quad S_L = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta^{\mu\nu}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = -\sigma^{\nu\mu}$$

Expliciete uitdrukkingen

$$\sigma^{0k} = \tfrac{i}{2}\left(\gamma^0\gamma^k - \gamma^k\gamma^0\right) = i\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{Boosts: } k = 1,2,3$$

$$\sigma^{ij} = \tfrac{i}{2}\left(\gamma^i\gamma^j - \gamma^j\gamma^i\right) = \tfrac{i}{2}\begin{pmatrix} -[\sigma_i,\sigma_j] & 0 \\ 0 & -[\sigma_i,\sigma_j] \end{pmatrix} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad \text{Rotaties: } (ij) = (12),(23),(31)$$

Met $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ volgt voor S_L^{-1}

γ^5 is een handige definitie voor de zwakke wisselwerking

$$S_L^{-1} = \left(1 - \frac{i}{4}\sigma_{\mu\nu}\Delta^{\mu\nu}\right)^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\Delta^{\mu\nu} = \gamma^0 S_L^\dagger \gamma^0 \quad \text{en} \quad \gamma^5 S_L = S_L \gamma^5$$

Specifiek geval: pariteit $S_P = S_P^\dagger = S_P^{-1}$

Bedenk: $a^\mu{}_\nu = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$S_P^{-1}\gamma^\mu S_P = a^\mu{}_\nu \gamma^\nu \quad \Rightarrow \quad \begin{cases} S_P^{-1}\gamma^0 S_P = +\gamma^0 \\ S_P^{-1}\gamma^k S_P = -\gamma^k \end{cases} \quad \Rightarrow \quad S_P = \gamma^0 \quad \text{en} \quad \gamma^5 S_P = -S_P\gamma^5$$

Spinoren: hoe transformeren ze?

Nu kun je volgende belangrijke relatie afleiden

$$\boxed{\bar{\psi}' = \psi'^{\dagger}\gamma^0 = \psi^{\dagger}S^{\dagger}\gamma^0 = \psi^{\dagger}\gamma^0 S^{-1} = \bar{\psi}S^{-1}}$$

En daarmee

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\psi'\psi' = \psi\, S^{-1}S\,\psi = \psi\,\psi \qquad \text{scalar}$$

$$\psi'\gamma^5\psi' = \psi\, S^{-1}\gamma^5 S\,\psi = \begin{cases} L: & +\psi\,\gamma^5\psi \\ P: & -\psi\,\gamma^5\psi \end{cases} \qquad \text{pseudo-scalar}$$

$$\psi'\gamma^{\mu}\psi' = \psi\, S^{-1}\gamma^{\mu}S\,\psi = \psi\, a^{\mu}{}_{\nu}\gamma^{\nu}\psi = a^{\mu}{}_{\nu}\psi\,\gamma^{\nu}\psi \qquad \text{vector}$$

$$\psi'\gamma^5\gamma^{\mu}\psi' = \psi\, S^{-1}\gamma^5\gamma^{\mu}S\,\psi = \begin{cases} L: & +a^{\mu}{}_{\nu}\psi\,\gamma^5\gamma^{\nu}\psi \\ P: & -a^{\mu}{}_{\nu}\psi\,\gamma^5\gamma^{\nu}\psi \end{cases} \qquad \text{pseudo-vector}$$

$$\psi'\sigma^{\mu\nu}\psi' = \cdots \qquad \text{tensor}$$

4x4=16 mogelijkheden hiermee geordend!

- γ^{μ} geen 4-vector; $\psi\,\gamma^{\mu}\psi$ wel!
- γ^5 belangrijk voor zwakke wisselwerking

Rotatie van een spinor rond z-as

z-as rotaties via J_3 operator

$$e^{-i\omega J_3} \rightarrow e^{-i\omega/2\sigma_3} = \cos(\omega/2) + i\sigma_3 \sin(\omega/2)$$

Voor Dirac spinoren

$$J_3 \rightarrow \tfrac{1}{2}\Sigma_3 = \tfrac{1}{2}\sigma_{12} = \tfrac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad \text{en} \quad \Sigma_3^2 = +1$$

$$\Rightarrow \begin{aligned} S_R &= \cos\tfrac{\omega}{2} + i\Sigma_3 \sin\tfrac{\omega}{2} = \cos\tfrac{\omega}{2} + \tfrac{i^2}{2}(\gamma^1\gamma^2 - \gamma^2\gamma^1)\sin\tfrac{\omega}{2} = \cos\tfrac{\omega}{2} + i\sigma^{12}\sin\tfrac{\omega}{2} \\ &\rightarrow 1 + i\sigma^{12}\tfrac{\omega}{2} \sim 1 + \tfrac{i}{2}\sigma^{12}\Delta_{12}, \quad ; \quad \Delta_{12} = -\Delta_{21} = \omega \\ &= 1 + \tfrac{i}{4}\left(\sigma^{12}\Delta_{12} + \sigma^{21}\Delta_{21}\right) \sim 1 + \tfrac{i}{4}\sigma^{\mu\nu}\Delta_{\mu\nu} \end{aligned}$$

Vgl. eerder gevonden S_L

$\vec{p} = \vec{0}$: niets

$\vec{p} \neq \vec{0}$: b.v. $\vec{p} = (p,0,0)$

$$\omega = \pi \quad \Rightarrow \quad \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p}{E+m} \end{pmatrix} e^{-iEt+ipx} \rightarrow \sqrt{E+m}\begin{pmatrix} i \\ 0 \\ 0 \\ \frac{-ip}{E+m} \end{pmatrix} e^{-iEt-ipx}$$

$$\omega = \tfrac{\pi}{2} \quad \Rightarrow \quad \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p}{E+m} \end{pmatrix} e^{-iEt+ipx} \rightarrow \sqrt{E+m}\begin{pmatrix} \frac{1+i}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1-i}{\sqrt{2}}\frac{p}{E+m} \end{pmatrix} e^{-iEt+ipy}$$

Boost van een spinor langs z-as

Voor Dirac spinoren

$$K_3 \to \frac{1}{2}\sigma_{03} = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \quad \text{en} \quad \sigma_{03}^2 = -1$$

$$\Rightarrow \quad S_L = \cosh(\omega/2) - i\sigma^{03}\sinh(\omega/2) = \begin{pmatrix} \cosh(\omega/2) & \sinh(\omega/2)\sigma_3 \\ \sinh(\omega/2)\sigma_3 & \cosh(\omega/2) \end{pmatrix}$$

Gonio:

$$\left.\cosh x = 2\cosh^2(x/2) - 1\right\} \Rightarrow \begin{cases} \cosh(\omega/2) = \sqrt{(1+\cosh\omega)/2} = \sqrt{\frac{E+mc^2}{2mc^2}} \\ \sinh(\omega/2) = \sqrt{\cosh^2(\omega/2) - 1} = \frac{p_3 c}{\sqrt{2mc^2(E+mc^2)}} \end{cases}$$

Dus

$$S_L \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}} & \frac{p_3 c}{\sqrt{2mc^2(E+mc^2)}}\sigma_3 \\ \frac{p_3 c}{\sqrt{2mc^2(E+mc^2)}}\sigma_3 & \sqrt{\frac{E+mc^2}{2mc^2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3 c}{E+mc^2} \\ 0 \end{pmatrix}$$

Met ook nog de e-macht zie je dus dat deze boost precies ψ geeft voor $p \neq 0$

$$\psi^{(1)}(x) = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \to \psi^{(1)}(x') = \sqrt{E'+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ 0 \end{pmatrix} e^{-ip'\cdot x'}$$

Inversie van een spinor in oorsprong

$$\Rightarrow \boxed{\psi \to \psi' = S_P \psi \quad \text{met} \quad S_P = \gamma^0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Want met $S_P \equiv \gamma^0$ voldaan aan :

$$a^\lambda{}_\nu \gamma^\nu = S_L^{-1} \gamma^\lambda S_L \to a^\lambda{}_\nu \gamma^\nu = S_P^{-1} \gamma^\lambda S_P = \begin{cases} +\gamma^0 \\ -\gamma^k \end{cases}$$

Kijk wat dit doet op:

fermion toestanden ($\psi^{(1)}$ en $\psi^{(2)}$)

antifermion toestanden ($\psi^{(3)}$ en $\psi^{(4)}$)

$$\boxed{\begin{cases} \psi^{(1-2)} \to S_P \psi^{(1-2)} = \gamma^0 \psi^{(1-2)} = +1 \times \psi^{(1-2)} \\ \psi^{(3-4)} \to S_P \psi^{(3-4)} = \gamma^0 \psi^{(3-4)} = -1 \times \psi^{(3-4)} \end{cases}}$$

En hiermee zie je dus:

intrinsieke pariteit fermion en anti-fermion toestanden tegengesteld

Ladingsconjugatie

Dirac vgl. elektron (lading −|e|) in e.m. veld $[\gamma^\mu(p_\mu + eA_\mu) - m]\,\psi = 0$

Dirac vgl. positron (lading +|e|) in e.m. veld $[\gamma^\mu(p_\mu - eA_\mu) - m]\,\psi_C = 0$

Er moet een relatie tussen ψ en ψ_C bestaan omdat we de "gewone" Dirac vgl. zullen gebruiken voor zowel elektronen als positronen. Deze relatie vind je zo:

$$\begin{aligned} 0 &= [\gamma^\mu(p_\mu + eA_\mu) - m]\,\psi \\ &= [\gamma^\mu(i\partial_\mu + eA_\mu) - m]\,\psi \Leftrightarrow \\ 0 &= [-\gamma^{\mu *}(i\partial_\mu - eA_\mu) - m]\,\psi^* \end{aligned}$$

Vind C die voldoet aan $-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu(C\gamma^0)$

Dan vind je uit de elektron Dirac vgl. de positron Dirac vgl.

$$\begin{aligned} 0 &= [-\gamma^{\mu *}(i\partial_\mu - eA_\mu) - m]\,\psi^* \Rightarrow \\ 0 &= (C\gamma^0) \times [-\gamma^{\mu *}(i\partial_\mu - eA_\mu) - m]\,\psi^* \\ &= [\gamma^\mu(i\partial_\mu - eA_\mu) - m]\,(C\gamma^0)\psi^* \end{aligned}$$

Relatie tussen elektron en positron oplossingen $\psi_C = (C\gamma^0)\psi^* = C\bar{\psi}^T$

Elektron & positron oplossingen

Expliciete uitdrukking $C\gamma^0$

$$C\gamma^0 = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

Dit voor mijn keuze van de γ^μ matrixes! Relaties elektron-positron oplossingen:

$$\begin{aligned}
\psi^{(1)} &= u^{(1)}e^{-ip\cdot x} &\rightarrow\quad (C\gamma^0)\psi^{(1)*} &= i\gamma^2\psi^{(1)*} = +u^{(4)}(-p)e^{+ip\cdot x} \equiv v^{(1)}(p)e^{+ip\cdot x} \\
\psi^{(2)} &= u^{(2)}e^{-ip\cdot x} &\rightarrow\quad (C\gamma^0)\psi^{(2)*} &= i\gamma^2\psi^{(2)*} = -u^{(3)}(-p)e^{+ip\cdot x} \equiv v^{(2)}(p)e^{+ip\cdot x} \\
\psi^{(3)} &= u^{(3)}e^{+ip\cdot x} &\rightarrow\quad (C\gamma^0)\psi^{(3)*} &= i\gamma^2\psi^{(3)*} = -u^{(2)}(-p)e^{-ip\cdot x} \\
\psi^{(4)} &= u^{(4)}e^{+ip\cdot x} &\rightarrow\quad (C\gamma^0)\psi^{(4)*} &= i\gamma^2\psi^{(4)*} = +u^{(1)}(-p)e^{-ip\cdot x}
\end{aligned}$$

deeltjes (elektronen)	$u^{(1-2)}(p)$	$0 = (\gamma_\mu p^\mu - m)u^{(1-2)}(p) = (\not{p} - m)u^{(1-2)}(p)$
anti-deeltjes (positronen)	$v^{(1-2)}(p)$	$0 = (\gamma_\mu p^\mu + m)v^{(1-2)}(p) = (\not{p} + m)v^{(1-2)}(p)$

Gebruiken:
"u" spinoren
"v" spinoren:

$$u^{(1)}(E,\vec{p}) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+m} \\ \frac{(p_x+ip_y)}{E+m} \end{pmatrix} \qquad v^{(1)}(E,\vec{p}) \equiv +u^{(4)}(-E,-\vec{p}) = +\sqrt{E+m}\begin{pmatrix} \frac{(p_x-ip_y)}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

$$u^{(2)}(E,\vec{p}) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \qquad v^{(2)}(E,\vec{p}) \equiv -u^{(3)}(-E,-\vec{p}) = -\sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{(p_x+ip_y)}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Normalisatie, orthogonaliteit en compleetheid

Normalisatie

$$\begin{cases} \bar{u}u = u^\dagger \gamma^0 u = u^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u = +2m \\ \\ \bar{v}v = v^\dagger \gamma^0 v = v^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = -2m \end{cases}$$

Orthogonaliteit

$$\begin{cases} \bar{u}^{(1)} u^{(2)} = u^{(1)\dagger} \gamma^0 u^{(2)} = 0 \\ \\ \bar{v}^{(1)} v^{(2)} = v^{(1)\dagger} \gamma^0 v^{(2)} = 0 \end{cases}$$

Compleetheid

$$\begin{cases} \displaystyle\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = u^{(1)} \bar{u}^{(1)} + u^{(2)} \bar{u}^{(2)} = \gamma^\mu p_\mu + m \equiv \not{p} + m \\ \\ \displaystyle\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = v^{(1)} \bar{v}^{(1)} + v^{(2)} \bar{v}^{(2)} = \gamma^\mu p_\mu - m \equiv \not{p} - m \end{cases}$$

Spin ½ Elektrodynamica

Ijkinvariantie en Dirac vergelijking

minimale substitutie
(via lokale ijkinvariantie)

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$$
$$\vec{p} = -i\vec{\nabla}; \qquad E = i\partial/\partial t$$

$$\Longrightarrow \underline{i\partial_\mu \to i\partial_\mu - qA_\mu}$$

$$A_\mu = (\phi, -\vec{A})$$
$$\partial_\mu = (\partial/\partial t, +\vec{\nabla})$$

Dirac vergelijking

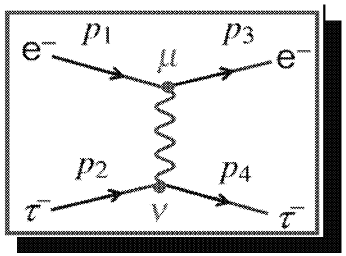
$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad \Longrightarrow \quad \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \quad \Longrightarrow \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma}.\vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = \gamma^0 \hat{H}\psi \quad = \quad m\psi - i\vec{\gamma}.\vec{\nabla}\psi + q\gamma^\mu A_\mu \psi$$

$$\times\gamma^0 : \qquad \hat{H}\psi \quad = \quad \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}.\vec{\nabla})}_{\text{Massa + KE}}\psi + q\underbrace{\gamma^0\gamma^\mu A_\mu}_{\text{Interactie: potentiële energie}}\psi$$

$$\hat{V}_D = q\gamma^0\gamma^\mu A_\mu$$



$$M = \langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - m_x^2} \langle \psi_d | V | \psi_b \rangle$$

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r} - Et)}$$
$$\sum_\lambda \varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^* = -g_{\mu\nu}$$

$$\boxed{M = [\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)]}$$

Dirac vergelijking en A^μ

Vrije Dirac-veld (elektronen) $\begin{cases} (\gamma_\mu p^\mu - m)\psi &= 0 \\ \psi &= u(p)e^{-ip\cdot x} \end{cases}$

Minimale substitutie (q_e=-e): $p^\mu \rightarrow p^\mu + eA^\mu$

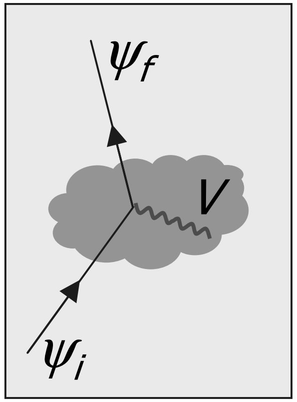
$$\begin{aligned} (\gamma_\mu[p^\mu + eA^\mu] - m)\psi &= 0 \Leftrightarrow \\ (\gamma_\mu p^\mu - m)\psi &= -e\gamma_\mu A^\mu \psi \equiv \gamma^0 V\psi \quad \text{waarbij} \quad \underline{V \equiv -e\gamma^0\gamma_\mu A^\mu} \end{aligned}$$

De overgangsamplitude wordt dus

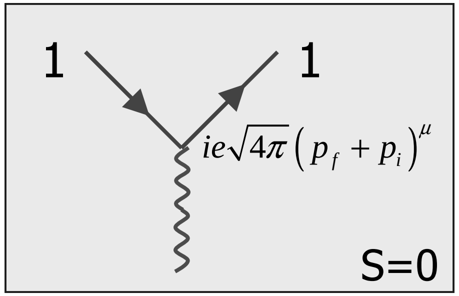
$$\begin{aligned} T_{fi} &= -i\int \psi_f^\dagger V \psi_i d^4x \\ &= -i\int \psi_f^\dagger[-e\gamma^0\gamma_\mu A^\mu]\psi_i d^4x \\ &= (ie)\int \bar{\psi}_f\gamma_\mu A^\mu \psi_i d^4x \\ &\equiv -i\int j_\mu^{fi} A^\mu d^4x \end{aligned}$$

