Lecture notes to a first year's university course

The Relativistic Quantum World

Maastricht University - Autumn 2017

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Abstract

In the beginning of the previous century our fundamental view of nature revolutionised with the birth of the theory of relativity and of quantum mechanics. In the course of the century a beautiful theoretical framework, called the Standard Model, was developed and its realisation in nature was step-by-step verified with the discovery of new, fundamental particles, culminating in the recent discovery of the Higgs boson. The lecture series contains a mini-course of eight one-hour lectures. We discuss the fundamental elements of the special and general relativity theory and quantum mechanics with a particular emphasis on the counter-intuitive aspects of both theories; the relativistic quantum world is fundamentally different from our macroscopic world. The course ends with a colloquium-like overview of recent developements in elementary particle physics.

Dedicated to
my parents

Contents

Introduction

"If you can't explain it simply, you don't understand it well enough."
-Albert Einstein

"There is nothing new to be discovered in physics now. All that remains is more and more precise measurements."
-Lord Kelvin

Around the year 1900, the situation in physics research can be best described with the above, famous quote from Lord Kelvin. The physics that Kelvin is referring to is what is currently called "classical" physics, which includes the mechanics of Isaac Newton, the theory of electromagnetism of James Maxwell and the statistical mechanics of Ludwig Boltzmann. There were at that time, however, two unsolved issues:

1. The mysterious existence of the æther: the measurements of the speed of light in the experiment of Michelson and Morley failed to detect the "aether", the medium that carries electromagnetic waves in vacuum.

2. The stability of the atom: if an atom consists of electrons circulating the nucleus, it should emit electromagnetic radiation and cannot be stable.

It turned out that these two puzzles led the way to a revolution in physics from "classical physics" to "modern physics". The absence of an aether is at the fundament of Einstein's theory of relativity; the stability of the atom is described by quantum mechanics. Both relativity and quantum mechanics have drawn a lot of attention and were hotly debated in a wide audience since they describe fundamental concepts in physics that may well be called counter-intuitive.

The theory of special relativity deals with objects that move with relative velocities much higher than we are used to in our everyday life experience. We will see that, based on two fundamental principles, our intuitive picture of time and space at low velocities will be changed drastically when we compare observers that move at relative velocities that approach the speed of light. The theory of general relativity introduces the equivalence between accelerated motion and gravitation leading to the notion of curved space-time and the phenomenon of gravitational waves.

Quantum Mechanics, in turn, deals with objects and phenomena that occur at much smaller sizes than we are used to in everyday life; at atomic and even sub-atomic scales.

The fundamental behaviour of matter and the character of the physics at the smallest scales is at least as strange as the theory of relativity.

Although both Einstein's theory of relativity as well as quantum mechanics have many counterintuitive aspects, Einstein's theory was never criticized by scientists, in contrast to the interpretation of quantum mechanics. The physical interpretation of the reality of a wave function is still a topic of discussion. However, until today the Copenhagen interpretation still stands, although in particular the "collapse" of the wave function is sometimes considered an unsatisfactory phenomenon.

It should be mentioned that the inventions of the theory of relativity and quantum mechanics do not make the classical theory "false". The old theory is still valid in what is called the "classical limit". In the limit of low velocities (compared to the speed of light) the Galilei laws of adding velocities works with a very high precision, while for macroscopic sized objects the Newtonian mechanics and Boltzmann statistics are very well applicable.

These lecture notes belong to a lecture series for first year university science students introducing the concepts of the theory of Relativity and Quantum Mechanics. Both for relativity as well as for quantum mechanics we investigate the nature of physics through several so-called *thought-experiments*, after the original German name *gedankenexperimenten*. These are hypothetical experiments that we could carry out assuming that our equipment has no limitations. Although such thought-experiments often cannot be carried out in practice today, there is no fundamental law in nature that prevents us to carry them out if our technology would be sufficiently advanced. The purpose of these examples is to illustrate the essence of the physics involved, if the experiments would be conducted under ideal circumstances and using perfect equipment.

The use of mathematics in the course is kept to a minimum. This implies that in particular the quantum mechanics theory is not presented in a rigorous and formal way. Also, although some historical facts are mentioned, I will not do justice to the correct historical developments of the theory, including many wrong turns (a very interesting topic in itself!). Instead, the main emphasis will be on conceptual aspects, trying to understand the character of physical law. There are a few sections of a slightly more advanced level, that perhaps involve a bit more mathematics. They are listed with an asterisk * and can be skipped without loss of continuity of the lectures.

These lecture notes are made in preparation of the lectures. They can be used by the audience, but should not be distributed. The original material presented in the courses is taken from books that are listed in the literature list below. Illustrations are often taken from the web.

Literature:

The following literature is used in the preparation of this course (the comments reflect my personal opinion):

Albert Einstein: “Relativity, the special and the general theory”, Routledge Classics, 2001.

Relativity presented by the hands of the master himself, first published in 1916. It is a concise text suited for beginning physics students, in which Einstein presents the special relativity theory in 60 pages, using only a minimum of formulae. The first lectures of this course are mainly based on the material presented in this book.

Sander Bais: ”De sublieme eenvoud van relativiteit”, Amsterdam University Press, 2007.

A visual introduction to the special theory of relativity. Instead of formulae, Sander Bais uses graphs to illustrate the concepts of simultaneity, time dilation and Lorentz transformations. Once you understand the diagrams, you never forget the concepts of relativity. As far as I know it is only available in Dutch.

Bernard F. Schutz: “A first course in general relativity”, Cambridge University Press, 1985.

A complete and excellent introduction on the theory of general relativity. Aimed at more advanced level undergraduate students, the book gives a thorough mathematical introduction to the Einstein equations and space-time curvature, gravitational waves, black holes and cosmology. Very suitable for a follow-up course for students interested in the mathematical formalism.

John Gribbin: “In search of Schrödinger’s Cat”, Black Swan edition, 1991.

A step by step introduction of quantum mechanics, telling the story of the old quantum theory of the atom up to the mysterious interpretation of quantum reality. A book without any formulae, explaining the developments in the previous century and expressing the discussions and doubts of the founders of the theory themselves. A very stimulating book on the interpretation of reality behind the laws of quantum mechanics.

Richard Feynman: “QED, the strange theory of light and matter”, Princeton University Press, 2006.

Feynman’s introduction to the theory of Quantum Electrodynamics (QED) using every day language and visualisations, including the famous Feynman diagrams. An excellent book to learn the concepts without any mathematics.

Richard Feynman, Robert Leighton, Matthew Sands: “The Feynman Lectures on Physics”, Addison-Wesley, 1966.

The 3-volume Feynman lectures are a classic for any physics student. Feynman’s unique and intuitive approach to physics is an eye-opener next to conventional textbooks on physics. For this course material of part III, ”Quantum Mechanics” was used: Feynman’s famous ”double slit” thought experiment. The Feynman lectures contain enough material for a full bachelor study on physics.

Paul Dirac: "The development of Quantum Theory", J. Robert Oppenheimer Memorial Prize Acceptance Speech at the University of Miami, Gordon and Breach science publishers, 1971.
A concise booklet including a speech in which Dirac reviews his own experience of the development of quantum mechanics in the first half of the previous century.

John Wheeler and Wojciech Zurech, editors: "Quantum Theory and Measurement", Princeton University Press, 1983.
An extensive collection on groundbreaking articles reviewing the development of quantum mechanics in the last century. It contains articles on the principles and interpretation of quantum mechanics, including the famous Bohr-Einstein dialogue, the concepts of hidden variables and complementarity and the role of the measurement process. It is a very complete set of original articles and commentaries at an advanced level.

In addition to these educational books I have enjoyed reading biographies of Einstein, Dirac and Schrödinger:

Walter Isaacson: "Einstein, his life and universe", Simon & Schuster UK Ltd, 2017.
An insightful description of the character and personal life of Albert Einstein. The biography extends from Einstein's childhood until his death. In the affectionately written text the reader gets to understand the ups and downs in Einstein's life. The ups include his insightfull moments developing special and general relativity theory, the downs his hesitation accepting the implications of quantum mechanics and his fruitless search for a unified field theory of electromanetism and gravity. The reader also gets a picture of Einstein's struggles in his personal life and unhappy marriages.

Abraham Pais: "Subtle is the Lord..., The science and the life of Albert Einstein", Oxford University Press, 1982.
A comprehensive scientific biography of the life of Einstein and his work, written by the Dutch physicist Pais, who joined Einstein in many walks during which they discussed physics. The book focuses on the scientific life of Einstein and is of rather technical level.

Graham Farmelo: "The Strangest Man, the hidden life of Paul Dirac, quantum genius", Faber and Faber Ltd, 2009.
A very entertaining text on the personal life of Paul Dirac, who is sometimes called the British Einstein. Dirac was the youngest theoretician ever to win the nobel prize for his prediction of the existence of antimatter. Farmelo brings the very introvert and strange personality of Dirac to life in a truly entertaining book.

John Gribbin: "Erwin Schrödinger and the quantum revolution", Black Swan, 2013.
The biography shows the personal life of Erwin Schrödinger, the inventor of the famous equation in his own name, describing fundamental particles as waves. His wave mechanics was the alternative interpretation of quantum mechanics to Heisenberg's more abstract matrix mechanics, until Dirac demonstrated that both can be seen as two views of the same fundamental objects. The book reviews Schrödinger's personal life, his career, and his hesitations on the interpretation of quantum mechanics.

Lecture 1

The Principle of Relativity and the Speed of Light

"Everything should be made as simple as possible, but not simpler"
-Albert Einstein

Albert Einstein and Annus Mirabilis 1905

The year 1905 is referred to as Einsteins "Annus Mirabilis" (miracle year). In that year Einstein published three groundbreaking papers in the scientific journal "Annalen der Physik". The papers were on three completely different topics:

1. The photo-electric effect: proposing an experiment to prove that light consists of photon quanta,

2. Brownian motion: explaining the microscopic motion of matter leading to the evidence that matter consists of atoms,

3. The special theory of relativity: demonstrating how space and time appear different for different observers.

Although Einstein is mostly known for his formula expressing the equivalence of matter and energy ($E = mc^2$), which is a consequence of the theory of relativity, he was rewarded the Nobel prize in 1922 in particular for his work on the photo-electric effect, a topic at the fundament of quantum mechanics. Interestingly, Einstein later argued against the interpretation of quantum mechanics as a correct fundamental theory. Before discussing quantum mechanics and its interpretation we first address relativity theory. We will try to understand how different observers, moving relatively to each other with a constant speed, observe elapsed time between two events and their separation in space in a different way. Interestingly, this follows by the requirement that the laws of physics for the two observers must be *the same*.

1.1 Reference systems and the Galilei transformation

Let us start by considering a first *gedankenexperiment*. Consider a boat sailing along a river with a speed of 15 km/h with respect to the land. On the deck of the boat Alice is running in the same direction with a speed of 10 km/h. What would be the velocity of Alice if it were measured by Bob, who is standing on the dike watching the boat and Alice pass by?

Our intuitive calculation tells us to calculate the velocity as: 15 km/h + 10 km/h = 25 km/h. This reasoning is called *Galilean addition of velocities*.

To express things a bit more formal we consider the boat as a so-called reference system or coordinate system and label it as S, which moves with a velocity v with respect to the land, which is a second reference system S'. For any object that moves with a velocity w in S, in the same direction of movement of S', the law of addition of velocities is from reference system S to S' :

$$w' = w + v \tag{1.1}$$

and the transformation of system S to system S' is called a Galilei transformation. We call the frames S and S' inertial frames.

For completeness we note that if Alice would be walking in the opposite direction as the speed of the boat, the law would of course not be the sum but the *difference* of the two velocities: $w' = v - w = -w + v$. Since a velocity can be represented by a number and a direction, represented by $\vec{v}$ or $\vec{w}$, the general transformation law is: $\vec{w}' = \vec{w} + \vec{v}$.

In a similar way we can calculate the speed at which two cars that approach each other from opposite directions, collide in a head-on accident. If the first car moves at 40 km/h and the second 60 km/h they approach each other and collide with 40 km/h + 60 km/h = 100 km/h. Alternatively, if they drive in the same direction with the faster car behind the slower one, they approach each other with 60 km/h - 40 km/h = 20 km/h. We know from our everyday experience that the Galilean law of adding velocities must be approximately correct in practical circumstances. However, perhaps surprisingly, Einstein teaches us this is *not exactly correct*, and we will derive the correct formula later in Section 3.4.

Let us now turn to the two postulates of the Special Relativity, which state that:

> For two observers in two so-called *inertial frames*, i.e. they move with a constant velocity relatively to each other:
>
> 1. The laws of physics for each observer are the same;
> 2. The speed of light in vacuum for each observer is the same.

We will discuss these postulates below.

1.2 The principle of relativity

Alice goes into her cabin inside the boat and takes a nap. As she wakes up, the water is perfectly calm and she can detect no movement. Her cabin has no windows and Alice wonders if there is a way *inside her cabin* to find out whether the boat is actually moving or not. (Since this is a gedankenexperiment we assume that the engine of the boat makes no sound etc.) To test it she throws a ball in the air and she detects nothing special in way the ball goes up and down. In particular, it does not "stay behind" with respect to the movement of the boat. The ball makes the same trajectory as it would if the boat would be at rest.

More generally, the principle of relativity states that there is *no way* that a *constant speed* can be detected from within the boat. This is the first postulate of Einstein's theory of relativity and it states that all laws of physics are identical in each inertial frame. To observe the *relative speed* with respect to the land she has to look outside.

*An Intriguing thought:
This postulate implies that only relative velocities can ever be measured and consequently the meaning of having an absolute velocity becomes unclear. How does one determine if something is absolutely at rest?

1.3 The speed of light

The second postulate of the theory of relativity is related to the speed of light. As we learn in school, the speed of light in vacuum is a constant: $c = 299\,792\,458$ m/s, about 300 000 km/s. This velocity does not depend on the colour of light, nor on the speed of the object emitting the light or on the speed of the observer. It is a physical constant and embedded into the laws of electromagnetism as the velocity with which electromagnetic waves travel through empty space. We also learn that nothing can travel faster than light, the light-speed is the ultimate velocity any object can ever have[1]. The constant speed of light, however, causes a crisis to the principle of relativity.

1.4 A contradiction between the principle of relativity and the speed of light?

Let us make a thought experiment of a rocket in space travelling at a very high speed. Inside the rocket Alice is looking out of the window to a star and she measures that her velocity with respect to the star is 250 000 km/s. From the opposite direction Bob approaches her in another rocket, and he measures his velocity to be 200 000 km/s. From this information Alice and Bob use Galilei's law of addition of velocities to calculate that

[1]Some time ago an experiment with neutrino particles reported on an observation of a speed just above the speed of light, but the measurement was found to be disturbed by a badly functioning optical connection in a cable.

they approach each other with a relative velocity of 250 000 km/s + 200 000 km/s = 450 000 km/s, significantly higher than the speed of light!

After Bob passed the star his rocket moves away from the star with 200 000 km/s, the light shining from the star travels at 300 000 km/s through space and passes Bob's rocket. As he measures the velocity of the light, what will he find? Is it 300 000 km/s − 200 000 km/s = 100 000 km/s? But that would be in contradiction with the postulate of the constant speed of light. *Any observer in any reference frame* should observe the same speed of light.

To make the contradiction even worse, consider that the rockets of Alice and Bob fly in the same direction, Bob leading with 200 000 km/s and Alice following him with 250 000 km/s in the same direction. Alice now takes out a flashlight and shines a beam of light out her front window, toward the rocket of Bob where it enters in the rear-window into his cabin. Bob measures the velocity of the light entering his rocket cabin. What will he find?

What is the velocity of light in the cabin of Alice? What is the velocity of the *same light beam* in the interstellar vacuum and finally in the cabin of Bob?

It seems that either the principle of relativity or the constancy of the speed of light cannot be held. Since the principle of relativity is so intuitive we expect that it is the constancy of the speed of light that cannot be maintained. Most probably it is due to the fact that in practice we never travel at such high speeds that we never observe differences in the speed of light.

Perhaps it is time to turn to a real experiment: the measurement of the speed of light.

1.5 The Michelson and Morley experiment

In 1849 Armand Fizeau was the first person to measure the speed of light and found 315 000 km/s, within 5% of the correct version. In 1862 Léon Foucault improved the method and obtained a value of 298 000 km/s. For illustration purpose, the method used by Fizeau and that of Foucault are shown in fig. 1.1.

From the theory of electromagnetism we know that light is a wave of an oscillating electric and magnetic field that propagates through vacuum, see the illustration in fig. 1.2. The speed of light is the speed at which electromagnetic waves propagate. It is related to the value of two constants: the electric permittivity of the vacuum (ε_0) and magnetic permeability of the vacuum (μ_0), via the relation $c = 1/\sqrt{\varepsilon_0 \mu_0}$. A detailed understanding of this relation is not required here apart from the point that an alternative measurement of the speed of light was possible by measuring the electric and magnetic constants ε_0 and μ_0. This measurement was done in 1856 by Wilhelm Weber and Rudolf Kohlraush and was found to be consistent with Foucault's measurement.

At the time, it was thought that empty space was filled with a so-called *æther* which carries the electromagnetic wave (e.g. compare to air carrying a sound wave). A first

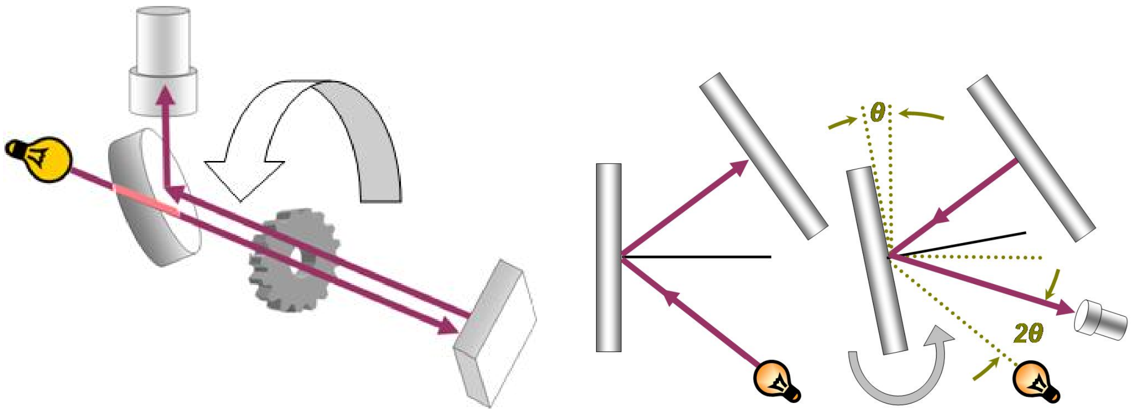


Figure 1.1: Schematic view of the apparatus of Fizeau (*left*) and that of Foucault (*right*). Both measure the time delay of a light beam reflected in a mirror at a distance of several km's. Fizeau projects a beam of light on a distant mirror using a rotating cogwheel which obscures the reflected beam by teeth, while Foucault uses a rotating mirror to project the beam on the distant mirror and measures the angle of the reflected light beam.

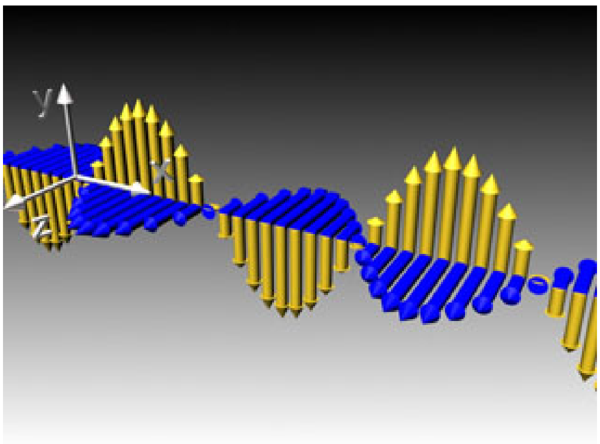


Figure 1.2: An illustration of a beam of light; perpendicular oscillations of electric and magnetic fields perpendicular to the direction of propagation of the beam.

attempt by Michelson to measure the motion of the earth through the reference frame that carried the æther failed due to limited accuracy of the apparatus. The measurement is based on the idea that the speed of light was a constant with respect to the reference frame in which the æther was at rest. Since the earth is moving through the æther in a specific direction, one can therefore measure the difference of the light-speed in different directions. In 1887 Michelson and Morley precisely determined the speed of motion of two perpendicular light beams using an improved version of Michelson's earlier failed experiment. The concept of the measurement is shown in fig. 1.3, while sketches of the first apparatus used by Michelson and the improved version of Michelson and Morley is shown in fig. 1.4.

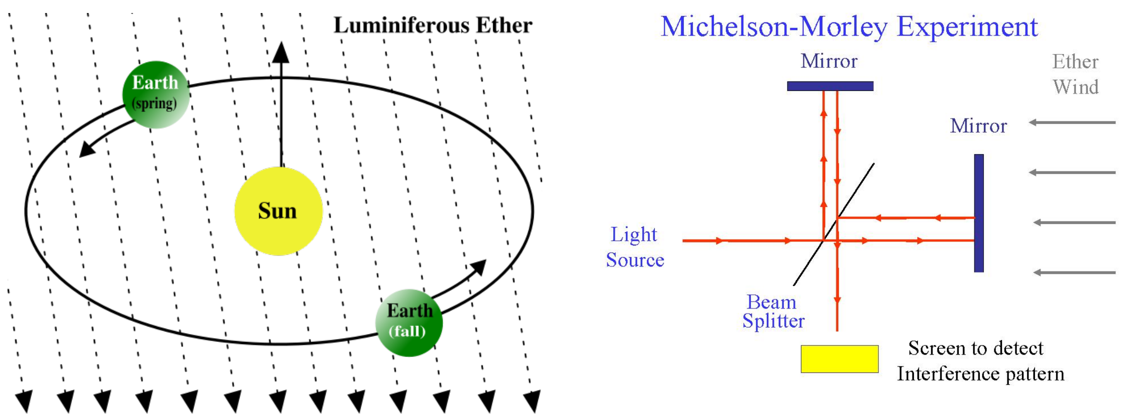


Figure 1.3: *Left:* If the universe is filled with an æther then the velocity of the earth through the æther would vary through the year. *Right:* An attempt to observe the æther via measurements of the light-speed in perpendicular arms of the Michelson Interferometer, assuming that the æther wind is from the right. A difference in the delay time of the two beams can be observed in the interference pattern of the reflected beams. Remember that it takes longer to swim up and down a distance of 100m in a river parallel to the direction of water flow than the same distance perpendicular to the flow.

It was expected that due to the earth's motion through the æther, the measured light speed would be different in the perpendicular arms. Also they expected to see a yearly cycle because of the earth's motion around the sun.

Exercise:

Calculate the expected delay time for the beams if the ætherwind has a velocity v along the direction of one of the arms of the interferometer, assuming the length of the interferometer is L.

Answer: $t_{\text{perpendicular}} = \frac{2L}{c} \frac{1}{\sqrt{1-v^2/c^2}}, t_{\text{parallel}} = \frac{2L}{c} \frac{1}{1-v^2/c^2}$

The Michelson-Morley experiment observed that there was no significant time difference. The velocity of light is the same constant in the two arms. It did not depend on day or night or the time of year (i.e. the position of the earth around the sun). At

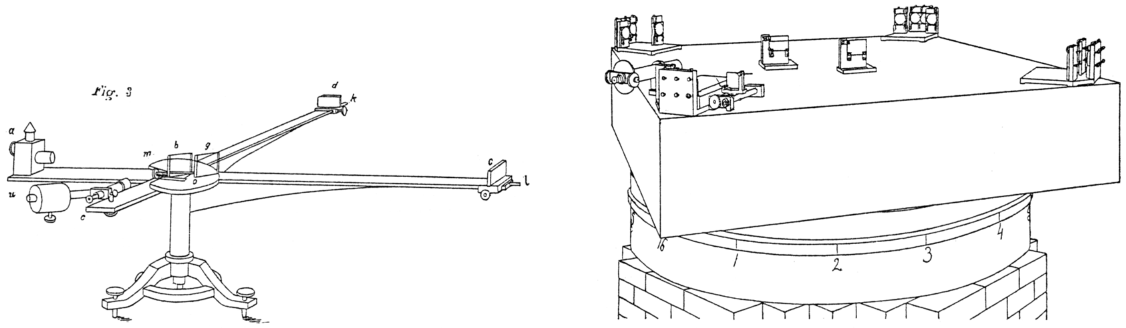


Figure 1.4: *Left:* A sketch of the original device used by Michelson. *Right:* A sketch of the improved measurement by Michelson and Morley.

the time, this could also be interpreted that the earth is "dragging" the æther along, so that the æther is "at rest" on earth.

To date, it has been verified that the observed speed of light is always the same: c is a constant independent of the frame of reference!

1.6 Does an absolute velocity exist?

Let us consider a beam of light in vacuum without the presence of an æther. As illustrated in fig. 1.2, in Maxwell's theory of electromagnetism, light consists of electromagnetic waves that propagate through the vacuum with the speed of light given by the relation: $c = 1/\sqrt{\varepsilon_0 \mu_0}$. The propagation takes place by oscillations of electric and magnetic fields perpendicular to the direction of motion ("transverse" oscillations). In the absence of an æther the propagation of waves occurs in empty space (vacuum) with a velocity c. But, can one consider a vacuum "at rest" or, alternatively, can one imagine a "moving" vacuum? How can the vacuum look different for me when I am "at rest" or when I move "through it"?

