Lecture 1

In the next three lectures the Standard Model of electroweak interactions will be introduced. We will do this via the principle of gauge invariance. The idea of gauge invariance forms now such a firm basis of the description of forces that I feel it is suitable to be discussed in these lectures. Some part of the material was already discussed in particle physics 1 course, and in the exercises we did. As these lectures are not part of a theoretical master course we will follow a utilitarian and hopefully intuitive approach. Certainly we will try to focus, as we did in the particle physics 1 course, on the concepts, rather then on formal derivations.
A good book on this topic is:
Chris Quigg, "Gauge Theories of the Strong, Weak, and Electromagnetic Interactions", in the series of "Frontiers in Physics", Benjamin Cummings.

1.1 Introduction

The reason why we chose the Lagrangian approach in field theory is that it is particularly suitable to discuss symmetry or invariance principles and conservation laws that they are related to. Symmetry principles play a fundamental role in particle physics. In general one can distinguish[1] in general 4 groups of symmetries. There is a theorem stating that a symmetry is always related to a quantity that is fundamentally unobservable. Some of these unobservables are mentioned below:

- permutation symmetries: Bose Einstein statistics for integer spin particles and Fermi Dirac statistics for half integer spin particles. The unobservable is the identity of a particle.

- continuous space-time symmetries: translation, rotation, acceleration, etc. The related unobservables are respectively: absolute position in space, absolute direction and the equivalence between gravity and accelleration.

- discrete symmetries: space inversion, time inversion, charge inversion. The unobservables are absolute left/right handedness, the direction of time and an asolute

[1] T.D. Lee: "Particle Physics and Introduction to Field Theory"

definition of charge. A famous example in this respect is to try and make an absolute definition of matter and anti-matter. Is it possible? This will be addressed in a later block of lectures in this course.

- unitary symmetries or internal symmetries: gauge invariances. These are the symmetries discussed in these lectures. As an example of an unobservable quantity we can mention the absolute phase of a quantum mechanical wave function.

We believe that all elementary interactions of the quarks and leptons can be understood as consequences of gauge symmetry priciples. The idea of local gauge invariant theory will be discussed in the first lecture and will be further applied in the unified electroweak theory in the second lecture. In the third lecture we will calculate the electroweak process $e^+e^- \to \gamma, Z\mu^+\mu^-$, using the techniques of the Particle Physics 1 course.

1.2 Lagrangian

In classical mechanics the Lagrangian may be regarded as the fundamental object, leading to the equations of motions of objects. From the Lagrangian, one can construct "the action" and follow Hamilton's principle of least action to find the physical path:

$$\delta S = \delta \int_{t_1}^{t_2} dt\ L\left(q, \dot{q}\right)$$

where $q, \dot{q}$ are the generalised coordinate and velocity.

Exercise 1:
Prove that satisfaction of Hamilton's principle is guaranteed by the Euler Lagrange equations:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

The classical theory does not treat space and time symmetrically as the Lagrangian might depend on the *parameter* t. This causes a problem if we want to make a relativisticaly covariant theory.

In a field theory the Lagrangian in terms of generalized coordinates is replaced $L(q, \dot{q})$ by a Lagrangian density in terms of fields $\phi\left(x\right)$ and their gradients:

$$\mathcal{L}\left(\phi\left(x\right), \partial_\mu \phi\left(x\right)\right) \quad \text{where} \quad L \equiv \int \mathcal{L}\left(\phi, \partial_\mu \phi\right)$$

The fields may be regarded as a separate generalized coordinate at each value of its argument: the space-time coordinate x. In fact, the field theory is the limit of a system of n degrees of freedom where n tends to infinity.

In this case the principle of least action becomes:

$$\delta \int_{t_1}^{t_2} d^4x \, \mathcal{L}\left(\phi, \partial_\mu \phi\right) = 0$$

where t_1, t_2 are the endpoints of the path.

This is guaranteed by the Euler Lagrange equantion:

$$\frac{\partial \mathcal{L}}{\partial \phi\left(x\right)} = \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\left(x\right)\right)}$$

which in turn lead to the equation of motion for the fields.
Note: If the Lagrangian is a Lorentz scalar, then the theory is automatically relativistic covariant.

What we will do next is to try and construct the Lagrangian for electromagnetic and weak interaction based on the idea of gauge invariance (or gauge symmetries).

1.3 Where does the name “gauge theory” come from?

The idea of gauge invariance as a dynamical principle is due to Hermann Weyl. He called it *“eichinvarianz”* (“gauge” = “calibration”). Hermann Weyl[2] was trying to find a geometrical basis for both gravitation and electromagnetism. Although his effort was unsuccesfull the terminology survived. His idea is summarized here.

Consider a change in a function $f(x)$ between point x_μ and point $x_\mu + dx_\mu$. If the space has a uniform scale we expect simply:

$$f(x + dx) = f(x) + \partial^\mu f(x) dx_\mu$$

But, if in addition the scale, or the unit of measure, for f changes by a factor $(1 + S^\mu dx_\mu)$ between x and $x + dx$, then the value of f becomes:

$$\begin{aligned} f(x+dx) &= \left(f(x) + \partial^\mu f(x) dx_\mu\right) \cdot \left(1 + S^\nu dx_\nu\right) \\ &= f(x) + \left(\partial^\mu f(x) + f(x) S^\mu\right) dx_\mu + O(dx)^2 \end{aligned}$$

So, to first order, the increment is:

$$\Delta f = \left(\partial^\mu + S^\mu\right) f \cdot dx_\mu$$

In other words Weyl introduced a modified differential operator by the replacement: $\partial^\mu \to \partial^\mu + S^\mu$.

One can see this in analogy in electrodynamics in the replacement of the momentum by the canonical momentum parameter: $p^\mu \to p^\mu - eA^\mu$ in the Lagrangian, or in Quantum Mechanics: $\partial^\mu \to \partial^\mu + ieA^\mu$, as was discussed in the lectures last semester. In this

[2] H. Weyl, *Z. Phys.* **56**, 330 (1929)

case the "scale" is $S^\mu = ieA^\mu$. If we now require that the laws of physics are invariant under a change:

$$(1 + S^\mu dx_\mu) \rightarrow (1 + ieA^\mu dx_\mu) \approx \exp\left(ieA^\mu dx_\mu\right)$$

then we see that the change of scale gets the form of a change of a phase. When he later on studied the invariance under phase transformations, he kept using the terminology of "gauge invariance".

1.4 Phase Invariance in Quantum Mechanics

The expectation value of a quantum mechanical *observable* is typically of the form:

$$\langle O \rangle = \int \psi^* O \psi$$

If we now make the replacement $\psi(x) \rightarrow e^{i\alpha}\psi(x)$ the expectation value of the observable remains the same. We say that we cannot measure the absolute phase of the wave function. (We can only measure *relative* phases between wavefunctions in interference experiments, see eg. the CP violation observables later on.)