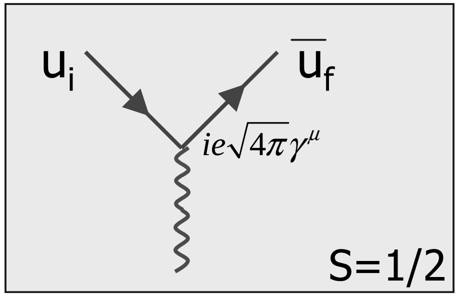
met:

$$\begin{aligned} j_\mu^{fi} &= -e\bar{\psi}_f\gamma_\mu\psi_i \\ &= -e\bar{u}_f\gamma_\mu u_i e^{i(p_f-p_i)\cdot x} \end{aligned}$$

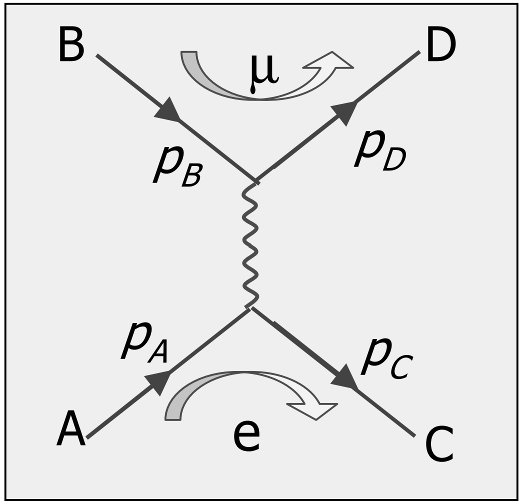


vergelijk nu de vertex factoren:





$e^-\mu^- \to e^-\mu^-$ verstrooiing



Relatie tussen stroom en vectorveld

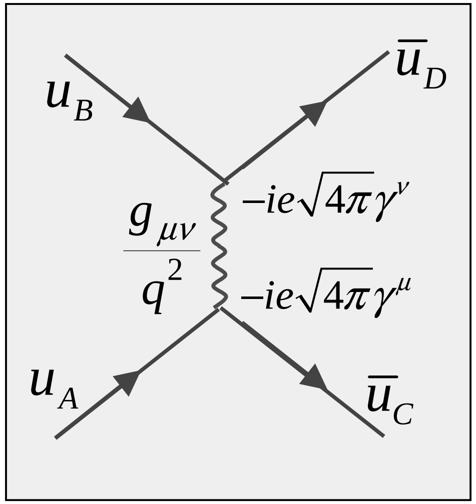
$$AC \text{ overgangsstroom} \qquad j_\mu^{CA} = -e\bar{u}_C\gamma_\mu u_A \times e^{i(p_C-p_A)\cdot x}$$

$$BD \text{ overgangsstroom} \qquad j_\mu^{DB} = -e\bar{u}_D\gamma_\mu u_B \times e^{i(p_D-p_B)\cdot x}$$

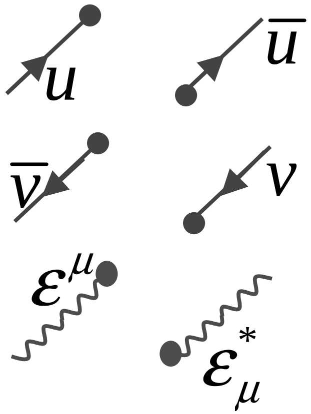
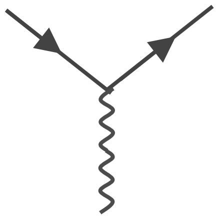
$$A^\mu \text{ t.g.v. } j_{DB}^\mu \qquad \Box A^\mu = 4\pi j_{DB}^\mu \Rightarrow A^\mu = \frac{-4\pi}{q^2} \times j_{DB}^\mu$$

En de overgangsamplitude wordt dus ($q=p_A-p_C=p_D-p_B$)

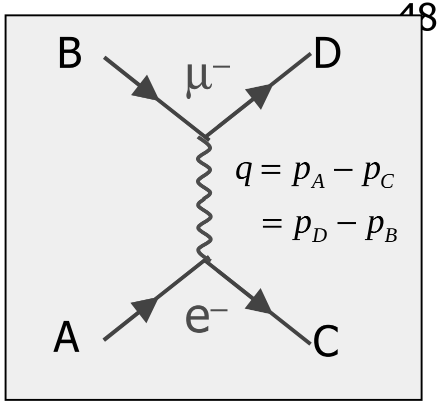
$$\begin{aligned} T_{fi} &= -i\int j_\mu^{CA} \times \frac{-4\pi}{q^2} j_{DB}^\mu d^4x \\ &= -i\int [-e\bar{u}_C\gamma_\mu u_A \times e^{i(p_C-p_A)\cdot x}] \times \frac{-4\pi}{q^2} \times [-e\bar{u}_D\gamma^\mu u_B \times e^{i(p_D-p_B)\cdot x}] d^4x \\ &= -i[-e\bar{u}_C\gamma_\mu u_A] \times \frac{-4\pi}{q^2} \times [-e\bar{u}_D\gamma^\mu u_B] \times (2\pi)^4\delta^4(p_A+p_B-p_C-p_D) \end{aligned}$$



Feynman regels voor QED (S=1/2)

Externe lijnen	Deeltje Anti-deeltje Foton		$\begin{cases} \text{inkomend:} & u \\ \text{uitgaand:} & \bar{u} \end{cases}$ $\begin{cases} \text{inkomend:} & \bar{v} \\ \text{uitgaand:} & v \end{cases}$ $\begin{cases} \text{inkomend:} & \epsilon_\mu \\ \text{uitgaand:} & \epsilon'^{*}_{\mu} \end{cases}$
Vertices			$-i(\sqrt{4\pi}e)\gamma_\mu$
Propagatoren	Deeltje		$\dfrac{i\left(\gamma_\mu q^\mu + m\right)}{q^2 - m^2}$
	Foton		$\dfrac{-ig_{\mu\nu}}{q^2}$

Relativistische spin



Relativistische spin $\qquad e^-\mu^- \rightarrow e^-\mu^- \quad$ (1 diagram)

$$
\begin{aligned}
-i\mathcal{M} &= [ie\sqrt{4\pi}\bar{u}_C\gamma^\mu u_A] \times \frac{-ig_{\mu\nu}}{(p_A-p_C)^2} \times [ie\sqrt{4\pi}\bar{u}_D\gamma^\nu u_B] \Rightarrow \\
\mathcal{M} &= -4\pi e^2[\bar{u}_C\gamma^\mu u_A] \times \frac{1}{q^2} \times [\bar{u}_D\gamma_\mu u_B] \qquad q = p_A - p_C
\end{aligned}
$$

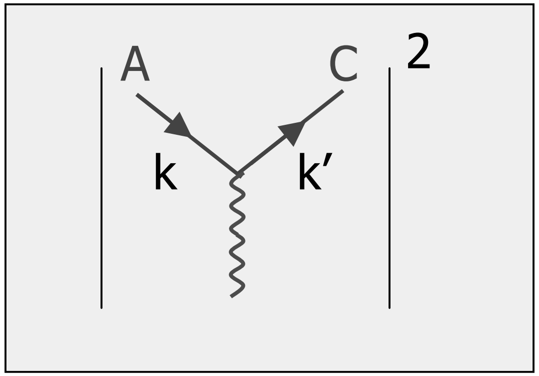
Voor |M|^2 volgt dus

$$
\begin{aligned}
|\mathcal{M}|^2 &= \left(\frac{4\pi e^2}{q^2}\right)^2 \times [\bar{u}_C\gamma^\mu u_A][\bar{u}_C\gamma^\nu u_A]^* \times [\bar{u}_D\gamma_\mu u_B][\bar{u}_D\gamma_\nu u_B]^* \Rightarrow \\
|\bar{\mathcal{M}}|^2 &= \left(\frac{4\pi e^2}{q^2}\right)^2 \times \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_C\gamma^\nu u_A]^* \times \frac{1}{2}\sum_{spin}[\bar{u}_D\gamma_\mu u_B][\bar{u}_D\gamma_\nu u_B]^* \\
&\equiv \left(\frac{4\pi e^2}{q^2}\right)^2 \times L^{\mu\nu}_{elektron} \times L^{muon}_{\mu\nu}
\end{aligned}
$$

lepton tensor

$$
\boxed{
\begin{aligned}
L^{\mu\nu}_{elektron} &= \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_C\gamma^\nu u_A]^* = \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_C\gamma^\nu u_A]^\dagger \\
&= \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][u_A^\dagger\gamma^{\nu\dagger}\bar{u}_C^\dagger] = \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_A\gamma^0\gamma^{\nu\dagger}\gamma^0 u_C] \\
&= \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_A\gamma^\nu u_C]
\end{aligned}
}
$$

De elektron lepton tensor



$$
\begin{aligned}
L^{\mu\nu}_{elektron} &= \frac{1}{2}\sum_{spin}[\bar{u}_C\gamma^\mu u_A][\bar{u}_A\gamma^\nu u_C] \rightarrow \frac{1}{2}\sum_{spin}[\bar{u}(k')\gamma^\mu u(k)][\bar{u}(k)\gamma^\nu u(k')] \\
&= \frac{1}{2}\sum_{spin}[\bar{u}(k')_{\alpha'}\gamma^\mu_{\alpha'\alpha}u(k)_\alpha][\bar{u}(k)_\beta\gamma^\nu_{\beta\beta'}u(k')_{\beta'}] \\
&= \frac{1}{2}\underbrace{\left[\sum_s u(k)^{(s)}_\alpha \bar{u}(k)^{(s)}_\beta\right]}_{(\not{k}+m)_{\alpha\beta}} \times \underbrace{\left[\sum_{s'} u(k')^{(s')}_{\beta'}\bar{u}(k')^{(s')}_{\alpha'}\right]}_{(\not{k}'+m)_{\beta'\alpha'}} \times \gamma^\mu_{\alpha'\alpha}\gamma^\nu_{\beta\beta'} \\
&= \frac{1}{2}\times(\not{k}'+m)_{\beta'\alpha'}\times\gamma^\mu_{\alpha'\alpha}(\not{k}+m)_{\alpha\beta}\times\gamma^\nu_{\beta\beta'} \\
&= \frac{1}{2}\mathrm{Tr}\;[(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\nu]
\end{aligned}
$$

$$\not{k} \equiv k^\mu \gamma_\mu$$

Hoewel je van deze uitdrukking op het eerste gezicht niet vrolijk wordt, geldt wel

1) Geen spinoren meer: via compleetheid relaties
2) De spoorberekening is rechtoe-rechtaan: m.b.v. enkele regels

\`Casimir’s trick’

Sporen met γ-matrices

Gebruik altijd:

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

producten van γ matrices

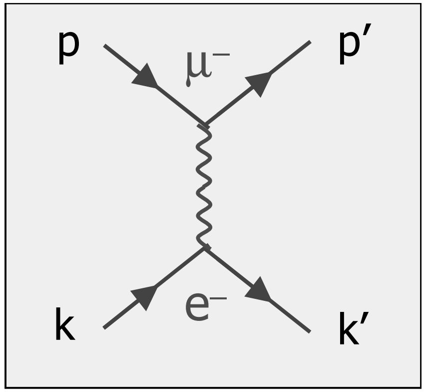
1. $\gamma^\mu\gamma_\mu = 4$
2. $\gamma^\mu \not{a}\gamma_\mu = \gamma^\mu\gamma^\alpha a_\alpha\gamma_\mu = -4\gamma^\alpha a_\alpha + 2g^{\mu\alpha}a_\alpha\gamma_\mu = -2\not{a}$
3. $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b$
4. $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$

sporen van producten van γ matrices

1. $\mathrm{Tr}\ 1 = 4$
2. Tr van *oneven* aantal γ matrices identiek 0
3. $\mathrm{Tr}\ \not{a}\not{b} = \mathrm{Tr}\ \gamma^\alpha\gamma^\beta a_\alpha b_\beta = -\mathrm{Tr}\ \not{a}\not{b} + \mathrm{Tr}\ 2a\cdot b \Rightarrow$
   $\mathrm{Tr}\ \not{a}\not{b} = \mathrm{Tr}\ a\cdot b = 4a\cdot b$
4. $\mathrm{Tr}\ \not{a}\not{b}\not{c}\not{d} = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$
5. $\mathrm{Tr}\ \gamma^5 = \mathrm{Tr}\ i\gamma^0\gamma^1\gamma^2\gamma^3 = 0$
6. $\mathrm{Tr}\ \gamma^5\not{a}\not{b} = 0$
7. $\mathrm{Tr}\ \gamma^5\not{a}\not{b}\not{c}\not{d} = 4i\epsilon_{\alpha\beta\gamma\delta}\times a^\alpha b^\beta c^\gamma d^\delta$

Zwakke wisselwerking

Voorbeeld $e^-\mu^- \to e^-\mu^-$



Toepassen spoor identiteiten geeft

$$
\begin{aligned}
L^{\mu\nu}_{elektron} &= \frac{1}{2}\mathrm{Tr}\ [(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] \\
&= \frac{1}{2}\left(\mathrm{Tr}\ \ \not{k}'\gamma^\mu\ \not{k}\gamma^\nu + \mathrm{Tr}\ m^2\gamma^\mu\gamma^\nu\right) \\
&= \frac{1}{2}\left(4[k'^\mu k^\nu + k'^\nu k^\mu - k'\cdot k g^{\mu\nu}] + m^2 4 g^{\mu\nu}\right) \\
&= 2\left[k'^\mu k^\nu + k'^\nu k^\mu + (m^2 - k'\cdot k)g^{\mu\nu}\right]
\end{aligned}
$$

Voor de muon tensor geldt hetzelfde

$$
L^{\mu\nu}_{muon} = 2\left[p'^\mu p^\nu + p'^\nu p^\mu + (M^2 - p'\cdot p)g^{\mu\nu}\right]
$$

Zodat

relativistische limiet: $M^2, m^2 \to 0$

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \left(\frac{4\pi e^2}{q^2}\right)^2 \times L^{\mu\nu}_{elektron} \times L_{\mu\nu}^{muon} \\
&= \left(\frac{4\pi e^2}{q^2}\right)^2 \times 8 \times \left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - \underbrace{\cancel{m^2 p'\cdot p}}_{0} - \underbrace{\cancel{M^2 k'\cdot k}}_{0} + \underbrace{\cancel{2m^2M^2}}_{0}\right] \\
&\to \left(\frac{4\pi e^2}{(k-k')^2}\right)^2 \times 8 \times [(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')]
\end{aligned}
$$

Merk op dat in de extreem relativistische limiet geldt

$$
\boxed{
\begin{aligned}
s &= (k+p)^2 \approx +2k\cdot p = +2k'\cdot p' \\
t &= (k-k')^2 \approx -2k\cdot k' \approx -2p\cdot p' \\
u &= (k-p')^2 \approx -2k\cdot p' = -2k'\cdot p
\end{aligned}
}
$$

$e^- \mu^- \rightarrow e^- \mu^-$

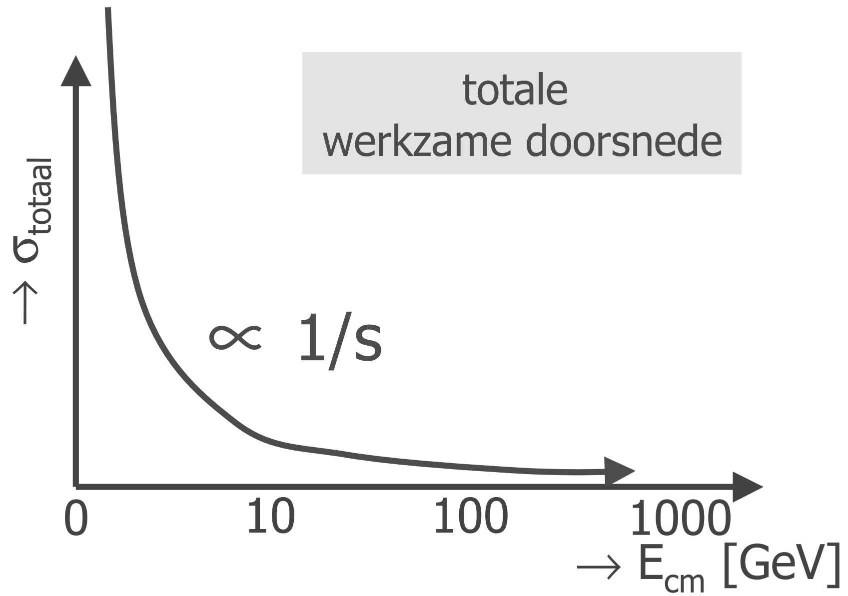
Hiermee wordt het matrix element uiteindelijk