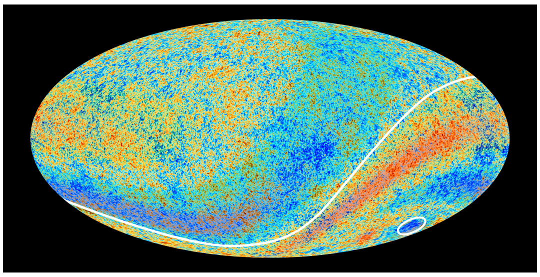
The principle of relativity states that there is no difference of the vacuum as seen by two relatively moving observers. Empty space is the same for both observers. One might phrase this alternatively by saying that an absolute velocity with respect to the vacuum does not exist. There is no way to measure an absolute speed since all laws of physics for two observers (think of Alice on the boat) are the same, including the laws of Maxwell and the propagation of the electromagnetic fields. Since there is no difference of a vacuum seen by a person "in rest" and a vacuum seen by a person "in motion", both observers will conclude that the speed of light in "their" vacuum is the same, independent of the fact that they move relatively to each-other. This is the essence of the theory of special relativity.

1.6.1 *The cosmic microwave background

Models of the Big Bang origin of the universe predict that an isotropic remnant electromagnetic radiation with a temperature of 3^oK should be present. In 1964 Arno Penzias and Robert Wilson accidentally discovered this so-called cosmic microwave background (CMB) radiation as a strong background in their attempts to communicate to satellites using radio waves.

The measurement of the CMB has been carried out in increasing precision by satellite experiments: first COBE, later WMAP and recently Planck. A detailed sky map of the temperature (or wavelength) of the radiation has been made. After correction for the movement of the earth the radiation is seen to be the same across the whole sky (i.e. it is isotropic), confirming theories of the Big Bang origin of the universe. Before correcting for the movement of the earth, a strong so-called *dipole effect* is visible as consequence of the movement of the earth through the CMB "wind".

Food for thought: apparently a special inertial frame in the universe does exist: that in which the CMB has no dipole effect, i.e. the frame in which it is *isotropic*.



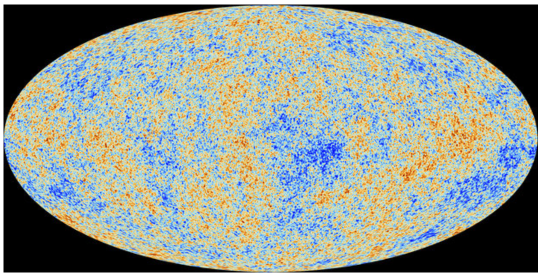


Figure 1.5: The temperature spectrum of the cosmic microwave background as observed by the Planck satellite. *Left:* The spectrum as seen after subtraction of the signals caused by our own milky way. The structure seen in red and blue is the so-called "di-pole" effect, caused by the movement of the earth around the sun and around the galaxy centre. *Right:* The spectrum as observed after correcting for the "di-pole" effect. The average temperature is 2.726 Kelvin while the temperature difference between the blue and red points is less than 1 mKelvin.

Lecture 2

Time Dilation and Lorentz Contraction

"When you are courting a nice girl an hour seems like a second. When you sit on a red-hot cinder a second seems like an hour. That's relativity."
-Albert Einstein

2.1 Relativity of Simultaneity

The Galilean transformation law assumes that time is universal for all observers in their reference systems. What this means is that the occurrence of two simultaneous events for one observer (Alice) are also simultaneous for another observer (Bob). Before we investigate this further we have to ask ourselves how to precisely define a simultaneous occurrence of events that do not take place at the same position. Since we now accept that the speed of light is an absolute universal constant for all observers we can use it to define simultaneity of distant events as explained below.

This time we ask Alice to board a fast train speeding along a straight track while Bob stands alongside the track, watching the train pass by (see fig. 2.1). At a given point in time Bob observes that lightning strikes at two places at the same time; one at a distance of 10 km along the train track in one direction (position A), and one at a distance of 10 km along the train track in the other direction (position B). How does he know that these events occur at the same time? The light from the lightning at A takes 10/300 000 seconds to reach his eye and the same is true for the light of the lightning at B. Since the distance to A and B is the same and also the light-speed is the same for both signals, the delay between the occurrence of each event and Bob's observation is the same. If Bob sees, from the corner of his eyes (or by using two mirrors), both lightning strikes at the same time, he concludes that the two events are simultaneous.

We now take the view of Alice aboard the train. Consider that Alice and Bob happen to be at the same position at the time that the lightning strikes according to Bob. By the time that the light has travelled 10 km, Alice, on the train, has moved in the direction of the velocity of the train so that she is no longer in the middle of positions A and

B. Let us say she moves away from A and towards B. Therefor she will conclude that the two lightning strokes are *not* simultaneous. Who is right? Are they simultaneous or not?

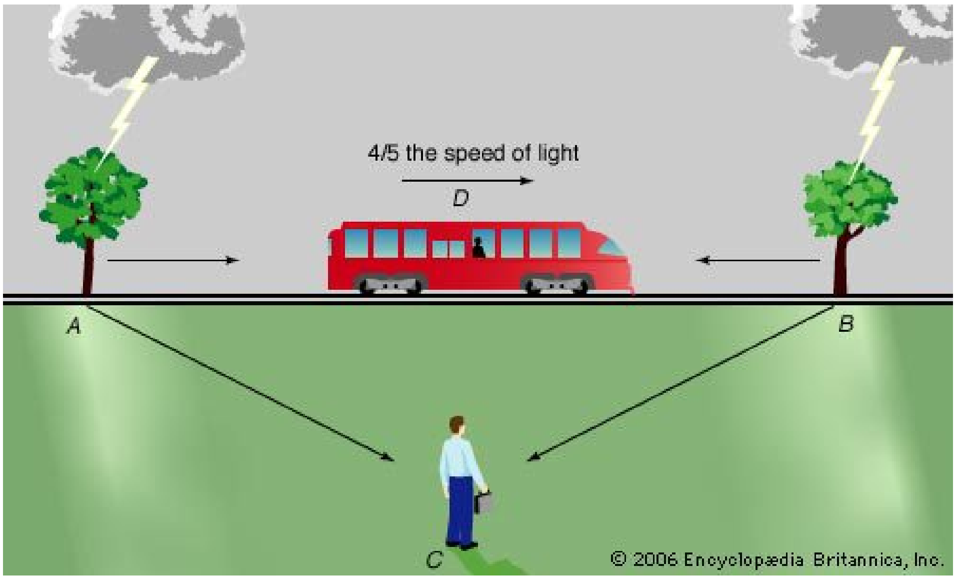


Figure 2.1: Defining the concept of simultaneity. Bob, standing at position C in the middle of A and B, sees the lightning strokes at A and B simulteneously. But Alice, at position D, does not!

Intuitively, we are tempted to say that Bob is right since he is not moving with respect to the earth. But let us reconsider the experiment in the vacuum of space and replace the train with rockets. In this case there is no external reference system and the vacuum seen by Alice is completely equivalent to that seen by Bob. Who is moving and who is standing still? Both will conclude that the light travels towards them from A and B with the same speed, only one of them will call the events simultaneous while the other does not.

The conclusion is that simultaneity of events depends on the observer, or the coordinate system. Before relativity we have assumed that time is a global constant for all frames. In relativity it is not. Each reference frame has its own time label.

2.1.1 Relativity of Distance

We ask Alice in the train to measure the length of the train. The simplest method for her would be to go through the train with a ruler and measure the distance from the front to rear ends. She here uses a different method that allows her to compare her results to a measurement of Bob. We ask her to *simultaneously* make two tick-marks at the tracks corresponding to the front and back of the train (remember it is a thought-experiment - she might use some sort of pen-on-a-stick system). To make a comparison, we now ask Bob, standing aside at the tracks, to measure the length of the passing train. He also does this by setting *simultaneous* marks at the front-end end rear-end of the train and subsequently measuring the distance of the two marks with a ruler. Since we have seen before that Alice and Bob *do not agree* on simultaneity of two events, Alice will protest that Bob does not set the tick marks at the same time at the front end rear end of the train and as a consequence the two measured lengths will not agree.

We conclude that in making a Galilei transformation we have to make two corrections in intuitive axiom's:

- a time interval between two events depends on the reference frame,
- a space interval ("distance") between two events depends on the reference frame.

2.2 Time Dilation

Since the light speed is a perfect constant of nature, independent of the observer, we can use it to construct a perfect clock to measure time intervals. Take two flat mirrors and place mirror B at a distance of 1 meter above mirror A, with the two reflecting sides facing towards each other as indicated in fig.2.2. A light beam that is injected in-between the mirrors will be reflected infinitely many times between the mirrors, assuming a thought-experiment of perfect mirrors. In this way we can use it as a clock, since the distance between the mirrors is a constant of 1 meter and the speed of light is constant. The unit of time ("a tick of the clock") can be defined as the time it takes between two reflections of the light beam (1 [m] / 3×10^8 [m/s], or phrased otherwise, one second is the time it takes for the light beam to be reflected 1.5×10^8 times at the top mirror A.

Consider such a "light-beam clock" in the same fast train, sitting next to Alice. As the train takes off and reaches a very high speed Alice notices nothing unusual as she is in rest relatively to the clock.

But let us turn to Bob instead. He observes that the two mirrors move with the speed of the train and accordingly the light beams have to travel a distance longer than a meter, which can be calculated simply using the Pythagorean theorem. Let us say that the fast train travels with half the speed of light, so that it has moved 0.5 meter during the time that the light has travelled from mirror A to mirror B. In that case the distance of the light travelled, according to Bob, is $\sqrt{1^2 + 0.5^2}\,\text{m} = \sqrt{1.25}\,\text{m} \approx 1.12\,\text{m}$. Since Bob knows that the speed of light is a constant of 3×10^8 m/s, he will conclude that the clock on board of the train has slowed down. This phenomenon is referred to as time dilation.

We proceed to calculate the exact formula for the time dilation. Say that the train moves at velocity v and that the time it takes the light beam to travel from mirror A to B, separated by a distance d, according to Alice is Δt while the corresponding time for Bob is $\Delta t'$. Bob will see that in the time $\Delta t'$ it requires the light to go from A to B, the train has moved by a distance $v \cdot \Delta t'$, such that the distance travelled by the light

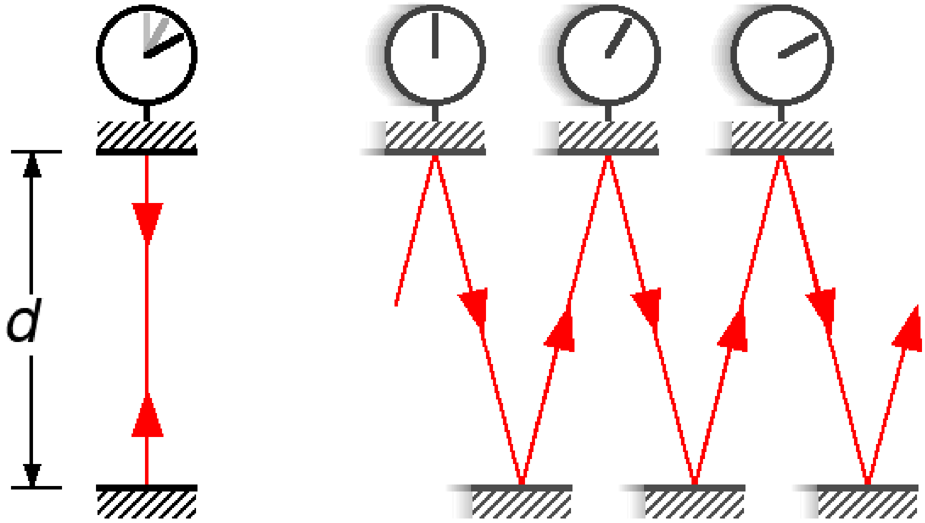


Figure 2.2: A clock made out of two mirrors. *Left:* The mirrors are separated a distance d from each other, such that the time used by a light-beam to go up and down is $\Delta t_{\rm left} = \Delta t = 2d/c$. *Right:* The same clock viewed moving to the right relatively to the observer with high speed. The distance travelled by the light beam now becomes larger and therefore $\Delta t_{\rm right} = \Delta t' > 2d/c$. The clock will tick slower for this observer.

from A to B, according to him, is:

$$d' = \sqrt{d^2 + (v\,\Delta t')^2} \quad . \tag{2.1}$$

For Alice the time elapsed is $\Delta t = d/c$, while for Bob the time elapsed is $\Delta t' = d'/c$. Substituting $d = c\,\Delta t$ and $d' = c\,\Delta t'$ in the above equation we get:

$$c\,\Delta t' = \sqrt{(c\,\Delta t)^2 + (v\,\Delta t')^2} \tag{2.2}$$

and subsequently:

$$(c\Delta t')^2 = (c\Delta t)^2 + (v\Delta t')^2 \quad \Rightarrow \quad (c^2 - v^2)(\Delta t')^2 = c^2(\Delta t)^2 \quad \Rightarrow \quad (\Delta t')^2 = \frac{c^2}{(c^2 - v^2)}(\Delta t)^2 \tag{2.3}$$

from which the time dilation factor γ follows by taking the square root of the equation:

$$\boxed{\Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}} \cdot \Delta t \quad \equiv \gamma \cdot \Delta t.}$$

Since Bob concludes that time has slowed down at the train, does it mean that consequently Alice concludes time runs faster for Bob? The answer is no, as can be understood by the way Alice observes a similar clock at the train-station. Relatively to

Alice, this clock moves with the same velocity v in the opposite direction and its ticking is slowed down with the same amount. According to the theory of relativity the only thing that matters is the relative speed and both Alice and Bob will conclude that the clock of the other has slowed down with the same factor of γ [1].

2.2.1 Muon particles

One might wonder whether the slow-down of the clock for an observer actually corresponds in reality to time proceeding slower. The following example shows the effect for elementary particles.

The earth atmosphere is bombarded constantly with cosmic ray particles, mainly consisting of protons. These high energy particles collide with atoms in the atmosphere and in the cases where the energy of the incoming particle is sufficiently high, new particles called secondaries, are created. One of these new particles is the muon, which is a particle that is well known from studies in laboratories. It is an unstable particle that has a half-life of 1.56 μs. This means that after 1.56 μs a muon has 50% probability to quantum-mechanically decay into other particles. Alternatively phrased: the number of surviving muon particles is 50% after 1.56 μs, 25% after 3.12 μs, etc.

These muons can be often produced with a velocity of 98% of the speed of light at an altitude of 10 km in the atmosphere. According to classical mechanics, this implies that it would take them about 34 μs to reach the surface of the earth, corresponding to about 22 times their half-life. This means that in case a million muons would be created, less than one muon (the calculated average is 0.24) would be expected to arrive at the surface of the earth.

However, observed from the earth the time-dilation factor for such a high speed muon is $\gamma = 1/\sqrt{1-0.98^2} \approx 5$, such that the observed half-life is 5 times longer $t_{\mathrm{obs}} = 7.79\mu$s. In that case the expectation of the number of surviving muon particles out of 1 million is about 48 000. The situation is depicted in fig. 2.3. Measurements of the number of arriving muons at the surface of the earth unambiguously show that the relativistic result is correct.

The following subsection describes a practical application of the relativisitic correction in navigation technology.

2.2.2 GPS Navigation

GPS navigation devices communicate via electromagnetic waves to satellites. By measuring the delay-times of radio-signals using synchronised clocks at the satellites, distances are measured. The fact that these satellites travel with speeds of about 40 000 km/h must be taken into account. A speed of 40 000 km/h, or 11 km/s corresponds to a small time dilation γ that would add up to 60 microseconds per day. Since time is used to measure the distances of the radio-waves, which travel with the speed of light, this

[1] Compare it to the following: if you look at someone in the distance he/she appears smaller. If that person looks back at you, you also appear to look smaller.

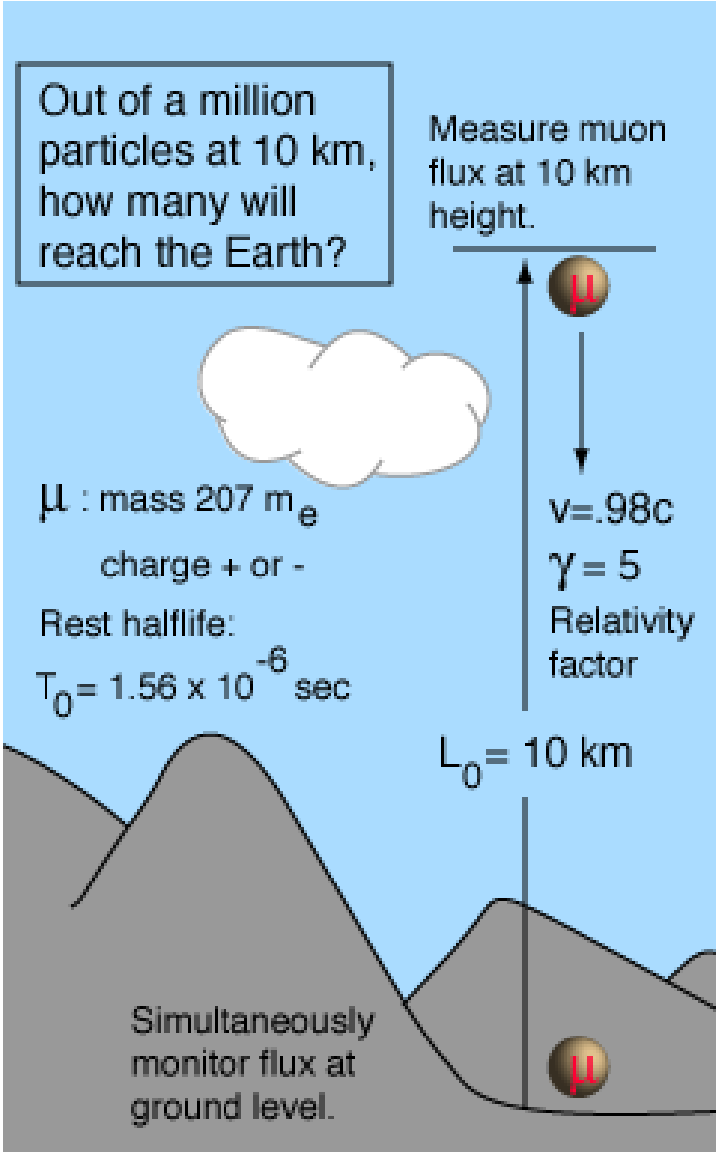


Figure 2.3: Muon particles are typically created with a velocity of 98% of the lightspeed at an altitude of 10 km. Since they have a half-life of 1.56 μs they are not expected to arrive at ground level in classical theory. Special relativity, however, gives the correct prediction.

means that each day an incremental miscalculation of 17 km would be made, if classical theory would be used. The real accuracy of the devices is about 10 m.

2.3 Lorentz Contraction

2.3.1 Travelling to the stars

Let us consider the consequences for length measurements. Alice now boards a spaceship and she takes off to the nearest star: proxima centauri. The number of ticks of her clock during the travel is registered by a counter that records the number of times the light has reflected in mirror A, leading to a total travel time t'. Knowing her relative velocity with respect to the stars Alice now calculates the distance L' to proxima centauri as:

$$L' = v \cdot t' \tag{2.4}$$

Bob, on earth, agrees with the number of clock ticks that are registered, but he claims that each tick of Alice's clock is slowed down by the time dilation factor γ such that according to him the total travel time is:

$$t = \gamma t' \tag{2.5}$$

Since Bob and Alice agree on their relative velocity, Bob will claim the distance from earth to proxima centauri is:

$$L = v \cdot t = v \cdot \gamma t' = \gamma L' \tag{2.6}$$

Since $\gamma > 1$, it turns out that Alice sees a shorter distance L' than Bob L. The fact that distances shrink at high speeds is called Lorentz contraction. The amount of the contraction is given by:

$$\boxed{L' = L\sqrt{1 - v^2/c^2} \quad .}$$

These equations express that an observer looking at occurrences and objects that are moving with a constant velocity with respect to him, sees that the progress of time slows down and that distances and sizes of objects become shorter as compared to when they are at rest. In the next lecture we will see how this can lead to paradoxes if not evaluated very carefully. The amount of time dilation and distance contraction is the relativistic factor γ and depends on the relative velocity involved. For velocities we are accustomed to in every day life the effect is very small. It becomes significant when the velocities become of the order of the light speed. The graph in fig. 2.4 shows the value of the factor γ as a function of the velocity v.

The following subsection gives an alternative derivation of the Lorentz contraction, but can be skipped without loss of continuity.

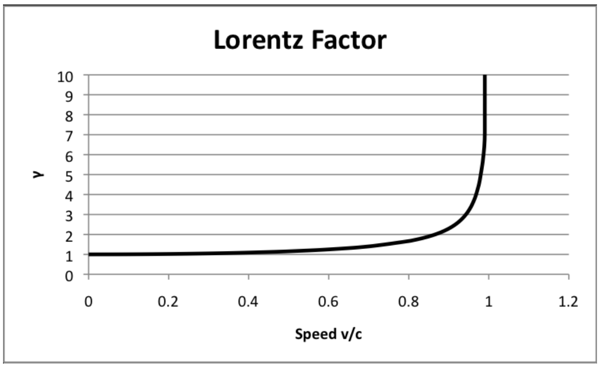


Figure 2.4: The relativistic time dilation and distance contraction factor γ as a function of the speed $\beta = v/c$. Notice that for low velocities (close to "0") there is almost no effect, while for velocities approaching the speed of light (close to "1") the effect becomes very large.

2.3.2 *The length of a speeding train

An alternative way to derive the Lorentz contraction formula can be done using the thought experiment of the speeding train. Alice stands at the rear end of the train and she sends a light beam across the train from the rear to the front, where it is reflected by a mirror. She measures the delay time of the light beam and calculates the length of the train, L, using the formula: $2L = c\,\Delta t = c\,\Delta t_1 + c\,\Delta t_2$. Although she does not measure the individual travel times from rear to front (Δt_1) and for the return from front to rear (Δt_2) she will assume they are identical.

Now, let us see how Bob, standing at the platform calculates the length of the train: L'. During the time the light takes Alice's light beam to travel from the back to the front of the train ($\Delta t'_1$), the train has actually moved a small additional distance ($\Delta L'_1 = v\,\Delta t'_1$). For the time it takes the light to go back to the rear ($\Delta t'_2$) the distance is reduced by ($\Delta L'_2 = v\,\Delta t'_2$). So, in total Bob calculates that the time from back to front and front to back are given by:

$$\Delta t'_1 = \frac{L' + v\,\Delta t'_1}{c}$$
$$\Delta t'_2 = \frac{L' - v\,\Delta t'_2}{c}$$

Eliminating $\Delta t'_1$ and $\Delta t'_2$ gives the expression for the total expected time:

$$\Delta t'_1 = \frac{L'}{c - v} \quad ; \quad \Delta t'_2 = \frac{L'}{c + v} \tag{2.7}$$

and for the total time:

$$\Delta t' = \Delta t'_1 + \Delta t'_2 = \frac{L'(c+v)}{c^2 - v^2} + \frac{L'(c-v)}{c^2 - v^2} = \frac{2L'}{c}\,\frac{1}{(1 - v^2/c^2)} \tag{2.8}$$

Alternatively, Bob observes that time is running slower for Alice according to the time dilation formula so that he knows that his time interval is related to that of Alice according to:

$$\Delta t' = \Delta t \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{2L}{c} \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.9}$$

Since the two expressions for $\Delta t'$ are valid we can relate L' to L:

$$\frac{2L'}{c} \frac{1}{(1 - v^2/c^2)} = \frac{2L}{c} \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.10}$$

and therefor that

$$\boxed{L' = L\sqrt{1 - v^2/c^2} \quad .}$$

In other words, the length of the moving train as seen by Bob is *shorter* than the length as seen by Alice on the train. The fact that distances shrink at high velocities is called Lorentz contraction.

The Lorentz contraction and time dilation are often expressed in terms of the relative speed ratio $\beta = v/c$ and the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$:

$$\boxed{\begin{aligned} L' &= L \cdot \sqrt{1 - v^2/c^2} = L \cdot \sqrt{1 - \beta^2} = L/\gamma \\ \Delta t' &= \Delta t \cdot \frac{1}{\sqrt{1 - v^2/c^2}} = \Delta t \cdot \frac{1}{\sqrt{1 - \beta^2}} = \Delta t \cdot \gamma \end{aligned}}$$

2.3.3 Muon particles revisited

Let us turn our attention back to the muon particles that are created at a height of 10 km in the earth's atmosphere. In the reference system of the particles, i.e. considering a muon itself as an observer, the half-life of these unstable particles is 1.56 μs. How do these particles manage to survive the distance of 10 km between their point of creation and their point of decay? For an observer on earth their half-life is 7.79 μs, but that is not true as seen by the muon particle. The muon particle relatively sees the surface of the earth approaching with high velocity (a value of $\gamma = 5$). Therefore it will see the distance from the point of creation to the point of decay as contracted to 10 km / $\gamma =$ 2 km.

So the observer on the earth concludes that the muon particles live longer by a factor γ so that it can arrive at the earth, while the observer travelling with the muon particles concludes that the distance reduces by a factor γ. Comparing their conclusions, they will both, however, predict the correct number of particles arriving at the surface of the earth.

It seems there is arbitrariness when we try to define sizes of objects or length of time spans as they depend on the reference frame of the observer. As a general rule the size of objects and the lifetime of particles are always specified for an observer that is relatively *in rest* with respect to the object. We call this the rest frame and we speak of *proper time* and *proper length*.

Lecture 3

The Lorentz Transformation and Paradoxes

"Imagination is more important than knowledge."
-Albert Einstein

3.1 The space-time coordinate system

The occurrence of an "event" (e.g. the start of a football match) is specified by giving its position (the Geusselt-stadium in Maastricht) and time of occurrence (Octobre 1, 2014, 20h00). In physics we indicate the location of an event by three space coordinates (x, y, z) with respect to a given reference point $(0, 0, 0)$ and a single time coordinate (t) with respect to an origin in time. The reference system for space-time is called Minkowski-space, after the German mathematician Hermann Minkowski. In such a reference frame an event "a" is specified by four numbers (t_a, x_a, y_a, z_a). We have learned that according to the relativity principle the laws of physics are the same in reference systems that move with constant relative velocities. All such reference frames are equivalent.