But, are we allowed to choose a different phase convention on, say, the moon and on earth, for a wave function $\psi(x)$? In other words, we want to introduce the concept of *local* gauge invariance. This means that the physics observable stays invariant under the replacement:

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$$

The problem that we face is that the Lagrangian density $\mathcal{L}\left(\psi(x), \partial_\mu \psi(x)\right)$ depends on both on the fields $\psi(x)$ and on the derivatives $\partial_\mu \psi(x)$. The derivative term yields:

$$\partial_\mu \psi(x) \rightarrow \partial_\mu \psi'(x) = e^{i\alpha(x)}\ \left(\partial_\mu \psi(x) + i\partial_\mu \alpha(x)\psi(x)\right)$$

The second term spoils the fact that the transformation is simply an overall (unobservable) phase factor. It spoils the phase invariance of the theory. But, if we replace the derivative ∂_μ by the gauge-covariant derivative:

$$\partial_\mu \rightarrow D_\mu \equiv \partial_\mu + ieA_\mu$$

and we require that the field A_μ at the same time transforms as:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

then we see that we get an overall phase factor for the covariant derivative term:

$$\begin{aligned} D_\mu \psi(x) \rightarrow D_\mu \psi'(x) &= e^{i\alpha(x)}\left(\partial_\mu \psi(x) + i\partial_\mu \alpha(x)\psi(x) + ieA_\mu(x)\psi(x) - ie\frac{1}{e}\partial_\mu \alpha(x)\psi(x)\right) \\ &= e^{i\alpha(x)} D_\mu \psi(x) \end{aligned}$$

As a consequence, quantities like $\psi^* D_\mu \psi$ will now be invariant under local gauge transformations.

1.5 Phase invariance for a Dirac Particle

Exercise 2a:
Show that the Euler Lagrange equations of the Lagrangian

$$\mathcal{L} = \mathcal{L}_{free} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x)$$

lead to the Dirac equation:

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$

and its adjoint.

We are going to replace:

$$\partial_{\mu} \to D_{\mu} \equiv \partial_{\mu} + iqA_{\mu}(x)$$

What happens to the Lagrangian?

$$\begin{aligned} \mathcal{L} &= \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\ \psi \\ &= \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - qA_{\mu}\bar{\psi}\gamma^{\mu}\psi \\ &= \mathcal{L}_{free} - \mathcal{L}_{int} \end{aligned}$$

with:

$$\mathcal{L}_{int} = J^{\mu}A_{\mu} \qquad \text{and} \qquad J^{\mu} = q\bar{\psi}\gamma^{\mu}\psi$$

which is the familiar current we discussed in the previous semester.

Exercise 2b)
Show that the Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi$$

is invariant under local gauge transformations:

$$\psi(x) \to e^{iq\alpha(x)}\psi(x) \qquad ; \qquad \bar{\psi}(x) \to e^{-iq\alpha(x)}\bar{\psi}(x)$$

if simultaneously A_{μ} transforms as:

$$A_{\mu}(x) \to A_{\mu}(x) - \partial_{\mu}\alpha(x)$$

In fact, the full QED Lagrangian includes also the so-called kinetic term of the field (the free fotons):

$$\mathcal{L}_{QED} = \mathcal{L}_{free} - J^{\mu}A_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $F^{\mu\nu} = \partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu}$, where the A fields are given by solutions of the Maxwell equations (see PP1):

$$\partial_{\mu}F^{\mu\nu} = 4\pi J^{\nu} \quad .$$

1.6 Interpretation

What does it all mean?
We started from a free field Lagrangian which describes Dirac particles. Then we required that the fields have a U(1) symmetry which couples to the charge q. In other words: the physics does not change if we multiply by a unitary phase factor:

$$\psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x)$$

However, in order to obtain this symmetry we *must* then introduce a gauge field, the photon, which *couples* to the charge q:

$$D_\mu = \partial_\mu + iqA_\mu(x)$$

and which transforms simultaneously as:

$$A'_\mu(x) = A\mu(x) - \partial_\mu\alpha(x)$$

This symmetry is called gauge invariance under U(1) transformations.

While ensuring the gauge invariance we have obtained the QED Lagrangian that describes the interactions between electrons and photons!
Note:
If the photon would have a mass, the corresponding term in the Lagrangian would be:

$$\mathcal{L}_\gamma = \frac{1}{2}m^2 A^\mu A_\mu$$

This term obviously violates local gauge invariance, since:

$$A^\mu A_\mu \to (A^\mu - \partial^\mu\alpha)(A_\mu - \partial_\mu\alpha) \neq A^\mu A_\mu$$

Conclusion: the photon must be massless. Later on in the course it will be discussed how masses of vector bosons can be generated in the Higgs mechanism.

1.7 Yang Mills Theories

The concept of *non abelian* gauge theories is introduced here in a somewhat historical context as this helps to also understand the origin of the term weak iso-spin and the relation to (strong-) isospin.

Let us look at an example of the isospin system, i.e. the proton and the neutron. Let us also for the moment forget about the electric charge (we switch off electromagnetism and look only at the dominating strong interaction) and write the free Lagrangian for nucleons as:

$$\mathcal{L} = \bar{p}\ (i\gamma^\mu\partial_\mu - m)\ p + \bar{n}\ (i\gamma^\mu\partial_\mu - m)\ n$$

or, in terms of a composite spinor $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$:

$$\mathcal{L} = \bar{\psi}\ (i\gamma^\mu\, I\, \partial_\mu - I\, m)\ \psi \quad \text{with} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we now, instead of a phase factor as in QED, make a *global* rotation in isospin space:

$$\psi \to \psi' = \exp\left(i\frac{\vec{\tau}\cdot\vec{\alpha}}{2}\right)\psi$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the usual Pauli Matrices [3]. We have introduced a SU(2) phase transformation of special unitary 2x2 transformations (i.e. unitary 2x2 transformations with det=+1).

What does it mean? We state that, if we forget about their electric charge, the proton and neutron are indistinguishable, similar to the case of two wavefunctions with a different phase). It is just convention which one we call the *proton* and which one the *neutron*. The Lagrangian does not change under such a *global* SU(2) phase rotation.

However, as this is a global gauge transformation, it implies that once we make a definition at given point in space-time, this convention must be respected anywhere in space-time. This restriction seemed unnatural to Yang and Mills in a local field theory.

Can we also make a *local* SU(2) gauge transformation theory? So, let us try to define a theory where we chose the isospin direction differently for any space-time point.

To simplify the notation we define the gauge transformation as follows:

$$\begin{aligned} \psi(x) \to \psi'(x) &= G(x)\psi(x) \\ \text{with } G(x) &= \exp\left(\frac{i}{2}\,\vec{\tau}\cdot\vec{\alpha}(x)\right) \end{aligned}$$

But we have again, as in the case of QED, the problem with the transformation of the derivative:

$$\partial_\mu\psi(x) \to G\,(\partial_\mu\psi) + (\partial_\mu\, G)\ \psi$$

(just write it out yourself).

So, also here, we must introduce a new gauge field to keep the Lagrangian invariant:

$$\mathcal{L} = \bar{\psi}\ (i\gamma^\mu\, I\, D_\mu - Im)\ \psi \quad \text{with} \quad \psi = \begin{pmatrix} p \\ n \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we introduce the new covariant derivative:

$$I\partial_\mu \to D_\mu = I\partial_\mu + igB_\mu$$

where g is a new coupling constant that replaces the charge e in electromagnetism. The object B_μ is now a (2x2) matrix:

$$B_\mu = \frac{1}{2}\,\vec{\tau}\cdot\vec{b}_\mu = \frac{1}{2}t^a b^a_\mu = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$$

[3] a representation is: $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$

$\vec{b}_\mu = (b_1, b_2, b_3)$ are now three gauge fields. We need now 3 fields rather then 1, one for each of the generators of the symmetry group of SU(2): τ_1, τ_2, τ_3.