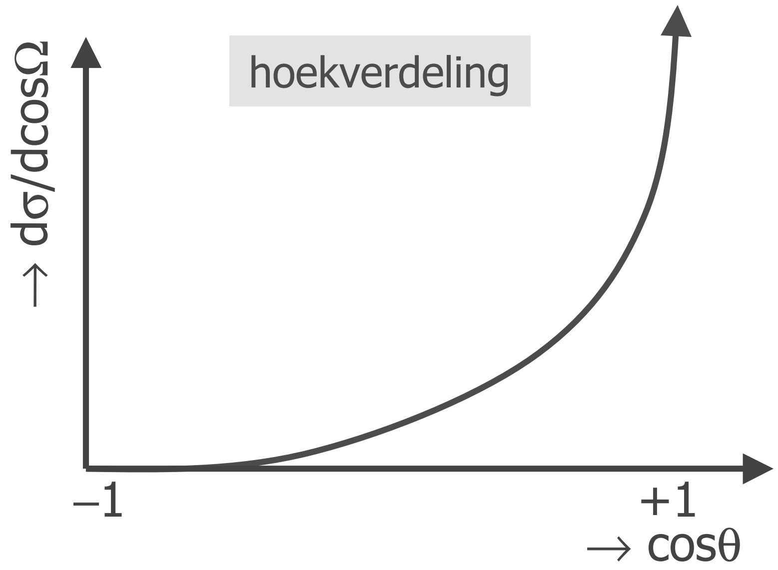
$$|\bar{\mathcal{M}}|^2 \approx (4\pi e^2)^2 \times 2 \times \frac{s^2 + u^2}{t^2}$$

$$\begin{cases} s \approx 4E^2 \\ t \approx -2E^2(1-\cos\theta) \\ u \approx -2E^2(1+\cos\theta) \end{cases}$$

En de werkzame doorsnede

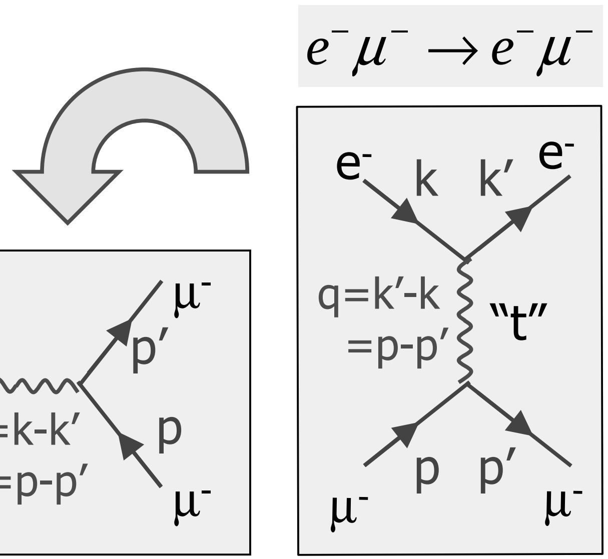
$$\frac{d\sigma}{d\Omega} \approx \frac{1}{64\pi^2 s}|\bar{\mathcal{M}}|^2 = \frac{1}{64\pi^2 s}(4\pi e^2)^2 \times 2 \times \frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s} \times \frac{s^2+u^2}{t^2}$$

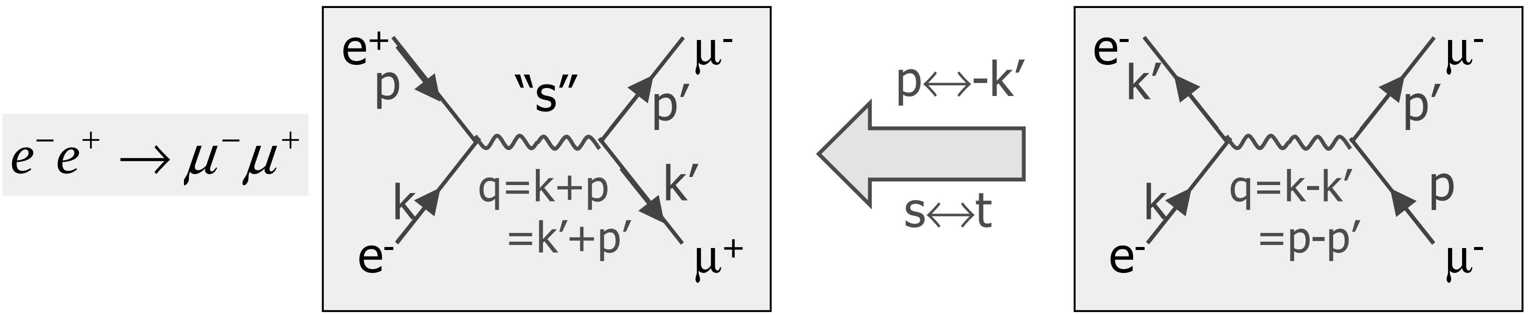




Voorbeeld (`crossing') $e^- e^+ \to \mu^- \mu^+$

er is een relatie tussen de amplituden voor $\begin{cases} e^-\mu^- \longrightarrow e^-\mu^- \\ e^-e^+ \longrightarrow \mu^-\mu^+ \end{cases}$





$e^+e^-\to\mu^+\mu^-$ amplitude $|\bar{\mathcal{M}}|^2 \approx (4\pi e^2)^2 \times 2 \times \dfrac{s^2+u^2}{t^2} \longrightarrow |\bar{\mathcal{M}}|^2 \approx (4\pi e^2)^2 \times 2 \times \dfrac{t^2+u^2}{s^2}$

en daarmee de werkzame doorsnede

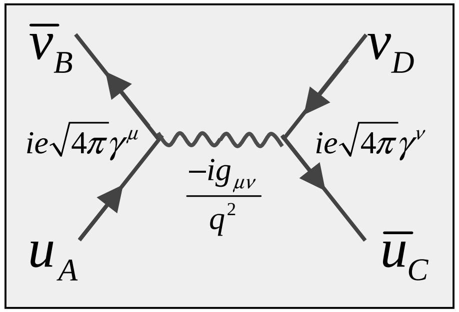
$$\frac{d\sigma}{d\Omega} \approx \frac{1}{64\pi^2 s}|\bar{\mathcal{M}}|^2 = \frac{1}{64\pi^2 s}(4\pi e^2)^2 \times 2 \times \frac{t^2+u^2}{s^2} = \frac{\alpha^2}{4s} \times 2 \times \frac{t^2+u^2}{s^2}$$

gebruik

$$\begin{cases} s \approx 4E^2 \\ t \approx -2E^2(1-\cos\theta) \\ u \approx -2E^2(1+\cos\theta) \end{cases}$$

$$\boxed{\begin{aligned} \frac{d\sigma}{d\Omega} &\approx \frac{\alpha^2}{4s} \times (1+\cos^2\theta) \\ \sigma_{tot} &= \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\alpha^2}{4s} \times 2\pi\frac{8}{3} = \frac{4\pi\alpha^2}{3s} \end{aligned}}$$

Directe berekening e⁻ e⁺ → $\mu^-\mu^+$



Je kunt het proces ook direct uitrekenen

$$
\begin{aligned}
-i\mathcal{M} &= [ie\sqrt{4\pi}\bar{v}_B\gamma^\mu u_A] \times \frac{-ig_{\mu\nu}}{(p_A+p_B)^2} \times [ie\sqrt{4\pi}\bar{u}_C\gamma^\nu v_D] \Rightarrow \\
\mathcal{M} &= -4\pi e^2[\bar{v}_B\gamma^\mu u_A] \times \frac{1}{q^2} \times [\bar{u}_C\gamma_\mu v_D] \qquad q = p_A + p_B
\end{aligned}
$$

De spin algebra geeft weer een spoor

$$
\begin{cases} P_A \to k \\ P_B \to p \\ P_C \to k' \\ P_D \to p' \end{cases}
$$

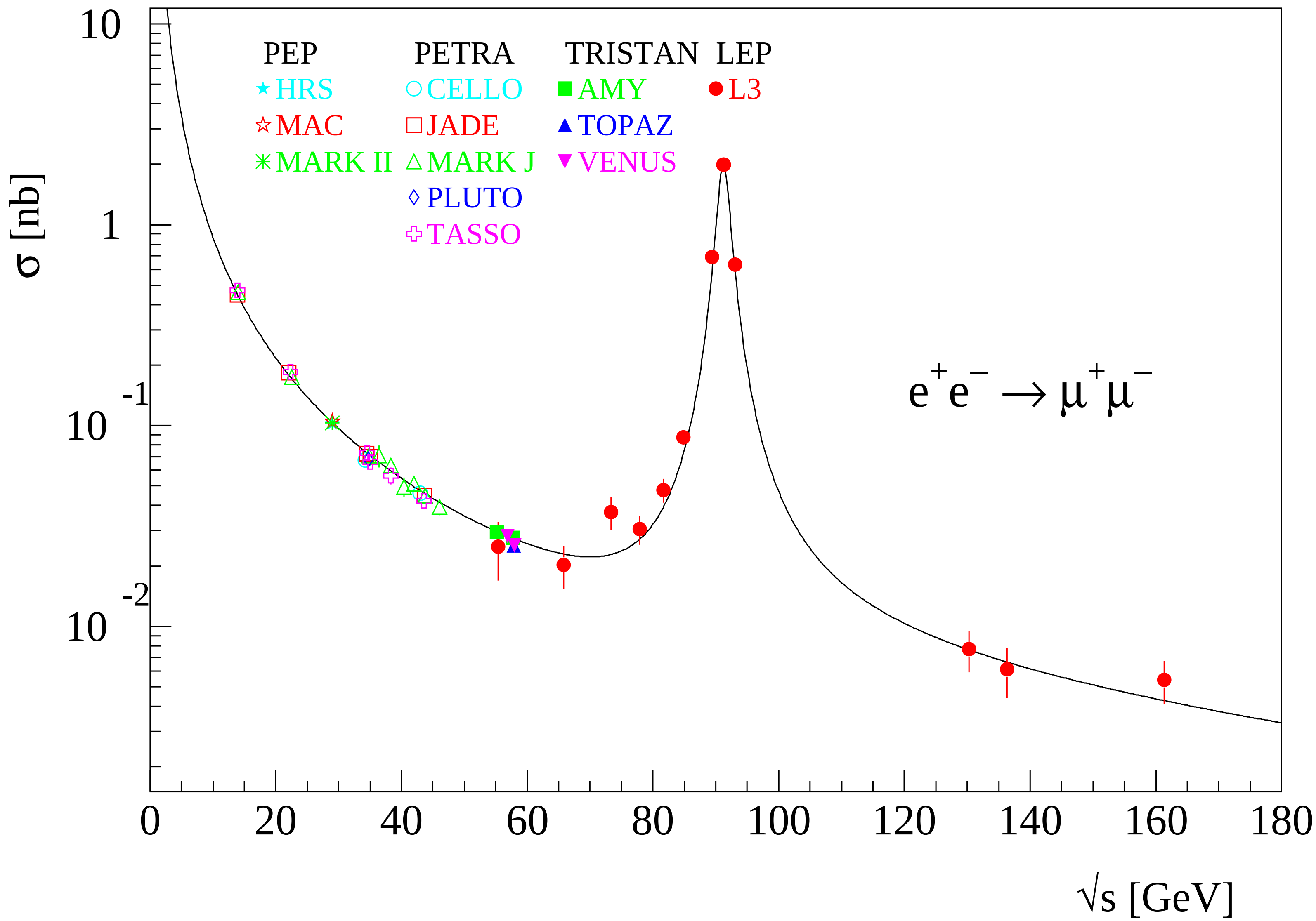
$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \left(\frac{4\pi e^2}{(k+p)^2}\right)^2 \times \frac{1}{4}\sum[\bar{v}(p)\gamma^\mu u(k)] \times [\bar{u}(k')\gamma_\mu v(p')] \times [\bar{v}(p')\gamma_\nu u(k')] \times [\bar{u}(k)\gamma^\nu v(p)] \\
&= \left(\frac{4\pi e^2}{(k+p)^2}\right)^2 \times \frac{1}{4} \times (\mathrm{Tr}\,[(\not{p} - M)\gamma^\mu(\not{k} + m)\gamma^\nu]) \times (\mathrm{Tr}\,[(\not{p}' - M)\gamma_\nu(\not{k}' + m)\gamma_\mu]) \\
&= \left(\frac{4\pi e^2}{(k+p)^2}\right)^2 \times \frac{1}{4} \times (\mathrm{Tr}\,[\not{p}\gamma^\mu\ \not{k}\gamma^\nu] - 4mMg^{\mu\nu}) \times (\mathrm{Tr}\,[\not{p}'\gamma_\nu\ \not{k}'\gamma_\mu] - 4mMg_{\mu\nu}) \\
&= \left(\frac{4\pi e^2}{(k+p)^2}\right)^2 \times 4\,(p^\mu k^\nu + p^\nu k^\mu - [p\cdot k + mM]g^{\mu\nu}) \times \left(p'_\mu k'_\nu + p'_\nu k'_\mu - [p'\cdot k' + mM]g_{\mu\nu}\right) \\
&= \left(\frac{4\pi e^2}{(k+p)^2}\right)^2 \times 4\,(2\underbrace{(p\cdot p')(k\cdot k')}_{t^2/4} + 2\underbrace{(p\cdot k')(p'\cdot k)}_{u^2/4} + \ldots\ldots)
\end{aligned}
$$

En dit wordt in de extreem relativistische limiet

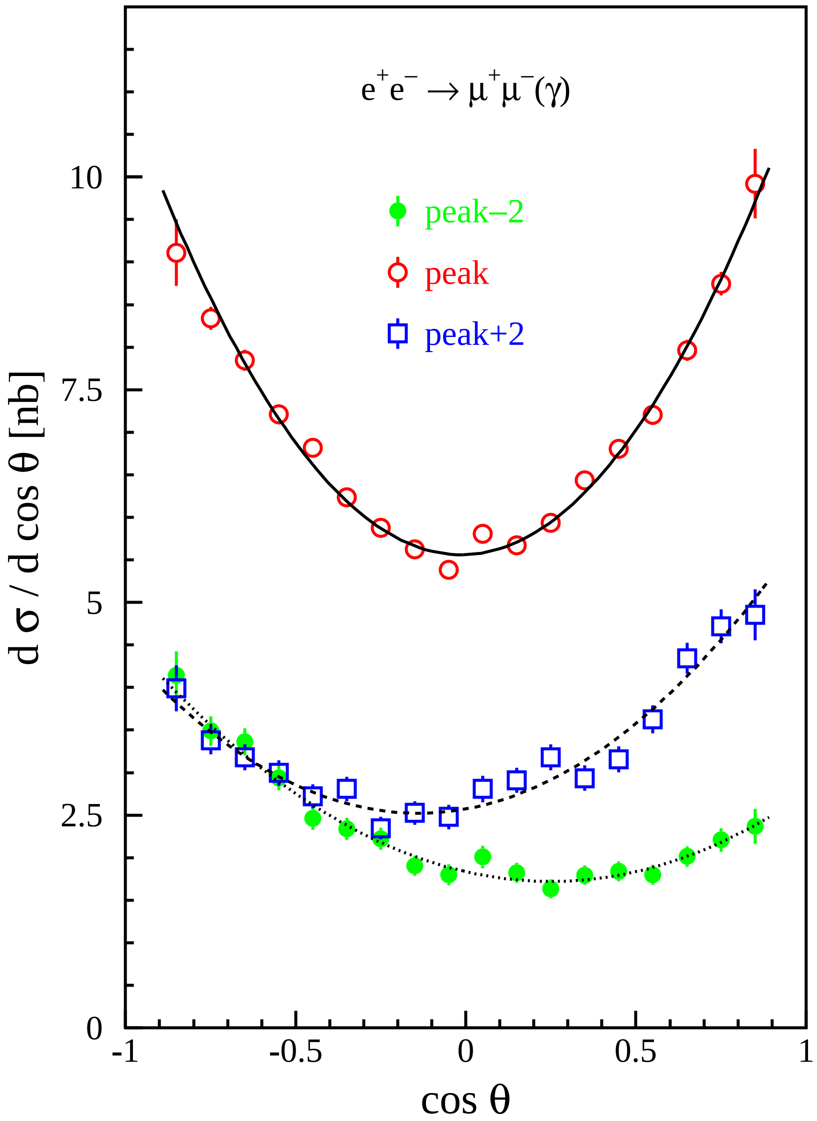
$$
\boxed{|\bar{\mathcal{M}}|^2 \approx \left(\frac{4\pi e^2}{s}\right)^2 \times 2 \times (t^2 + u^2)}
$$

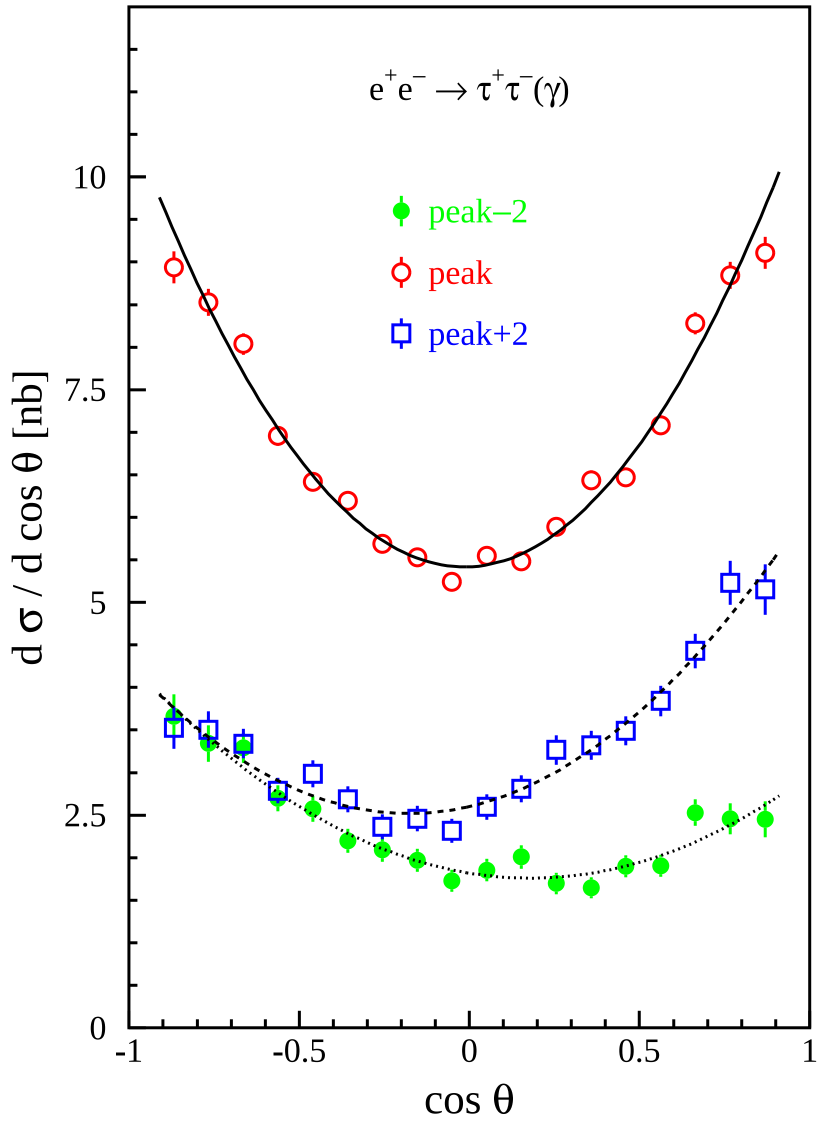
(gelijk eerdere resultaat)

Werkzame doorsnede $e^-e^+ \to \mu^-\mu^+$



Hoekverdelingen: $e^-e^+ \to \mu^-\mu^+$ and $\tau^-\tau^+$





En nu heb je ook $e^-e^+ \to q\bar{q}$ gedaan!