To simplify the discussion we assume here that movements occur along the direction of the spatial x-axis. This is not a loss of generality as we can always choose the axis in the direction we wish.

What we have done in the previous lecture is to compare a given event (eg. a lightning strike) in two different reference frames (eg. Alice's: a moving train, and Bob's: the platform). Imagine the lightning strikes at position x and time t in Alice's reference system S, we can try to find the corresponding coordinates x' and t' in Bob's reference system S' moving with relative velocity v along the x-direction.

In general, if an event occurs in reference system S at the space-time location (t, x, y, z), we ask what would be the corresponding co-ordinates in reference system S', which moves at a velocity v along the x-direction. The situation is illustrated in fig. 3.1.

If physics would behave according to Galilei's reasoning, the transformation would

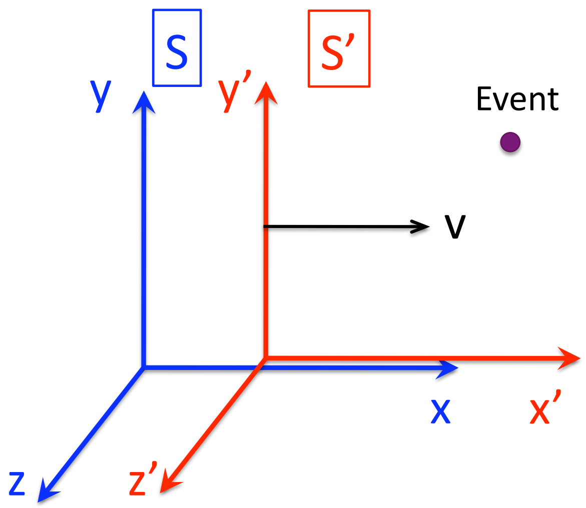


Figure 3.1: The same event has different coordinates in two reference systems S and S'. The translation from S to S' is given by the Galilei transformation in classical theory and by the Lorentz translation in special relativity.

be:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned} \tag{3.1}$$

The first line refers to the universality of time in Galilei transformations, while the second line gives the ”intuitive” addition (or subtraction) of velocities. We have seen in the previous lecture, however, that time is not universal and that relative stretches of time and space are different for different observers. What is then the corresponding transformation for an event going from frame S to S'? The answer is called the Lorentz transformation:

$$\begin{aligned} t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \end{aligned} \tag{3.2}$$

Using the Lorentz transformations we specify how the coordinates of a given event as seen by observer S: (t, x, y, z) translate to the coordinates for an observer S': (t', x', y', z') who is moving relative to S with a velocity v in the x-axis direction.

The Lorentz transformations can be written in a more concise way introducing the parameters β and γ:

$$\begin{aligned}\beta &= v/c \\ \gamma &= \frac{1}{\sqrt{1-v^2/c^2}} \quad ,\end{aligned}$$

where β is the fractional speed with respect to the light speed and γ is the relativistic factor we encountered in lecture 2. The Lorentz transformations then become:

$$\boxed{\begin{aligned}x' &= \gamma\,(x - \beta ct) \\ ct' &= \gamma\,(ct - \beta x)\end{aligned}}$$

and the inverse transformations are:

$$\boxed{\begin{aligned}x &= \gamma\,(x' + \beta ct') \\ ct &= \gamma\,(ct' + \beta x')\end{aligned}}$$

From experience in everyday life we are accustomed to observers who move with low relative velocities, which means that v is much smaller than c, such that $\beta \approx 0$. In that case the numerical value of the relativistic factor γ is very close to 1. As a consequence the Lorentz transformation becomes equal to the Galilei transformation. In other words, our every day life observations are well in agreement with the Galilei transformation, which is the reason that we intuitively feel that this must be correct.

However, when the velocity c approaches the speed of light: so $v/c = \beta \to 1$, the relativistic factor γ becomes a large number and the transformation is very different from what we are used to!

We conclude that the classical theory remains correct in the limit of low velocities. Only when we consider observers that move with relative speeds approaching the speed of light, our intuitive (i.e. classical) transformation is no longer correct.

3.2 *Derivation of the Lorentz Transformation

We have seen before that the principle of relativity and a constant light-speed lead to Lorentz contraction and time dilation. We will use this to derive in a simple way the coordinate transformation law between two reference systems S and S'. Let us start with the case of the Galilei transformation law. The transformation, and the reverse transformation, are:

$$\begin{aligned}x' &= x - vt \\ x &= x' + vt' \qquad (3.3)\end{aligned}$$

These equations are in agreement with the principle of relativity but do not respect the law of constant light-speed. Let us make a minimal modification by multiplying these equations by a factor γ, which at this point is unknown:

$$\begin{aligned} x' &= \gamma\,(x - vt) \\ x &= \gamma\,(x' + vt') \end{aligned} \tag{3.4}$$

Next, consider a beam of light in system S. Since it indeed travels with the speed of light we should have the equation $x = ct$ for the position of the front of the light-beam. Seen from system S' the same light-beam is described by the corresponding equation: $x' = ct'$, since it also travels with the same speed c in S'. If we now substitute these equations in the transformation laws of x' and x we find:

$$\begin{aligned} ct' &= \gamma\,(ct - vt) \\ ct &= \gamma\,(ct' + vt') \end{aligned} \tag{3.5}$$

Eliminating t' and t we obtain:

$$\begin{aligned} t' &= \gamma \frac{(c - v)}{c}\, t \\ t &= \gamma \frac{(c + v)}{c}\, t' \end{aligned} \tag{3.6}$$

By substituting the first into the second we find:

$$t = \gamma^2 \frac{(c + v)}{c}\,\frac{(c - v)}{c}\, t \tag{3.7}$$

from which we can derive the equation for γ by deviding by t:

$$1 = \gamma^2 \frac{(c^2 - v^2)}{c^2} \tag{3.8}$$

and the expression for γ follows:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.9}$$

We therefore see that the transformation law that keeps the speed of light constant for different obervers S and S' is:

$$x' = \gamma\ (x - vt) = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{3.10}$$

$$x = \gamma\ (x' + vt') = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{3.11}$$

Subsequently, we derive the transformation laws for time t by solving eqs. 3.10 and 3.11 for t and t':

$$\begin{aligned} \gamma vt &= \gamma x - x' \\ \gamma vt' &= x - \gamma x' \end{aligned} \tag{3.12}$$

By substituting $x' = \gamma x - \gamma vt$ (the first equation) into the second equation it follows that:

$$\gamma vt' = \left(1 - \gamma^2\right) x + \gamma^2 vt \tag{3.13}$$

which, using eq 3.9 and therefore $(1-\gamma^2)/\gamma = -\gamma\, v^2/c^2$, gives:

$$t' = \gamma \left(t - vx/c^2\right) = \frac{1}{\sqrt{1 - v^2/c^2}} \left(t - vx/c^2\right) \tag{3.14}$$

Similarly, the reverse transformation leads to:

$$t = \gamma \left(t' + vx'/c^2\right) = \frac{1}{\sqrt{1 - v^2/c^2}} \left(t' + vx'/c^2\right) \tag{3.15}$$

3.3 Paradoxes

Contrary to Galilei transformations, simultaneity of events and consequently the length of objects and time intervals are not the same for observers that move relatively to each other. This leads to amusing apparent contradictions, or paradoxes. A few of these are discussed in the remainder of this chapter. In all cases the paradox can be solved by a careful consideration of the Lorentz transformations between the different observers.

3.3.1 Causality

Imagine a crime scene situation in which Alice enters a room, sees Bob at the far end of the room and shoots him (remember, it is only a gedankenexperiment!). Sherlock Holmes, standing next to Alice, observes that Alice has shot the gun at a position x_A and a time t_A. He also sees that Bob gets shot and dies at a position in the far end of the room x_B, and at a time t_B. Since the gun is fired before Bob is hit he sees: $t_B > t_A$. Watson instead, is observing the crime scene from a high speed train, moving relative to Sherlock with a speed of 60% of the light speed ($v = 0.6c$ or $\beta = 0.6$). What does he conclude?

The answer can be found directly from the Lorentz transformations. Let us simplify the calculation by assuming that Sherlock sees Alice firing the gun at a time $t_A = 0$ and that the position at the doorstep is x_A= 0.

Since Watson is speeding along with $v = 0.6c$ he observes things with a γ factor given by $\gamma = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - 0.36} = 1.25$. Using the equation $ct' = \gamma\,(ct - \beta x)$ and substituting $t = t_A = 0$ and $x = x_A = 0$ Watson will conclude that Alice fires the gun at $t'_A = 0$, the same as Sherlock. Subsequently, substituting $t = t_B$ and $x = x_B$ he will conclude that Bob dies at a time corresponding to: $ct'_B = 1.25\ (ct_B - 0.6x_B)$.

But then Watson observes a peculiar phenomenon: if $ct_B - 0.6x_B$ is less then zero (remember $t_A = 0$), i.e. if the distance x_B is more than $ct_B/0.6$, then the time that Bob dies is actually *before* Alice fires the gun! Clearly, Alice cannot have killed Bob before she shot the gun!

Sherlock, as usual, solves the riddle on second thought. In the situation he observed, the distance between Alice and Bob was actually larger than the distance travelled by a light beam in a time between the firing of the shot t_A and the arrival of the bullet t_B. So, even if the bullet had travelled with the light-speed, it could not have reached Bob within this time-span. That means that the original conclusion of Sherlock that Alice has killed Bob was not correct. Perhaps Bob just had a heart-attack.

3.3.2 A ladder in a barn

In a next example consider that Alice is running towards a barn and horizontally caries a long ladder of 5 meter length. Imagine that she approaches the barn with a speed $v = 0.8c$. Bob, standing next to the barn sees that the barn is 4 meters deep. However, when he measures the length of the ladder carried by Alice, he sees that it is shortened (Lorentz contracted) by a factor $1/\gamma = \sqrt{1 - 0.8^2} = 0.6$. Bob concludes that the ladder is 5m x 0.6 = only 3 meters long and that it will comfortably fit in the barn.

Alice instead sees the barn approaching relative to her with $v = 0.8c$, and that the barn is reduced in length by a factor 0.6. Instead of 4 m she observes it to be only 2.4 meters deep, and concludes that there is no way that her 5 m long ladder will fit.

Bob decides to do the following experiment: Alice keeps on running into the shed and at the *same time* that the front of her ladder hits the back-wall of the shed he slams the entrance door closed. What will happen? Is Alice with her ladder inside or not? The situation is depicted in fig. 3.2.

The essential point is Bob's statement that he closes the door *at the same time* that the ladder reaches the back wall. Let us calculate this time for both Bob (reference frame S) and Alice (reference frame S').

The coordinates in S (Bob's view) when Alice's ladder reaches the entrance of the shed are taken to be $x_1 = 0$ and $t_1 = 0$. Alice moves in the direction of the positive x-axis with a velocity $v = 0.8c$ and it follows from the Lorentz transformations that Alice (system S') also calls this space-time point $x'_1 = 0$ and $t'_1 = 0$.

But when does the front of the ladder reach the back wall of the shed? According to Bob this is the spacetime point $x_2 = 4\text{m}$ and $t_2 = 4\text{m}/0.8c = 5\text{m}/c$. The corresponding time as seen by Alice is found using the Lorentz transformation with $\beta = 0.8$ and $\gamma = 1.67$:

$$ct'_2 = \gamma\,(ct_2 - \beta x_2) = 1.67\ (5\text{m} - 3.2\text{m}) = 3\text{m} \tag{3.16}$$

Since the relative speed of Alice and the shed is 0.8 c, Alice at this time t'_2 has only entered the shed by a distance of $3\text{m}/c \cdot 0.8c = 2.4\text{m}$. So, Bob wants to shut the door at a time when Alice has only entered 2.4 m, much less than the 5 m ladder. Alice and Bob don't agree on the simultaneity of the time the door must be closed.
In conclusion, lengths can only be compared in the same coordinate frame.

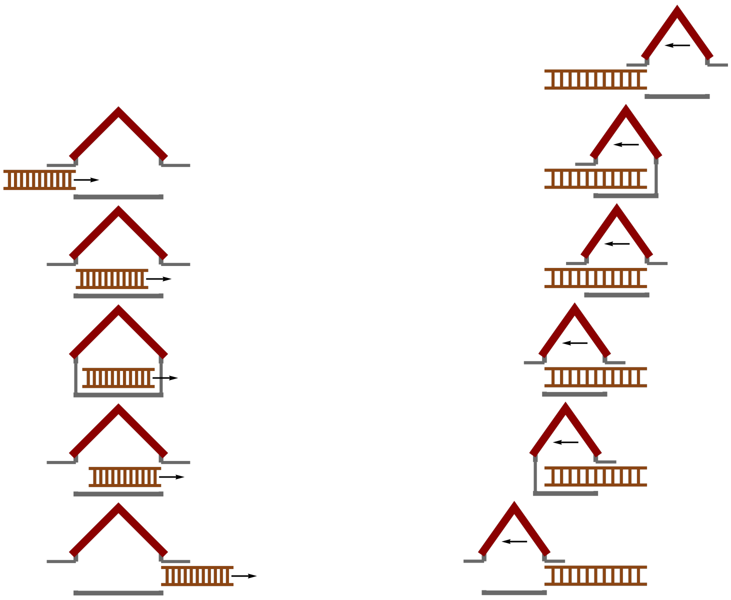


Figure 3.2: *Left:* Bob sees a Lorentz contracted ladder approaching the barn and concludes that at some point it will be fully in the bar, such that the bar can be fully closed at a given time. *Right:* Alice sees a Lorentz contracted garage approaching and concludes the ladder will not fit at any point in time. In her view the back-door already opens before the front door closes.

***Compressed ladder**

A careful reader will note that, independent of the simultaneity of the closure of the two doors, in Bob's case the ladder is inside while in Alice's case it is not. The details of the paradox are hidden in the fact that we consider the stopping ladder as a perfectly rigid object. Relativistically it is not. When the front of the ladder stops by the force of the wall, the stopping-force is transmitted by the ladder molecules from the front to the back of the ladder with a speed less than the speed of light. In other words, when the front stops, the back does not know it yet and continues to move, such that the ladder compresses and becomes shorter. Objects do not stay perfectly rigid under special relativity.

3.3.3 Twin Paradox

A particular famous paradox is known as the twin paradox. In this case we consider identical twins Jules and Jim. Jules (after Jules Verne) is the more adventurous of the two and decides to travel with very high speed (about 99.5% of the light speed) to the nearest star: proxima centauri. It takes him a year to get there and a year to return to earth. When he returns to earth he finds that his twin brother, Jim, has become about 9 years older than he is. What happened?

From the point of Jim on earth, Jules travels with high speed to a star at a distance

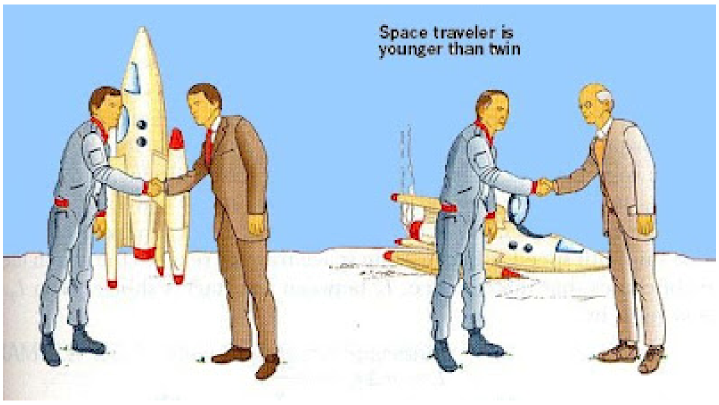


Figure 3.3: One half of an identical twin travels with high speed to proxima centauri and returns to earth. The time elapsed for the travelling twin can be a year, while the time elapsed for the twin remaining at earth is many years.

of about 5 years from earth. If he travels almost at the speed of light it will take Jules 5 years to get there and 5 years to return, such that Jim has aged 10 years when Jules comes back. Jim, on earth, has studied relativity theory and understands that his brother has only aged one year, since Jules travels at 99.5% of the light-speed, time goes slower for him with a factor $\gamma = 10$ such that Jules aged only one year while Jim aged 10.

However, Jules may argue as follows: the only thing that matters is the *relative* speed between Jim and Jules. So when Jules, from his spaceship looks back at earth, he sees that time on earth goes slower. He therefore concludes that Jim should be *younger* than him when he returns, rather then older! Their conclusion is opposite!

The answer of the paradox is hidden in the fact that we have only considered observers moving at a *constant* velocity. However, Jules speeds to proxima centauri, where he has to *slow down* turn around and *accelerate* back to earth. A more careful analysis shows that during his turnaround, time on earth progresses very fast, such that in the end both Jim and Jules agree on the same conclusion: Jules will be older when he returns to earth.

3.3.4 An Experimental Verification

In October 1971, Joseph C. Hafele, a physicist, and Richard E. Keating, an astronomer, took cesium-beam atomic clocks aboard commercial airliners (see fig. 3.4) to test the theory of relativity. They flew twice around the world, first eastward (with the rotation of the earth), then westward (against the rotation of the earth), and compared the clocks to another one that remained at the United States Naval Observatory. When reunited, the three clocks were found to disagree consistent with the predictions of relativity theory.
(The difference between the clock travelling eastward and westward was about 300 nsec.)

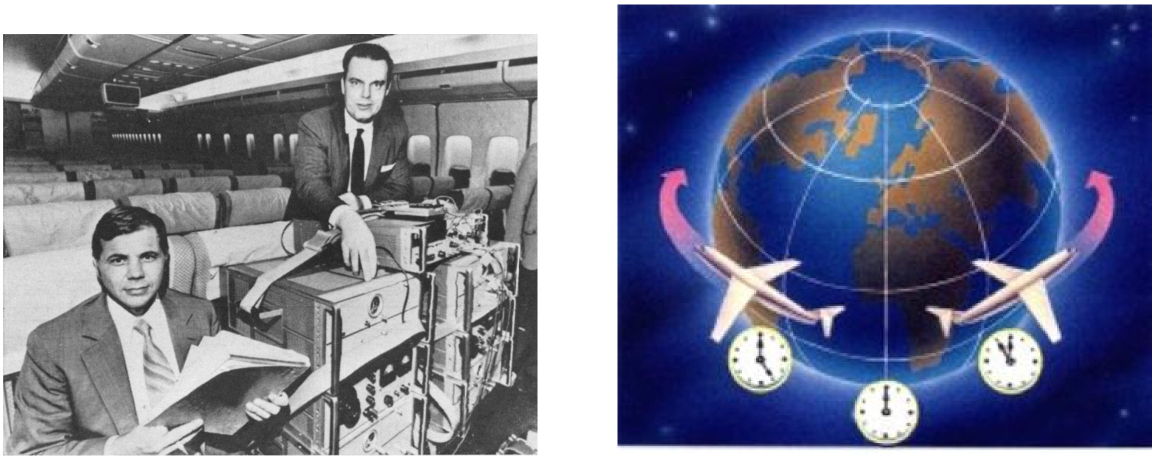


Figure 3.4: Atomic clocks were synchronized and taken on board of planes flying eastward and westward around the earth. After arrival they were compared and the prediction of relativity was seen to be correct.

<u>Exercises</u>
Calculate the relativistic reduction in distances for the following cases.

1. Driving by car with a speed of 120 km/h from Amsterdam to Maastricht, assuming the distance is exactly 200.0 km for an observer in rest with the surface of the earth, when driving by car with a speed of 120 km/h.

2. Flying in a rocket with a speed of 40 000 km/h from the earth to the moon, assuming the distance is exactly 384 000 km for an observer in rest relatively to the earth and the moon.

3. Following a neutrino particle over a distance from the sun to the nearest star, Proxima Centauri, a distance of 4.243 lightyears when relatively in rest with respect to the sun, when the neutrino, a very, very light particle, travels at a speed that is only 0.0000001% less than the speed of light.

4. Following a light particle across the universe.

3.4 Addition of Velocities

Finally we are ready to address the law of addition of velocities as we set out to do in Lecture 1. If Alice runs with 10 km/h on the deck of a boat sailing with 15 km/h, what is the observed speed of Alice with respect to Bob, who is standing alongside the river? Intuitively, using the Galilei translation, we expected this to be 15 km/h.

However, we saw that the relativity principle is in contradiction with the Galilean law of addition of velocities, since the light-speed is always the same for each observer, be it Alice on the boat, or Bob on land.

Are we sure the intuitive law of adding velocities is exactly valid? In the following we will see that this law is *almost correct for low relative velocities* (compared to the speed of light), while it *does not work at all for high relative velocities* (close to the speed of light).

Let us first see how to derive the law of adding velocities for a Galilei transformation. We say that the moving boat has a velocity v and Alice runs with velocity w on the boat, assuming that both velocities are in the x-direction. Therefore, the position of Alice on the boat (in reference system S) is given as: $x = wt$. The corresponding position of Alice with respect to Bob on the shore is: $x' = w't'$. How can we calculate the speed w' with respect to the shore (in reference system S') when we know the velocity of Alice on the boat w, and the speed of the boat v?

Using the Galilei transformation laws:

$$x' = x + vt \tag{3.17}$$

$$t' = t \tag{3.18}$$

we can make the following substitution steps:

$$\begin{aligned} x &= w\,t \quad \Rightarrow \quad x = w\,t' \quad \Rightarrow \quad x' - v\,t = w\,t' \\ \Rightarrow \quad x' - v\,t' &= w\,t' \quad \Rightarrow \quad x' = (v+w)\,t' \Rightarrow \quad x' = w'\,t' \quad . \end{aligned}$$

The last step is the equation of moving at a speed w' equal to $v + w$, which is our intuitive law for adding velocities.

We can repeat the same exercise, but now we use the rules of special relativity. Again we will start with Alice running on the boat according to $x = wt$, but this time using the Lorentz transformation:

$$x' = \gamma\ (x + v\,t) \tag{3.19}$$

$$t' = \gamma\ \left(t + \frac{w}{c^2}\,x\right) \tag{3.20}$$

The idea is the same as in the Galilei transformation, only the mathematics is a bit more complicated. (The derivation of the formula can be skipped.) We start with the Lorentz transformations:

$$x' = \gamma\,x + \gamma\,v\,t \tag{3.21}$$

$$t' = \gamma\,t + \gamma\,\frac{v}{c^2}\,x \tag{3.22}$$

Substituting $x = wt$ gives:

$$x' = \gamma\,(v + w)\ t \tag{3.23}$$

$$t' = \gamma\left(1 + \frac{vw}{c^2}\right)\ t \tag{3.24}$$

Inverting eq. 3.24 gives:

$$t = \frac{1}{\gamma}\left(\frac{1}{1+\frac{vw}{c^2}}\right) t' \tag{3.25}$$

and substituting this in eq. 3.23 finally leads to:

$$x' = \frac{v+w}{1+\frac{vw}{c^2}}\, t' \equiv w't' \tag{3.26}$$

such that we arrive at the law for adding velocities:

$$\boxed{w' = \frac{w+v}{1+\frac{wv}{c^2}}}$$

This means that the intuitive addition of velocities is almost correct if w and v are small compared to the speed of light c. In that case the correction factor $1 + vw/c^2$ is approximately equal to 1 and we find back the Galilei transformation.

Let us return to the example of lecture 1 where Bob and Alice approach each other in two rockets, Alice with 250 000 km/s and Bob from the other side with 200 000 km/s, the speed with which they approach each other is:

$$w' = \frac{250\,000 + 200\,000}{1 + 0.555} = 289\,000 \text{ km/s} \quad ,$$

less than the speed of light!

Similarly, in case Alice is travelling in a rocket with a speed of 99% speed of light and if she shoots a bullet inside the rocket close to the speed of light (again, say 99%), the intuitive law changes completely. In that case Bob, on a planet, does not see the bullet flying almost two times the speed of light, but with a speed of 0.99995 times the speed of light! (Try it using the formula.)

Even in the extreme case where we take $w = c$ and $v = c$ we find $w' = 2c/2 = c$! A velocity higher than the speed of light never occurs!

We conclude that our intuitive law for adding velocities is not correct. For all practical purposes in everyday life it gives almost the correct result. However, at high velocities a relativistic correction is needed and as a result nothing can travel faster than the speed of light!

3.5 *The law $E = mc^2$

Perhaps the most famous equation of all time is Einstein's equivalence of energy and mass equation: $E = mc^2$. Let us, as a final subject in the relativity lectures, reproduce Einstein's elegant reasoning to derive this relation. Einstein considered a floating box somewhere in the vacuum of deep space. He then imagined that inside the box a bit of light is being emitted by the left wall of the box. The light travels to the right of the

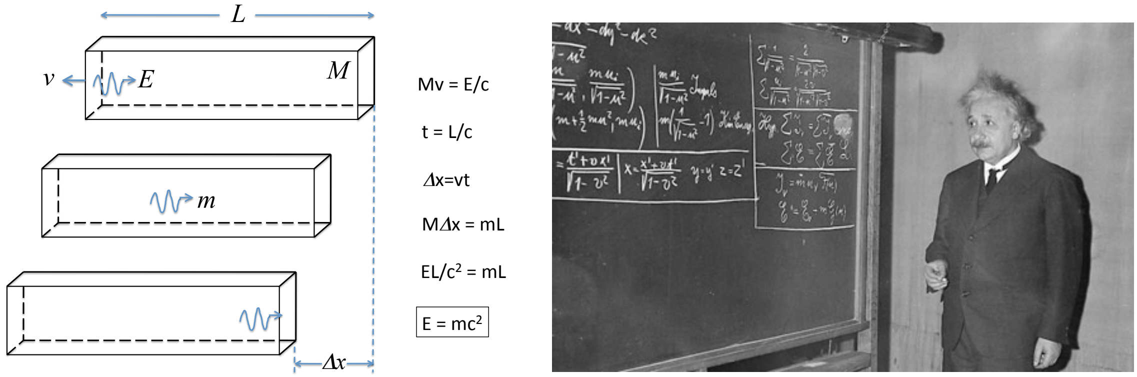


Figure 3.5: *Left:* The derivation of the law $E = mc^2$, see text. *Right:* Photograph of Einstein during a lecture when he derives his famous equation.

box where it is absorbed by the right side wall. The situation is sketched in fig. 3.5, which also shows a picture of Einstein as he is deriving his famous formula.