We want get again a behaviour:

$$D_\mu \psi \to D'_\mu \psi' = G\,(D_\mu \psi)$$

because in that case the Lagrangian $\bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$ is invariant for local gauge transformations. If we write out the covariant derivative term we get:

$$\begin{aligned} D'_\mu \psi' &= \left(\partial_\mu + igB'_\mu\right)\,\psi' \\ &= G\,(\partial_\mu \psi) + (\partial_\mu G)\,\psi + igB'_\mu\,(G\psi) \end{aligned}$$

If we compare this to the desired result:

$$\begin{aligned} D'_\mu \psi' &= G\,(\partial_\mu \psi + igB_\mu)\,\psi \\ &= G\,(\partial_\mu \psi) + igG\,(B_\mu \psi) \end{aligned}$$

then we see that the desired behaviour is obtained if the gauge field transforms simultaneously as:

$$igB'_\mu\,(G\psi) = igG\,(B_\mu \psi) - (\partial_\mu G)\,\psi$$

which must then be true for all values of the nucleon field ψ. Multiplying this operator equation from the right by G^{-1} we get:

$$B'_\mu = GB_\mu G^{-1} + \frac{i}{g}\,(\partial_\mu G)\,G^{-1}$$

Although this looks rather complicated we can again try to interpret this by comparing to the case of electromagnetism, where $G_{em} = e^{iq\alpha(x)}$.
Then:

$$\begin{aligned} A'_\mu &= G_{em} A_\mu G^{-1}_{em} + \frac{i}{q}\,(\partial_\mu G_{em})\,G^{-1}_{em} \\ &= A_\mu - \partial_\mu \alpha \end{aligned}$$

which is exactly what we had before.

Exercise 3:
Consider an infinitesimal gauge transformation:

$$G = 1 + \frac{i}{2}\vec{\tau}\cdot\vec{\alpha} \qquad\qquad |\alpha_i| << 1$$

Use the general transformation rule for B'_μ and use $B_\mu = \frac{1}{2}\vec{\tau}\cdot\vec{b}_\mu$ to demonstrate that the fields transform as:

$$\vec{b}'_\mu = \vec{b}_\mu - \vec{\alpha}\times\vec{b}_\mu - \frac{1}{g}\partial_\mu\vec{\alpha}$$

(use: the Pauli-matrix identity: $(\vec{\tau}\cdot\vec{a})(\vec{\tau}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\tau}\cdot(\vec{a}\times\vec{b})$*).*

So for isospin symmetry the b^a_μ fields transform as an isospin rotation and a gradient term. The gradient term was already present in QED. The rotation term is new. It arises due to the non-commutativity of the 2x2 isospin rotations. If we write out the gauge field transformation formula in components:

$$b'^l_\mu = b^l_\mu - \epsilon_{jkl}\,\alpha^j\,b^k - \frac{1}{g}\partial_\mu\alpha^l$$

we can see that there is a coupling between the different components of the field. This is called self-coupling of the field. The effect of this becomes clear if one also considers the kinetic term of the isospin gauge field (analogous to the QED case):

$$\mathcal{L}_{SU(2)} = \bar{\psi}\;(i\gamma^\mu D_\mu - m)\;\psi - \frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}$$

Introducing the field strength tensor:

$$F_{\mu\nu} = \frac{1}{2}\vec{F}_{\mu\nu}\cdot\vec{\tau} = \frac{1}{2}F^a_{\mu\nu}\tau^a$$

the Lagrangian is usually written as (using the Pauli identity $tr(\tau^a\tau^b) = 2\delta^{ab}$):

$$\mathcal{L}_{SU(2)} = \bar{\psi}\;(i\gamma^\mu D_\mu - m)\;\psi - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

with individual components of the field strength tensor:

$$F^l_{\mu\nu} = \partial_\nu b^l_\mu - \partial_\mu b^l_\nu + g\;\epsilon_{jkl}\;b^j_\mu\;b^k_\nu$$

The consequence of the last term is that the Lagrangian term $F_{\mu\nu}F^{\mu\nu}$ contains contributions with 2, 3 and 4 factors of the b-field. These couplings are respectively referred to as bilinear, trilinear and quadrilinear couplings. In QED there's only the bilinear photon propagator term. In the isospin theory there are self interections by a 3-gauge boson vertex and a 4 gauge boson vertex.

1.7.1 What have we done?

We modified the Lagrangian describing isospin 1/2 doublets $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$:

$$\mathcal{L}^{free}_{SU(2)} = \bar{\psi}\;(i\gamma^\mu\partial_\mu - m)\;\psi$$

We made the replacement $\partial_\mu \to D_\mu = \partial_\mu + igB_\mu$ with $B_\mu = \frac{1}{2}\vec{\tau}\cdot\vec{b}_\mu$, to obtain:

$$\begin{aligned}
\mathcal{L}_{SU(2)} &= \bar{\psi}\;(i\gamma^\mu D_\mu - m)\;\psi \\
&= \mathcal{L}^{free}_{SU(2)} - \frac{g}{2}\,\vec{b}_\mu\cdot\bar{\psi}\gamma^\mu\vec{\tau}\psi \\
&= \mathcal{L}^{free}_{SU(2)} - \mathcal{L}^{interaction}_{SU(2)} \\
&= \mathcal{L}^{free}_{SU(2)} - \vec{b}_\mu\cdot\vec{J}^\mu
\end{aligned}$$

where $\vec{J}^\mu = \frac{g}{2}\bar{\psi}\gamma^\mu\vec{\tau}\psi$ is the isospin current.

Let us compare it once more to the case of QED:

$$\mathcal{L}_{U(1)} = \mathcal{L}_{U(1)}^{free} - A_\mu \cdot J^\mu$$

with the electromagnetic current $J^\mu = q\bar{\psi}\gamma^\mu\psi$

We have neglected here the kinetic terms of the fields:

$$\mathcal{L}_{SU(2)} = \bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi - \frac{1}{2} tr F_{\mu\nu}F^{\mu\nu}$$

which contains self-coupling terms of the fields.

1.7.2 Assessment

We see a symmetry in the $\begin{pmatrix} p \\ n \end{pmatrix}$ system: the isospin rotations.