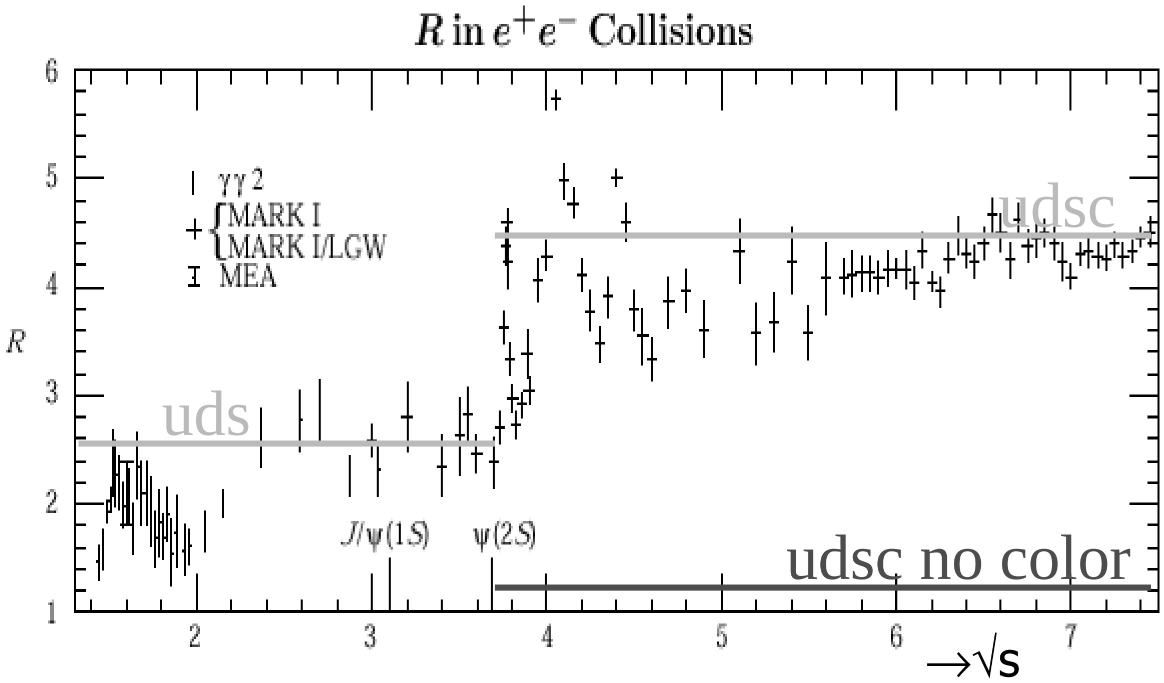
Met drie kleuren

u, c, t $\quad |q| = \frac{2}{3}: \quad \frac{d\sigma}{d\Omega} = \frac{4}{9} \times \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \Rightarrow \sigma_{tot} = \frac{16\pi\alpha^2}{27s} \quad \sum_{colour} \frac{16\pi\alpha^2}{27s} \to \frac{16\pi\alpha^2}{9s}$

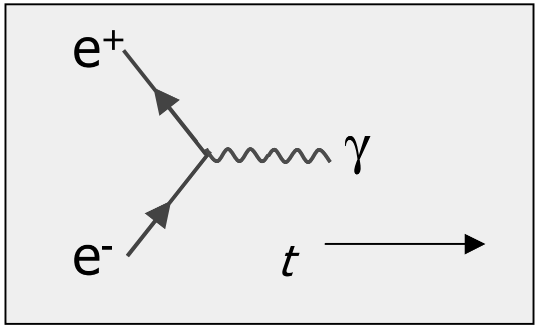
d, s, b $\quad |q| = \frac{1}{3}: \quad \frac{d\sigma}{d\Omega} = \frac{1}{9} \times \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \Rightarrow \sigma_{tot} = \frac{4\pi\alpha^2}{27s} \quad \sum_{colour} \frac{4\pi\alpha^2}{27s} \to \frac{4\pi\alpha^2}{9s}$

$$R \equiv \frac{\sigma_{qq}}{\sigma_{\mu\mu}} = \frac{\sum_{quarks} e^+e^- \to q\bar{q}}{e^+e^- \to \mu^+\mu^-}$$

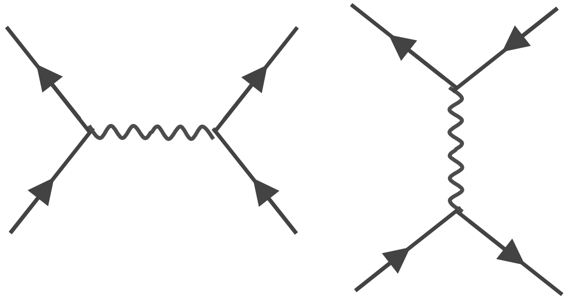
$$\sigma_{\mu\mu} = \frac{4\pi\alpha^2}{3s}$$



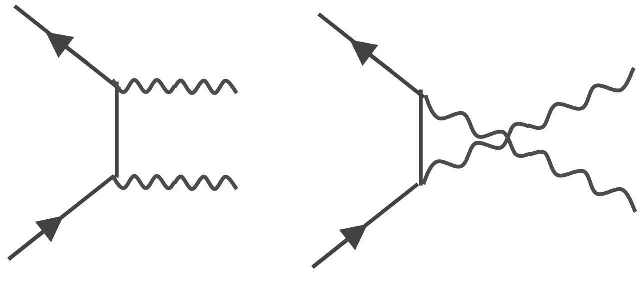
QED



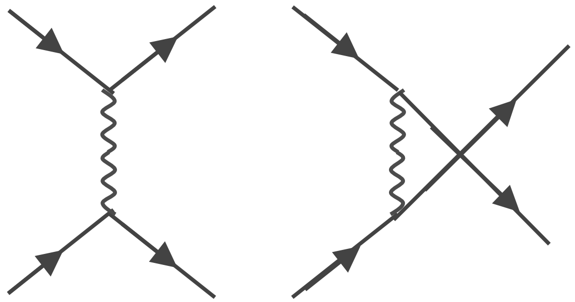
Fundamentele interacties:



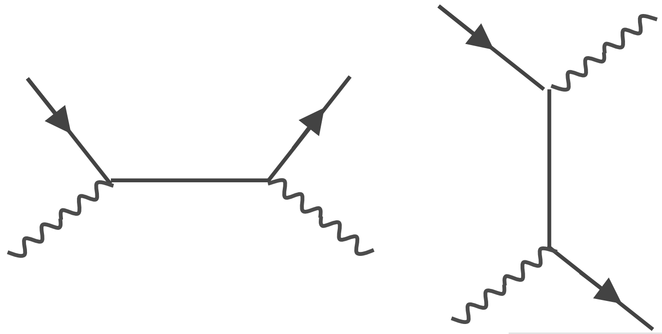
Bhabha verstrooiing $e^+e^- \rightarrow e^+e^-$



paar annihilatie/creatie $e^-e^+ \rightarrow \gamma\gamma$

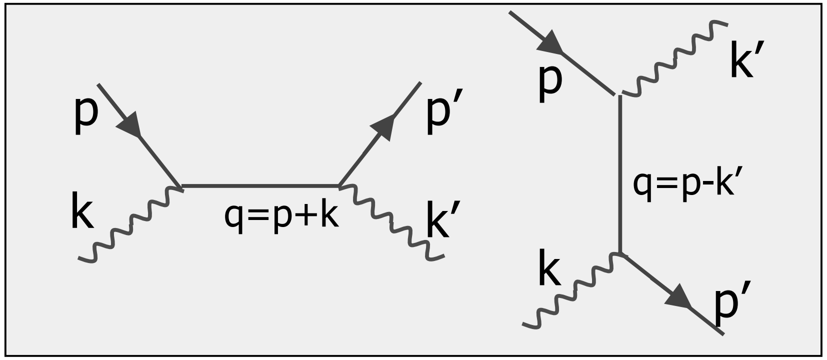


Möller verstrooiing $e^-e^- \rightarrow e^-e^-$



Compton verstrooiing $e^-\gamma \rightarrow e^-\gamma$

Compton verstrooiing $e^{-}\gamma \rightarrow e^{-}\gamma$



$$
\begin{aligned}
-i\mathcal{M}_{\dashv} &= [ie\sqrt{4\pi}\bar{u}(p')\gamma^{\mu}\epsilon'^{*}_{\mu}(k')] \times \frac{i(\not{q}+m)}{q^2-m^2} \times [ie\sqrt{4\pi}\gamma^{\nu}u(p)\epsilon_{\nu}(k)] \quad q=p+k=p'+k' \\
-i\mathcal{M}_{\lfloor} &= [ie\sqrt{4\pi}\bar{u}(p')\gamma^{\mu}\epsilon_{\mu}(k)] \times \frac{i(\not{q}+m)}{q^2-m^2} \times [ie\sqrt{4\pi}\gamma^{\nu}u(p)\epsilon'^{*}_{\nu}(k')] \quad q=p-k'=p'-k \\
-i\mathcal{M} &= -i\mathcal{M}_{\dashv} - i\mathcal{M}_{\lfloor}
\end{aligned}
$$

ijk-invariantie geeft $\mathcal{M} = \epsilon'^{*}_{\mu}(k')\epsilon_{\nu}(k)T^{\mu\nu} \;\Rightarrow\; \begin{cases} k_{\nu}T^{\mu\nu} = 0 \\ k'_{\mu}T^{\mu\nu} = 0 \end{cases}$

relativistische limiet,
d.w.z. verwaarloos
rustmassas

$$
\begin{aligned}
\mathcal{M}_{\dashv} &\rightarrow 4\pi e^2 \epsilon_{\nu}(k)\epsilon'^{*}_{\mu}(k')\bar{u}(p')\gamma^{\mu} \times \frac{\not{p}+\not{k}}{s} \times \gamma^{\nu}u(p) \\
\mathcal{M}_{\lfloor} &\rightarrow 4\pi e^2 \epsilon'^{*}_{\nu}(k')\epsilon_{\mu}(k)\bar{u}(p')\gamma^{\mu} \times \frac{\not{p}-\not{k}'}{u} \times \gamma^{\nu}u(p) \\
&= 4\pi e^2 \epsilon_{\nu}(k)\epsilon'^{*}_{\mu}(k')\bar{u}(p')\gamma^{\nu} \times \frac{\not{p}-\not{k}'}{u} \times \gamma^{\mu}u(p)
\end{aligned}
$$

relabel: μ↔ν

kwadraat $\overline{|\mathcal{M}|^2} = \overline{|\mathcal{M}_a + \mathcal{M}_b|^2} = \overline{|\mathcal{M}_a|^2} + \overline{|\mathcal{M}_{\lfloor}|^2} + \overline{\mathcal{M}_a\mathcal{M}_b^{*}} + \overline{\mathcal{M}_a^{*}\mathcal{M}_b}$

Compton verstrooiing…

$$g_{\nu\nu'}g_{\mu\mu'}$$

$$\sum_T \varepsilon_\nu^T(k)\varepsilon_{\nu'}^{*T}(k) = -g_{\nu\nu'}$$

$$\not{p}$$

$$\not{p}'$$

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m \approx \not{p}$$

$$
\begin{aligned}
4\overline{|\mathcal{M}|}^2 &= \overline{|\mathcal{M}_a|}^2 + \overline{|\mathcal{M}_\lfloor|}^2 + \overline{\mathcal{M}_a\mathcal{M}_b^*} + \overline{\mathcal{M}_a^*\mathcal{M}_b} \\
&= \left(\frac{4\pi e^2}{s}\right)^2 \left\{\sum_{T,T'} \epsilon_\nu^T(k)\epsilon_\mu'^{T'*}(k') \times \epsilon_{\nu'}^{T*}(k)\epsilon_{\mu'}'^{T'}(k')\right\} \times \\
&\quad \left\{\sum_{s,s'} \bar{u}^{(s')}(p')\gamma^\mu(\not{p}+\not{k})\gamma^\nu u^{(s)}(p) \times \bar{u}^{(s)}(p)\gamma^{\nu'}(\not{p}+\not{k})\gamma^{\mu'}u^{(s')}(p')\right\} + \\
&\quad \left(\frac{4\pi e^2}{u}\right)^2 \left\{\sum_{T,T'} \epsilon_\nu^T(k)\epsilon_\mu'^{T'*}(k') \times \epsilon_{\nu'}^{T*}(k)\epsilon_{\mu'}'^{T'}(k')\right\} \times \\
&\quad \left\{\sum_{s,s'} \bar{u}^{(s')}(p')\gamma^\nu(\not{p}-\not{k}')\gamma^\mu u^{(s)}(p) \times \bar{u}^{(s)}(p)\gamma^{\mu'}(\not{p}-\not{k}')\gamma^{\nu'}u^{(s')}(p')\right\} + \\
&\quad \frac{(4\pi e^2)^2}{su} \left\{\sum_{T,T'} \epsilon_\nu^T(k)\epsilon_\mu'^{T'*}(k') \times \epsilon_{\nu'}^{T*}(k)\epsilon_{\mu'}'^{T'}(k')\right\} \times \\
&\quad \left\{\sum_{s,s'} \bar{u}^{(s')}(p')\gamma^\mu(\not{p}+\not{k})\gamma^\nu u^{(s)}(p) \times \bar{u}^{(s)}(p)\gamma^{\mu'}(\not{p}-\not{k}')\gamma^{\nu'}u^{(s')}(p')\right\} + \\
&\quad \frac{(4\pi e^2)^2}{su} \left\{\sum_{T,T'} \epsilon_\nu^T(k)\epsilon_\mu'^{T'*}(k') \times \epsilon_{\nu'}^{T*}(k)\epsilon_{\mu'}'^{T'}(k')\right\} \times \\
&\quad \left\{\sum_{s,s'} \bar{u}^{(s)}(p)\gamma^\nu(\not{p}+\not{k})\gamma^\mu u^{(s')}(p') \times \bar{u}^{(s')}(p')\gamma^{\nu'}(\not{p}-\not{k}')\gamma^{\mu'}u^{(s)}(p)\right\}
\end{aligned}
$$

Compton verstrooiing………

Converteren naar traces

$$
\begin{aligned}
4\overline{|\mathcal{M}|}^2 \quad = \quad & \left(\frac{4\pi e^2}{s}\right)^2 \mathrm{Tr}\ \{\not{p}'\gamma^\mu(\not{p}+\not{k})\gamma^\nu\ \not{p}\gamma_\nu(\not{p}+\not{k})\gamma_\mu\} + \\
& \left(\frac{4\pi e^2}{u}\right)^2 \mathrm{Tr}\ \{\not{p}'\gamma^\nu(\not{p}-\not{k}')\gamma^\mu\ \not{p}\gamma_\mu(\not{p}-\not{k}')\gamma_\nu\} + \\
& \frac{(4\pi e^2)^2}{su} \mathrm{Tr}\ \{\not{p}'\gamma^\mu(\not{p}+\not{k})\gamma^\nu\ \not{p}\gamma_\mu(\not{p}-\not{k}')\gamma_\nu\} + \\
& \frac{(4\pi e^2)^2}{su} \mathrm{Tr}\ \{\not{p}\gamma^\nu(\not{p}+\not{k})\gamma^\mu\ \not{p}'\gamma_\nu(\not{p}-\not{k}')\gamma_\mu\}
\end{aligned}
$$