From Maxwell's classical equations of electrodynamics Einstein knew that light carries momentum. At the time the light is emitted to the right, Newton's law of action and reaction states that the box should move a tiny bit to the left, just enough to compensate the momentum of the light to the right. Alternatively, when the light impacts on the right wall it will move the box, with the same amount to the right. However we know that, since no force is active on the total system of the box and the light, the centre of mass of the system stays in rest during the whole process. The following equations describe this thought experiment and lead to the famous equation of Einstein.

The momentum of light follows from Maxwell's equations [1] :

$$p_{\text{light}} = \frac{E}{c} \quad , \tag{3.27}$$

while the momentum of the box is given by:

$$p_{\text{box}} = Mv \quad , \tag{3.28}$$

where M is its mass and v its velocity. Let us call the size of the box (i.e. the distance of left to right wall) L, then the time it takes for the light to go from the left wall to the right wall is: $\Delta t = L/c$. During this time the box has moved a little bit to the left, say a distance Δx, such that the box had a velocity to the left, equal to:

$$v = \frac{\Delta x}{\Delta t} = \frac{\Delta x c}{L} \tag{3.29}$$

[1] It is perhaps surprising that a light particle, a photon, which is known to be massless can have momentum, since Newton's momentum is calculated as: $p = mv$. However, the correct relativistic equation for momentum is: $p = m\gamma v$, where γ is the relativistic factor. For light m is zero, but γ is infinite! In fact the product $m\gamma$ is the effective mass of the photon.

The law of conservation of momentum $p_{\text{light}} = p_{\text{box}}$ gives then from eqs. 3.27 and 3.28:

$$Mv = \frac{E}{c} \tag{3.30}$$

and substituting the result for v above:

$$M\Delta x = \frac{EL}{c^2} \tag{3.31}$$

Let us now assign a hypothetical mass m to the light particle and calculate the center of mass position, $\overline{x}$, for the box + light system:

$$\overline{x} = \frac{Mx_1 + mx_2}{M + m} \quad , \tag{3.32}$$

where x_1 and x_2 are the respective center positions of the box and the light. If we say that the photon starts at the left wall of the box and define this position to be $x_2 = 0$, then the center of mass at the time the light is emitted is:

$$\overline{x} = \frac{Mx_1}{M + m} \tag{3.33}$$

and at the time the light is absorbed at the right hand wall it is:

$$\overline{x} = \frac{M\left(x_1 - \Delta x\right) + mL}{M + m} \tag{3.34}$$

Relating the two above equations 3.33 and 3.34, we find:

$$Mx_1 = M\left(x_1 - \Delta x\right) + mL \quad , \tag{3.35}$$

or, perhaps as expected:

$$mL = M\Delta x \quad . \tag{3.36}$$

Now, using eq. 3.31, we obtain:

$$mL = \frac{EL}{c^2} \quad , \tag{3.37}$$

which, after eliminating L and rearranging, gives the final result:

$$\boxed{E = mc^2}$$

Lecture 4

General Relativity and Gravitational Waves

"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."
-Albert Einstein

4.1 The Ehrenfest Paradox

Paul Ehrenfest, an Austrian physicist and close friend of Einstein, was professor in Leiden on the position formerly held by Hendrik Antoon Lorentz. Ehrenfest was an excellent teacher, well known for his clarifying explanations of paradoxes in both relativity theory and quantum mechanics. Perhaps the most famous thought experiment paradox in relativity theory was named after him: the Ehrenfest paradox.

Consider a rigid rotating cylinder with a radius r, as shown in fig.4.1. Two measuring rods are laid out along the cylinder: one of them is attached to the cylinder, while the other is attached the floor just next to it, as is shown in the figure. The circumference of the circle C is given by the formula $C = 2\pi r$. If the cylinder rotates with angular velocity ω, where ω is the number of radians it rotates per second, then a fixed point on the edge of the cylinder rotates with a velocty $v = \omega r$. A person standing next to the cylinder ("Alice") will see that the ruler attached to the cylinder shrinks according to the Lorentz contraction factor $1/\gamma$, such that the circumference becomes *smaller* with a factor $1/\gamma$:

$$C^{Alice} = \frac{2\pi r}{\gamma} = \sqrt{1 - v^2/c^2}\, 2\pi r \; < \; 2\pi r \quad . \tag{4.1}$$

However, since the direction of the radius r is perpendicular to the motion and therefor does not contract in the rotating motion, this leads to a contradiction since the circumference of a circle is always $C = 2\pi r$. The paradox further deepens if we consider the point of view of a person sitting *on the disk* ("Bob"). Bob will see that the measuring rod connected to the floor outside the cylinder will move relative to him with an equal but opposite velocity v. He will conclude that the ruler on the *outside* undergoes Lorentz

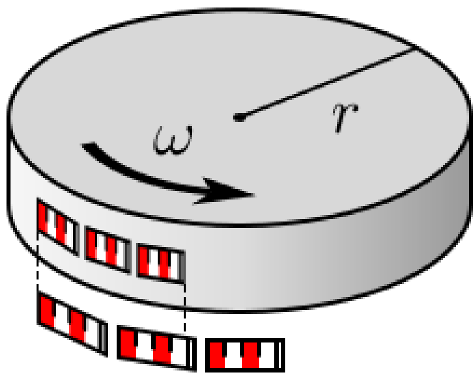


Figure 4.1: The Ehrenfest paradox of a rigid rotating disk. Atoms on the edge of the disk have a velocity equal to $v = \omega r$. The circumeference of the disk should not change as it is equal to $2\pi r$, but according to special relativity an observer outside the disk will measure that the circumference will shrink, while a person on the disk will measure that it becomes larger.

contraction such that more of them will fit along the circumference. He will find that the circurference is equal to:

$$C^{Bob} = \gamma\, 2\pi r = \frac{2\pi r}{\sqrt{1 - v^2/c^2}} \; > \; 2\pi r \quad . \tag{4.2}$$

The solution to the Ehrenfest paradox, and other similar ones, are that in special relativity we have always considered *inertial frames*, i.e. systems that move with relative *constant* velocity to each under and undergo no external force. For the rotating disk this is certainly not the case since the points on the edge of the fast rotating cylinder feel a strong centrifugal force: they undergo an acceleration that points towards the center of the disk.

4.2 The principle of Equivalence

In the first lecture the principle of relativity was introduced, which states that the laws of physics are the same for observers that move with a constant relative velocity, i.e. in *inertial frames*. These are observers on which no external forces act. On earth such frames do not exist since we will always feel the presence of gravity. In order to extend the notion of inertial systems to presence of gravitation Einstein considered a though experiment analogous to the one of Alice in the cabin of her boat, who failed to detect absolute motion.

In the his thought experiment Einstein now imagines that Bob boards a spaceship. The spaceship takes off and travels to interstellar space, far away from the presence of stars and planets. When the engines of the ship are switched off, Bob moves with constant velocity and it is impossible for him to detect absolute motion. In addition,

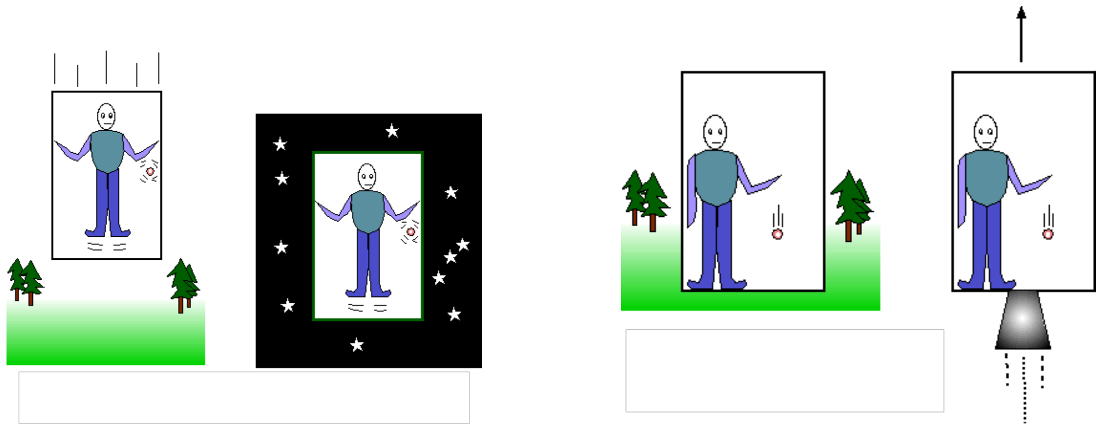


Figure 4.2: The principle of equivalence. *Left:* an observer in a closed spaceship cannot tell the difference whether he is freely falling in the earth gravitational field or floating in interstellar space, as all objects will make the same path. *Right:* an observer similarly cannot tell the difference between standing at rest in a gravitational field or by accelerating in a rocket in free space without the presence of gravity, as all objects fall in the same way.

as there now is no gravity present, he will float weigtless in his cabin: there is no force acting upon him.

Consider now that the direction of the ship happens to be precisely back to earth, and it enters earth's gravitational field. The velocity will no longer be constant, as the ship will accelerate in free fall towards the earth surface (please remember it is a though experiment and we may assume that parachutes will be employed to save our astronaut Bob in the end). Due to a unique aspect of the gravitational force, all objects fall with the same acceleration. Therefor, since all objects around Bob in his cabin are falling together with him, it turns out Bob feels no force acting and *there is no way that he can detect the difference between "floating in empty space" and "accelerating in free fall towards the earth"*. The two situations as illustrated on the left side of fig. 4.2 are fully equivalent. In a lecture that he gave in at Glasgow University Einstein refer's to this realization as his happiest thought: "If a person falls freely he will not feel his own weight".

Let us return to Bob in the interstellar spaceship and now assume that the engines of the spaceship are turned on. The situation is illustrated on the right side of in fig.4.2. As soon as the ship accelerates, all objects that were before floating freely inside the cabin, now fall to the ground with an acceleration equal and opposite to the acceleration of the spaceship (they stay "at rest" wrt. space as before). All objects will fall with the same acceleration. As Galilei had demonstrated, this is equivalent to how all objects fall on earth, ie. in a gravitational field. Phrased differently, due to the acceleration Bob will feel a force on his body directed to the floor of the spaceship, which is the same as a gravitational field from a planet. Again, while staying within the cabin of the spaceship, there is no experiment possible such that Bob can distinguish the two situations; they are *equivalent*.

4.3 The Eötvös Experiment

This principle of equivalence is often described by the statement: *the inertial mass is identical to the gravitational mass.* The inertial mass m_i of an object is defined by Newton's law of *inertia* which describes an object's resistence to acceleration:

$$F = m_i \, a \tag{4.3}$$

where F is the force applied and a is the resulting acceleration. Similarly the gravitational mass m_g is defined by Newton's law of *gravitation*:

$$G = m_g \, g \tag{4.4}$$

where G is the force of gravity and g the gravitational acceleration. The principle of equivalence implies that there is only one mass

$$m_i = m_g \equiv m \tag{4.5}$$

for all objects.

The equivalence of inertial mass and gravitational was verified with high precision by the Hongarian physicist Loránd Eötvös using a torsion balance. The experiment is sketched in fig. 4.3. The left side of the picture illustrates the directions of the forces felt by an object at a given position on earth. On the one hand the object feels the gravitational force G, which points towards the center of he earth. Since the earth is rotating around the polar axis, all objects on the center of the earth undergo also a centrifugal force F, which points outward and perpendicular to the axis of rotation, as is illustrated in the figure. The key point is to realize that the centrifugal force follows from Newtons's law of inertia:

$$F = m_i \, a \quad \text{with} \quad a = \frac{v^2}{R} \quad , \tag{4.6}$$

while the gravitational force follows from the force of gravity:

$$G = m_g \, g \quad \text{with} \quad g = \frac{GM_\oplus}{R^2} \quad , \tag{4.7}$$

where G is Newton's gravitational constant, $M_\oplus$ is the mass of the earth, and R is the radius of the earth. If inertial mass is identical to gravitational mass ($m_i = m_g$), then for different masses m_1 and m_2 one must always find that $F_1/G_1 = F_2/G_2$. This is tested in an elegant way in the Eötvös experiment using a large and a small mass connected by a rod as illustrated in the right side of fig.4.3. The system is suspended by a thin wire that can rotate freely, while the rotation is monitored by a mirror on the rod. If the ratio of the gravitational force and the centrifugal force of the two objects is not identical a net force will cause the system to turn along with the rotation of the earth.

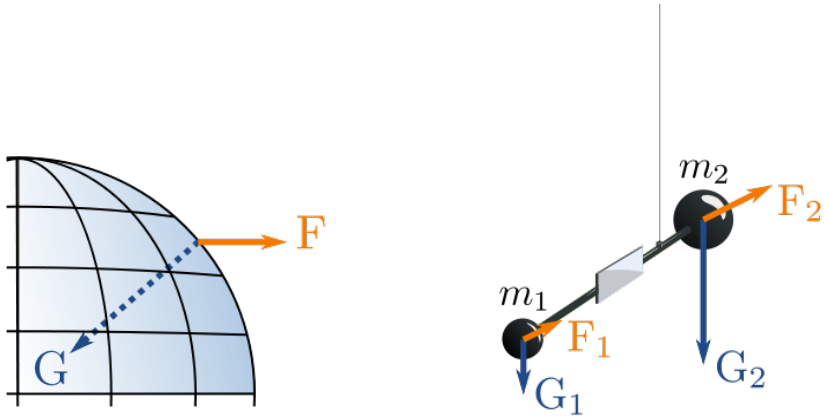


Figure 4.3: Illustration of the Eötvös experiment. *Left:* Illustation that shows the direction in which gravitational and centrifugal forces act on a mass on earth. *Right:* schematic setup of the experiment. If the ratio of forces F_1/G_1 and F_2/G_2 differ the resulting forces on m_1 and m_2 have a different direction and the setup will rotate with the earth rotation. The mirror together with a light beam is used to detect any such rotation, which in the end was not detected.

4.4 Bending of Light

The principle of equivalence has far reaching consequences. They can be seen, as we did in special relativity, again by considering a beam of light. Let us proceed with the thought experiment of astronauts in a spaceship. Imagine that a spaceship has transparant windows and a beam of light is shone from the outside of the spaceship through the window, as in fig. 4.4. Consider the following three situations where an astronaut outside the spaceship shines his torch light in the same way "horizontally" into the window of the rocket.

1. "zero motion": the spaceship is at rest next to an astronaut outside the ship who shines the beam of light into the ship. The view from an astronaut inside is indicated on the left of the picture; both for the astronaut outside as well as inside the lightbeam is represented by the horizontal beam.

2. "constant velocity": the spaceship passes the astronaut with constant velocity. By the time it takes for the beam of light to travel from the left side of the rocket to the right side, the rocket has moved some distance in the upward direction. While the astronaut outside claims the beam of light is horizontal (he shines it horizontally), the rocket passes him in the upward direction, such the astronaut *inside* the rocket will see a beam of light that travels in a straight line angled downward with respect to his own reference system.

3. "acceleration": the most interesting situation occurs when the rocket has the engines turned on and it passes the outside astronaut as it *accelerates* in upward direction. Again the outside observer simply sees the same horizontal beam, but

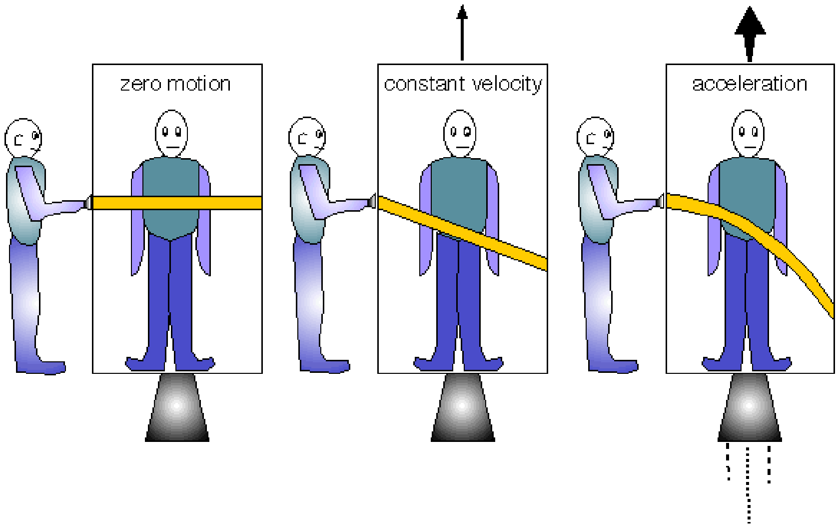


Figure 4.4: The path of a lightbeam inside a rocket in three situations: *Left:* the rocket in rest relative to the outside astronaut, *Middle:* the rocket passes the outside astronaut with constant velocity, *Right:* the rocket accelerates as it passes the astronaut. In the first two cases the beam of light is a straight line, in the third case the light makes a curved trajectory in the reference frame of the rocket.

for inside observer the beam of light changes more and more the angle as the rocket accelerates. The beam of light will follow a *curved path* in the reference frame of the person inside the rocket.

These three situations are illustrated in fig. 4.4. The result of this thought experiment becomes highly interesting when we consider the principle of equivalence: there is no difference between an accelerated rocket and a rocket resting in a gravitational field, as eg. on the surface of the earth. *The beam of light must also follow a curved path if the rocket in the final example would rest on the surface of the earth.*

Clearly, in daily life we do not see curved rays of light, since the acceleration required is very high, or alternatively, the gravitational field must be very strong. On May 29, 1919 Sir Arthur Eddington led an experiment that tested Einsteins statement of the bending of light in the gravitational field of the sun during a solar eclipse. During the eclipse he measured the positon of a star he observed in a direction close to the sun, and compared that with its regular position. The star seemed to have changed position! Einstein was proven right as the light from the star slightly curved in the gravitational field as it closely passed the sun as is shown in fig. 4.5. The result of the measurement was published in newspapers around the world and instantly made Einstein a world famous scientist.

Einstein suggested that light always choses the shortest path through space, but that space itself is curved near a massive object in a similar way that distances contract at large speeds relative to being in rest. A beam of light follows the shortest path

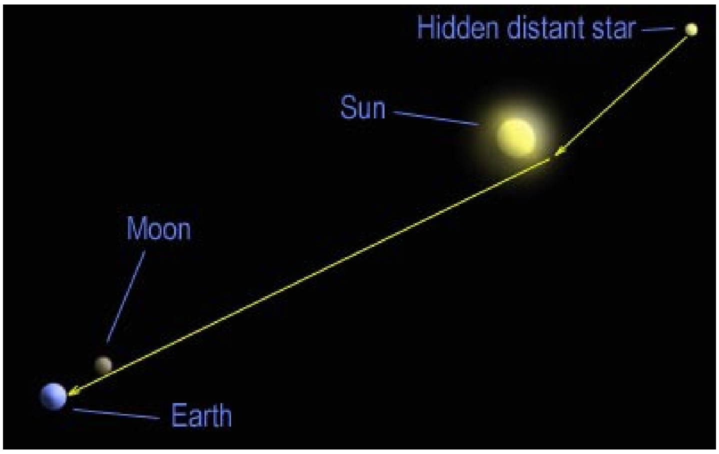


Figure 4.5: Illustration of the situation observed during the solar eclipse in 1919. As seen from the earth, the moon causes a solar eclipse such that we can look at stars in the direction close to the sun. A distant star that should have been hidden behind the sun turned out to be visible since the light from the star was curved around the sun by 1.75 arcseconds, in agreement with the theory of relativity.

through curved aspace, a so-called *geodesic*. A nice analogy of this effect, given by Stephen Hawking, is the comparison of light in curved space with the distorted path of the shadow of an airplane flying in a straight line over hilly terrain.

The phenomenon is currently well known and is often used in astrophysics imaging using a mechanism that is called *gravitational lensing*. A very massive object can bend the passing light from objects that are positioned beyond it in a similar way as a lens focusses light in a telescope. An example of gravitational lensing is shown in fig. 4.6.

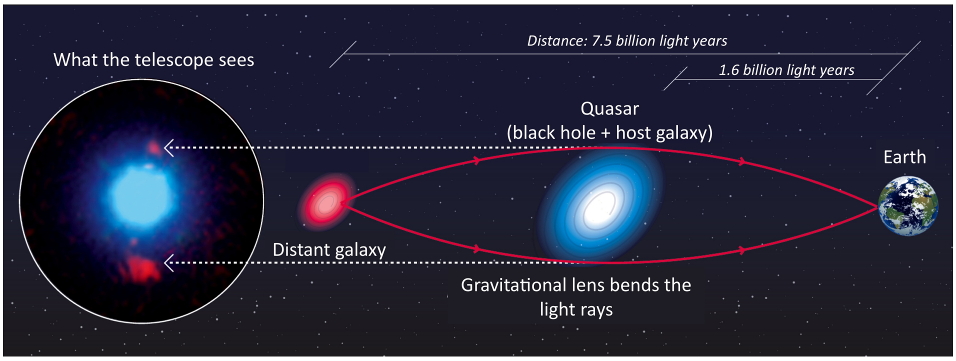


Figure 4.6: Example of gravitational lensing where images of a distant galaxy are observed through gravitational lensing of a massive quasar object in the line of sight between the distand object and the earth.

4.5 The Harvard Tower Experiment

Einstein proposed a thought experiment to examine the effect of a gravitational field on light. This thought experiment tests the so-called gravitational red-shift of light. Imagine, as is illustrated in fig. 4.7, that we drop a particle with mass m from a high tower with height h in the gravitational field of the earth. At the top the particle has an energy equal to:

$$E^m_{top} = mc^2 \quad . \tag{4.8}$$

As the particle falls down it increases its kinetic energy such that at the bottom it has an energy:

$$E^m_{bottom} = mc^2 + \frac{1}{2}mv^2 = mc^2 + m\,g\,h \quad , \tag{4.9}$$

where the second step follows after calculating the velocity at the bottom, $v = \sqrt{2gh}$ and g is the gravitational acceleration constant. Einstein, in his thought experiment, proposes a device that converts the particle at the bottom, *without loss of energy*, into a photon that is emitted at the bottom of the tower back towards the top. The energy of that photon is equal $E^\gamma_{botttom} = E^m_{bottom}$. Since a photon can only travel at the speed of light it would mean that at the top we would have $E^\gamma_{top} = E^\gamma_{bottom}$. If we convert that again into a particle according to $E^m_{top} = m'c^2$, we would end up with a new particle with mass $m' > m$, and we created a perpetuum mobile that clearly violates energy conservation.

The conclusion is that a photon loses energy as it climbs up in the gravitational field. The energy of a photon is given by the quantum mechanical relation with its frequency f: $E = hf$, where h = Planck's constant. Einsteins concludes that the photon reduces frequency as it climbs the gravitational field such that:

$$\frac{E^\gamma_{bottom}}{E^\gamma_{top}} = \frac{hf_{bottom}}{hf_{top}} = \frac{mc^2 + mgh}{mc^2} = 1 + \frac{gh}{c^2} \quad , \tag{4.10}$$

such that energy is again conserved at the top compared to the initial situation. In the above we assumed that the gravitational acceleration g does not change with height and that the particle velocity is relatively small (non-relativistic). The fraction of energy the photon loses ($\frac{gh}{c^2}$) as it climbs the gravitational field is therefore relatively small.

Einstein's prediction that the frequency of light should reduce as it climbs in the gravitational potential was tested by Robert Pound & Glen Rebka in 1960 in Jefferson lab in what is called the Pound-Rebka or the Harvard Tower Experiment. They used a source of iron (^{57}Fe) emitting gamma rays and detectors at both the bottom and the ceiling of the building with a height of 22.5 meters. To detect this tiny effect they had to remove any effect from the Doppler effect that is caused by (very low) relative velocities between the floor and the ceiling. To isolated the Doppler effect they placed the iron source in the center of a loudspeaker cone vibrating between 10 Hz and 50 Hz, such that they could determine both the Doppler effect and the gravitational components in the

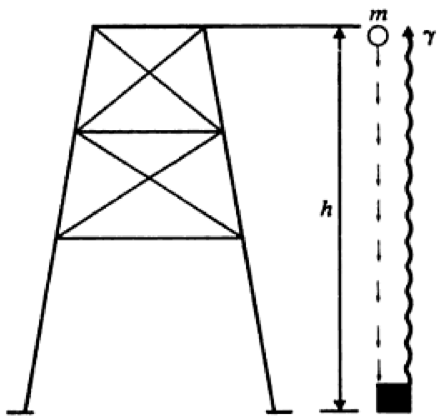


Figure 4.7: Einstein's thought experiment to prove that a photon *must* lose energy as it travels upward in a gravitational field. First a particle with mass m is dropped to gain kinetic energy; second it is converted at the bottom into a photon without loss of energy, and finally it is converted again to a particle at the top. If the photon would not lose energy a contradiction arises as the newly created particle at the top would have more energy than that of the originbal particle, violating energy conservation.

wavelength shift. The setup is illustrated in fig. 4.8. They found a result that was in agreement with Einsteins relativity theory.