- If we require local gauge invariance of such transdormations we need to introduce $\vec{b}_\mu$ gauge fields.
- But what are they? $\vec{b}_\mu$ must be three massless vector bosons that couple to the proton and neutron. It cannot be the π^-, π^0, π^+ since they are pseudoscalar particles rather then vector bosons. It turns out this theory does not describe the strong interactions. We know now that the strong force is mediated by massless gluons. In fact gluons have 3 colour degrees of freedom, such that they can be described by 3x3 unitary gauge transformations (SU(3)), for which there are 8 generators. The strong interaction will be discussed later on in the particle physics course. Next lecture we will instead look at the weak interaction and introduce the concept of weak iso-spin.
- Also, we have started to say that the symmetry in the p, n system is only present if we neglect electromagnetic interactions, since obviously from the charge we can absolutely define the proton and the neutron state in the doublet. In such a case where the symmetry is only approximate, we speak of a *broken symmetry* rather then of an *exact symmetry.*

Lecture 2

In the previous lecture we have seen how imposing of a local gauge symmetry requires a modification of the free Lagrangian such that a theory with interactions is obtained. We studied:

- local $U(1)$ gauge invariance:

$$\bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi = \bar{\psi}\ (i\gamma^\mu \partial_\mu - m)\ - \underbrace{q\bar{\psi}\gamma^\mu\psi}_{J^\mu}\ A_\mu$$

- local $SU(2)$ gauge invariance:

$$\bar{\psi}\ (i\gamma^\mu D_\mu - m)\ \psi = \bar{\psi}\ (i\gamma^\mu \partial_\mu - m)\ - \underbrace{\frac{g}{2}\bar{\psi}\gamma^\mu\vec{\tau}\psi}_{\vec{J}^\mu}\ \vec{b}_\mu$$

For the $U(1)$ symmetry we can identify the A_μ field as the photon and the Feynman rules for QED, as we discussed them in particle physics 1, follow automatically. For the $SU(2)$ case we hoped that we could describe the strong nuclear interactions, but this failed.

Let us now, instead of the strong isospin doublet $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ introduce the following doublets:

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{and} \quad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

and we speak instead of "weak isospin" doublets. Note that the fermion fields have an L index (for "left-handed"). These left handed states are defined as:

$$\nu_L = \frac{1}{2}(1-\gamma_5)\ \nu \qquad\qquad u_L = \frac{1}{2}(1-\gamma_5)\ u$$
$$e_L = \frac{1}{2}(1-\gamma_5)\ e \qquad\qquad d_L = \frac{1}{2}(1-\gamma_5)\ d$$

with the known projection operators (see the PP1 lectures):

$$\psi_L = \frac{1}{2}(1-\gamma_5)\ \psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma_5)\ \psi$$

(Remember: for massless particles" $\psi_L = \psi_{-\text{helicity}}$ and $\psi_R = \psi_{+\text{helicity}}$.)

The origin of the weak interaction lies in the fact that we now impose a local gauge symmetry in weak isospin rotations of left handed fermion fields. This means that if we "switch off" charge we cannot distinguish between a ν_L and a e_L or a u_L and a d_L state. The fact that we only impose this on left handed states implies that the weak interaction is completely left-right asymmetric. (Intuitively this is very difficult to accept: why would there be a symmetry for the left-handed states only?!) This is called *maximal violation of parity.*

It will turn out that the three vector fields (b_1, b_2, b_3 from the previous lecture) can later be associated with the carriers of the weak interaction, the W^+, W^-, Z bosons. However, these bosons are not massless. An explicit mass term ($\mathcal{L}_M = K b_\mu b^\mu$) would in fact break the gauge invariance of the theory. Their masses can be generated in a mechanism that is called spontaneous symmetry breaking and involves a new hypothetical particle: the Higgs boson. The main idea of the symmetry breaking mechanism is that the Lagrangian retains the full gauge symmetry, but that the ground state, i.e. the vacuum, is no longer at a symmetric position. The realization of the vacuum selects a preferred direction in isospin space, and thus breaks the symmetry. Future lectures will discuss this aspect in more detail.

To construct the weak $SU(2)_L$ theory we start again with the free Dirac Lagrangian and we impose $SU(2)$ symmetry (but now on the weak isospin doublets):

$$\mathcal{L}_{free} = \bar{\psi}_L \ (i\gamma^\mu \partial_\mu - m) \ \psi_L$$

Again we introduce the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + igB_\mu \quad \text{with} \quad B_\mu = \frac{1}{2}\,\vec{\tau} \cdot \vec{b}_\mu$$

then:

$$\mathcal{L}_{free} \to \mathcal{L}_{free} - \vec{b}_\mu \cdot J^\mu_{weak}$$

with the weak current:

$$J^\mu_{weak} = \frac{g}{2} \bar{\psi}_L \gamma^\mu \vec{\tau} \psi_L$$

This is just a copy from what we have seen in the strong isospin example.

The model for the weak interactions now contains 3 massless gauge bosons (b^1, b^2, b^3). However, in nature we have seen that the weak interaction is propagated by 3 massive bosons W^+, W^-, Z^0.

From the Higgs mechanism it turns out that the physical fields associated with b^1_μ and b^2_μ are the charged W bosons:

$$W^\pm_\mu \equiv \frac{b^1_\mu \mp i b^2_\mu}{\sqrt{2}}$$

2.1 The Charged Current

We will use the definition of the W-fields to re-write the first two terms in the Lagrangian of the weak current:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{free} + \mathcal{L}_{weak}^{int} \\ \text{with} \quad \mathcal{L}_{weak}^{int} &= -\vec{b}_\mu \cdot \vec{J}^\mu_{weak} = -b^1_\mu J^{1\mu} - b^2_\mu J^{2\mu} - b^3_\mu J^{3\mu} \end{aligned}$$

The charged current terms are:

$$\mathcal{L}_{CC} = -b^1_\mu J^{1\mu} - b^2_\mu J^{2\mu}$$

with:

$$J^{1\mu} = \frac{g}{2}\,\bar{\psi}_L\,\gamma^\mu\,\tau_1\,\psi_L \quad ; \quad J^{2\mu} = \frac{g}{2}\,\bar{\psi}_L\,\gamma^\mu\,\tau_2\,\psi_L$$

Exercise: *Show that the re-definition* $W^\pm_\mu = \frac{b^1_\mu \mp i b^2_\mu}{\sqrt{2}}$ *leads to:*

$$\begin{aligned} \mathcal{L}_{CC} &= -W^+_\mu J^{+\mu} - W^-_\mu J^{-\mu} \\ \textit{with:} \quad J^{+\mu} &= \frac{g}{2}\,\bar{\psi}_L\,\gamma^\mu\,\tau^+\,\psi_L \quad ; \quad J^{-\mu} = \frac{g}{2}\,\bar{\psi}_L\,\gamma^\mu\,\tau^-\,\psi_L \\ \textit{and with:} \quad \tau^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

So, for the physical fields W^+ and W^- the leptonic currents are:

$$J^{+\mu} = \frac{g}{\sqrt{2}}\,\bar{\nu}_L\,\gamma^\mu\,e_L \quad ; \quad J^{-\mu} = \frac{g}{\sqrt{2}}\,\bar{e}_L\,\gamma^\mu\,\nu_L$$

or written out with the left-handed projection operators:

$$J^{+\mu} = \frac{g}{\sqrt{2}}\,\bar{\nu}\,\frac{1}{2}\left(1+\gamma^5\right)\,\gamma^\mu\,\frac{1}{2}\left(1-\gamma^5\right)\,e \quad .$$