De trace theoremas geven je voor $|a|^2$ en $|b|^2$

$$
\begin{aligned}
& \mathrm{Tr}\ \{\not{p}'\gamma^\mu(\not{p}+\not{k})\gamma^\nu\ \not{p}\gamma_\nu(\not{p}+\not{k})\gamma_\mu\} = \\
& \mathrm{Tr}\ \{\gamma_\mu\ \not{p}'\gamma^\mu(\not{p}+\not{k})\gamma^\nu\ \not{p}\gamma_\nu(\not{p}+\not{k})\} = \mathrm{Tr}\ \{(-2\ \not{p}')(\not{p}+\not{k})(-2\ \not{p})(\not{p}+\not{k})\} = \\
& 4\times \mathrm{Tr}\ \{\not{p}'\ \not{k}\ \not{p}\ \not{k}+\not{p}'\ \not{p}\ \not{p}\ \not{k}+\not{p}'\ \not{k}\ \not{p}\ \not{p}+\not{p}'\ \not{p}\ \not{p}\ \not{p}\} \approx 32(p'\cdot k)(p\cdot k) \approx -8us
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{Tr}\ \{\not{p}'\gamma^\nu(\not{p}-\not{k}')\gamma^\mu\ \not{p}\gamma_\mu(\not{p}-\not{k}')\gamma_\nu\} = \\
& \mathrm{Tr}\ \{\gamma_\nu\ \not{p}'\gamma^\nu(\not{p}-\not{k})\gamma^\mu\ \not{p}\gamma_\mu(\not{p}-\not{k}')\} = \mathrm{Tr}\ \{(-2\ \not{p}')(\not{p}-\not{k}')(-2\ \not{p})(\not{p}-\not{k})\} = \\
& 4\times \mathrm{Tr}\ \{\not{p}'\ \not{k}'\ \not{p}\ \not{k}'-\not{p}'\ \not{p}\ \not{p}\ \not{k}'-\not{p}'\ \not{k}'\ \not{p}\ \not{p}+\not{p}'\ \not{p}\ \not{p}\ \not{p}\} \approx 32(p'\cdot k')(p\cdot k') \approx -8us
\end{aligned}
$$

Compton verstrooiing……………

En idem voor de twee kruistermen

$$
\begin{aligned}
&\mathrm{Tr}\ \{\not{p}'\gamma^{\mu}(\not{p}+\ \not{k})\gamma^{\nu}\ \ \not{p}\gamma_{\mu}(\not{p}-\ \not{k}')\gamma_{\nu}\} = \\
&-2\mathrm{Tr}\ \{\not{p}'\gamma^{\mu}(\not{p}+\ \not{k})(\not{p}-\ \not{k}')\gamma_{\mu}\ \ \not{p}\} = -8\mathrm{Tr}\ \{\not{p}'[(p+k)\cdot(p-k')]\ \ \not{p}\} = \\
&-32[p'\cdot p][(p+k)\cdot(p-k')] = -32[p'\cdot p]\left([p\cdot(p'+k')]-[k'\cdot(p+k)]\right) = \\
&-32[p'\cdot p]\left([p\cdot p']-[k'\cdot k]\right) \approx 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Tr}\ \{\not{p}\gamma^{\nu}(\not{p}+\ \not{k})\gamma^{\mu}\ \ \not{p}'\gamma_{\nu}(\not{p}-\ \not{k}')\gamma_{\mu}\} = \\
&-2\mathrm{Tr}\ \{\not{p}\gamma^{\nu}(\not{p}+\ \not{k})(\not{p}-\ \not{k}')\gamma_{\nu}\ \ \not{p}'\} = -8\mathrm{Tr}\ \{\not{p}[(p+k)\cdot(p-k')]\ \ \not{p}'\} = \\
&-32[p\cdot p'][(p+k)\cdot(p-k')] = -32[p\cdot p']\left([p\cdot(p'+k')]-[k'\cdot(p+k)]\right) = \\
&-32[p\cdot p']\left([p\cdot p']-[k'\cdot k]\right) \approx 0
\end{aligned}
$$

Hierbij gebruik gemaakt van
$$
\begin{cases}
s = (p+k)^2 \approx +2p\cdot k = +2p'\cdot k' \\
t = (p-p')^2 \approx -2p\cdot p' \approx -2k\cdot k' \\
u = (p-k')^2 \approx -2p\cdot k' = -2p'\cdot k
\end{cases}
$$

De totale amplitude wordt dus

$$
\overline{|\mathcal{M}|}^2 \approx \overline{|\mathcal{M}_{\dashv}|}^2 + \overline{|\mathcal{M}_{b}|}^2 \approx \frac{1}{4}\times(4\pi e^2)^2\times 8\left(-\frac{u}{s}-\frac{s}{u}\right) = 32\pi^2 e^4\left(-\frac{u}{s}-\frac{s}{u}\right)
$$

Compton verstrooiing.....................

We hebben voor het matrix element dus: $\overline{|\mathcal{M}|}^2 = 32\pi^2 e^4 \left(-\frac{u}{s} - \frac{s}{u}\right)$

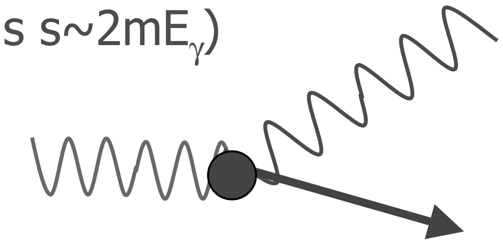
De differentiele werkzame doorsnede is dan: $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|}^2 = \frac{e^4}{2s}\left(-\frac{u}{s} - \frac{s}{u}\right)$

Voor berekening van totale werkzame doorsnede is alleen bijdrage van term –s/u van belang.
Om een eindig antwoord te krijgen is het noodzakelijk m^2 van elektron propagator te behouden:

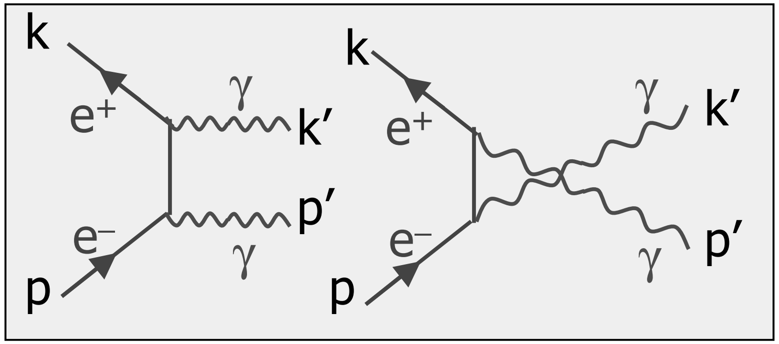
$$\begin{aligned} \sigma_{tot} &= \int \frac{d\sigma}{d\Omega} d\Omega \approx -\int \frac{e^4}{2s}\frac{s}{u-m^2} d\Omega \qquad u \approx \frac{-1}{2} s(1+\cos\theta) \\ &\approx \int \frac{e^4}{2s}\frac{s}{\frac{1}{2}s(1+\cos\theta)+m^2} d\Omega = \frac{2\pi e^4}{s} \ln\left(1+\cos\theta+\frac{m^2}{s}\right)\Bigg|_{\cos\theta=-1}^{\cos\theta=+1} \\ &\approx \frac{2\pi e^4}{s}\ln\frac{s}{m^2} = \frac{2\pi\alpha^2}{s}\ln\frac{s}{m^2} \end{aligned}$$

In lab-stelsel (elektron in rust, foton energie E_γ, en dus s~2mE_γ)

$$\sigma_{tot} \approx \frac{2\pi\alpha^2}{2mE_\gamma}\ln\frac{2mE_\gamma}{m^2} = \frac{\pi\alpha^2}{mE_\gamma}\ln\frac{2E_\gamma}{m} \propto \frac{1}{E_\gamma}$$

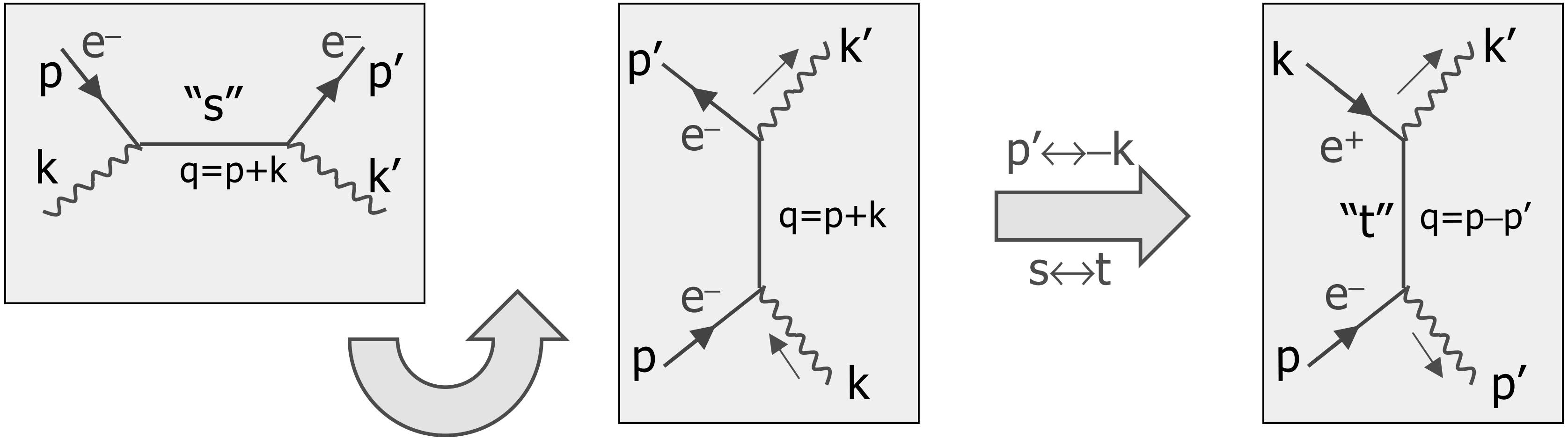


Paar annihilatie



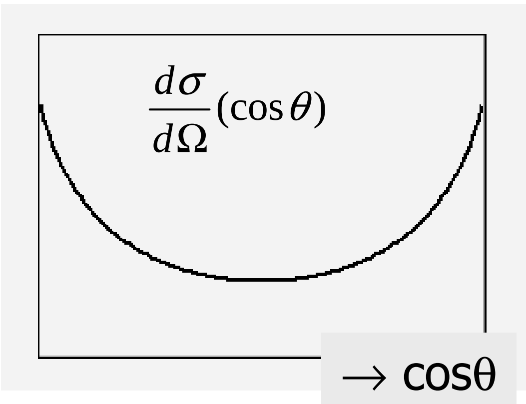
b.v.

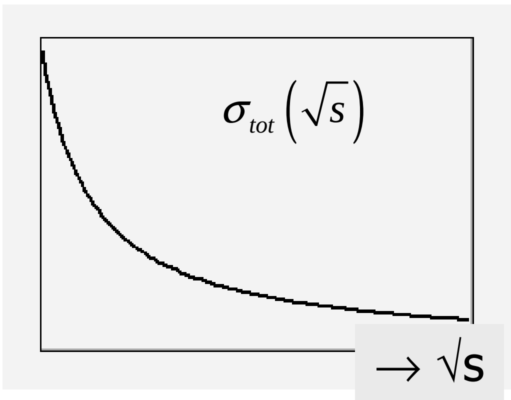
Amplitude volgt uit de amplitude voor e⁻γ→e⁻γ via "crossing":



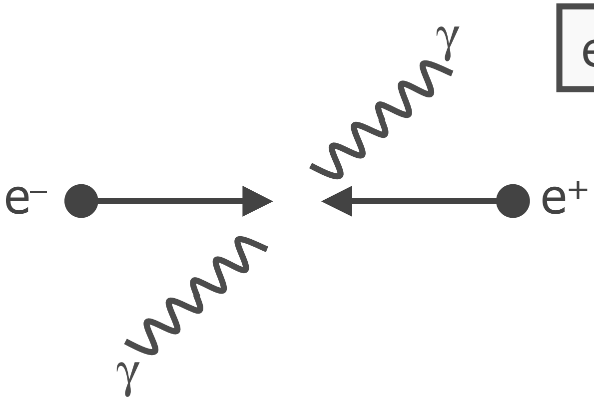
Dus: $\overline{|\mathcal{M}|^2} = 32\pi^2 e^4 \left(\frac{u}{t} + \frac{t}{u}\right) \quad \Longrightarrow$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2} \times \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2} \\
&= \frac{1}{2} \times \frac{1}{64\pi^2 s} 32\pi^2 e^4 \left(\frac{u}{t} + \frac{t}{u}\right) \\
&= \frac{\alpha^2}{4s}\left(\frac{u}{t} + \frac{t}{u}\right) \\
&= \frac{\alpha^2}{4s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \\
&= \frac{\alpha^2}{2s}\left(\frac{1}{1-\cos^2\theta}\right)
\end{aligned}$$



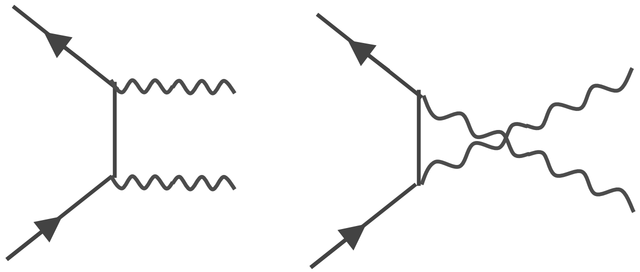


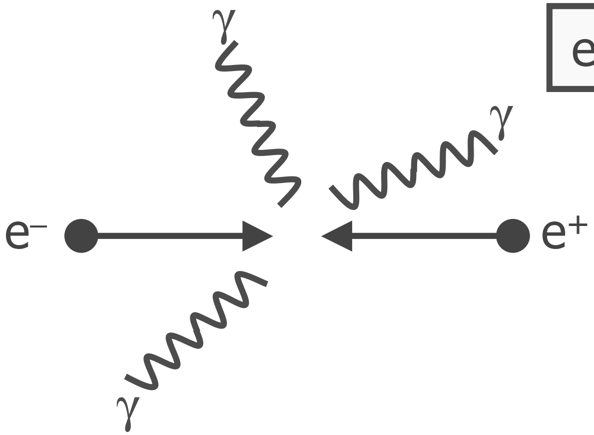
Na dit alles: computer programma FORM



$e^+e^- \to \gamma\gamma$

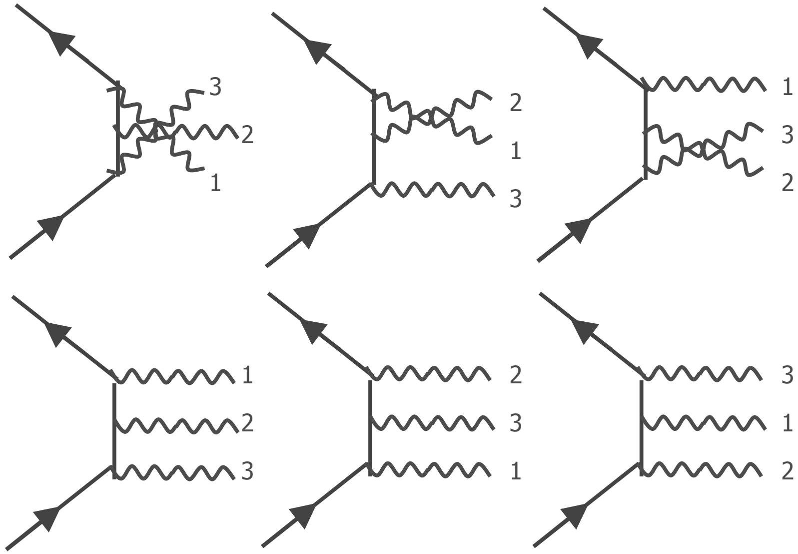
orde α^2





$e^+e^- \to \gamma\gamma\gamma$

orde α^3

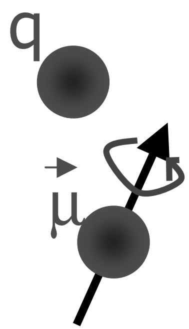


Meting van `g-2’

Magnetisch moment Dirac deeltje

In volgende college zal blijken dat een Dirac deeltje wisselwerkt via

- zijn elektrische lading $q = -|e|$ (net als een Klein-Gordon deeltje)
- zijn magnetisch moment $\vec{\mu} = \dfrac{-|e|\hbar}{2mc}\vec{\sigma} = \dfrac{-|e|}{mc}\vec{s}$ $\qquad (\vec{s} = 1/2\,\hbar\vec{\sigma})$



factor two!

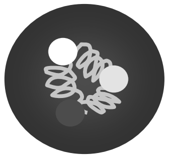
Conventies: Bohr magneton: $\mu_B = \dfrac{e\hbar}{2mc}$ (experiment)

Algemeen : $\vec{\mu} \equiv g\,\dfrac{\mu_B}{\hbar}\,\vec{s}$ (Lande g - factor) $\qquad \left(\text{Klassiek}: \quad \vec{M} = \dfrac{e}{2mc}\vec{L}\right)$

Dirac deeltje: $g = \dfrac{|\vec{\mu}|}{\mu_B|\vec{s}|} = 2$ (magie in pre-Dirac era)

g=2?