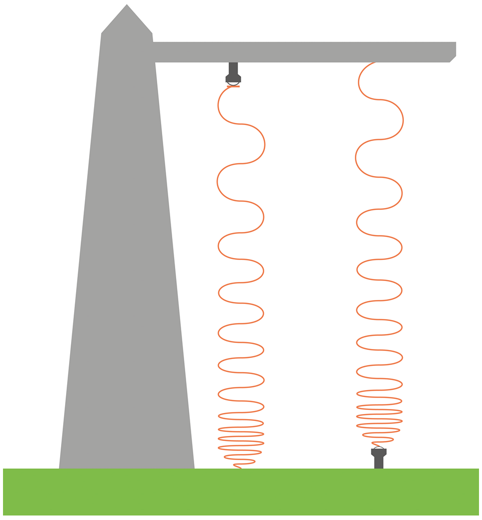
4.6 Space-Time Curvature

How can the photon become red-shifted as it climbs the gravitational potential? According to relativity theory not only space, but also time is curved in the presence of a gravitational field. Time slows down in a gravitational potential, such that a clock tick goes faster at high altitude and slower at low altitude, see the illustration in fig. 4.9.

The amount of curvature of space-time can be calculated as follows. From special relativity we have seen that time dilates and sizes contract when objects move at high speeds, where the Lorentz factor is equal to the factor $\sqrt{1 - v^2/c^2}$. Consider now an object of mass m that falls towards the earth with mass $M_\oplus$. Assuming that the mass starts with velocity $v = 0$, what will it's velocity be when it reaches the surface? The potential energy U of the object w.r.t. the earth is given by Newton's law:

$$U = F \cdot r = -G\frac{M_\oplus m}{r} \quad , \tag{4.11}$$

where G is newton's gravitational constant, $M_\oplus$ is the mass of the earth and r the distance towards the center of the earth. Conservation of energy implies that at all times the object will have a velocity such that the sum of kinetic and potential energy



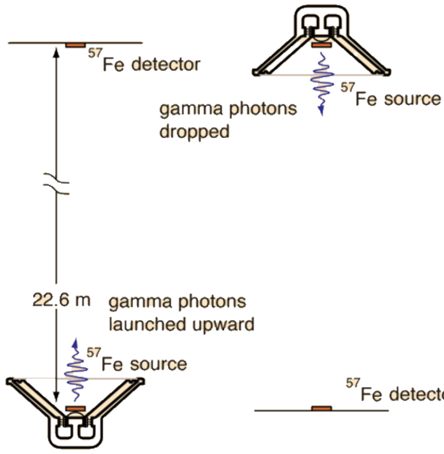


Figure 4.8: *Left:* Principle of the Pound-Rebka experiment: a photon travelling upward in the earth gravitational field reduces in frequency (is "redshifted"), a photon travelling downward increases in frequency ("blueshifted"). *Right:* The set-up used in the experiment where radioactive sources where placed in the center of loudspeakers, such that the wavelength shifting Doppler effects could be precisely determined and extracted.

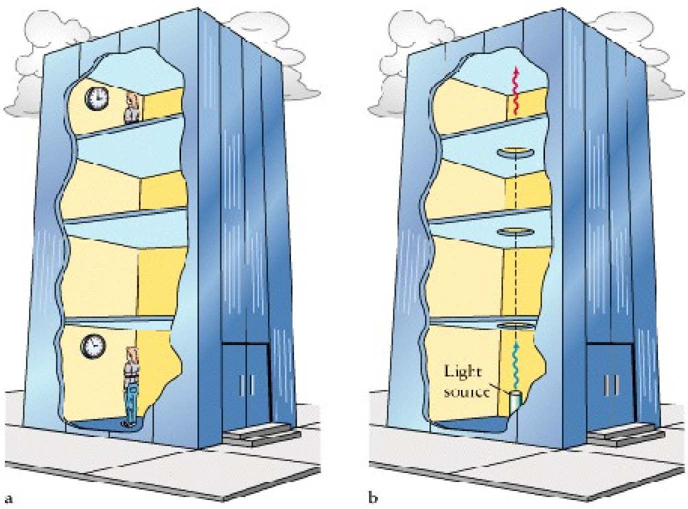


Figure 4.9: Two equivalent statements: (*Left*) a clock at higher altitude ticks faster than an identical clock at lower altitude, (*Right*) a photon travelling upward in the Pound-Rebka experiment is red-shifted.

is zero:

$$\frac{1}{2}mv^2 - G\frac{M_\oplus m}{r} = 0 \ , \tag{4.12}$$

from which we obtain that the velocity of the object as it approaches the earth is:

$$v = \sqrt{\frac{2GM_\oplus}{r}} = v_{\rm esc} \ . \tag{4.13}$$

Reversing the trajectory we see that this velocity is exactly equal to the minimal escape velocity ($v_{\rm esc}$) an object requires to be able to escape from the earth, which is about 40 000 km/s. If we plug in the expression of the escape velocity $v_{\rm esc}$ into the Lorentz factor we obtain that time slows down and space contracts close to the earth according to the factor:

$$\sqrt{1 - v^2/c^2} = \sqrt{1 - \frac{2GM_\oplus}{rc^2}} \ . \tag{4.14}$$

4.7 A Black Hole

A black hole is a region of space-time which causes such strong gravitational field that not even light can escape. In that case we insert $v_{\rm esc} = c$ in the equation above to find that these conditions happen at a distance of a given mass M of:

$$1 - \frac{2GM}{rc^2} = 0 \quad \text{or} \quad r = \frac{2GM}{c^2} \tag{4.15}$$

This distance is called the Schwarzschild radius, named after Karl Schwarzschild, who calculated it first in 1916. It can be calculated for any object of given mass M. The Schwarzschild radius of the earth is about 9 mm. An object becomes a black hole if all of its mass is contained within the Schwarzschild radius, which clearly is not the case for earth.

4.8 Gravitational Waves

It is beyond the scope of this course to discuss the Einstein equations and derive the propagation of gravitational waves. Let us, instead, consider their existance in equivalence to electrodynamics. In the theory of electrodynamics we learn that when charged particles are accelerated they cause electric and magnetic fields that change in time and cause electromagnetic waves. Electromagnetic waves are oscillating electric and magnetic fields that carry energy through the vacuum and propagate with the speed of light. Similarly, gravitational waves are caused by accelerated massive objects. Known astronomical objects of sufficient mass that may produce observable gravitational waves are systems of binary stars; in particular binary systems involving neutron stars or black holes. In these systems massive objects rotate relatively closely around each other with high velocities, such that the varying gravitational fields cause waves analogous to electromagnetic waves.

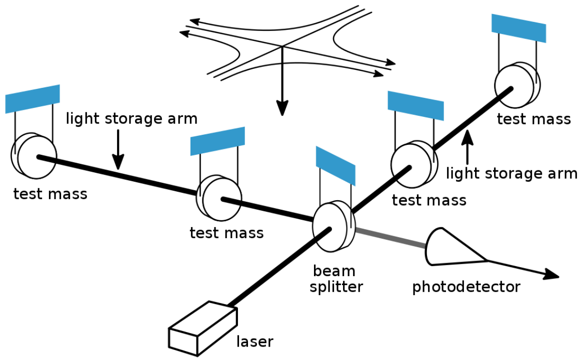


Figure 4.10: Schematic diagram of a gravitational wave interferometer. The interferometer detects a change in the length of the light storage arms between the test masses. The length of the arms is tuned such that the beams interfere destructively in the photodetector (the so-called "dark fringe"). Passage of a gravitational wave will cause an oscillation in the length of the arms such that the light-waves of the two arms no longer fully cancel each other.

4.9 Detection of Gravitational Waves

In the case of electromagnetic radiation electric and magnetic fields oscillate perpendicular (i.e. *transverse*) to the direction of propagation. Similarly, in gravitational deformations of space occur transverse to the propagation of the waves. The passing of a gravitational wave can be observed by detecting the change of distance between two free falling test-masses in the temporary contracting and expanding space dimensions.

A relatively simple device to detect gravitational waves is called a Weber bar, named after Joseph Weber. It consists of a large solid metal bar that is shielded from outside vibrations. A detection occurs if a passing gravitational wave temporarily deforms the bar and excites the bar's resonant frequency, which can be amplified and detected. Several of these devices exist but their sensitivity is such that only the most powerfull gravitational waves may be detected.

The most sensitive gravitational wave detectors are interferometers that measure transverse minute changes in the distances of kilometer-long arms of their set-up, using interference of laser beams. The setup is conceptually the same as the Michelson Morley experiment, apart form the fact that the lightspeed is now known to be constant and the experimental sensitivity is much higher. The principle of operation is illustrated in fig. 4.10.

The Laser Interferometer Gravitational Wave Observatory (LIGO) in the US includes two interferometers with light storage arms of 4 km; one located in Livingston, Louisiana and one in Hanford, Washington. Gravitational waves were first detected on September 14, 2015, 100 years after Einstein predicted their existence. The observed event was caused by a collision followed by a so-called merger of two black holes of respectively 29 and 36 times the mass of the Sun. The event took place about 1.3 billion years ago and a total energy of about 3 times the total mass of the Sun was converted into gravitational

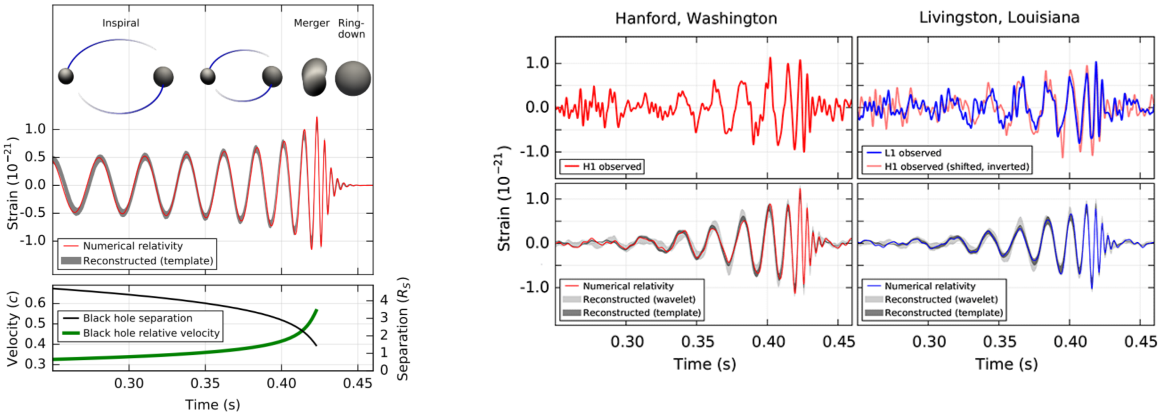


Figure 4.11: *Left:* The expected gravitational signal of two inspiralling black hole, where the strain is the relative deformation of space per unit of length. *Right:* The top-left plot labelled H1 is the observed signal of Hanford, the top-right plot is the observed signal of Louisiana with the H1 signal superposed. The bottom plots show the reconstructed signal together with the calculation of numerical relativity theory.

wave energy. The observed signal of the LIGO detectors is shown in fig. 4.11. Recently the Virgo detector, located near Pisa, has joined in a combined detection run with LIGO.

The LIGO and Virgo detectors have a sensitivity to observe, in addition to massive black hole mergers, also merging neutron stars in the relative vicinity of our galaxy. The Einstein Telescope (ET) is a so-called third generation gravitational wave detector that has the possibility to observe such mergers all through the visible universe.

The LIGO and Virgo detectors have a sensitivity to observe, in addition to massive black hole mergers, also merging neutron stars in the relative vicinity of our galaxy. The Einstein Telescope (ET) is a so-called third generation gravitational wave detector that has the possibility to observe such mergers all through the visible universe. Currently, one of the candidate locations being investigated for this facility is in the "heuvelland" in south Limburg.

Lecture 5

The Early Quantum Theory

"If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet."
-Niels Bohr

"Gott würfelt nicht (God does not play dice)."
-Albert Einstein

5.1 A Deterministic Universe?

The founding father of the physics we now refer to as classical physics is Isaac Newton (1642 - 1727). At the foundations of classical physics are his three laws:

1. The law of inertia: a body in rest moves with a constant speed.
2. The law of force and acceleration: F=m a
3. The law: Action = - Reaction

These laws, together with the idea that everything in the universe consists of fundamental constituents of matter ("particles") leads to a philosophical dilemma that, once given the initial position of all particles and their velocities, one can in principle predict the future of any system like a pre-programmed clock. In such a deterministic universe there is little room for something like the concept of free-will. As we will see quantum mechanics introduces a fundamental principle of chance into the laws of physics, allowing only *statistical* predictions. It has a fundament based on in-determinism: the so-called Heisenberg uncertainty principle. Einstein did not like this fundamental statistical uncertainty principle in the laws of nature, which is the reason of his famous quote on quantum mechanics. The uncertainty principle is linked to a fundamental constant of nature, similar to the speed of light, and is called Planck's constant. The discovery of its existence goes back to the nature of light: does light consist of particles or is light waves?

5.2 The Nature of Light: Particle or Wave

The authoritative Isaac Newton believed that light consisted of a stream of particles, called corpuscles. He maintained that these light-particles travelled faster in dense media, similar to the fact that sound travels faster in water than in air, and using this he could explain diffraction of light (eg. look at a pencil sticking into a glass of water).

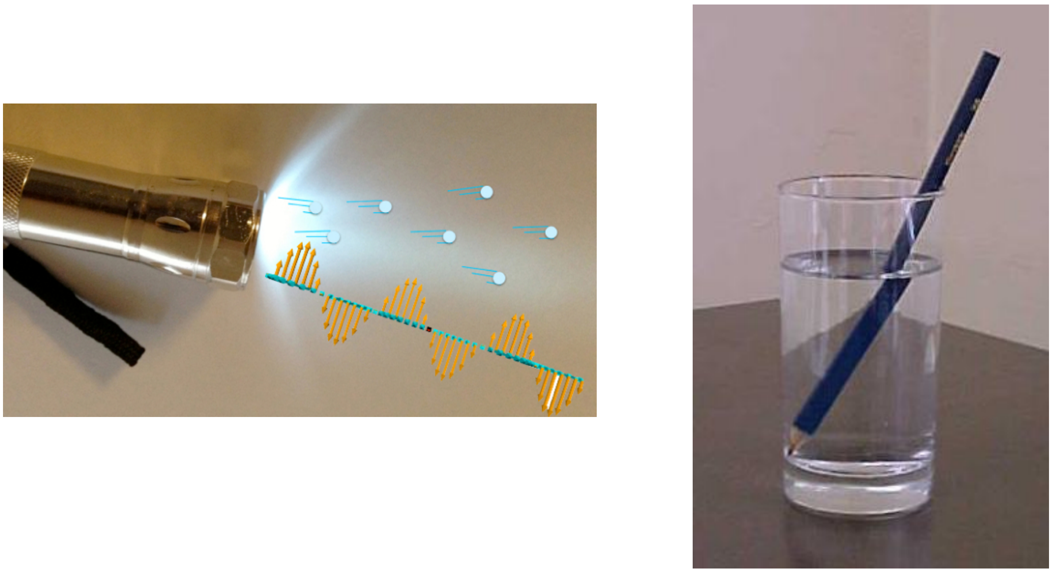


Figure 5.1: *Left:* The nature of light: does it consist of particles (Newton) or waves (Huygens)? *Right:* With both hypotheses, the phenomenon of refraction can be explained.

Alternatively, Christiaan Huygens, 13 years older then Newton, developed the theory that light consisted of transverse waves, comparable to those that move across the surface of water. Instead of water, light used *the æther* as the medium that oscillates.

In 1803 Thomas Young performed a decisive experiment by interfering waves, clearly illustrating that light has a wave character. To understand the nature of this measurement let us first have a closer look at the nature of waves.

5.2.1 Water Waves

It is perhaps easiest to consider water waves that are created if we drop a pebble in a lake. The water molecules make a vertical oscillation motion and via the molecular forces acting between the water molecules the wave propagates horizontally along the surface of the water, causing a spreading circle. In other words, the waves travel in a direction perpendicular (or *transverse*) to the vertical oscillations of the molecules. A water wave is schematically illustrated in the left picture of fig. 5.2. In physics class we learn that the wavelength of the water waves (λ) is related to the oscillation frequency of the molecules (f) via the equation: $\lambda = v/f$, where v is the velocity of propagation of the wave.

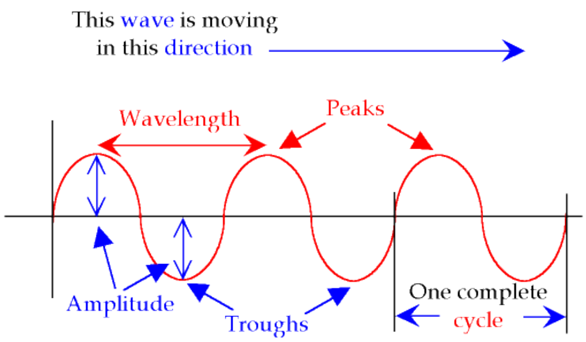


Figure 5.2: *Left:* a picture of a wave and its characteristics. *Right* a picture how waves can interfere: where peak meets peak the wave is amplified, where peak meets trough the wave is cancelled.

An essential aspect of waves is that they can cancel each other. Consider the picture on the right of fig. 5.2, where two stones are dropped in the water next to each other. Each causes a circle of waves, but where the two waves meet they can either reinforce (where peak meets peak) or cancel each other (where peak meets trough).

We are familiar with a similar phenomenon of sound waves cancelling one another in head-phones with noise-cancellation functionality. Sound-waves are caused by oscillating atoms of air that in this case are longitudinally propagating pressure waves. In noise-cancellation devices the sound-waves in the environment are used to create additional sound waves with opposite amplitudes, as illustrated on the left in fig. 5.3. Adding together the two opposite waves, the end result is no waves, or silence!

5.2.2 Light Waves

How does this work for light? Light is a wave formed by oscillations of the electromagnetic field as was illustrated in fig. 1.2. Similar to water waves an electric field and a magnetic field oscillate perpendicular to each other and generate a light wave that propagates through empty space with the speed of light c, in the direction perpendicular to both the electric and magnetic field oscillations.

Thomas Young's experiment, illustrated on the right of fig. 5.3, showed that one can create darkness by adding two sources of light, a phenomenon which was not possible to explain with light consisting of classical particles, and as such this was seen as the convincing argument that light must consist of waves. The particle-wave debate seemed finally settled. This was further worked out by Maxwell who gave a full description of light as oscillations of electromagnetic field waves. Similarly to sound waves, which are longitudinal oscillations of a medium (air), it was originally expected that light

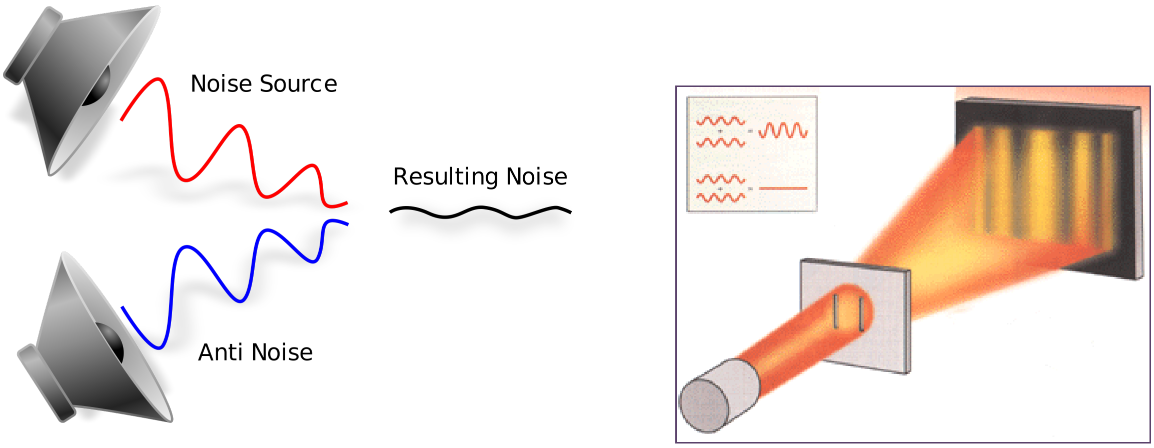


Figure 5.3: *Left:* Active Noise Reduction: sound waves cancel each other. *Right:* The interference experiment of Young, showing light and dark areas on a screen resulting from light passing through two slits in screen, seems to make the decisive answer: light consists of waves.

waves would be longitudinal oscillations of the æther. However, in Maxwell's theory of electromagnetism, light waves are transverse oscillations of electric and magnetic fields that propagate in vacuum.

In all cases, water waves, air waves, light waves, the energy or the intensity of the wave signal is equal to the square of the amplitude of the oscillations: $I \propto A^2$. Two waves with the same phase (peak meets peak) sum up to a wave with double amplitude, so-called *constructive interference*, creating more light, sound, etc., while two waves with opposite phase (peak meets trough) cancel each other to zero amplitude, so-called *destructive interference*, creating darkness, silence, etc. The interference aspect is essential for the wave-light character of phenomena.

5.2.3 Quantized Light

The first discovery leading to the development of quantum mechanics was related to the frequency of light emitted by heated material. The spectrum of electromagnetic radiation emitted by a heated so-called black-body object could not be explained using the classical theory. The German Physicist Max Planck discovered in the year 1900 that the spectrum was very well described if one assumed that the radiation could only be emitted as multiples of an elementary unit: the *quanta*. At the time it was not believed yet that light *consisted* of quanta, but that somehow it was always *emitted* in discrete quanta. The relation of the energy in a given quantum and the corresponding frequency of the radiation is:

$$E = hf \quad , \tag{5.1}$$

where h is the famous Planck constant of action. Planck, to his own frustration, never managed to understand the origin of the quantized radiation.

The value of Planck's constant is very small, 6.6262×10^{-34} Js, which is the reason that it only plays a role at an atomic level rather than in macroscopic processes. One interesting aspect to notice is that its unit is a unit [Js] is the unit of action, which is energy $\times$ time. This means that the numerical value of Planck's constant h, similar to the speed-of-light constant c, is the same for any observer in any difference frame. (A unit of energy would depend on the chosen reference frame.)

5.2.4 The Photon as a particle

In the same year he published the special theory of relativity Einstein also published a paper to explain the nature of a phenomenon known as the *photo-electric effect*. In this phenomenon, discovered in 1887 by Heinrich Hertz, electrons are emitted in certain materials if ultraviolet light was shone on them, see the left side of fig. 5.4. A surprising fact was discovered by Philipp Lenard in 1902 that the energy of the emitted electrons did not depend on the *intensity* of the light source, but only on its *frequency*. Under a given threshold frequency, no electrons emerged from the material, while for frequencies above the threshold light frequency, the energy of the emerging electrons was proportional to the energy or frequency of the light, according to the formula of Planck. Again, such behaviour could not be explained using Maxwell's description of electromagnetic radiation, since in a wave-theory the energy depends on the amplitude of the waves, thus on the intensity of light source.

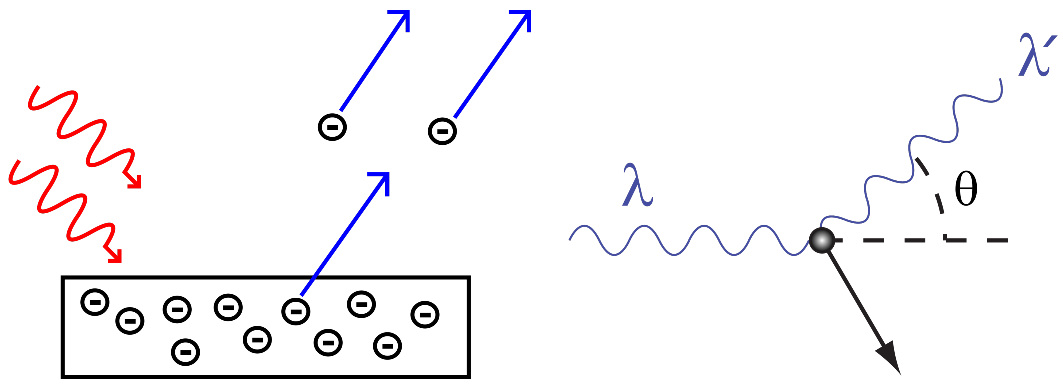


Figure 5.4: *Left:* Illustration of the photoelectric effect. Depending on their frequency photons can knock out electrons of a material. *Right:* The Compton effect. A photon of wavelength λ comes in from the left, collides with an electron and another photon of wavelength λ' comes out, together with the electron. The energies of the incoming and outgoing photons and the electron agree with the laws of scattering particles.

Einstein boldly posed in his Nobel prize winning paper that this is explained by the *nature* of light: light consists of individual *quanta* of light which individually cause emission of the electrons from the material. For light of high frequency (e.g. ultraviolet), the energy of the quantum is high and electron is knocked out at high speed; for light of low frequency (e.g. infrared), the energy is low and insufficient to knock out an electron.

The Newton vs Huygens debate of the nature of light, particle vs wave, was open again! Only after Arthur Holly Compton in 1923 (Nobelprize 1927) performed measurements that showed that the energy of emerging electrons was perfectly described by a process of a colliding photon particle with an electron (see the right side of fig. 5.4, the community was convinced: light must sometimes be described as a stream of particles! Einstein had derived that the relation between the energy of the photon and the frequency of light is: $E = hf$, and similarly that photons have a momentum $p = E/c$, such that:

$$p = \frac{hf}{c} \tag{5.2}$$

Since the relation between wavelength and frequency for light is $\lambda = c/f$ (and therefore $f = c/\lambda$), we can translate Einstein's relation between momentum frequency into $p = \frac{h}{\lambda}$, usually written as:

$$\lambda = \frac{h}{p} \quad , \tag{5.3}$$

showing that the wavelength of light is inversely proportional to the momentum of the photon.

5.3 The Wave Nature of Elementary Particles

5.3.1 Instability of the classical atom

Rutherford had shown that the atom consists of a small, dense, positively charged nucleus surrounded by negatively charged electrons, similar to planets in an orbit around the sun. What remained a mystery was how atoms could in this case be stable. What keeps the positively charged nucleus together? Why don't the surrounding electrons radiate energy in the form of electromagnetic waves as predicted by Maxwell's theory?