Note that we have the identity:

$$\begin{aligned} \left(1+\gamma^5\right)\gamma^\mu\left(1-\gamma^5\right) &= \gamma^\mu + \gamma^5\gamma^\mu - \gamma^\mu\gamma^5 - \gamma^5\gamma^\mu\gamma^5 \\ &= \gamma^\mu - 2\gamma^\mu\gamma^5 + \left(\gamma^5\right)^2\gamma^\mu \\ &= 2\gamma^\mu\left(1-\gamma^5\right) \end{aligned}$$

such that we get for the leptonic charge raising current (W^+):

$$\boxed{J^{+\mu} = \frac{g}{2\sqrt{2}}\,\bar{\nu}\,\gamma^\mu\,(1-\gamma^5)\,e}$$

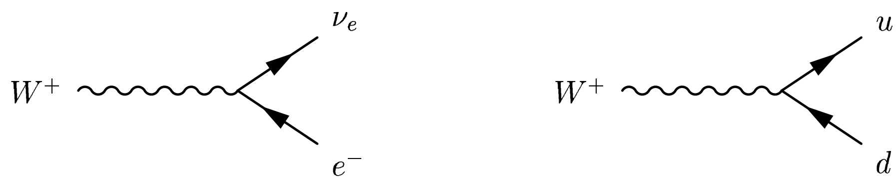
and for the leptonic charge lowering current (W^-):

$$\boxed{J^{-\mu} = \frac{g}{2\sqrt{2}}\,\bar{e}\,\gamma^\mu\,(1-\gamma^5)\,\nu} \quad .$$

Remembering that a vector interaction has an operator γ^μ in the current and an axial vector interaction a term $\gamma^\mu\gamma^5$, we recognize in the charged weak interaction the famous "V-A" interaction.

The same is true for the quark-currents and we can recognize the following currents in the weak interaction:

Charge raising:



Charge lowering:

2.2 The Neutral Current

2.2.1 Empirical Appraoch

The Lagrangian for weak and electromagnetic interactions is:

$$\begin{aligned}
\mathcal{L}_{EW} &= \mathcal{L}_{free} - \mathcal{L}_{weak} - \mathcal{L}_{EM} \\
\mathcal{L}_{weak} &= W^+_\mu J^{+\mu} + W^-_\mu J^{-\mu} + b^3_\mu J^\mu_3 \\
\mathcal{L}_{EM} &= a_\mu J^\mu_{EM}
\end{aligned}$$

Let us again look at the interactions for leptons ν, e, then:

$$\begin{aligned}
J^\mu_3 &= \frac{g}{2}\,\bar{\psi}_L\,\gamma^\mu\,\tau^3\,\psi_L = \frac{g}{2}\,\bar{\nu}_L\,\gamma^\mu\,\nu_L - \frac{g}{2}\,\bar{e}_L\,\gamma^\mu\,e_L \qquad \left(\text{we used} : \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
J^\mu_{EM} &= q\,\bar{e}\,\gamma^\mu\,e = q\,(\bar{e}_L\,\gamma^\mu\,e_L) + q\,(\bar{e}_R\,\gamma^\mu\,e_R)
\end{aligned}$$

Exercise: *Show explicitly that:*

$$\bar{\psi}\,\gamma^\mu\,\psi = \bar{\psi}_L\,\gamma^\mu\,\psi_L + \bar{\psi}_R\,\gamma^\mu\,\psi_R$$

making use of $\psi = \psi_L + \psi_R$ *and the projection operators* $\frac{1}{2}(1-\gamma_5)$ *and* $\frac{1}{2}(1+\gamma_5)$

Experiments have shown that in contrast to the charged weak interaction, the neutral weak current associated to the Z-boson is *not* purely left-handed, but:

$$J^{\mu f}_{NC} = \frac{g}{2}\,\bar{\psi}^f\,\gamma^\mu\,\left(C^f_V - C^f_A\gamma^5\right)\,\psi^f$$

where C^f_V and C^f_A are no longer equal to 1, but they are constants that express the relative strength of the vector and axial vector components of the interaction. Their value depends on the type of fermion f, as we will see below.

Taking again the leptons $\psi = \begin{pmatrix}\nu \\ e\end{pmatrix}$ we get:

$$J^\mu_{NC} = \frac{g}{2}\,\bar{\nu}\,\gamma^\mu\,\left(C^\nu_V - C^\nu_A\gamma^5\right)\,\nu + \frac{g}{2}\,\bar{e}\,\gamma^\mu\,\left(C^e_V - C^e_A\gamma^5\right)\,e$$

At this point we introduce the left-handed and right-handed couplings:

$$C_R \equiv C_V - C_A \qquad\qquad C_V = \frac{1}{2}\left(C_R + C_L\right)$$

$$C_L \equiv C_V + C_A \qquad\qquad C_A = \frac{1}{2}\left(C_L - C_R\right)$$

then:

$$\left(C_V - C_A\gamma^5\right) = \underbrace{C_V - C_A}_{C_R}\left(\frac{1+\gamma^5}{2}\right) + \underbrace{C_V + C_A}_{C_L}\left(\frac{1-\gamma^5}{2}\right)$$

For neutrino's we have $C^\nu_L = 1$ and $C^\nu_R = 0$. So, for leptons the observed neutral current can be written as:

$$J^\mu_{NC} = \frac{g}{2}\,\left(\bar{\nu}_L\,\gamma^\mu\,\nu_L\right)\;+\;\frac{g}{2}\,\left(C^e_L\,\bar{e}_L\,\gamma^\mu\,e_L\right)\;+\;\frac{g}{2}\,\left(C^e_R\,\bar{e}_R\,\gamma^\mu\,e_R\right)$$

We had for the electromagnetic current:

$$J^\mu_{EM} = \qquad\qquad q\,\left(\bar{e}_L\,\gamma^\mu\,e_L\right)\;+\;q\,\left(\bar{e}_R\,\gamma^\mu\,e_R\right)$$

and for the SU(2) current:

$$J^\mu_3 = \frac{g}{2}\,\left(\bar{\nu}_L\,\gamma^\mu\,\nu_L\right)\;-\;\frac{g}{2}\,\left(\bar{e}_L\,\gamma^\mu\,e_L\right)$$

We now insert that J^μ_3 is in fact a linear combination of J^μ_{NC} and J^μ_{EM}:

$$J^\mu_3 = a\cdot J^\mu_{NC} + b\cdot J^\mu_{EM}$$

- look at the ν_L terms: $a = 1$
- look at the e_R terms: $\frac{g}{2}C^e_R + q\cdot b = 0 \qquad \Rightarrow C^e_R = -\frac{2qb}{g}$

- look at e_L terms : $\frac{g}{2}C^e_L + q \cdot b = -\frac{g}{2}$ $\Rightarrow C^e_L = -1 - \frac{2qb}{g}$

Therefore:

$$
\begin{aligned}
C_V &= \frac{1}{2}\,(C_R + C_L) &\Rightarrow\qquad C^e_V &= -\frac{1}{2} - \frac{2q}{g}b \\
C_A &= \frac{1}{2}\,(C_L - C_R) &\Rightarrow\qquad C^e_A &= -\frac{1}{2}
\end{aligned}
$$

The vector coupling now contains a constant b which gives the ratio in which the $SU(2)$ current ($\frac{g}{2}$) and the electromagnetic current (q) are related. The constant b is a constant of nature and is written as $b = \sin^2\theta$: where θ represents the *weak mixing angle*.
We will study this more carefully below.