Baryonen: p & n



$g_p \approx +5.6$

$g_n \approx -3.8$

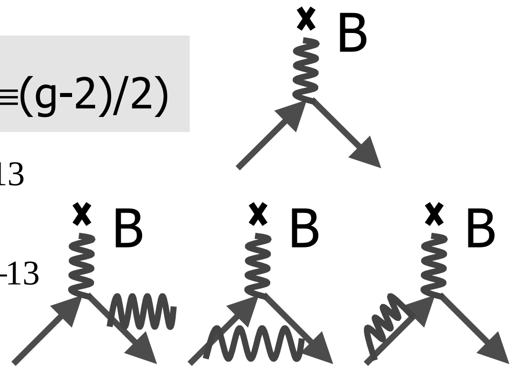
quark sub-structuur

Leptonen: $e^{\pm}$ & $\mu^{\pm}$ ($a \equiv (g-2)/2$)

$$a_{e-} = (11596521884 \pm 43) \times 10^{-13}$$

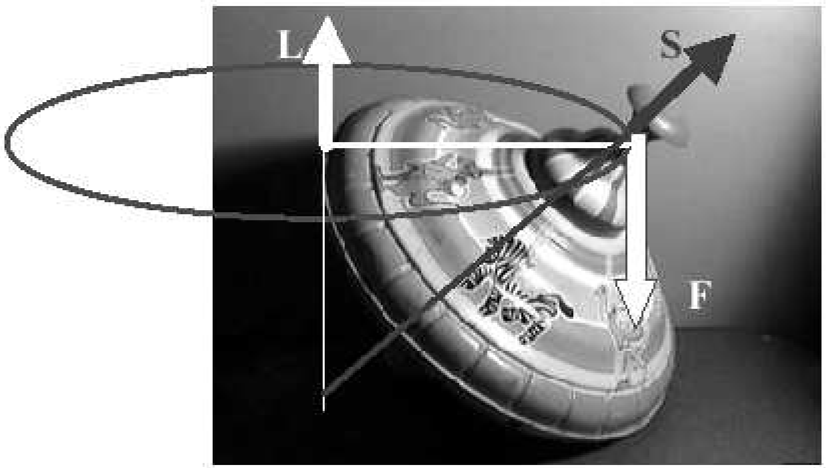
$$a_{e+} = (11596521879 \pm 43) \times 10^{-13}$$

$$a_{\mu} = (11659203 \pm 15) \times 10^{-10}$$



sub-structuur vacuüm

Classical precession: spinning top

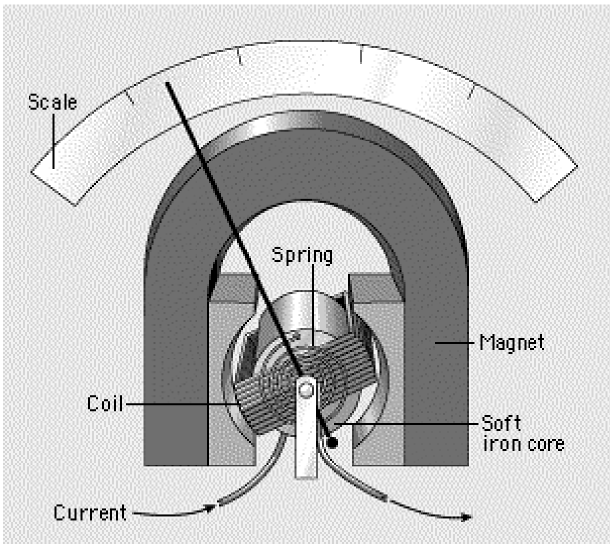


$$\text{Newton}: \; \vec{F} = \frac{d\vec{p}}{dt} \rightarrow m\frac{d\vec{v}}{dt} \quad \left(\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}\right)$$

$$\text{Angular momentum}: \; \vec{L} \equiv \vec{r} \times \vec{p} \;\Rightarrow$$

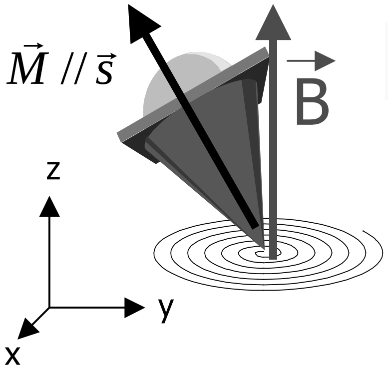
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r}\times\vec{p}\right) = \overbrace{\cancel{\frac{d\vec{r}}{dt}\times\vec{p}}}^{=0} + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F} \equiv \vec{T} \quad \text{(torque)}$$



Torque on magnetic moment $\vec{M}$ in $\vec{B}$ - field :

$$\vec{T} = \vec{M}\times\vec{B} \rightarrow g\frac{e}{2mc}\vec{s}\times\vec{B}$$



Apply to an electron or muon

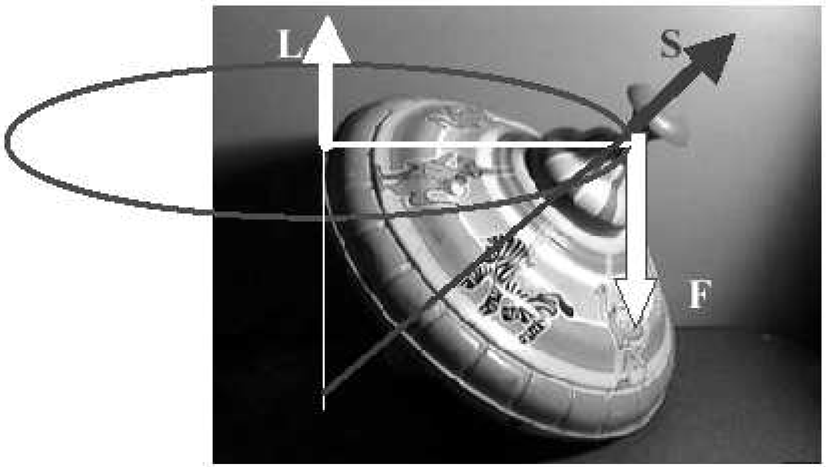
$$\frac{d\vec{L}}{dt} = \vec{T} \quad \rightarrow \quad \frac{d\vec{s}}{dt} = g\frac{e}{2mc}\vec{s}\times\vec{B}$$

$$\ddot{s}_x = -\left(g\frac{eB}{2mc}\right)^2 s_x$$

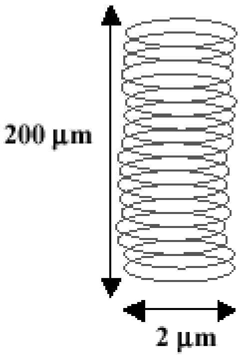
$$\frac{d}{dt}\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = g\frac{e}{2mc}\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ s_x & s_y & s_z \\ 0 & 0 & B \end{pmatrix} = g\frac{eB}{2mc}\begin{pmatrix} +s_y \\ -s_x \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} s_x(t) \\ s_y(t) \end{pmatrix} \propto \begin{pmatrix} \cos\left(g\frac{eB}{2mc}t\right) \\ \sin\left(g\frac{eB}{2mc}t\right) \end{pmatrix} \quad \Rightarrow \text{frequency}: \; \omega \equiv g\frac{eB}{2mc}$$

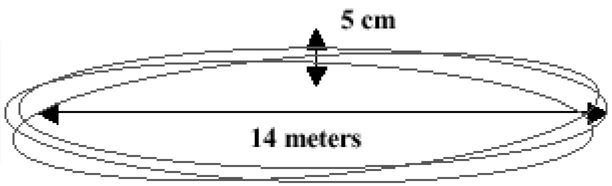
Principe



Elektron



Muon



$\vec{B}$

"Penning" trap (N_e=1)

E≈1 meV (T≈4.2 K)

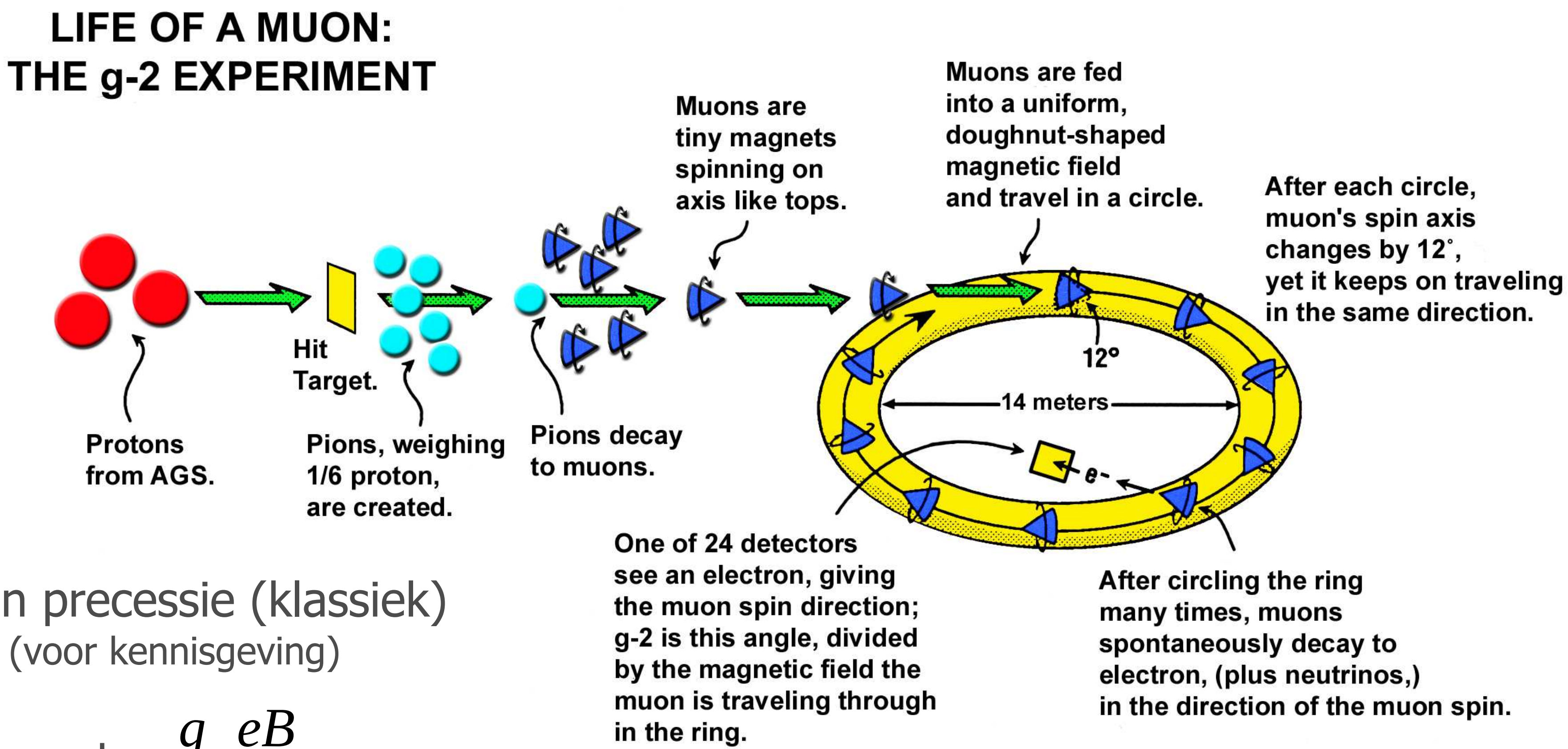
$\omega=(g/2)\times(eB/mc)$

Storage Ring ($N_\mu \approx 10^4$)

E≈3 GeV ($\gamma \approx 30$)

$\Delta\omega=((g-2)/2)\times(eB/mc)$

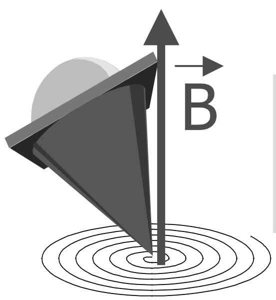
Principe meting van “g-2” muon



Spin precessie (klassiek)
(voor kennisgeving)

μ in rust: $\dfrac{g}{2}\dfrac{eB}{m_\mu c}$

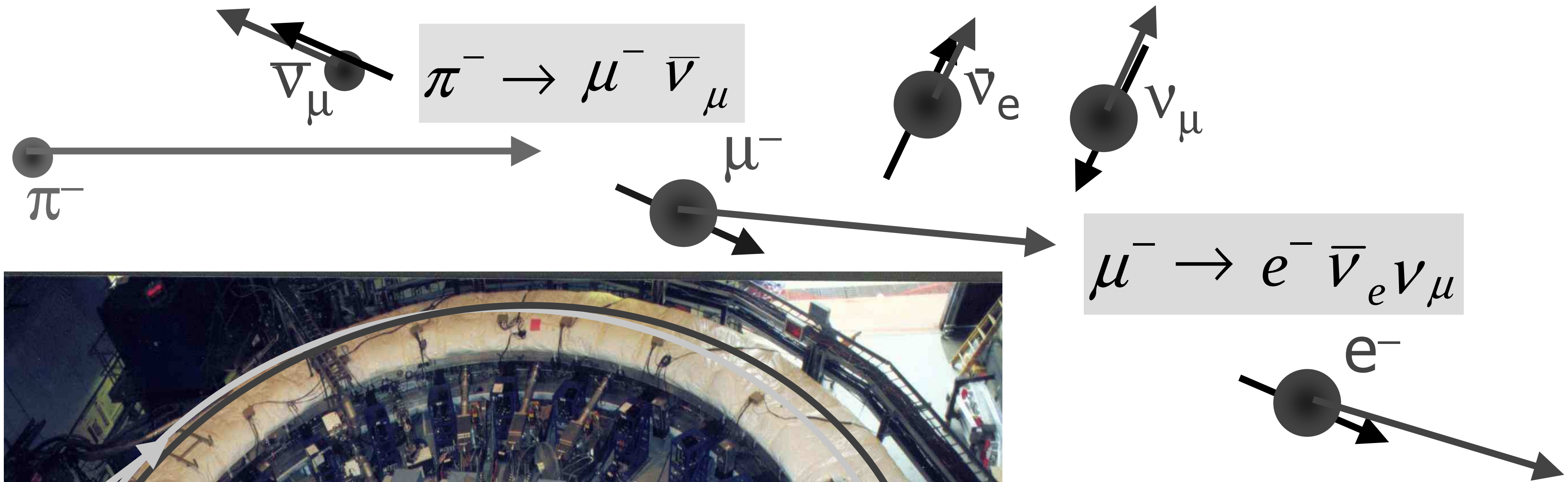
μ in orbit: $\left(1+\gamma\dfrac{g-2}{2}\right)\dfrac{eB}{\gamma m_\mu c}$



$$\Delta\omega=\left(1+\gamma\frac{g-2}{2}\right)\frac{eB}{\gamma m_\mu c}-\frac{eB}{\gamma m_\mu c}=\frac{g-2}{2}\frac{eB}{m_\mu c}$$

Realiteit "g-2" muon

$$\pi^- \rightarrow \mu^- \, \overline{\nu}_\mu$$



$$\mu^- \rightarrow e^- \, \overline{\nu}_e \nu_\mu$$

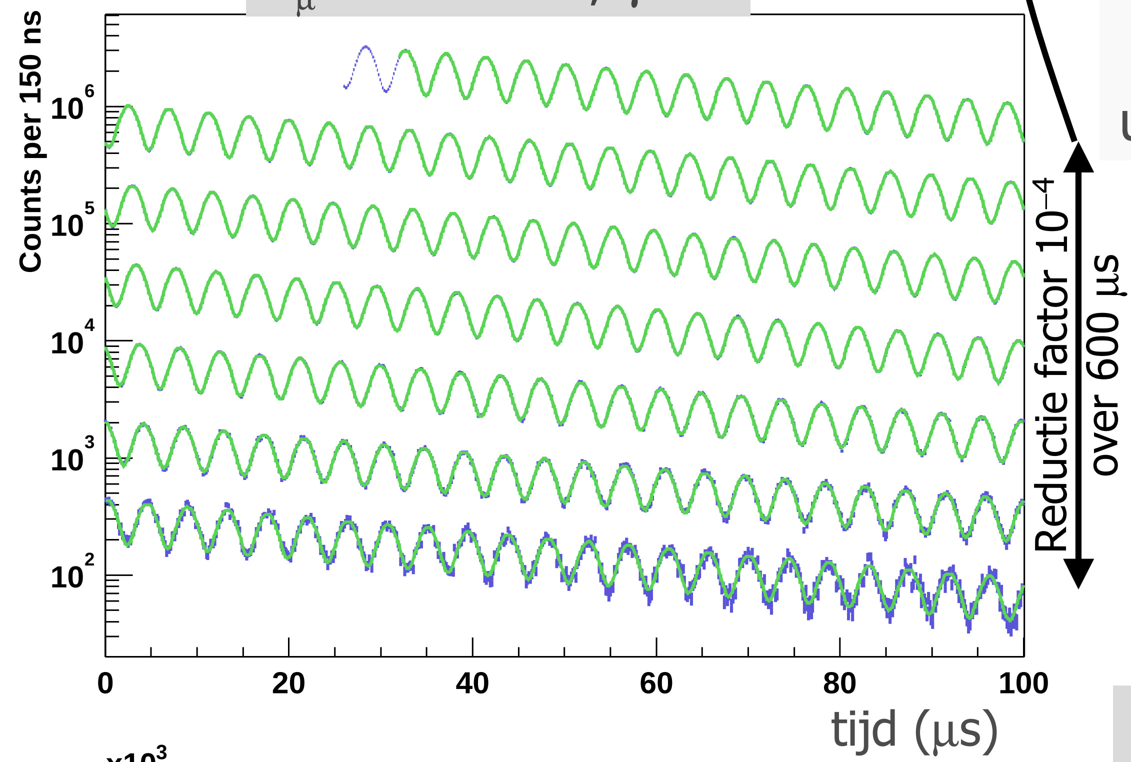
<u>Sterke correlatie:</u>

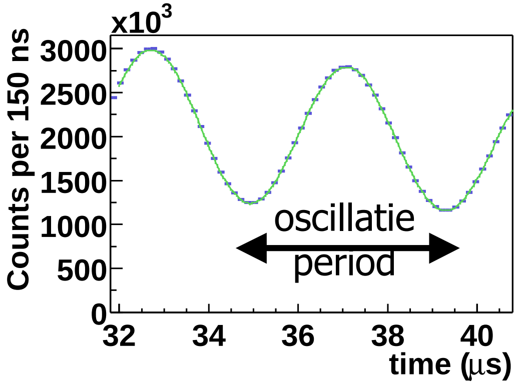
muon spin
&
elektron energie

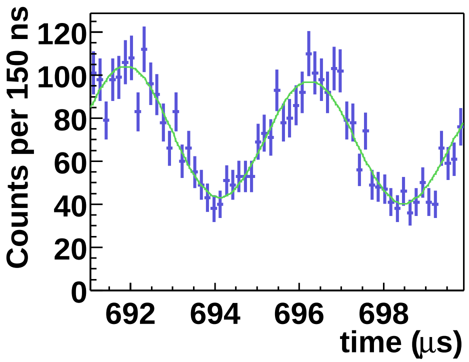
Het resultaat!