An electrically charged particle is surrounded by an electric field. If such a particle undergoes an accelerated motion, it will cause a changing electric field and according to the laws of Maxwell it will generate electromagnetic waves. In the classic atomic theory, the electrons are therefore expected to radiate electromagnetic waves and lose energy. In fact, according to the theory of Maxwell, it would only take a millionth of a millionth of a second until the electron would collapse into the nucleus of an atom. In other words: stable atoms cannot exist according to classical theory.

5.3.2 De Broglie waves

In 1924 the French physicist Louis de Broglie suggested that the wave-particle dualism that applies to light also applies to particles. He proposed that the same formula applicable to photons would also apply to particles:

$$\lambda = \frac{h}{p} \quad \text{or} \quad \lambda = \frac{h}{mv} \quad . \tag{5.4}$$

The consequences of this idea are far reaching: not only light is described by both particles and wave characteristics, but also particles: electrons and protons. Unlike in the cases of water waves, sound waves or electromagnetic field waves, it was not at all clear *what* is oscillating in these particle waves.

De Broglie's hypothesis was confirmed in 1927 when it was seen that electrons, similar to light wave show a diffraction behaviour, and therefore indeed behave like waves. The nature of these waves is still a mystery. In the next chapter we discuss in detail the interference character of particle waves.

Exercises:

- Calculate the wavelength of an electron ($m = 9.1 \times 10^{-31}$ kg) travelling at 10% of the lightspeed. (Planck's constant = 6.6×10^{-34}Js). Compare the result to the size of an atom, typically 10^{-10}m .
  Answer: $\lambda_{\text{electron}} = 0.024$ nm $= 0.24 \times 10^{-10}$m.

- The wavelength of visible light is 400 nm to 700 nm. Can you explain why it takes an electron microscope to see atoms?

- Calculate the wavelength of a fly (m=0.1 g) travelling at a speed of 10 m/s. Why are quantum effects more relevant for electrons than for macroscopic objects?

5.3.3 Bohr's quantum mechanical atom

We celebrated recently the 100-th anniversary of the Bohr model of the atom, which actually was a mixture of quantum theory and classical theory. He assumed that only for specific orbits of the electron, given by quantum numbers, the atom would be stable. The electron could jump from one orbit to another by either emitting or absorbing a quantum of light.

Bohr's model of the atom was understood by Erwin Schrödinger using De Broglie's idea of the wave nature of electrons. An electron wave circling in many orbits around the nucleus would interfere with itself as it goes around and around. If the length orbit is such that it is *exactly* equal to n=1, 2, 3 etc... times the wavelength of the electron, the electron wave would positively interfere with itself, resonating in a standing wave around the orbit. In this case there is no longer a picture of an accelerated charge in orbit around a nucleus, but a stable standing wave with a given energy level of the atom.

The requirement for the first standing wave is that the wavelength of the electron is equal to the circumference of the orbit: $2\pi r = \lambda$. But also orbits with the following condition $2\pi r = 2\lambda$, or in general $2\pi r = n\lambda$, meet the requirements for positive interference: peak meets peak. In all other conditions, e.g. $2\pi r = 1.2\lambda$, the result after many revolutions is that the negative interference cancels the wave.

In classical mechanics we learn that the angular momentum is defined as $L = r \times p$, such that substituting this for r in $2\pi r = 2\lambda$ and also using $\lambda = h/p$, the Bohr quantization rule follows:

$$\boxed{2\pi L = nh \quad \text{or} \quad L = n\hbar}$$

where the so-called reduced Planck constant $\hbar = h/2\pi$ is introduced. This is the main quantum rule of the Bohr atom.

Using the quantisation of the orbits Bohr predicted the corresponding energy levels belonging to each of the orbits. Electrons could sit in different discrete energy levels in an atom and they could jump from one level to the next by either emitting or absorbing a quantum of energy corresponding to the difference in energy between the orbits. Jumping from a high energy level to a low energy level, the atom would emit a photon; jumping from a low energy level to a high energy level, the atom would need to absorb a photon of the corresponding energy. The atomic model of Bohr was a great success as the jumping of electrons joined by the absorption or emission of photons described the observed light spectrum of hot atomic gases. In fig. 5.5 the Bohr model for the atom is illustrated and the spectrum of hydrogen gas, with the so-called Balmer lines, is shown.

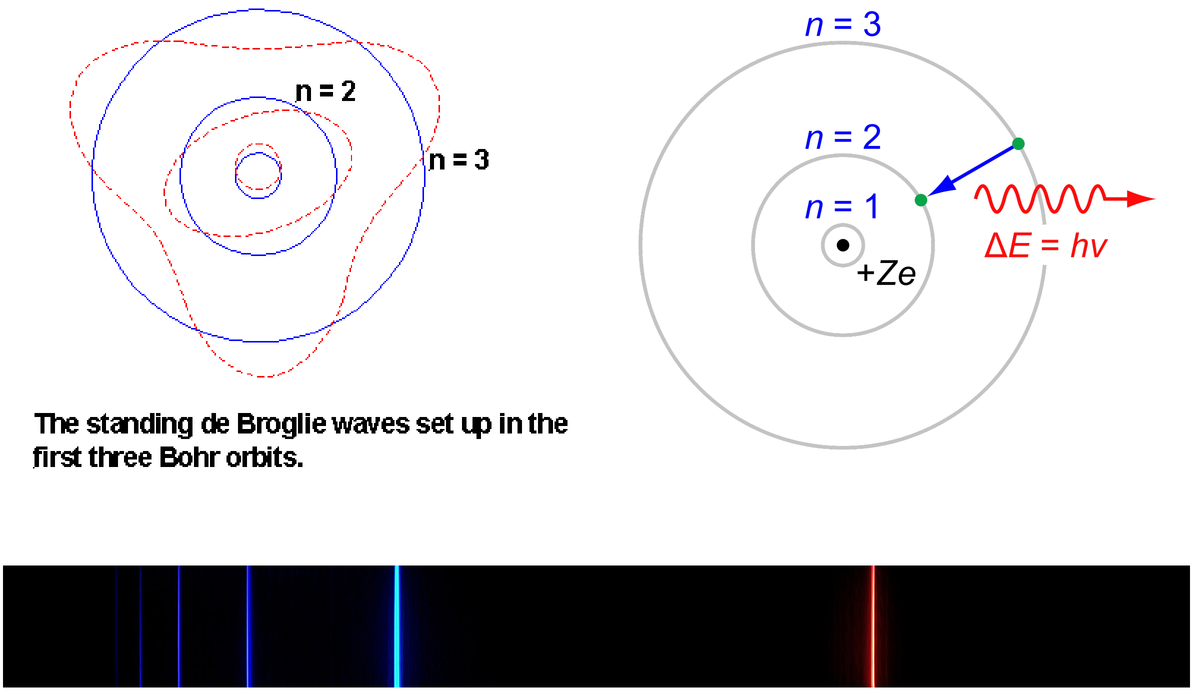


Figure 5.5: The Bohr atom. *Top Left:* The electron waves illustrated for the lowest three orbits in an atom. *Top Right:* Electrons can jump between orbits by either emitting or absorbing a photon of just the right energy. *Bottom:* The Balmer spectrum: the light emission lines of hydrogen, which were explained by Bohr's model of the atom.

Not only did Bohr's model allow to explain the emission of photons, it also predicted how many electrons could sit in each orbit for different elements. Wolfgang Pauli had discovered that no two electrons could occupy the same quantum state, which is known by the Pauli exclusion principle. As a consequence all orbits of electrons of different atoms could be calculated. For chemical reactions between elements, or bonds between atoms, the number of electrons in the outermost shell are most important. The "inner shells" are hidden and play no role in chemistry. As such, the Bohr model led to the statement: "chemistry is explained."

5.4 Particle Wave Duality

Subatomic matter does not behave like waves and it does not behave like particles. It does not behave like anything we know from the macroscopic world. Fortunately, electrons and light at least behave in a similar way. We say that on a subatomic scale matter behaves like particle waves. The bad news is that we can't *explain* quantum behaviour. At the end of the lectures you will probably not understand it. The reason is that nobody really understands it. We can describe how it works using mathematics. But, *why* it behaves completely different than macroscopic matter nobody knows. Alternatively, why would fundamental particles behave the same as macroscopic particles? The fact that they are fundamental, ie. indivisible, makes them different from anything we know by our experience or intuition.

5.4.1 *Matrix Mechanics of Heisenberg vs Wave Mechanics of Schrödinger

In 1925, before De Broglie's wave hypothesis, quantum theory was in chaos. Quantum rules seemed to work, but no-one understood why. In that year Werner Heisenberg got the idea to describe atomic phenomena with the obscure mathematics of so-called non-commuting numbers. Algebraically this means that he used numbers a and b for which $a \times b$ is *not* the same as $b \times a$. Such quantities are called matrices. Heisenberg found that he could derive the atomic energy levels if he would make the assumption that the variables for space (x) and momentum (p) would behave according to the following equation:

$$\boxed{px - xp = \frac{\hbar}{i}}$$

where $\hbar$ is Planck's constant and i is the so-called imaginary number $\sqrt{-1}$. This is the famous Heisenberg uncertainty relation. The equation implies that if we first measure the momentum and then the position of a particle, we don't get the same as first measuring position and then momentum, or, we cannot know at the same time the momentum and the position. The usual formulation of the Heisenberg uncertainty relation expresses this:

$$\boxed{\Delta x \, \Delta p \geq \frac{\hbar}{2}}$$

Heisenberg and others found that by replacing the classical numbers of x and p in Newton's equations by matrix numbers they obtained a new theory that explained quantum atoms. At the same time the new theory included the old Newton theory for macroscopic objects, in a similar way that the relativistic theory of Einstein includes the Galilei theory for low velocities. The uncertainty relation is at the heart of quantum mechanics and is related to the particle-wave complementarity of quantum mechanics, as we will illustrate below.

While the abstract matrix mechanics of Heisenberg was being developed, the Austrian physicist Erwin Schrödinger learnt of particle wave nature of De Broglie and replaced the classical description of Newton's second law of motion by a wave equation, which is now known as Schrödinger's equation. He assigned the symbol ψ to the wavefunction, which is also often called the ψ-function. Schrödinger's wave mechanics gives a more intuitive picture to illustrate the uncertainty relation then Heisenberg's matrix mathematics.

Within a year after Heisenberg and Schrödinger presented their description of quantum mechanics the brilliant Englisch theoretical physicist Paul Dirac presented a third, and even more fundamental description of quantum mechanics and showed that both Heisenberg's matrix mechanics and Schrödinger's wave mechanics are two aspects of the same fundamental physics. We will limit ourselves here to Schrödinger's description.

5.4.2 Particle waves and the uncertainty relation

Erwin Schrödinger assumed that the underlying nature of particles were physical waves of some kind. Figure 5.6 shows how we can try to unite particle and wave concepts in Schrödigers picture. Consider a wave, as shown in black on the left in the figure. If you add it together with another wave, shown in blue, with a somewhat different wavelength, the picture on the right side emerges. In the center, the two waves interfere constructively, while at the sides they cancel each other. Using the interpretation that the amplitude of the wave is related to the probability of presence of the particle, we arrive at the following picture.

- On the left we see pure waves, they have a large amplitude anywhere and have a precisely defined momentum p given by De Broglie: $p = \frac{h}{\lambda}$.
- On the right we see a sum of waves of different wavelength. They only have a large amplitude in the center with a spread in position, and their momentum is not exactly known as it is a sum of different wavelengths.

In general we can add many waves together. The more different waves we add, the better the position of the particle is limited, but the more unknown is the momentum. Alternatively, the less waves we add, the better we know the particle's momentum, but the more unknown becomes the momentum. This is Heisenberg's uncertainty relation: *we cannot define at the same time the position and the momentum of a particle.*

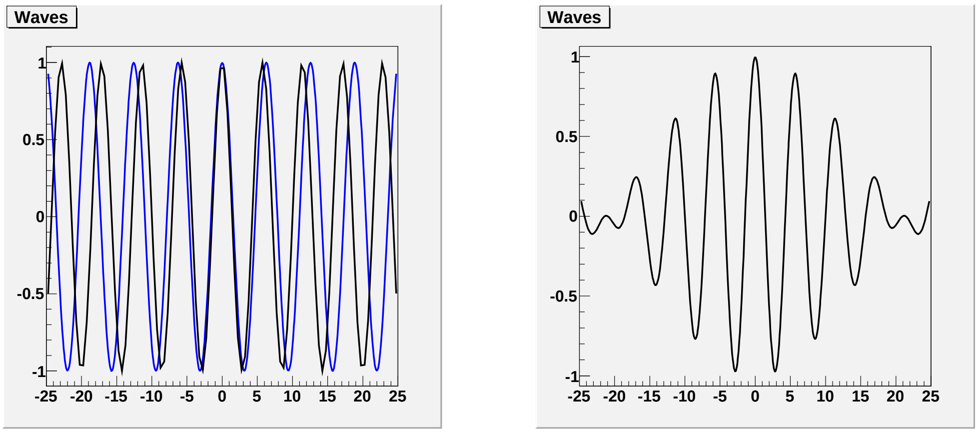


Figure 5.6: *Left:* Two cosine waves with slightly different wavelength. *Right:* The sum of the two cosine functions, a wave packet limited in space.

5.4.3 Diffraction of light and the uncertainty relation

The same uncertainty also holds for light. It can be demonstrated by a relatively simple example. Consider a beam of laser light as is generated by a simple laser pointer. If we shine the beam directly on a screen we will see a clean spot of light. However if we let the laser light-beam pass through a narrow slit, we will notice that the light spot on the screen *widens* exactly in the narrow direction of the slit. The narrower we make the slit, the wider becomes the light spot on the screen. This is illustrated in fig. 5.7.

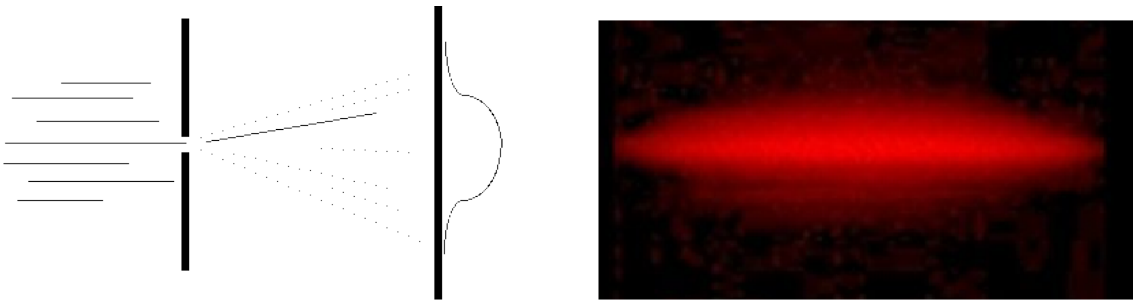


Figure 5.7: *Left:* The setup of a simple experiment using a light beam shining from the left through a thin slit onto a screen. *Right:* The resulting image on the screen. The narrower the slit, the wider the image on the screen, as predicted by the uncertainty relation.

What happens is that the narrow slit determines the position (x) of the photons. If we do that, as a consequence of the uncertainty relation, the momentum along the x-

direction becomes undefined. This means that by restricting the position of the photons, they are now composed of a wide range of different momenta, in other words, the direction of the photons becomes less defined. The narrower the slit, the wider becomes the spot of light on the screen.

Interestingly, this phenomenon can be correctly calculated classically using the interference of waves as well as quantum mechanically using the uncertainty relation. They both predict an identical effect.

5.5 Conclusion

The uncertainty relation is a fundamental aspect of quantum nature. It is not a consequence of our limitations in the measurement, it is a fundamental aspect of quantum reality. Although Schrödinger believed that particles consisted of a physical wave, the German physicist Max Born could not accept this existence of a ”real” wave. He stated that the only physical meaning of the wave was that the square of its amplitude is equal to the probability of finding the particle at that particular point. This is what is called the famous *Copenhagen interpretation*, strongly advocated by Niels Bohr and Max Born. This interpretation brings back a fundamental role of probability and chance into physics laws and removes the determinism which is present in classical physics. Einstein and Schrödinger resented this statistical aspect in the laws of physics.

The mathematics for the *probability-wave* function ψ was similar to the intensity of classical waves, apart from the fact that the wave function was a *complex number*, while the real probability was found as:

$$\mathrm{Prob}(x,t) = |\psi(x,t)|^2 = \psi(x,t)\ \psi^*(x,t) \quad .$$

This equation states in words that the probability to observe a quantum mechanical object at a given point in space, x, and a given point in time, t, is given by the *square* of the amplitude of the wave-function at that point in time and space. The wave function itself cannot be observed directly, only its square. Although it leads to very exciting consequences in physics, we will here ignore the complication of complex numbers.

In the next lecture we will be confronted by the surprising nature of the complementary wave-particle description of quantum mechanics via Feynman’s beautiful thought experiment of the double slit experiment.

Lecture 6

The Double Slit Experiment

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment it's wrong."
-Richard Feynman

6.1 Introduction

On an atomic level, matter and light behave like nothing we have any direct experience with. In the previous lecture we have seen that electrons and photons have particle properties, but also wave properties. The double slit experiment is a famous thought experiment that illustrates the essential aspects of quantum mechanics; in particular it beautifully demonstrates the wave particle duality. In the words of Feynman, the experiment illustrates the *only* mystery of quantum mechanics which nobody truly understands. We can only consistently describe it and use it to make statistical predictions for the outcome of experiments.

In the following three "cases" we will repeat similar thought experiments with respectively "bullets", "waves" and "electrons", to demonstrate the behaviour of respectively a classical particle, a classical wave and finally a quantum particle. The experiment with the electron will turn out to be impossible to be described in any classical way. It demonstrates the true mystery of quantum mechanics and also shows the role of the observer in the experiment.

6.2 Case 1: "Bullets"

The first experiment is illustrated in Fig. 6.1. Consider a particle gun that shoots bullets in a random direction. At some distance of the gun there is a wall with two adjacent thin openings, labelled 1 and 2, which are wide enough to let the bullets pass. Behind the wall there is a backstop which can "absorb" the bullets as they arrive, for example a wall of wood. This wall contains a "detector" (e.g. a box of sand) that can count the bullets as they arrive at the backstop. By placing the detector at each position along the backstop for some time and counting the number of bullets that arrive we can reconstruct the probability of the arrival bullets per unit time as a function of the coordinate x, the distance from the centre.

The question we ask ourselves in this experiment is: "What is the probability that a bullet arrives at the backstop at a distance x from the centre?". To find the experimental answer we count the bullets at each position and then divide that number by the total amount of bullets that were fired.

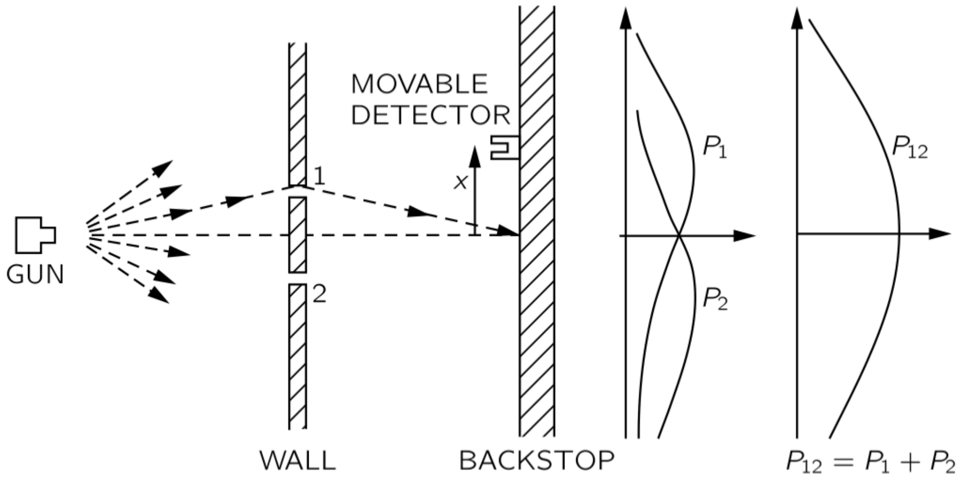


Figure 6.1: Feynman's double slit experiment with classical particles.

Let us first consider the case where only opening 1 is present and opening 2 is closed. The bullets that go through the opening may scatter from the sides of it and we observe the probability distribution given as P_1 in the figure. The maximum of the distribution is in a straight line behind the opening and there is some scattering to the sides. The distribution reflects the probability distribution where a bullet, to be fired, may arrive. Alternatively, if opening 1 is closed and 2 is opened, we observe a similar probability distribution P_2, which now has a maximum straight after slit 2. If in a third and final case both opening 1 and 2 are opened, we expect to observe the distribution labelled as P_{12}, and we can check the result:

$$P_{12} = P_1 + P_2 \quad . \tag{6.1}$$

The probability distribution for the two open holes is given by the sum of the probability distributions for each of the two cases with one slit open. This means that if we were to calculate the probability for a subsequent bullet to arrive at a given position x, we would take into account probability distribution P_{12}. The fact that P_{12} is the sum of the probabilities P_1 and P_2 of independent experiments is translated in words by saying that there is no interference between the processes of bullets going through 1 and bullets going through 2.

6.3 Case 2: "Waves"

Next, we replace the gun and bullets by a generator of waves. This could be water waves, sound waves, or electromagnetic light waves. Let us use the case of water waves and construct a similar setup as before, depicted in Fig. 6.2. On the left there is a source of waves, generating circular waves e.g. by dropping a stone in the water. The stone will make the water molecules oscillate up and down where it hits the water and generate circular waves. The waves spread out and arrive again at a wall that has two openings. The water in the openings 1 and 2 will move up and down and in turn operate as new sources generating waves behind the wall. Again there is an absorber wall, which contains a detector. In this case the detector measures the "intensity" of the arriving waves at a given position at the wall. This implies that the detector is sensitive to the height h of the oscillating wave, but it is calibrated in such a way that it registers the *square* of the amplitude of the wave. Remember that in waves the intensity (or energy) is related to the height of the wave as: $I = |h|^2$, such that the detector measures the amount of energy that is carried by the wave to a given position on the screen.

Immediately we note a difference with the case of bullets. Whereas the bullets arrive in quantum lumps, one by one, there is no such "lumpiness" or quantum behaviour of the waves. The further away we put the screen, the lower become the observed amplitudes, since the energy of the waves is spread out in a larger circle, such that the amplitude at a given position reduces with the distance from the source. Contrary to the case of bullets, the intensity of the waves can have any value, even an infinitely small one.

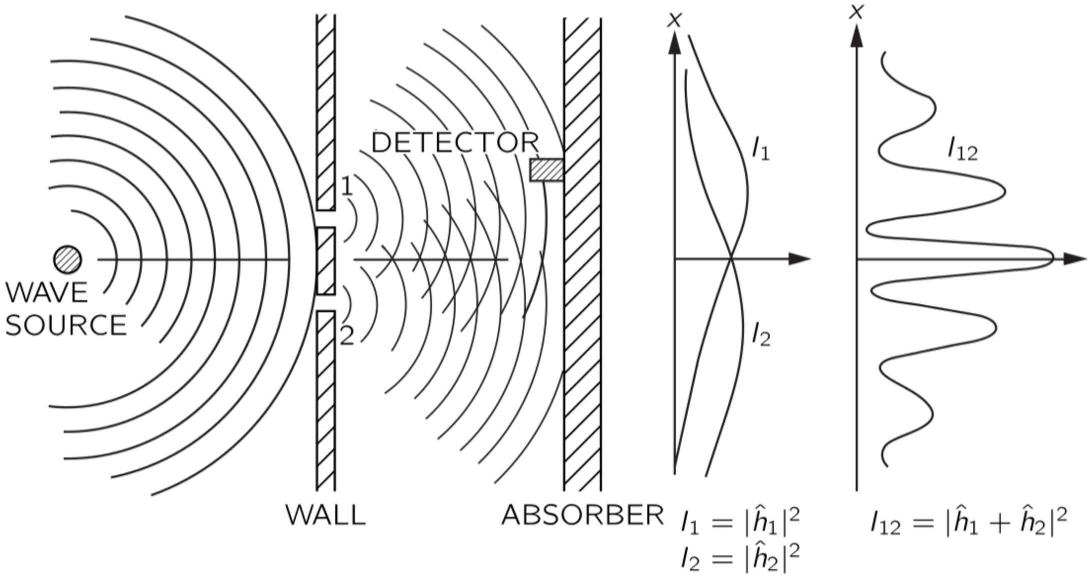


Figure 6.2: Feynman's double slit experiment with classical waves.