2.2.2 Hypercharge vs Charge

Again, we write down the electroweak Lagrangian, but this time we pose a different U(1) symmetry (see H&M[1], Chapter 13):

$$
\boxed{\mathcal{L}_{EW} = \mathcal{L}_{free} - ig\,\vec{J}^{\mu}_{SU(2)} \cdot \vec{b}_\mu - i\tfrac{g'}{2} J^\mu_Y a_\mu}
$$

where Y is the so-called *hypercharge* quantum number.
The $U(1)$ gauge invariance in now imposed on the quantity hypercharge rather the charge, and it has a coupling strength $g'/2$.

As before we have the physical charged currents:

$$
W^\pm_\mu = \frac{b^1_\mu \mp i b^2_\mu}{\sqrt{2}} \quad .
$$

For the neutral currents we say that the physical fields are the following linear combinations:

$$
\begin{aligned}
A_\mu &= a_\mu \cos\theta_w + b^3_\mu \sin\theta_w \qquad &\text{(massless)} \\
Z_\mu &= -a_\mu \sin\theta_w + b^3_\mu \cos\theta_w \qquad &\text{(massive)}
\end{aligned}
$$

and the origin of the name *weak mixing angle* for θ_w becomes clear.

We can now write the terms for b^3_μ and a_μ in the Lagrangian:

$$
\begin{aligned}
-igJ^\mu_3 b^3_\mu - i\frac{g'}{2}J^\mu_Y a_\mu &= -i\left(g\sin\theta_w J^\mu_3 + g'\cos\theta_w \frac{J^\mu_Y}{2}\right)\,A_\mu \\
&\quad -i\left(g\cos\theta_w J^\mu_3 - g'\sin\theta_w \frac{J^\mu_Y}{2}\right)\,Z_\mu \\
&\equiv -iqJ^\mu_{EM}A_\mu - ig_Z J^\mu_{NC} Z_\mu
\end{aligned}
$$

[1] Halzen and Martin, Quarks & Leptons: “An Introductory Course in Modern Particle Physics”

The weak hypercharge is introuced in complete analogy with the strong hypercharge, for which we have the famous Gellmann - Nishijima relation: $Q = I_3 + \frac{1}{2}Y_S$. In the electroweak theory we use: $Q = T_3 + \frac{1}{2}Y$ which means:

$$\boxed{J^\mu_{EM} = J^\mu_3 + \tfrac{1}{2}J^\mu_Y}$$

then, indeed, for the A_μ field we have:

$$-ig\sin\theta_w \left(J^\mu_3 + \frac{g'\cos\theta_w}{g\sin\theta_w} \cdot \frac{1}{2} J^\mu_Y \right) = -ieJ^\mu_{EM} \quad ,$$

provided the following relation holds:

$$g\sin\theta_w = g'\cos\theta_w = e \quad .$$

The weak mixing angle is defined as the ratio of the coupling constants of the $SU(2)_L$ group and the $U(1)_Y$ group:

$$\tan\theta_w = \frac{g}{g'} \quad .$$

For the Z-currents we then find:

$$\begin{aligned} & -i\left(g\cos\theta_w J^\mu_3 - \frac{g'}{2}\sin\theta_w \cdot 2\left(J^\mu_{EM} - J^\mu_3\right) \right) Z_\mu \\ = & \ \ldots \\ = & -i\frac{e}{\cos\theta_w \sin\theta_w} \left(J^\mu_3 - \sin^2\theta_w J^\mu_{EM} \right) Z_\mu \end{aligned}$$

So we see that:

$$\boxed{J^\mu_{NC} = J^\mu_3 - \sin^2\theta_w \, J^\mu_{EM}}$$

which is in agreement with what we had obtained earlier:

$$J^\mu_3 = a \cdot J^\mu_{NC} + b \cdot J^\mu_{EM} \quad \text{with} \quad a = 1 \ \text{ and } \ b = \sin^2\theta_w$$

2.2.3 Assessment

We introduce a symmetry group $SU(2) \otimes U(1)_Y$ and describe electroweak interactions with:

$$-i\left(g\, \vec{J}^\mu_L \cdot \vec{b}_\mu + \frac{g'}{2} J^\mu_Y \cdot a_\mu \right)$$

The coupling constants g and g' are free parameters (we can also take e and $\sin^2\theta_w$). The electromagnetic and weak currents are then given by:

$$\begin{aligned} J^\mu_{EM} &= J^\mu_3 + \frac{1}{2}J^\mu_Y \\ J^\mu_{NC} &= J^\mu_3 - \sin^2\theta_w J^\mu_{EM} = \cos^2\theta_w J^\mu_3 - \sin^2\theta_w \frac{J^\mu_Y}{2} \end{aligned}$$

and the interaction term in the Lagrangian becomes:

$$-i\left(eJ^{\mu}_{EM}\cdot A_{\mu}+\frac{e}{\cos\theta_w\sin\theta_w}J^{\mu}_{NC}\cdot Z_{\mu}\right)$$

in terms of the physical fields A_μ and Z_μ.

2.3 The Mass of the W and Z bosons

In the electroweak model as introduced here, the gauge fields must be massless, since explicit mass terms ($\sim \phi_\mu\phi^\mu$) are not gauge invariant. In the Standard Model the mass of all particles are generated in the mechanism of spontaneous symmetry breaking, introducing the Higgs particle (see later lectures.) Here we just give an empirical argument to predict the mass of the W and Z particles.

1. Mass terms are of the following form:

$$M^2_\phi = \langle\phi\,|H|\,\phi\rangle \qquad \text{for any field } \phi$$

2. From the comparison with the Fermi 4-point interaction we find:

$$\frac{G_F}{\sqrt{2}}=\frac{g^2}{8M^2_W} \qquad \Rightarrow \qquad M^2_W=\frac{\sqrt{2}g^2}{8G_F}=\frac{\sqrt{2}}{8G_F}\,\frac{e^2}{\sin^2\theta}$$

Thus, we get the following predictions:

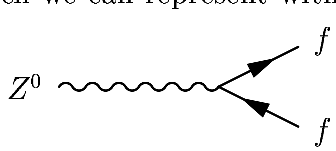
$$\begin{aligned} M_W &= \sqrt{\frac{\sqrt{2}}{8G_F}}\,\frac{e}{\sin\theta_w}=81\text{ GeV}\\ M_Z &= M_W\ (g_z/g)=M_W/\cos\theta=91\text{ GeV}\end{aligned}$$

2.4 The Coupling Constants for $Z\to f\bar{f}$

For the neutral Z-current interaction we have in general:

$$\begin{aligned} -ig_Z\,J^{\mu}_{NC}\,Z_\mu &= -i\frac{g}{\cos\theta_w}\left(J^\mu_3-\sin^2\theta_w J^\mu_{EM}\right)\,Z_\mu\\ &= -i\frac{g}{\cos\theta_w}\,\bar{\psi}_f\gamma^\mu\underbrace{\left[\frac{1}{2}\left(1-\gamma^5\right)T_3-\sin^2\theta_w Q\right]}_{\frac{1}{2}\left(C^f_V-C^f_A\gamma^5\right)}\psi_f\cdot Z_\mu\end{aligned}$$

which we can represent with the following vertex:



$$-i\,\frac{g}{\cos\theta_w}\,\gamma^\mu\,\frac{1}{2}\left(C^f_V-C^f_A\gamma^5\right)$$

with:

$$
\begin{aligned}
C_L^f &= T_3^f - Q^f \sin^2\theta_w \\
C_R^f &= -Q^f \sin^2\theta_w
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
C_V^f &= T_3^f - 2Q^f \sin^2\theta_w \\
C_A^f &= T_3^f
\end{aligned}
$$

fermion	T_3	Q	Y	C_A^f	C_V^f
$\nu_e\ \nu_\mu\ \nu_\tau$	$+\frac{1}{2}$	0	-1	$\frac{1}{2}$	$\frac{1}{2}$
$e\ \ \mu\ \ \tau$	$-\frac{1}{2}$	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_w$
$u\ \ c\ \ t$	$+\frac{1}{2}$	$+\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$
$d\ \ s\ \ b$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$

Table 2.1: The neutral current vector and axial vector couplings for each of the fermions in the Standard Model.