$\tau_\mu \approx 2.2\ \mu s$

$E_\mu \approx 3.096$ GeV, $\gamma \approx 30$







Bepaal de muon spin richting
Als functie van de tijd
uit de gemeten elektron energie

Spectrum qualitatief:

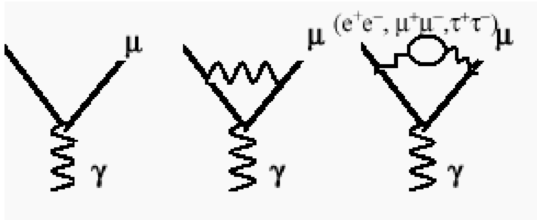
- exponentieel gedrag: muon verval
- oscillaties: spin precessie

Spectrum quantitatief:

$$N(t) = N e^{-t/\gamma\tau}(1 + A\cos\Delta\omega t)$$

$$\boxed{\Rightarrow \quad \Delta\omega = \frac{g-2}{2}\frac{eB}{m_\mu c}}$$

Wat zegt de theorie?



Bottom line elektron: uitrekenen! $\boxed{a_e^{the} \approx 11596521880 \times 10^{-13}}$

1e orde correctie: enige dimensieloze variabele: α

$$\frac{g-2}{2} \approx 0.5 \frac{\alpha}{\pi}$$

$$\boxed{a_e^{exp} \approx 11596521870 \times 10^{-13}}$$

2e+3e+4e orde correcties

$$\frac{g-2}{2} \approx 0.5 \frac{\alpha}{\pi} - 0.32847844400 \left(\frac{\alpha}{\pi}\right)^2 + 1.181234017 \left(\frac{\alpha}{\pi}\right)^3 - 1.5098 \left(\frac{\alpha}{\pi}\right)^4$$

Bottom line muon: uitrekenen! $\boxed{a_\mu^{the} \approx 116584706 \times 10^{-11}}$

1e orde correctie: enige dimensieloze variabele: α

$$\frac{g-2}{2} \approx 0.5 \frac{\alpha}{\pi}$$

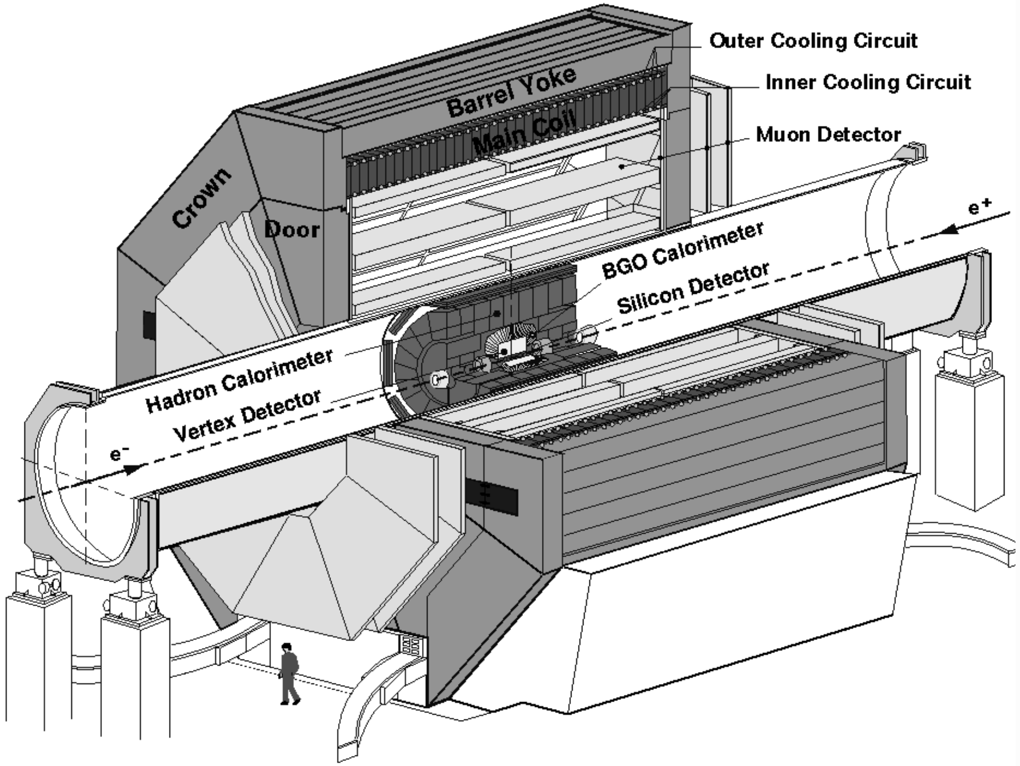
$$\boxed{a_\mu^{exp} \approx 116591600 \times 10^{-11}}$$

2e+3e+4e orde correcties (groter dan voor elektron vanwege $m_\mu / m_e \approx 40000$)

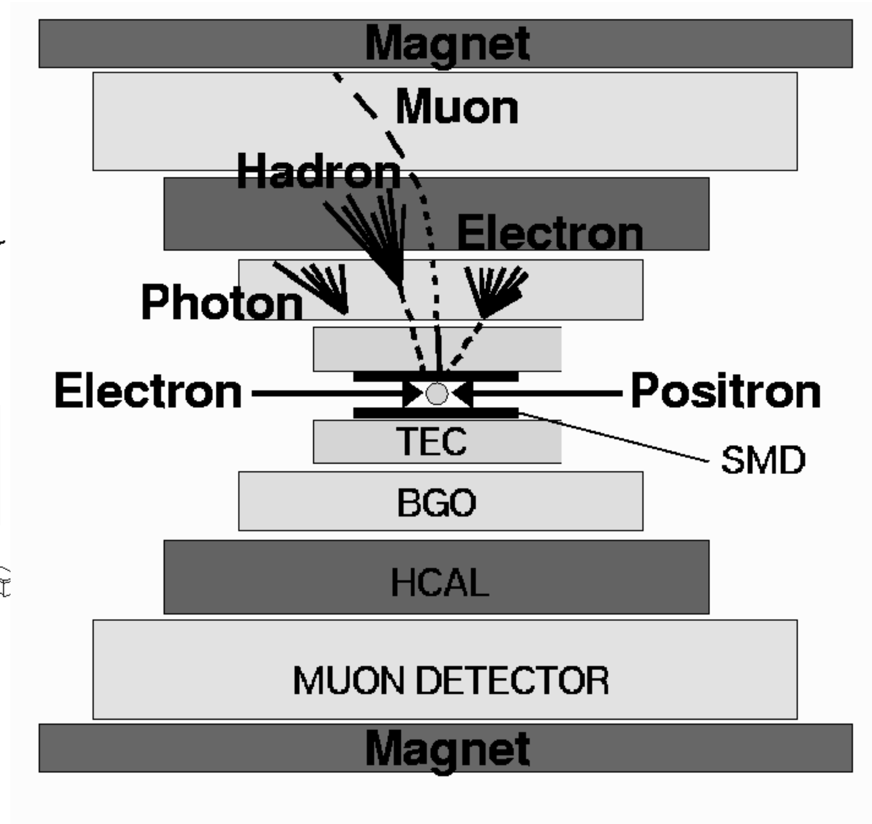
$$\frac{g-2}{2} \approx 0.5 \frac{\alpha}{\pi} + 0.765857376 \left(\frac{\alpha}{\pi}\right)^2 + 24.05050898 \left(\frac{\alpha}{\pi}\right)^3 + 126.07 \left(\frac{\alpha}{\pi}\right)^4$$

Coming soon!

Analyse voorbeeld: normalisatie



detector



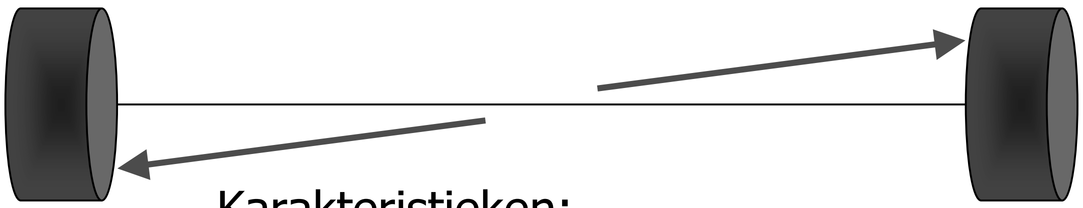
identifikatie

Bhabha verstrooiing: kleine hoeken

Voor elk process:

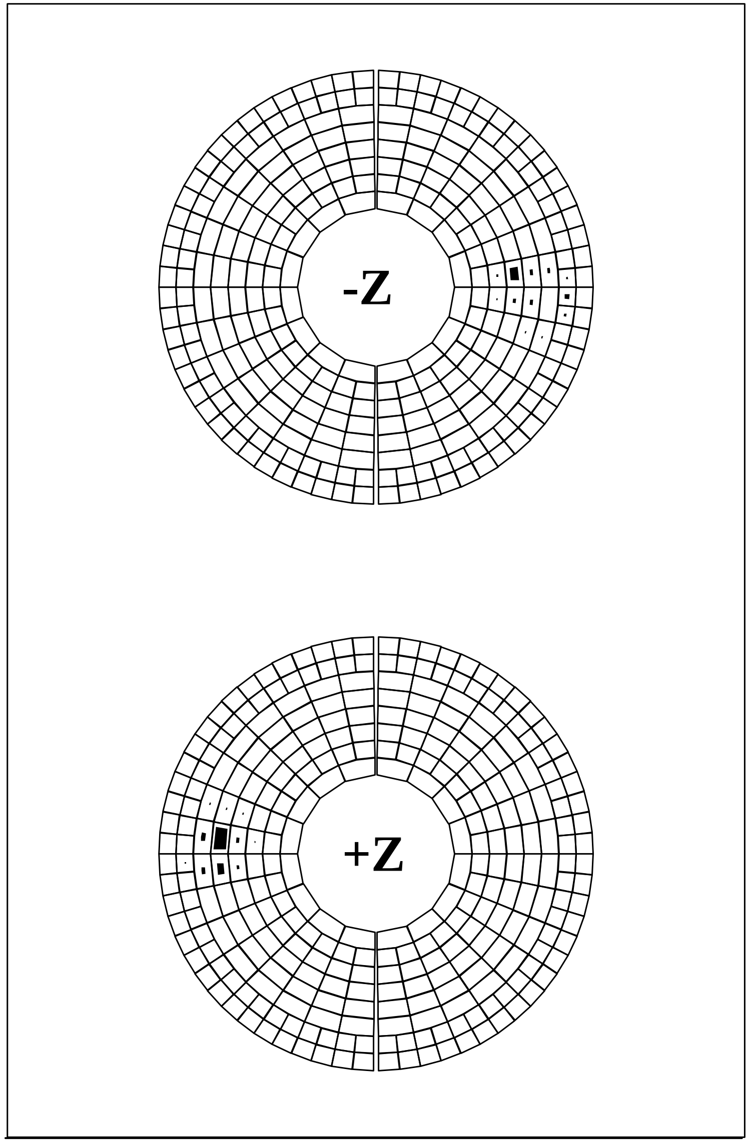
$$\sigma = N_{events}/\text{Normalisatie}$$

Luminositeits detector



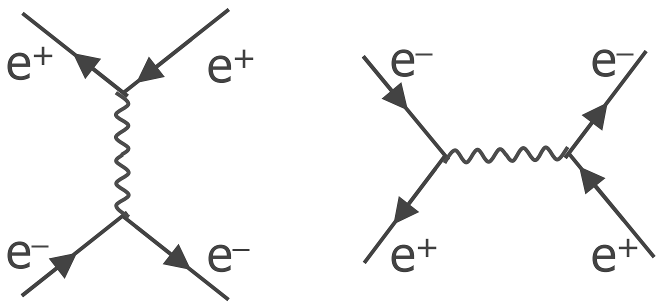
Karakteristieken:

- "back-to-back"
- energie = E_{beam}
- kleine hoeken

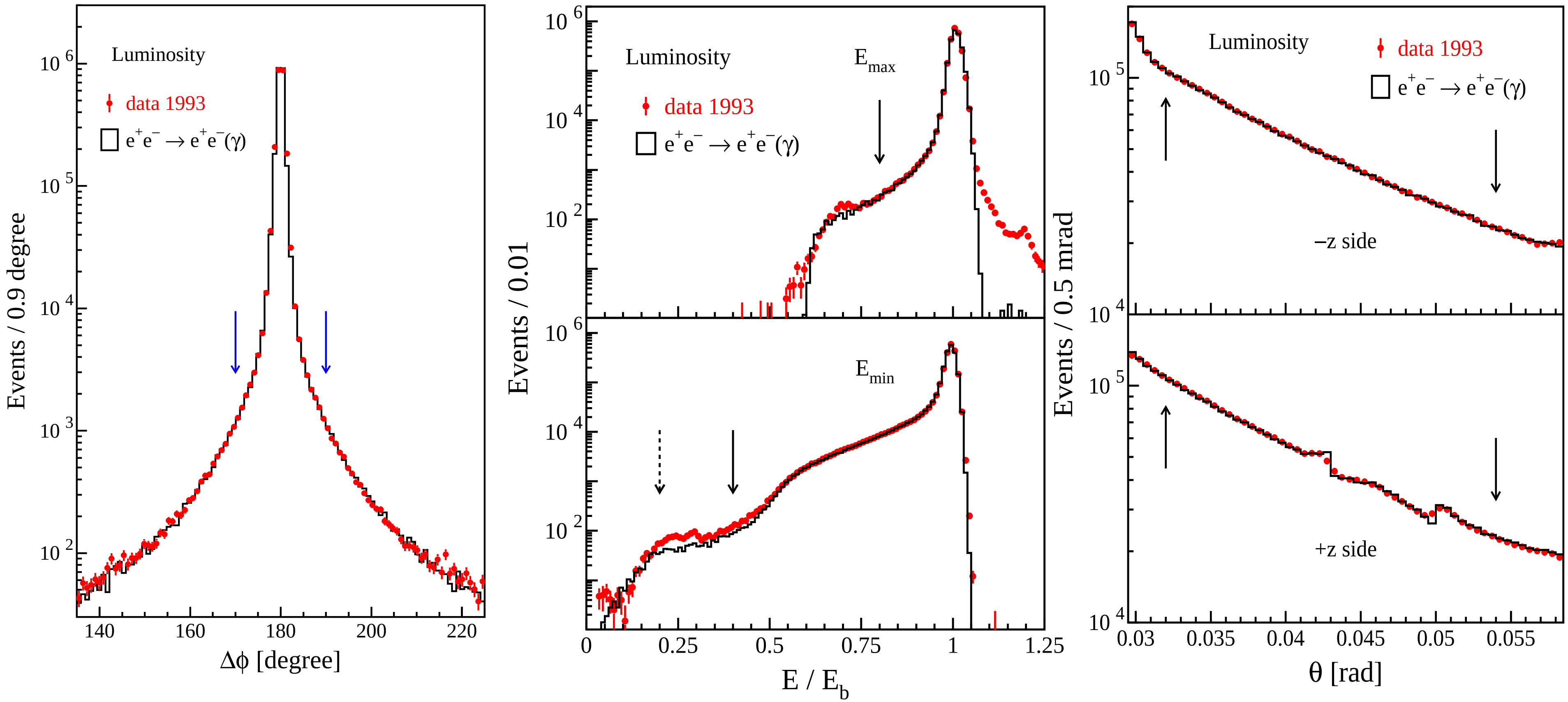


huiswerk, bereken:

$$e^- e^+ \rightarrow e^- e^+$$



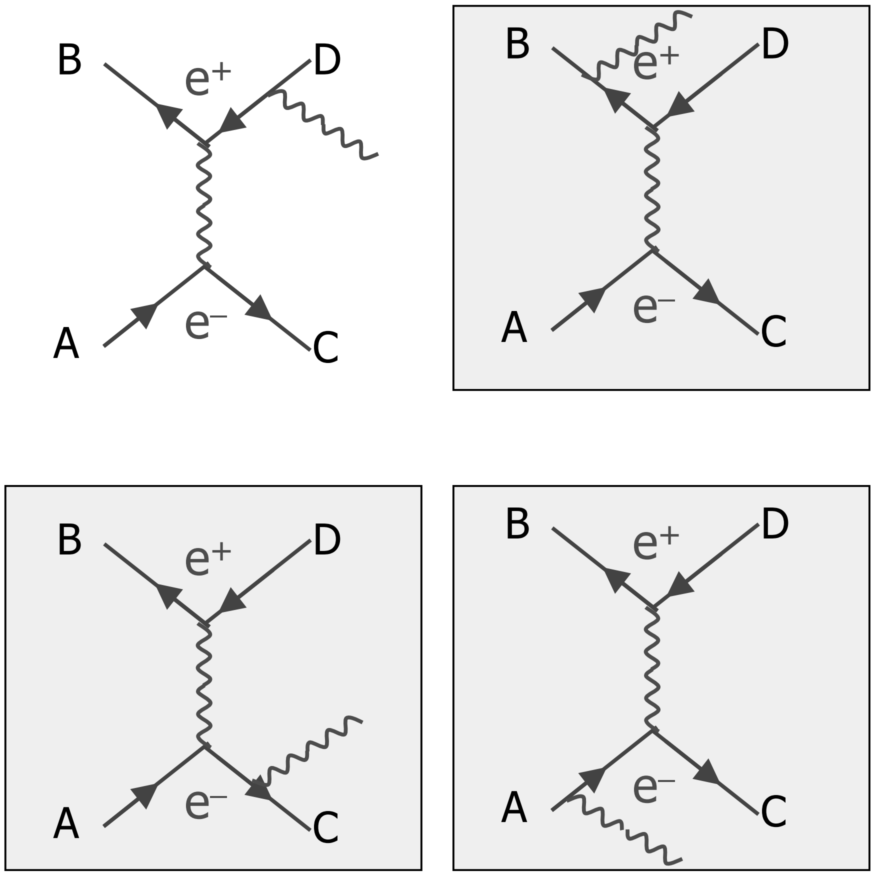
Resultaat

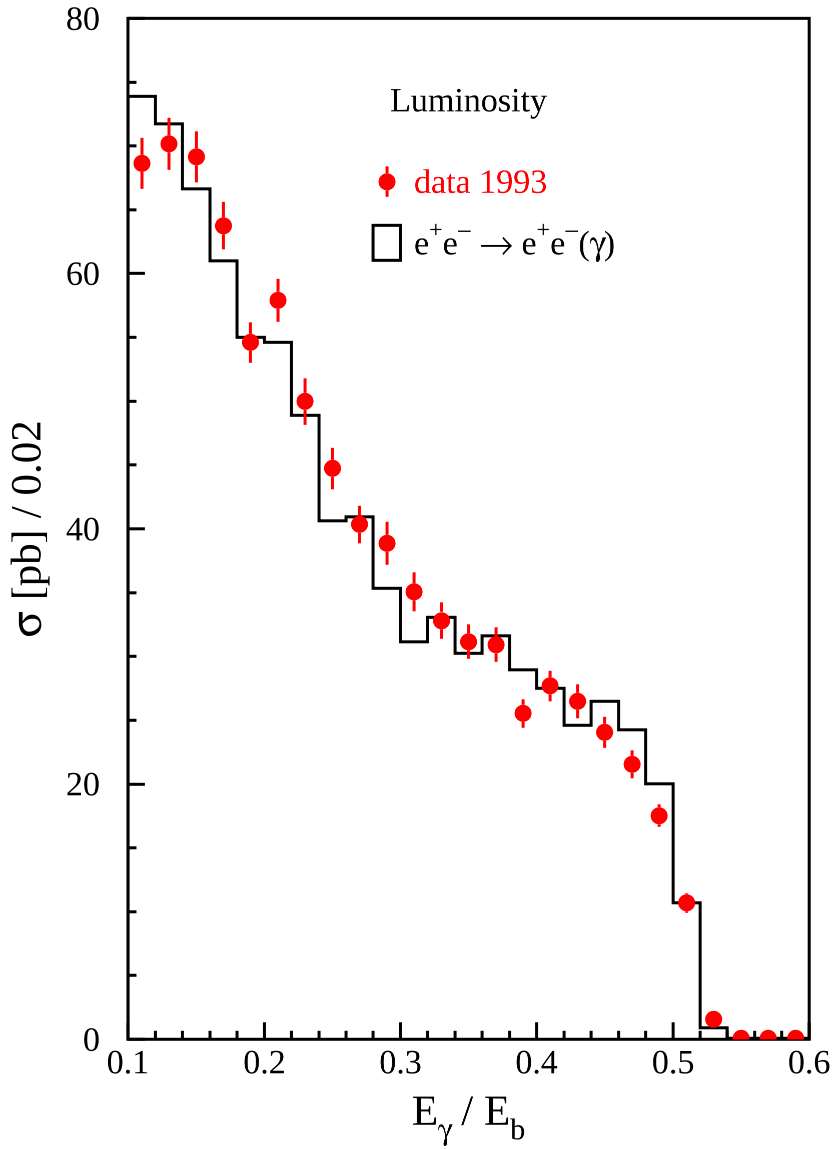


voor ieder ander process:

$$\sigma = \text{Konstante} \times N_{events}/N_{luminosity}$$

Hogere orde correcties





Werkzame doorsneden

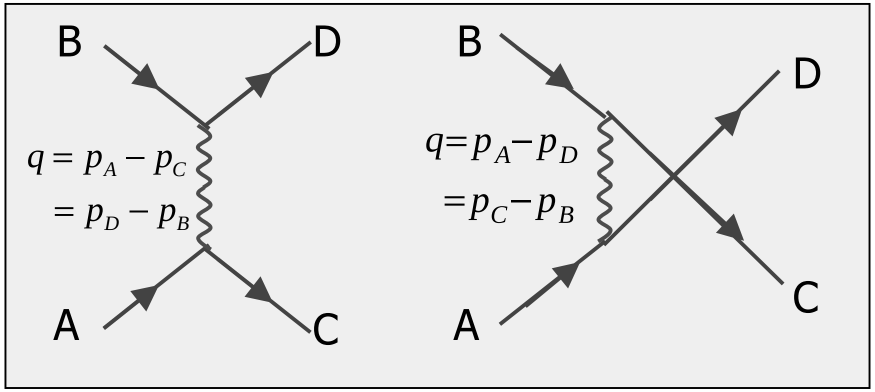
Experiment (vaak): spin toestanden van in- en uit-komende deeltjes onbekend

Uitgaande deeltjes: spins sommeren
Inkomende deeltjes: spins middelen

$$\boxed{|\mathcal{M}|^2 \longrightarrow |\bar{\mathcal{M}}|^2 \equiv \frac{1}{(2s_A+1)(2s_B+1)} \sum_{spin} |\mathcal{M}|^2}$$

beschouw dit eens voor een concreet voorbeeld:

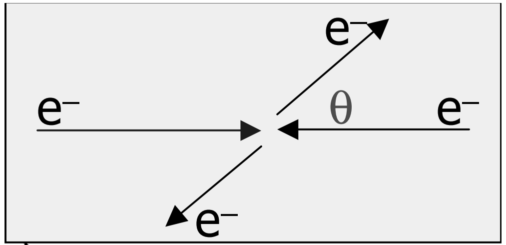
$$e^- e^- \to e^- e^-$$



$$\begin{aligned} -i\mathcal{M} &= [ie\sqrt{4\pi}\bar{u}_C\gamma^\mu u_A] \times \frac{-ig_{\mu\nu}}{(p_A-p_C)^2} \times [ie\sqrt{4\pi}\bar{u}_D\gamma^\nu u_B] - \\ & \quad [ie\sqrt{4\pi}\bar{u}_D\gamma^\mu u_A] \times \frac{-ig_{\mu\nu}}{(p_A-p_D)^2} \times [ie\sqrt{4\pi}\bar{u}_C\gamma^\nu u_B] \Rightarrow \\ \mathcal{M} &= -4\pi e^2 \frac{[\bar{u}_C\gamma^\mu u_A]\times[\bar{u}_D\gamma_\mu u_B]}{(p_A-p_C)^2} + 4\pi e^2 \frac{[\bar{u}_D\gamma^\mu u_A]\times[\bar{u}_C\gamma_\mu u_B]}{(p_A-p_D)^2} \end{aligned}$$

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_C)^2$$
$$u = (p_A - p_D)^2$$

Niet relativistische limiet



Niet relativistische limiet: lim|**p**|→0
$$\begin{cases} \text{Inkomend:} & u^s \to \sqrt{2m}\begin{pmatrix}\chi^s \\ 0\end{pmatrix} \\ \text{Uitgaand:} & \bar{u}^s \to \sqrt{2m}\left(\chi^{s\dagger}, 0\right)\end{cases}$$

$$\gamma^0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \text{ en } \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

De vectoren worden
$$\bar{u}^s\gamma^\mu u^s = \begin{cases} 2m & \text{voor } \mu = 0 \\ 0 & \text{voor } \mu \neq 0 \end{cases}$$
$$\bar{u}^s\gamma^\mu u^{s'} = 0 \quad \forall\mu \text{ indien } s \neq s'$$

d.w.z. de spins verandert niet. Dus de termen met matrix-element $\neq 0$:

$$\begin{aligned}
\mathcal{M}(++\to++) &= \mathcal{M}(--\to--) = -4\pi e^2 4m^2(\tfrac{1}{t} - \tfrac{1}{u}) \\
\mathcal{M}(+-\to+-) &= \mathcal{M}(-+\to-+) = -4\pi e^2 4m^2 \tfrac{1}{t} \\
\mathcal{M}(+-\to-+) &= \mathcal{M}(-+\to+-) = +4\pi e^2 4m^2 \tfrac{1}{u}
\end{aligned}$$

$$t = -4|\vec{p}|^2 \sin^2\frac{\theta}{2}$$
$$u = -4|\vec{p}|^2 \cos^2\frac{\theta}{2}$$

En de amplitude na sommatie en middelen

sommeren!

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{2}{4}(4\pi e^2 4m^2)^2\left[(\frac{1}{t} - \frac{1}{u})^2 + \frac{1}{t^2} + \frac{1}{u^2}\right] \\
&\to 128\pi^2\alpha^2 m^4 \frac{1}{16|\vec{p}|^4}\left[\frac{2}{\sin^4\theta/2} + \frac{2}{\cos^4\theta/2} - \frac{2}{\sin^2\theta/2\cos^2\theta/2}\right]
\end{aligned}$$

middelen!

vgl. weer met Rutherford!

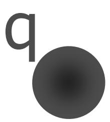
$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \times \frac{1}{64\pi^2 s}|\bar{\mathcal{M}}|^2 = \frac{\alpha^2 m^2}{16|\vec{p}|^4}\left[\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} - \frac{1}{\sin^2\theta/2\cos^2\theta/2}\right]$$

identieke deeltjes!

QED: Lading & magnetisch moment

M.b.v. de *Gordon decompositie* wordt de S=1/2 vertex herschreven:

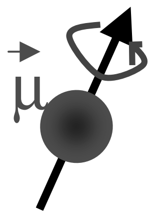
$$
\begin{aligned}
i\bar{u}_f\sigma^{\mu\nu}(p_f-p_i)_\nu u_i &= \tfrac{1}{2}\bar{u}_f(\gamma^\nu\gamma^\mu-\gamma^\mu\gamma^\nu)(p_f-p_i)_\nu u_i \\
&= \tfrac{1}{2}\bar{u}_f\left[\gamma^\nu\gamma^\mu p^f_\nu-\gamma^\nu\gamma^\mu p^i_\nu-\gamma^\mu\gamma^\nu p^f_\nu+\gamma^\mu\gamma^\nu p^i_\nu\right]u_i \\
&= \tfrac{1}{2}\bar{u}_f\left[\gamma^\nu\gamma^\mu p^f_\nu+\gamma^\mu\gamma^\nu p^i_\nu-2g^{\mu\nu}p^i_\nu\right]u_i+ \\
&\quad\ \tfrac{1}{2}\bar{u}_f\left[\gamma^\nu\gamma^\mu p^f_\nu+\gamma^\mu\gamma^\nu p^i_\nu-2g^{\mu\nu}p^f_\nu\right]u_i \\
&= -\bar{u}_f\left[p^\mu_f+p^\mu_i\right]u_i+\bar{u}_f\left[\gamma^\nu\gamma^\mu p^f_\nu+\gamma^\mu\gamma^\nu p^i_\nu\right]u_i \\
&= -\bar{u}_f\left[p^\mu_f+p^\mu_i\right]u_i+\bar{u}_f\left[\gamma^\mu m+\gamma^\mu m\right]u_i \\
&= -\bar{u}_f\left[p^\mu_f+p^\mu_i\right]u_i+\bar{u}_f\left[2\gamma^\mu m\right]u_i \Rightarrow
\end{aligned}
$$



De eerste term is identiek aan het Klein-Gordon resultaat; ladings-interactie

Gewoon: stug door rekenen!

$$
\boxed{\bar{u}_f\gamma^\mu u_i = \frac{1}{2m}\bar{u}_f\left[(p^\mu_f+p^\mu_i)+i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u_i}
$$



De tweede term is de magnetische moment interactie, d.w.z. $\vec{\mu}\cdot\vec{B}=\vec{\mu}\cdot\vec{\nabla}\times\vec{A}$

De overgangswaarschijnlijkheid:

$$
\begin{aligned}
T_{fi} &\rightarrow -2\pi i\delta(E_i-E_f)\int j^{fi}_\mu A^\mu d^3x \\
&= 2\pi ie\delta(E_i-E_f)\int\frac{1}{2m}\bar{\psi}_f\left[(p^\mu_f+p^\mu_i)+i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]\psi_i A^\mu d^3x
\end{aligned}
$$

Waarom magnetisch moment?

$$\frac{-e}{2m}\int \bar{\psi}_f\, i\,\sigma_{\mu\nu}\,(p_f-p_i)^\nu\, A^\mu\,\psi_i\, d^3x$$

In de niet-relativistische limiet geeft deze term, $A^\mu=(0,\mathbf{A})$

$$\rightarrow \frac{-e}{2m}\, i\int \bar{\psi}_f\,\left[\sigma_{12}(p_f-p_i)^2A^1+\sigma_{13}(p_f-p_i)^3A^1+\sigma_{21}(p_f-p_i)^1A^2+\sigma_{23}(p_f-p_i)^3A^2+\ldots\right]\psi_i\, d^3x$$

$$= \frac{-e}{2m}\, i\int \bar{\psi}_f\,\left[\sigma_3(p_f-p_i)^2A^1-\sigma_2(p_f-p_i)^3A^1-\sigma_3(p_f-p_i)^1A^2+\sigma_1(p_f-p_i)^3A^2+\ldots\right]\psi_i\, d^3x$$

$$= \frac{-e}{2m}\, i\int \bar{\psi}_f\,\left[(\vec{p}_f-\vec{p}_i)\cdot\vec{\sigma}\times\vec{A}\right]\psi_i\, d^3x$$

$$\frac{-e}{2m}\, i\int \bar{\psi}_f\,\left[(\vec{p}_f-\vec{p}_i)\cdot\vec{\sigma}\times\vec{A}\right]\psi_i\, d^3x$$

$$= \frac{-e}{2m}\, i\int \bar{u}_f\,[\vec{\sigma}\times\vec{A}]u_i\cdot e^{+ip_fx}\left(\vec{p}_f-\vec{p}_i\right)e^{-ip_ix}\, d^3x$$

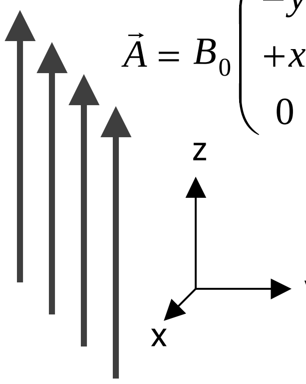
$$= \frac{-e}{2m}\, i\int \bar{u}_f\,[\vec{\sigma}\times\vec{A}]u_i\cdot\vec{\nabla}\left(e^{+ip_fx}e^{-ip_ix}\right)d^3x$$

partieel integreren

$$= \frac{+e}{2m}\, i\int \left(e^{+ip_fx}e^{-ip_ix}\right)\;\bar{u}_f\,[\vec{\sigma}\cdot\vec{\nabla}\times\vec{A}]u_i\, d^3x$$

$$= \frac{+e}{2m}\, i\int \bar{\psi}_f\,[\vec{\sigma}\cdot\vec{\nabla}\times\vec{A}]\psi_i\, d^3x \equiv \frac{+e}{2m}\, i\int \bar{\psi}_f\,[\vec{\sigma}\cdot\vec{B}]\psi_i\, d^3x$$

$$\vec{B}=\begin{pmatrix}0\\0\\B_0\end{pmatrix}\qquad \vec{A}=B_0\begin{pmatrix}-y\\+x\\0\end{pmatrix}$$



$$\Rightarrow \vec{\mu}_e=\frac{-e}{2m}\,\vec{\sigma}$$