Let us consider first the case that only slit 1 is open. The water inside the slit (or the

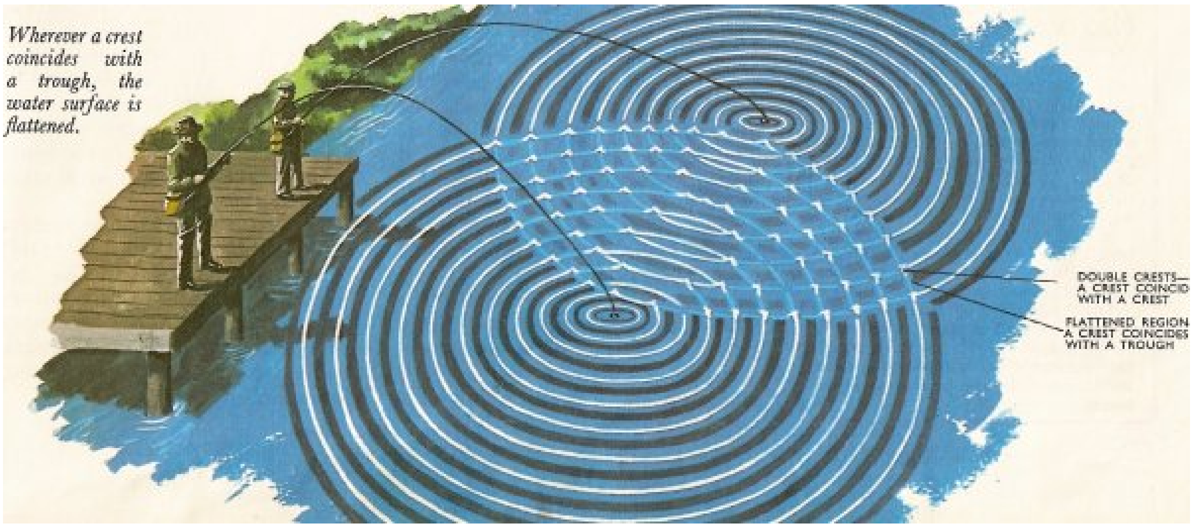


Figure 6.3: Interference of water waves. The fishing rod of each fisher creates circular waves; where they meet interference occurs.

electromagnetic light wave in an equivalent wave experiment) will oscillate and behave as a new point source of waves, emitting again circular waves. The detector will register an intensity curve given by $I_1 = |h_1|^2$ as shown in the figure. Similar to the case of the bullets, the intensity is highest directly behind the opening and becomes smaller further away to the sides. Again, opening only the second slit by itself would give a similar Intensity curve, denoted $I_2 = |h_2|^2$ in the figure.

But, consider next the case where both slides are open at the same time. Both points 1 and 2 are emitting circular waves but the phenomenon of *interference* occurs, as was already discussed in the previous lecture. Where the maxima of the waves meet the amplitude of the waves is increased ("constructive interference") and where a maximum meets a minimum the waves cancel each other ("destructive interference"). Perhaps such interference patterns are familiar for adjacent sources of water waves as shown in Fig. 6.3.

What is important to notice is that the instantaneous heights of the interfering waves are added, while the detector at the screen records the *intensity* of the resulting wave. In that case:

$$I = |h_1 + h_2|^2 \quad , \tag{6.2}$$

which is crucially different than eq. 6.1. At some places at the screen no intensity of waves are detected while at other places the intensity more than doubles.

6.3.1 *Calculation of the Interference

Let us try to calculate the interference. For a given point in the water at the detection screen the wave oscillation is represented by a cosine oscillation in time, where the position of the water as a function of time is given by:

$$h \cos(\omega t + \phi) \quad . \tag{6.3}$$

Here, h is the maximal amplitude of the oscillation, $\omega = 2\pi f$ where f is the frequency of the oscillation, and finally ϕ is the so-called phase, which depends on the position of the oscillation in the cycle at time $t = 0$. Since we have two independent sources of light we have to add up the individual waves into a resulting wave $R(t)$:

$$R(t) = h_1 \cos(\omega t + \phi_1) + h_2 \cos(\omega t + \phi_2) \tag{6.4}$$

where the phases ϕ_1 and ϕ_2 now depend on the distance from respectively slit 1 and slit 2 to the position at the screen.

For simplicity we assume that the height of wave 1 is the same as that of wave 2: $h_1 = h_2 = h$ and we can use the trigonometry rule:

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A - B)\ \cos \frac{1}{2}(A + B) \tag{6.5}$$

to find that:

$$R(t) = 2h \cdot \cos\left(\frac{1}{2}(\phi_1 - \phi_2)\right)\ \cos\left(\omega t + \frac{1}{2}(\phi_1 + \phi_2)\right) \tag{6.6}$$

This means that there is an oscillation taking place with a resulting amplitude h':

$$\begin{aligned} R(t) &= h' \cos\left(\omega t + \frac{1}{2}(\phi_1 + \phi_2)\right) \\ \text{with} \quad h' &= 2h \cdot \cos\left(\frac{1}{2}(\phi_1 - \phi_2)\right) \end{aligned}$$

and the observed intensity is the square of this:

$$I = R^2 = h'^2 \cos^2\left(\omega t + \frac{1}{2}(\phi_1 + \phi_2)\right) \tag{6.7}$$

The amplitude h'^2 can be further reduced using the trigonometry rule that states: $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$, to obtain:

$$h'^2 = 2h^2 + 2h^2 \cos(\phi_1 - \phi_2) \quad . \tag{6.8}$$

In the more general case that $h_1 \neq h_2$ the expression is:

$$h'^2 = h_1^2 + h_2^2 + 2h_1 h_2 \cos(\phi_1 - \phi_2) \quad . \tag{6.9}$$

The term $\phi_1 - \phi_2$ is called the phase difference of the interfering waves and follows from the difference δ in path lengths of each of the interfering waves to the detector, multiplied by one full oscillation phase of 2π:

$$\phi_1 - \phi_2 = \frac{d_1 - d_2}{\lambda}\, 2\pi \quad . \tag{6.10}$$

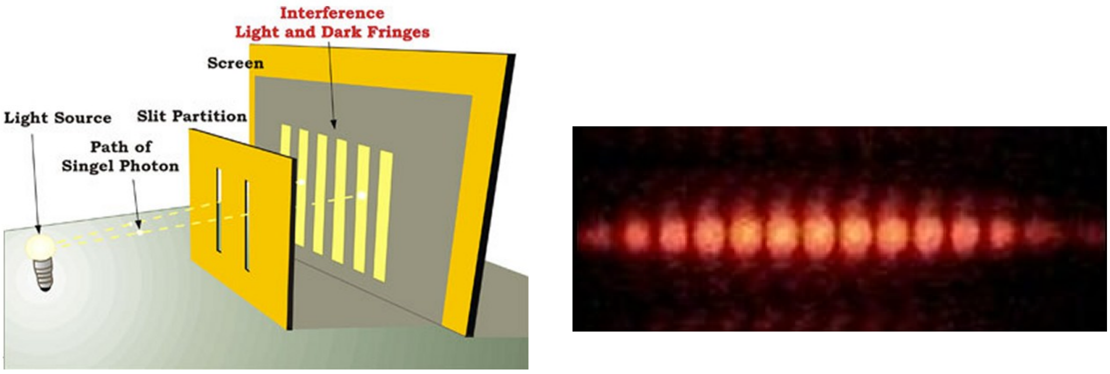


Figure 6.4: Young's experiment of interfering light waves. *Left:* Artistic view of the setup and the light and darkness pattern that such experiment might show. *Right:* Real picture of a double slit interference pattern obtained with a laser.

We see that if the difference in path length divided by the wavelength is equal to an integer number of wavelengths ($n = 0, 1, 2, ..$), there is constructive interference, while if it is a half-integer $(2n + 1)/2$ there is destructive interference. Also note that in the case there is constructive interference the amplitude is given by $|h'| = 2|h|$, such that the intensity of the resulting wave $I = 4|h|^2$, which is more than the sum of the individual intensities! At the screen we see a pattern of fast changing signals of high and low intensities (see Fig. 6.3), which is certainly not equal to the sum of the individual intensities I_1 and I_2.

This experiment was carried out with light in the famous experiment by Young (see Fig. 6.4) and was considered to be the decisive argument that light must indeed consist of (electromagnetic) waves.

6.4 Case 3: Electrons

Instead of classical particles or waves we are now ready to repeat the experiment with quantum particles: electrons. By heating a tungsten wire electrons are emitted. Putting the heated wire in a box with a small hole gives us an electron gun similar to the bullet gun in case 1 above. In Fig. 6.5 we have again a sketch of a similar double slit setup as before. The detector in this case could be eg. a geiger counter, which produces a "click" each time an electron is detected at a given position at the screen.

Immediately we notice that the counter produces clear "clicks". There is no continuous signal arriving and an electron is always registered at one specific point at the screen, unlike a wave, which spreads out and arrives at many places at the same time. Our first conclusion therefore is that electrons "come in lumps", quoting Richard Feynman.

Similar to the case of bullets we can scan the counting rate of the arriving bullets as function of the coordinate x. In other words, we measure the probability for electrons to arrive at position x. Again, we first do the measurements with only one slit open

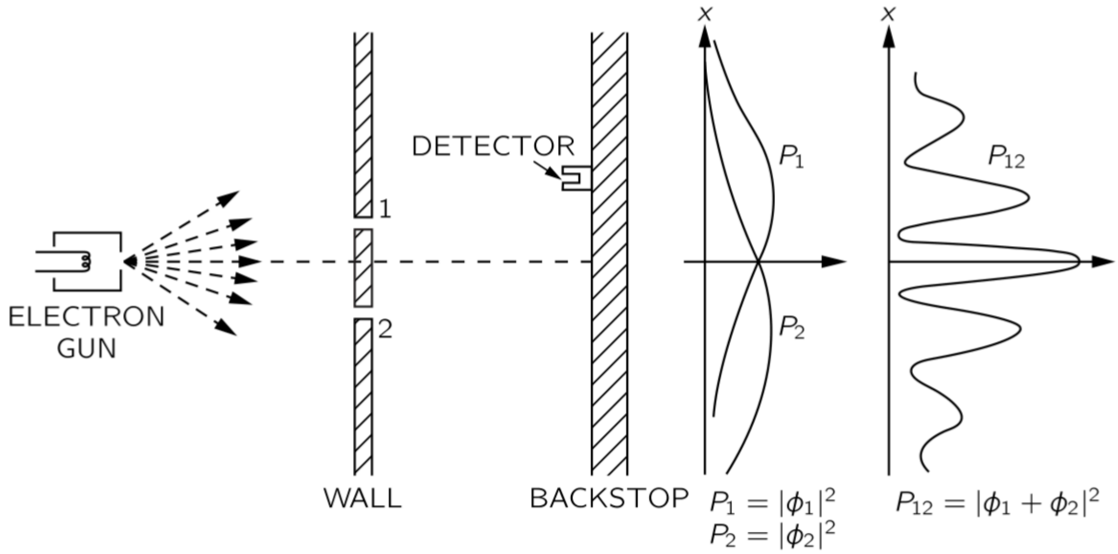


Figure 6.5: Feynman's double slit experiment with electrons.

and reproduce the probability curves P_1 and P_2, which are similar to the case of bullets. Since we have seen that electrons "come in lumps" we expect that they *either* pass through slit 1 *or* through slit 2 and that opening both slits gives us the same result as the case of the bullets: adding the individual probabilities corresponding to slit 1 and slit 2.

However, when we repeat the experiment with both slits open, surprisingly, we observe the curve P_{12}, which has the pattern of interfering waves, even though the electrons still arrive in lumps.[1] Clearly, in this case $P_1 + P_2 \neq P_{12}$.

In fact, the observed spectrum of the number of arriving electrons is exactly the same as that of the intensity of waves. Similar to the water waves we can assign a mathematical wave to the electrons and calculate the curves by adding up the amplitudes of the waves from slit 1 and slit 2. In case both slits are open we find:

$$A_{12} = A_1 + A_2 \tag{6.11}$$

where A_1 and A_2 are the amplitude of the individual electron waves (similar to eq. 6.4):

$$A_{12} = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) \tag{6.12}$$

and the probability curves are obtained by

$$P_{12} = |A_{12}|^2 = |A_1 + A_2|^2 \tag{6.13}$$

[1] It should be noted that in practice this is a difficult experiment to carry out in reality since the distance of the slits has to be very small, but it has been done!

which is of the same form as eq. 6.2 of case 2: the interfering waves. Therefore the result will be the same interference pattern that we calculated for waves in section 6.3.1.

Surprisingly, with both slits open the electrons in this experiment behave exactly like the waves. The first suspicion is that perhaps the individual electrons interfere with another, so we reduce the average rate at which electrons are fired to make sure we have single electrons at a time, for example by shooting one electron per second. However, the pattern remains the same.

The experiment is consistent with De Broglie's idea that electrons have a quantum wave behaviour, where the wave goes partly through slit 1 and partly through slit 2, and the amplitudes are added up again at any point in space after the slits. The probability to observe the electron at any place is found as the square of the wave function.

The question remains: what *are* these waves?

6.5 Case 4: Watching the Electrons

Let us try to be smarter and simply *observe* through which hole the electron goes by looking closely at the individual slits. Since electrons scatter light we can do this by shining light toward the slits and by seeing a photon in detector D_1 or D_2 we can reconstruct whether the electron actually went through slit 1 or slit 2. The set-up is shown in Fig. 6.6 and Fig. 6.7.

What happens is that each time a photon arrives at the detector screen we will also see a photon in *either* D_1 *or* D_2. We *never* see simultaneous signals in D_1 and D_2. So, when we watch them it turns out that electrons either pass slit 1 or slit 2, and never does some electron wave go through two holes at the same time! But how do we get the interference, then? In fact, by our action of watching a slit we have killed the interference pattern on the detection plane! Our experiment now gives the same results as that achieved with the classical bullets: now the total probability P_{12} is now equal to the sum of the individual probabilities $P'_{12} = P_1 + P_2$. By watching the electrons passing through the slits we have destroyed the interference!

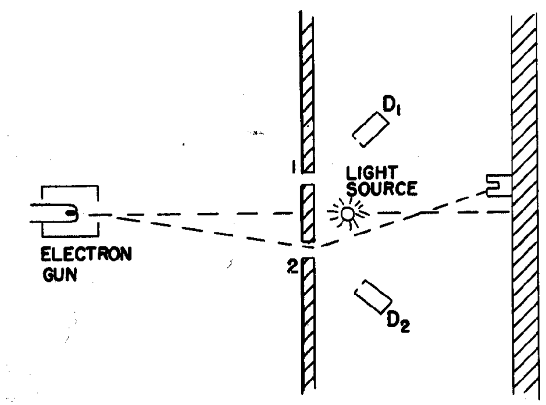


Figure 6.6: The double slit experiment with a light source and two detectors D_1 and D_2 added, such that we can "see" through which slit the electron passes.

Maybe the light source is too intense and it disturbs the photons, so let us turn down the intensity of the light. Since there are now only a few photons available sometimes

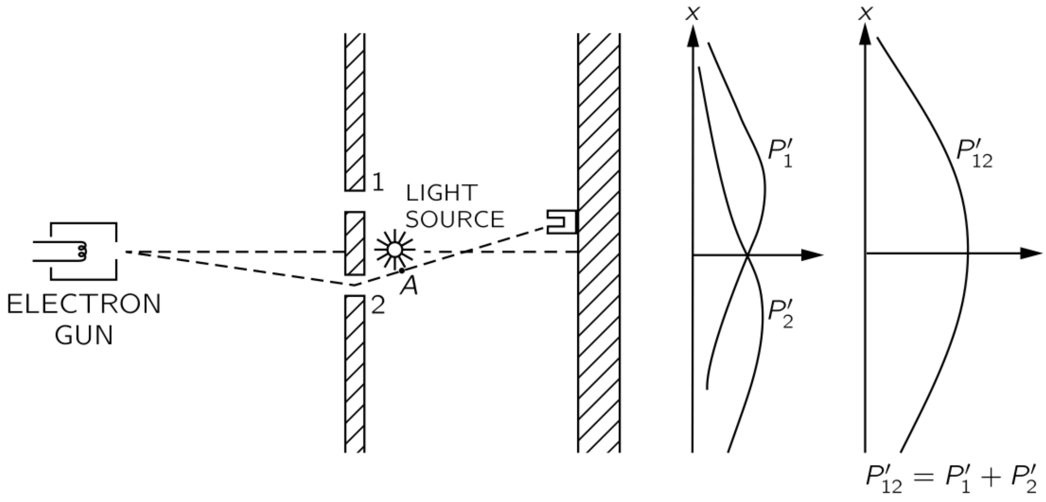


Figure 6.7: Feynman's double slit experiment with electrons while they are being watched.

no signal is seen *in either* of the detectors D_1 and D_2 and we don't know whether the electron went through slit 1 or slit 2. For all these cases the observed probability spectrum is like interfering waves. For the cases where we do observe which slit the electron went through the probability curve does not show interference!

Perhaps the photon disturbs the electron due to the "kick" it gives it? Let us make the photon "softer" by choosing a different wavelength of the light (remember $p = h/\lambda$). What happens then?

As long as the wavelength of the photon is smaller than the distance between the slits 1 and 2 nothing changes and we observe the classical pattern. As soon as the photon wavelength is larger than the distance between the gaps we see the interference pattern appear again, independent on whether the photon is observed in D_1 or in D_2. However, due to the limited resolution of light of this wavelength we can no longer relate the signal in one of the detectors to slit 1 or slit 2!

Finally we can only come to one conclusion: *if we do not observe through which slit the electron passes there is interference; if we observe through which slit the electron passes there is no interference.*

It is important to realise that this is not a limitation of any equipment but a fundamental aspect of nature. This is a specific example of the uncertainty principle of Heisenberg invoked by the measurement of the position (measuring through which slit the particle passed) and the momentum of the electron (represented by the direction of the electron and leading to the interference pattern).

6.6 Conclusion

At quantum level both ”particle” and ”wave” are equally valid, complementary properties of the same reality. This aspect is referred to as *complementarity* in the Copenhagen interpretation of quantum mechanics. Under some measurement circumstances an observer can decide which property becomes reality. If he chooses not to put the detectors there, then the ”wave”-like interference pattern will become reality; if he does put the detectors there, then the ”particle”-like paths will become reality. The fact that an observer can not observe at the same time particle-like and wave-like characteristics is guaranteed by the uncertainty relation. Nature does not allow us to measure them at the same time.

Finally, it is important to realise that quantum mechanics only predicts probabilities. In the case where the observer choses to measure the path of the particle, the choice whether the particle goes through slit 1 or slit 2 is completely random and contrary to classical theory quantum mechanics is not deterministic. Nature herself does not know which way the electrons go.

Lecture 7

The Delayed Choice Experiment and the EPR Paradox

"Your theory is crazy, but not crazy enough to be true."
- Niels Bohr

"I don't like it, and I'm sorry I ever had anything to do with it."
- Erwin Schrödinger, on the interpretation of quantum mechanics.

7.1 The delayed choice experiment

The way a quantum particle behaves is related to the question we ask it. If we ask the quantum: "Do you behave like a particle and go through one slit?", it will answer: "Yes!". If we ask the quantum: "Do you behave like a wave and pass through both slits?", it will also answer: "Yes!".

The complementarity principle states that the particle has both wave-like and particle-like properties, which are implicitly included in the wave-function, as long as the system developes without disturbance from an observer. As soon as we disturb it by making an observation, we force the system to chose a reality, either particle-like or wave-like. In the Copenhagen interpretation the act of the measurement causes what is called *the collapse of the wave function.*

To investigate this further John Wheeler in 1978 proposed a new modification of the double slit experiment. Let the particles *first pass undisturbed* through the double slit, and wait until later before asking the question: "Are you a particle?" or "Are you a wave?" In terms of Feynman's example, what he proposes is to put the measurement screen at a large distance from the two slits, and to put two telescopes *behind* the screen to watch whether the particle goes to slit 1 or slit 2. Since, in this setup, the screen is blocking the view of the telescopes, the particles are not watched and the usual interference pattern will occur caused by the quantum particle "seeing" both slits. However, if we could suddenly make the screen transparent at a time *after* the photon has already passed the slits, we could still *afterwards reconstruct* whether it has gone

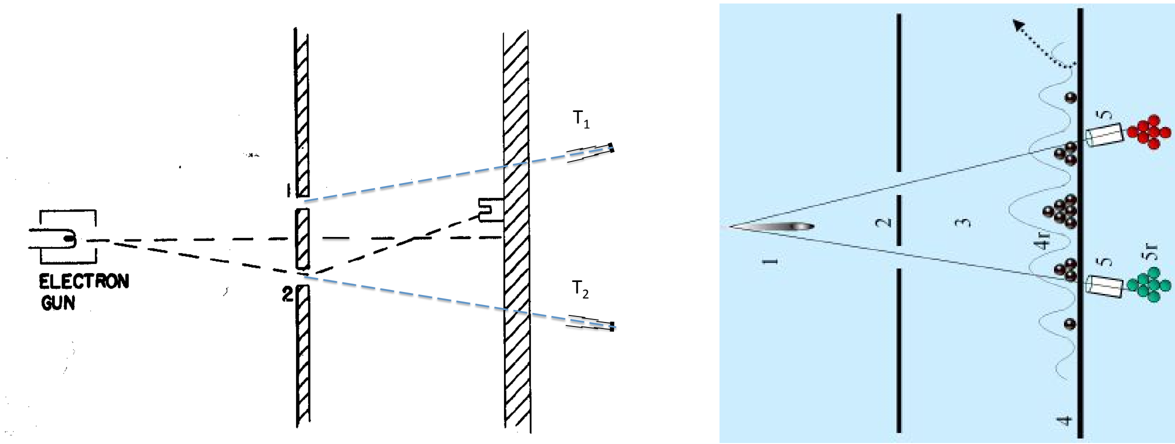


Figure 7.1: Delayed choice experiment of Wheeler. *Left:* The detectors are replaced by telescopes T_1 and T_2 positioned behind the screen, which focus on slit 1 and slit 2 respectively. *Right:* If the screen is in place the particle distribution will follow the black interference pattern, if the screen is removed the telescopes will observe the classical distribution without interference.

through slit 1 or slit 2. What happens in this case?

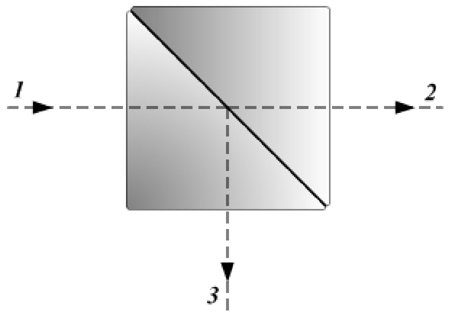
The double slit setup is modified as shown in Fig. 7.1. In this case the detectors D_1 and D_2, which look at the slit are now replaced by telescopes that are positioned *behind* the screen. The telescope T_1 is focussed on slit 1 and T_2 on slit 2. However, with the detection screen in place the telescopes see nothing and we have Feynman's double slit experiment with interference. However, if the screen would be removed the telescopes detect which slit the particle went through and there would be no interference.

But, does this make sense if we could make the distance between the slits and the screen very large and randomly decide *long after the particle went through the slits* to suddenly make the screen transparent? How would the particle know to behave while passing the slits?

7.1.1 The Experiment of Aspect

The delayed choice experiment of Wheeler has recently been carried out by the French physicist Alain Aspect and his colleagues, in 2007. The setup is somewhat different then in the case of Feynman's thought experiment, but in essence it is the same. In Aspect's experiment photons are used as quantum particles and the double slit is replaced by a so-called beam-splitter, an optical device that splits a beam of light in two paths as is illustrated in Fig. 7.2. For a single photon quantum it means that it has a 50%-50% probability to be reflected or to be transmitted.

The idea of the experiment of Aspect is shown in Fig. 7.3. Single photon pulses enter the setup from the left side and are split in two paths at beamsplitter BS_{input}: "half"



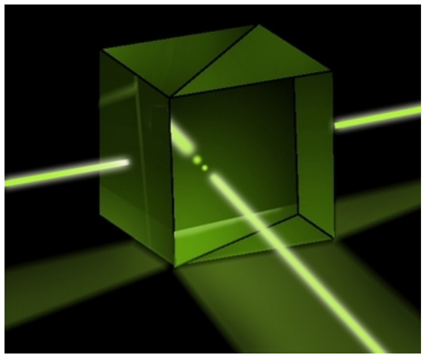
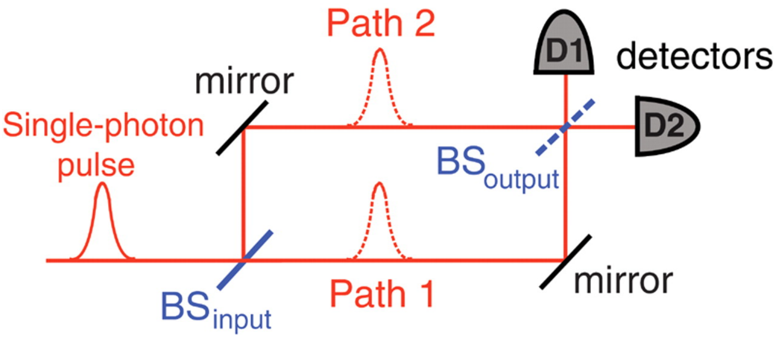


Figure 7.2: A beam splitter: half of the incoming photons (path 1) carry on along path 2 and half of them are reflected along path 3. For a single photon this reproduces the 50%-50% choice of the double slit experiment.



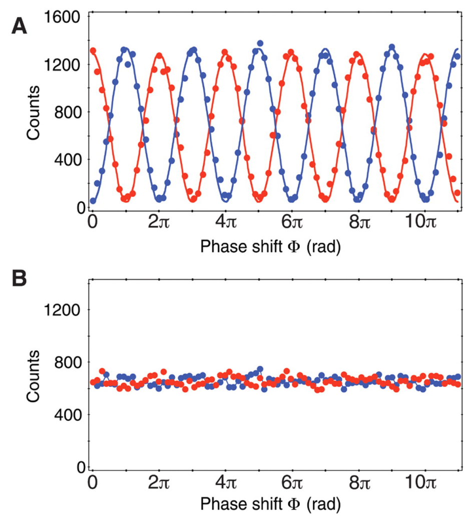


Figure 7.3: *Left:* Conceptual setup of the Aspect experiment where two photon wave functions travel along path 1 and path 2 and interfere before entering the detectors. *Right:* Graph A, on top, shows the situation with interference, when BS_{output} is switched on. Graph B, below, shows the situation with no interference, when BS_{output} is switched off.

goes straight and "half" is reflected vertically, in an equivalent way as it would chose between slits 1 and 2 in the double slit experiment. Using a mirror the photons are sent along "Path 1" and "Path 2", which are long distances. A second mirror brings the photons back to a second beam-splitter BS_{output}. Here again the photon can go straight or be deflected. But, now the waves that follow path 1 and go upward to detector D1 will meet and interfere with the waves from path 2 that are deflected upward and go to detector D1. So detector D1 will see an interference pattern from the light coming from the two paths. A similar interference is seen at detector D2. If the two paths 1 and 2 are of exactly equal length the waves interfere coherently ("crest meets crest"), if the path-lengths differ by half a wavelength they interfere destructively ("crest meets trough"). This can be tested by slightly changing the relative path length of Path 1 and Path 2.