Lecture 3

3.1 The Cross Section of $e^-e^+ \to \mu^-\mu^+$

Equipped with the Feynman rules of the electroweak theory we now proceed to calculate the cross section of the electroweak process: $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$. We assume the following kinematics: There are two Feynman diagrams that contribute to the process:

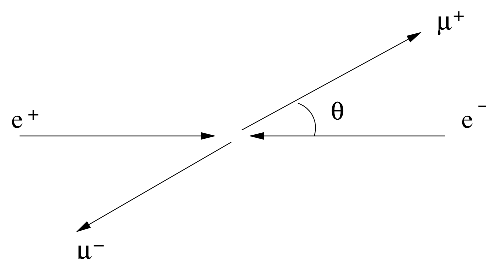


Figure 3.1: Kinematics of the process $e^-e^+ \to \mu^-\mu^+$.

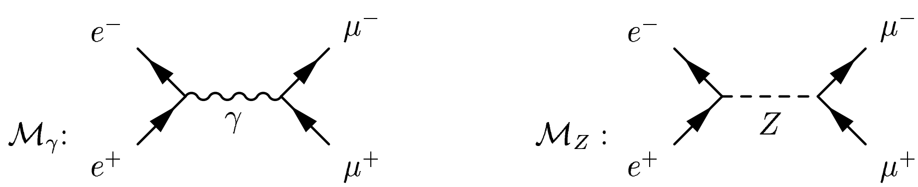


Figure 3.2: Feynman diagrams contributing to $e^-e^+ \to \mu^-\mu^+$

In complete analogy with the calculation of the QED process $e^+e^- \to e^+e^-$ (see PP1 lectures) we obtain the cross section using Fermi's Golden rule:

$$d\sigma = \frac{\left|\overline{\mathcal{M}}\right|^2}{F}\, dQ$$

With the phase factor dQ flux factor F:

$$\begin{aligned} dQ &= \frac{1}{4\pi^2}\frac{p_f}{4\sqrt{s}}\, d\Omega \\ F &= 4p_i\sqrt{s} \end{aligned}$$

$$\sigma\left(e^-e^+ \to \mu^-\mu^+\right) = \frac{1}{64\pi^2}\cdot\frac{1}{s}\cdot\left|\overline{\mathcal{M}}\right|^2$$

The Matrix element now includes:

$$\begin{aligned}\mathcal{M}_\gamma &= -e^2\left(\bar{m}\gamma^\mu m\right)\cdot\frac{g_{\mu\nu}}{q^2}\cdot\left(\bar{e}\gamma^\nu e\right)\\ \mathcal{M}_Z &= -\frac{g^2}{4\cos^2\theta_w}\left[\bar{m}\gamma^\mu\left(C_V^m - C_A^m\gamma^5\right)m\right]\cdot\frac{g_{\mu\nu}-q_\mu q_\nu/M_Z^2}{q^2-M_Z^2}\cdot\left[\bar{e}\gamma^\nu\left(C_V^e - C_A^e\gamma^5\right)e\right]\end{aligned}$$

The propagator for massive vector bosons (Z-boson) is discussed in Halzen & Martin §6.11 and §6.12. The wave equation of a massless spin-1 particle is:

$$\begin{aligned}\Box^2 A_\mu &= 0 &\Rightarrow&\quad i\frac{-g_{\mu\nu}}{q^2}\\ \left(\Box^2+M^2\right)Z_\mu &= 0 &\Rightarrow&\quad i\frac{-g_{\mu\nu}+q_\mu q_\nu/M^2}{q^2-M^2}\end{aligned}$$

We can simplify the propagator of the Z if we ignore the lepton masses. In practice this means that we work in the limit of high-energy scattering. In that case the Dirac equation becomes:

$$\left(i\partial_\mu\gamma^\mu - m\right)\bar{\psi}_e = 0 \quad\Rightarrow\quad \left(\gamma^\mu P_{\mu,e}\right)\bar{\psi}_e = 0$$

Since $P_e = \frac{1}{2}q$ we also have:

$$\frac{1}{2}\left(\gamma^\mu q_\mu\right)\bar{\psi}_e = 0 \quad\Rightarrow\quad q_\mu\cdot q_\nu/M_z^2 = 0$$

Thus the propagator simplifies:

$$\frac{g_{\mu\nu}-q_\mu q_\nu/M_Z^2}{q^2-M_Z^2} \to \frac{g_{\mu\nu}}{q^2-M_Z^2}$$

Thus we have for the Z-exchange matrix element the expression:

$$\mathcal{M}_Z = \frac{-g^2}{4\cos^2\theta_w}\,\frac{1}{q^2-M_Z^2}\cdot\left[\bar{m}\,\gamma^\mu\left(C_V^m - C_A^m\gamma^5\right)\,m\right]\left[\bar{e}\,\gamma_\mu\left(C_V^e - C_A^e\gamma^5\right)\,e\right] \quad .$$

To calculate the cross section by summing over $\mathcal{M}_\gamma$ and $\mathcal{M}_Z$ is now straightforward but a rather lengthy procedure: applying Casimir's trick, trace theorems, etc. Let us here try to follow a different approach.