So-far we are repeating the double slit experiment. We might, however, decide to suddenly remove the second beam-splitter BS_{output} from the set-up. In practice this is done by a fast switch of the optical properties of the BS_{output} beam-splitter such that it becomes fully transparent. In that case there are no joining light waves in the detectors and the detector D1 just sees the light that travels along path 1 and the detector D2 only sees the light that travels along path 2.

The graphs on the right-hand side of Fig. 7.3 represent the results seen if there is interference (A), or if there is no interference (B). If the length of paths 1 and 2 are made slightly different, the oscillation behaviour of graph A clearly shows that the two paths interfere, while the constant count rate of graph B shows that there is no interference.

The essence of the delayed choice experiment is to make the distance along paths 1 and 2 very long and only decide at the last moment, *after the photon has long passed* BS_{input} whether to remove BS_{output} or not. In practice it is even done in a completely random way. The striking observation is that we get the same result as in the standard double slit experiment. It does not matter if we ask the question before or after the photon passed the beam splitter.

From the point of view of the photon the situation can be represented as if it asks the observer when it passes BS_{input}: "OK, what's your question: should I behave as a particle or as a wave?", but the experimenter says: "I'm not yet going to tell you. First pass through and I will ask my question later." Finally, when the photon arrives at BS_{output} the observer asks: "Were you a wave when passing BS_{input}?", and the photon will answer: "Yes". Alternatively, if the observer asks: "Were you a particle when passing BS_{input}?", it will also answer: "Yes!" because detectors D1 and D2 never see anything at the same time in that case.

In the real experiment of Aspect in 2007, schematically shown in Fig. 7.4, a 48 meter long path is used and the removal of the second beamsplitter is accomplished with technology that uses light polarisation, of which the details are not relevant here. The essential point is that the experiment can switch between a situation with or without BS_{output} present while the photon is underway, but has already passed BS_{input}.

The experiment demonstrates that the predictions of quantum mechanics are correct.

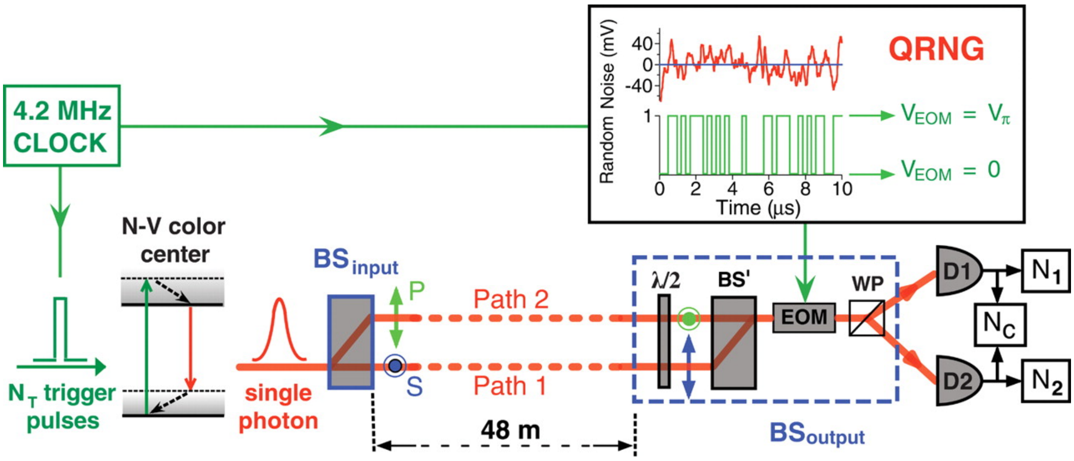


Figure 7.4: A more detailed representation of the Aspect experiment, taken from the publication of Aspect and co-workers.

7.2 Schrödinger's Cat

The way reality is described in the so-called Copenhagen interpretation, advocated by Niels Bohr and others, is that the quantum-mechanical wave-function keeps all possibilities of an experiment "open". Only at the time a measurement is done the wave function is forced to "collapse" into one of the possible outcomes. That outcome is not deterministic, only a probability can be given corresponding to the square of the amplitude of the wave function. This led to the famous quote of Einstein: "Gott würfelt nicht!" (God does not play dice!)

In an attempt to show that this leads to ridiculous consequences Erwin Schrödinger invented the famous cat-paradox, which leads to a cat that is at the same time alive and dead. In his view the concept of the collapse of the wave function could not be held.

Imagine you lock a cat in a closed box in which there is no way to make a measurement inside. In the same box there is a radiative source which might trigger a mechanism to break a glass containing poison causing the cat's death. The radioactive source is of such intensity that the probability that it radiates is exactly 50% after 15 minutes. This means that a 50%-50% quantum decision, whether an atom has emitted radiation leading to a dead cat or not, only becomes reality at the time when we open the box. Since this is a delayed quantum decision experiment, the cat must be considered in a quantum state of both alive and dead before the measurement of looking into the box is carried out.

This problem is generally referred to as the measurement problem in more philosophical physics debates. A topic of debate has been whether it requires human observation or not. Perhaps the cat is already an observer that can collapse the wave function. Alternatively, one can consider the experiment being carried out by one person in a closed laboratory. Only when he comes out and announces his results to the press the wave-function for the outside world collapses, etc. etc.

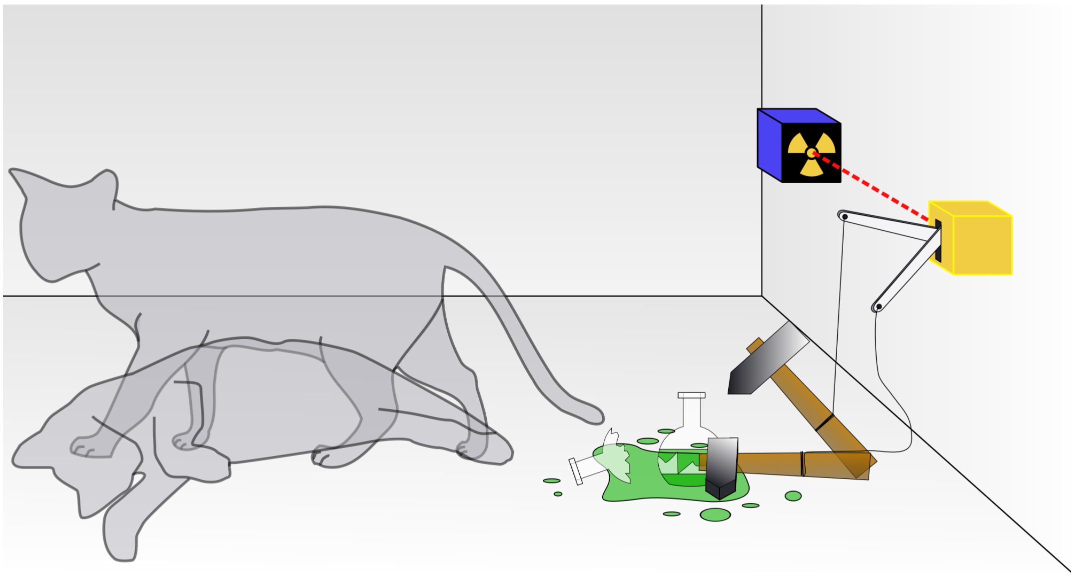


Figure 7.5: Representation of the quantum wave-function that simultaneously includes both possible realizations of dead and alive cat.

The discussion, which is mainly of a philosophical nature, is not fully settled. The interpretation of Schrödingers cat thought experiment is still a topic of discussion for physicists. The issue is purely our intuition, since mathematically quantum mechanics gives correct predictions for experiments. The discussion becomes of a relatively philosophical nature.

In the second half the last century Hugh Everett, a student of John Wheeler, published in his thesis the so-called *many worlds interpretation*. He maintains that each possible outcome of a measurement becomes reality, but all these realities exist in parallel worlds or disconnected universes. Each quantum decision leads to a split-off of a new world.

However, recently also less radical theories are being developed that show that while for atomic phenomena these quantum effects play a large role, they become less and less relevant for more complex, macroscopic systems. Such that in a natural way macroscopic objects do not experience superpositions of quantum states. The larger the system, the more susceptible it becomes to a disturbance from the outside to collapse the wave function.

7.3 Einstein revisited: the EPR Paradox

The result of the delayed choice experiment confirms the interpretation that quantum mechanics can only make a prediction of a probability in any experiment. Einstein considered the Copenhagen interpretation, including the wave function collapse not reasonable, and believed that there must exist some, yet unknown, fundamental quantity based on which the quantum particle behaves in a deterministic way.

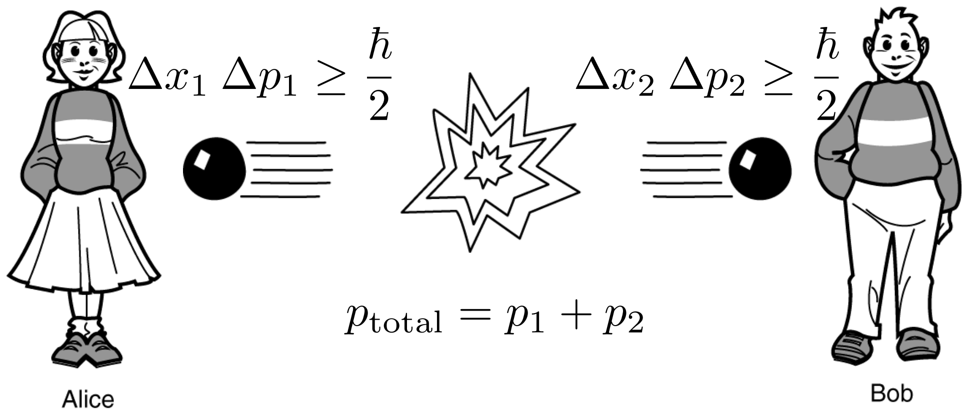


Figure 7.6: Alice and Bob carrying out an EPR experiment. Since the total momentum of the two particles is known a measurement of Alice on particle 1 instantaneously effects the quantum state of Bob's particle 2, even though they are separated by a large distance.

In 1935, Einstein and his colleagues Podolsky and Rosen wrote a famous paper in which they proposed an experiment that was intended to illustrate that the Copenhagen interpretation could not be correct. The thought experiment is illustrated in Fig. 7.6. Imagine that two particles are produced and the sum of their total momentum is known; this is well possible according to the rules of quantum mechanics. These particles fly a long distance apart until a measurement on one of these particles is done. Imagine that Alice measures the momentum of "her" particle with great precision, then automatically the momentum of Bob's particle is known, since $p_1 + p_2 = p_{\text{total}}$. Quantum mechanics dictates that Alice, in doing a precise measurement of the momentum, she has destroyed the information on her particles position, due to Heisenberg's uncertainty relation:

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

But, since now the momentum of Bob's particle is also known, she also destroyed the position information of Bob's particle.

Along the same line of reasoning, if Alice decides to measure the position of her particle its momentum becomes unknown, and therefore also the momentum of Bob's particle. What seemed completely unreasonable to Einstein was that a decision of Alice to measure the choice of measuring either the momentum or position of her particle would instantaneously affect Bob's particle, using some "spooky" action at a distance in contradiction with causality. Einstein, Podolsky and Rosen stated that *no reasonable version of reality could be expected to permit this.*

This "reasonable" version of reality is usually referred to as local realism. It implies that all objects must have pre-existing values for any possible measurement that is made. In the Copenhagen interpretation of quantum mechanics the wave function of the system does not contain a pre-existing value for a measurement. At the time of the measurement the outcome occurs randomly and the wave function instantaneously collapses.

7.3.1 *An EPR Experiment

In 1964 John Steward Bell proposed an experiment to carry out the EPR test based on spin measurements on pairs of entangled electrons. The idea is similar to the measurement of momentum and position, but uses the measurement of two spin directions of two particles, see Fig. 7.7, which are limited by the same fundamental Heisenberg uncertainty relation. Bell designed mathematical criteria (the famous Bell inequalities) to performa a decisive measurement to answer the question: does nature behave according to *local realism* or is there instantaneous "spooky" *action at a distance*?

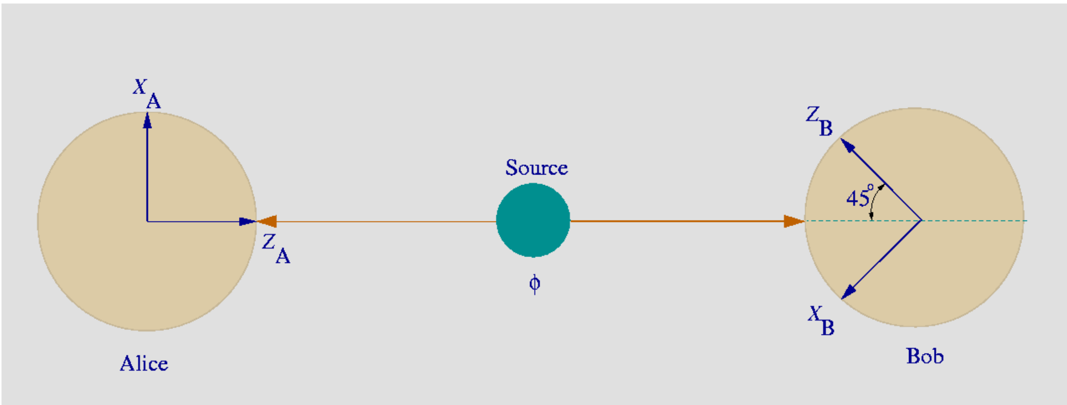


Figure 7.7: Alice and Bob carrying out an EPR experiment with two particles that have opposite spin. If Alice measures spin along the z-direction, Bob's spin point also along the z-direction. However, if Bob decides to measure along another axis, the outcome will be uncertain.

In the mean time several such experiments have been carried out and the outcome is clear: nature does not follow the "reasonable" laws of local realism. The "spooky action at a distance" is what occurs. Einstein has unfortunately not lived to see the outcome of these experiments. Being an honest man, he would certainly have been persuaded by the outcome of these experiments.

7.4 Antimatter

In 1928, Dirac combined Einsteins equation $E = mc^2$ of special relativity with the wave equation of Schrödinger into a relativistic wave equation, now called the Dirac equation. This equation, which was purely derived using a mathematical symmetry of space and time for different observers (see the relativity lectures), had two unexpected features:

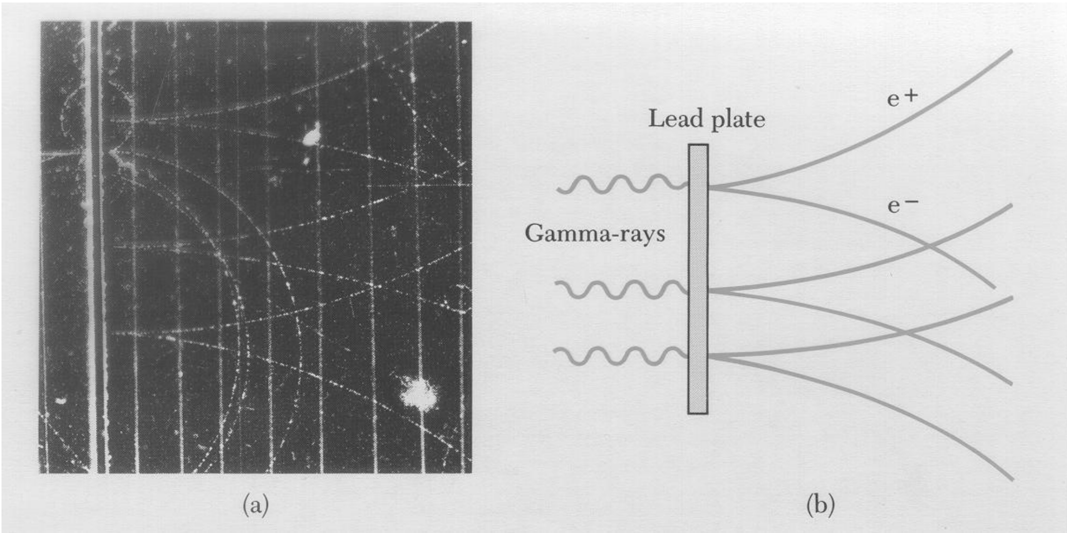


Figure 7.8: Electron positron pair creation from gamma rays. *Left:* A photograph from electron - positron creation events obtained with a particle detector. *Right:* Schematic explanation of the photograph on the left.

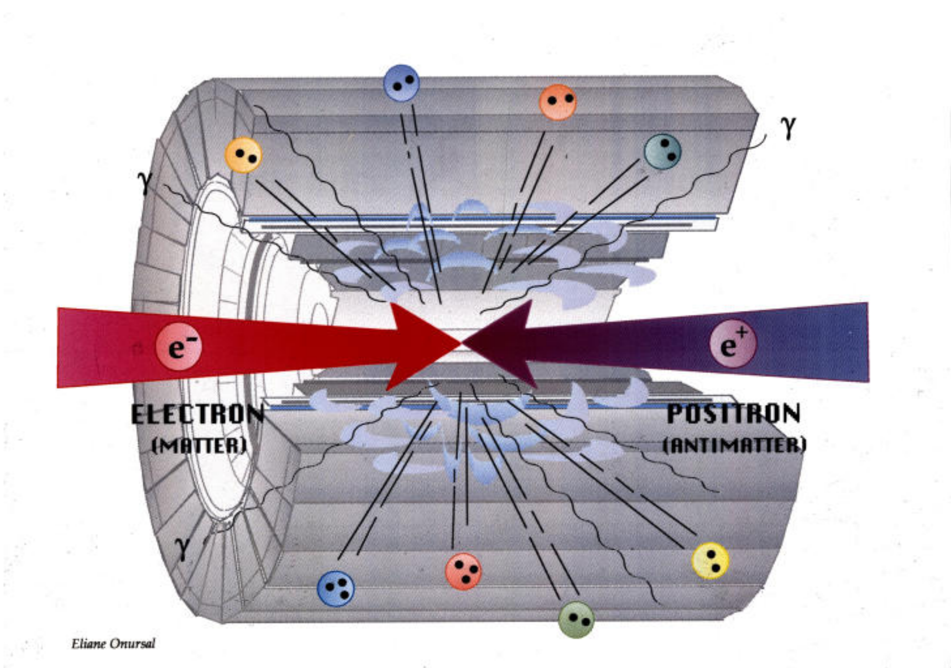


Figure 7.9: Illustration of annihilation of electrons and positron in a particle collider. First the electrons and positrons are annihilated into pure energy. The energy in turn creates different particle and antiparticle pairs that are detected in the apparatus.

- it predicted correctly that electrons should have a spin=1/2 quantum number behaviour.
- it predicted, from its symmetry, that for each particle solution of the equation there should also exist an antimatter mirror particle.

The last point was a complete surprise and was received with scepticism, until in 1932 Anderson discovered the positron, the anti-particle of the electron, when he was studying cosmic rays. A particle and its anti-particle conjugate have exactly equal mass, but all internal quantum numbers, like charge, are opposite. As a consequence, pairs of a particle plus an antiparticle can be spontaneously created as soon as enough energy is available. For example, since the mass of an electron (and also positron) is 9.1×10^{-31}kg, it requires $2 \times 8.2 \times 10^{-14}$J of energy to create a electron-positron particle pair out of pure energy. The inverse is also true: if a particle meets its anti-particle, they annihilate into an amount of energy that corresponds to the sum of their masses.

7.5 Feynman Diagrams

Using the symmetry of Dirac's description of particles and anti-particles Richard Feynman discovered that an anti-particle is mathematically identical to an ordinary particle that is travelling backwards in time. If we would be able to let time proceed backwards an electron suddenly becomes a positron in the time-reversed world.

Feynman subsequently introduced a new intuitive way to display interactions of particles as well as antiparticles using space-time diagrams. He plotted the world lines of the particles and photons as they undergo an interaction. An example of such a plot is shown in Fig. 7.10. In this diagram the event of an electron and positron accidentally colliding and annihilating into a gamma-ray photon is instead seen as an electron that emits a gamma ray and suddenly turns back in time!

John Wheeler suggested once in a telephone call to Richard Feynman the following idea: maybe the whole universe contains just one single electron that is moving forwards and backwards in time. At least that would explain why they all have exactly the same electric charge.

Why does this not work? It would require that we see the same number of electrons and anti-electrons (positrons) in the universe. The observed large matter anti-matter asymmetry is one of the puzzles in particle physics today.

7.6 Quantum Field Theory: the vacuum in turmoil

The Heisenberg uncertainty for momentum and space ($\Delta x \; \Delta p \geq \hbar/2$) also holds for time and energy: $\Delta E \, \Delta t \geq \hbar/2$. The equation implies that if a system is considered for a very short time span Δt, its amount of energy during that time period is not precisely defined, similar to the case that the momentum is not well defined for a very short distance scale.

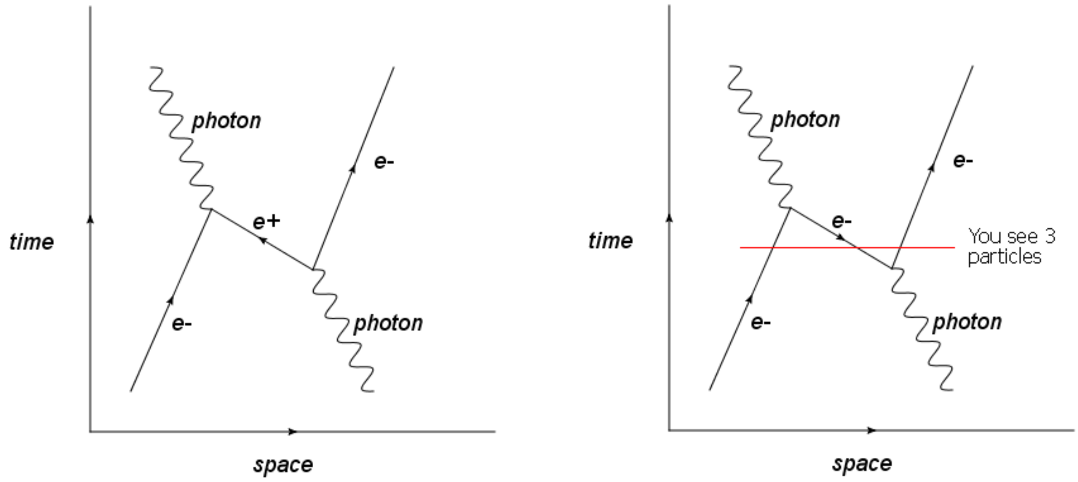


Figure 7.10: Two different views of the same process: *Left: In "Time" order:* initially, an electron and a photon exist. The photon creates a positron and an electron. The positron meets the first electron and annihilates into a new photon. Finally only a photon and an electron exist *Right: in "Feynman" order:* An electron comes in and decides to radiate off a photon and turn back in time. Subsequently it radiates off another photon and turns again in the positive time direction.

A curious possibility arises if the *uncertainty* of energy becomes so large that it allows a temporarily creation of a particle anti-pair: the vacuum creates a particle - antiparticle pair, which a very short time later annihilate again! This is referred to as virtual creation of particles. Looking at the vacuum for such very short periods of time, it is not "empty" at all: many particles are created and annihilated in a continuous dance. The shorter the timescale, the more exotic is the vacuum.

In the theory of quantum electrodynamics charged particles interact with each other via the exchange of a photon. The corresponding Feynman diagram for two electrons that interact in the field of each other's charge is shown in Fig. 7.11. The diagram shows that the force occurs via the exchange of a virtual photon, which is temporarily "borrowed" from the vacuum. The photon is called the force-carrier particle of the electromagnetic interaction.

Since experiments carried out at CERN in 1984 we know that radioactivity is another fundamental force, which is generated by the exchange of Z, W^+, W^- bosons. These force carriers are similar to photons apart from the fact that in contrast to the massless photons, they have a high mass. As a consequence of their high mass a radioactive interaction requires to borrow more energy from the vacuum, which can only be done at shorter time scales. As a result the radioactive interaction is weaker than electromagnetism and generally referred to as the weak interaction. The recently discovered Higgs particle provides an explanation for the mass of the weak force carriers: the W and Z bosons.

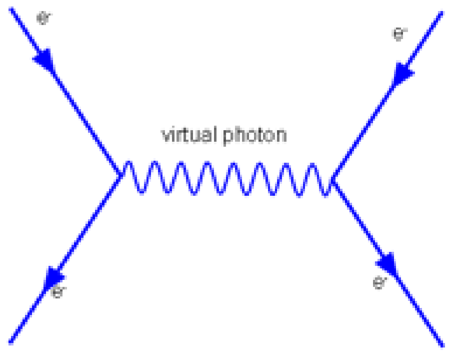


Figure 7.11: A Feynman diagram of an electromagnetic interaction between two electrons. The electrons approach each-other until they are close enough to exchange a photon that is created within the energy restrictions of the Heisenberg uncertainty relation. This is how the particles "feel" each others field.

The strong nuclear force, in turn, is generated by massless gluons. Gluons differ from photons due to the fact that they interact with themselves such that the net effect at a distance is that there is no interaction. A photon has no electric charge and therefore does not "feel" itself. Although the strong force dominates over electromagnetism inside an atomic nucleus (it is the binding force of atomic nuclei), it has no effect outside it.

Finally, the last fundamental force of nature is gravity. Einstein has spent the second half of his later career to find a combined field theory for gravity and electromagnetism, but did not succeed. The current theoretical efforts to create a quantum theory for gravity are still at a hypothetical level.