We rewrite the $\mathcal{M}_Z$ matrix element in terms of right-handed and left-handed couplings, using the definitions: $C_R = C_V - C_A$; $C_L = C_V + C_A$. As before we have:

$$\left(C_V - C_A\gamma^5\right) = \left(C_V - C_A\right)\cdot\frac{1}{2}\left(1+\gamma^5\right) + \left(C_V + C_A\right)\cdot\frac{1}{2}\left(1-\gamma^5\right) \quad .$$

Thus:

$$\left(C_V - C_A\gamma^5\right)\psi = C_R\psi_R + C_L\psi_L \quad .$$

Let us now look back at the QED process:

$$\mathcal{M}_\gamma = \frac{-e^2}{s}\,(\bar{m}\gamma^\mu m)\,(\bar{e}\gamma_\mu e)$$

with (see previous lecture):

$$\begin{aligned}(\bar{m}\gamma^\mu m) &= (\bar{m}_L\gamma^\mu m_L) + (\bar{m}_R\gamma^\mu m_R)\\ (\bar{e}\gamma_\mu e) &= (\bar{e}_L\gamma_\mu e_L) + (\bar{e}_R\gamma_\mu e_R)\end{aligned}$$

The fact that there are no terms connecting L-handed to R-handed $(\bar{m}_R\gamma^\mu m_L)$ actually implies that we have helicity conservation for high energies (i.e. neglecting $\sim m/E$ terms) at the vertices:

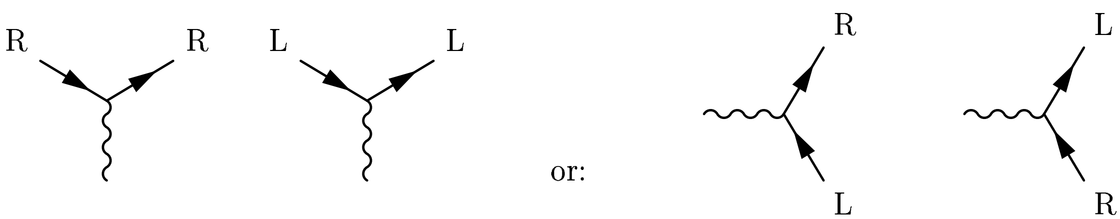


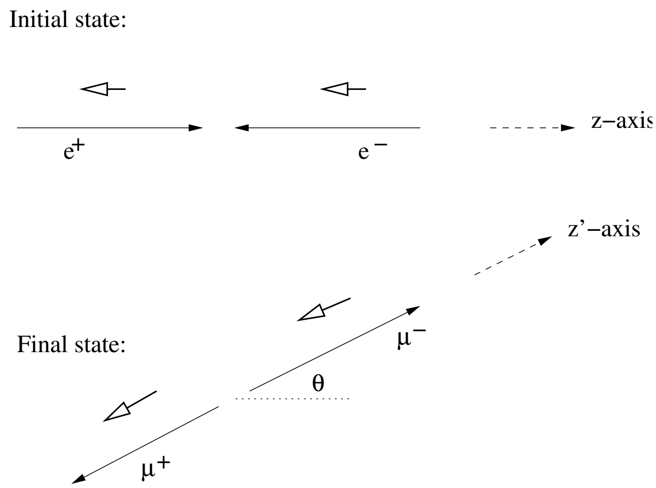
Figure 3.3: Helicity conservation. *left:* A right-handed incoming electron scatters into a right-handed outgoing electron and vice versa in a vector or axial vector interaction . *right:* In the crossed reaction the energy and momentum of one electron is reversed: i.e. in the e^+e^- pair production a right-handed electron and a left-handed positron (or vice versa) are produced. This is the consequence of a spin=1 force carrier. (In all diagrams time increases from left to right.)

As a consequence we can decompose the unpolarized QED scattering process as a sum of 4 cross section contributions:

$$\begin{aligned}\frac{d\sigma}{d\Omega}^{\text{unpolarized}} = \frac{1}{2}\Big\{ &\frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_L\mu^+_R\right) + \frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_R\mu^+_L\right)\\ &\frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_L\mu^+_R\right) + \frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_R\mu^+_L\right)\Big\}\end{aligned}$$

where we *average* over the incoming spins and *sum* over the final state spins.

Let us look in more detail at the helicity dependence (H&M §6.6):



In the initial state the e^- and e^+ have opposite helicity (as they produce a spin 1 γ).

The same is true for the final state μ^- and μ^+.

So, in the center of mass frame, scattering proceeds from an initial state with $J_Z = +1$ or -1 along axis $\hat{z}$ into a final state with $J'_Z = +1$ or -1 along axis $\hat{z}'$. Since the interaction proceeds via a photon with spin $J = 1$ the amplitude for scattering over an angle θ is then given by the rotation matrices[1].

$$d^j_{m'm}(\theta) \equiv \left\langle jm'|e^{-i\theta J_y}|jm\right\rangle$$

where the y-axis is perpendicular to the interaction plane.

In the example we have $j = 1$ and $m, m' = \pm 1$

$$\begin{aligned} d^1_{1\,1}(\theta) &= d^1_{-1-1}(\theta) = \frac{1}{2}(1+\cos\theta) \\ d^1_{1-1}(\theta) &= d^1_{-1\,1}(\theta) = \frac{1}{2}(1-\cos\theta) \end{aligned}$$

From this we can see that:

$$\begin{aligned} \frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_L \mu^+_R\right) &= \frac{\alpha^2}{8s}(1+\cos\theta)^2 = \frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_R \mu^+_L\right) \\ \frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_R \mu^+_L\right) &= \frac{\alpha^2}{8s}(1-\cos\theta)^2 = \frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_L \mu^+_R\right) \end{aligned}$$

Indeed the unpolarised cross section is obtained as the spin-averaged sum over the allowed helicity combinations: $\frac{1}{2}\cdot[(1)+(2)+(3)+(4)] =$

$$\frac{d\sigma}{d\Omega}^{\text{unpol}} = \frac{\alpha^2}{8s}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \quad .$$

[1] See H&M§2.2:

$$e^{-i\theta J_2}\,|j\;m\rangle = \sum_{m'} d^j_{m\,m'}(\theta)\,|j\;m'\rangle$$

and also appendix H in Burcham & Jobes

Now we go back to the γ, Z scattering. We have the individual contributions of the helicity states, so let us compare the expressions for the matrix-elements $\mathcal{M}_\gamma$ and $\mathcal{M}_Z$:

$$\begin{aligned} \mathcal{M}_\gamma &= -\frac{e^2}{s}\,[\,(\bar{m}_L\gamma^\mu m_L)+(\bar{m}_R\gamma^\mu m_R)\,]\cdot[\,(\bar{e}_L\gamma_\mu e_L)+(\bar{e}_R\gamma_\mu e_R)\,] \\ \mathcal{M}_Z &= -\frac{g^2}{4\cos^2\theta_w}\,\frac{1}{s-M_Z^2}\,[\,C_L^m\,(\bar{m}_L\gamma^\mu m_L)+C_R^m\,(\bar{m}_R\gamma^\mu m_R)\,]\cdot[\,C_L^e\,(\bar{e}_L\gamma_\mu e_L)+C_R^e\,(\bar{e}_R\gamma_\mu e_R)\,] \end{aligned}$$

Since the helicity processes do not interfere, we can see that:

$$\begin{aligned} \frac{d\sigma}{d\Omega}_{\gamma,Z}\left(e_L^-e_R^+\to\mu_L^-\mu_R^+\right) &= \frac{\alpha^2}{4s}\,(1+\cos\theta)^2\cdot|1+r\,C_L^mC_L^e|^2 \\ \frac{d\sigma}{d\Omega}_{\gamma,Z}\left(e_L^-e_R^+\to\mu_R^-\mu_L^+\right) &= \frac{\alpha^2}{4s}\,(1-\cos\theta)^2\cdot|1+r\,C_R^mC_L^e|^2 \end{aligned}$$

with:

$$r=\frac{g^2}{e^2}\,\frac{1}{4\cos^2\theta_w}\,\frac{s}{s-M_z^2}=\frac{\sqrt{2}G_FM_Z^2}{e^2}\,\frac{s}{s-M_Z^2}\quad .$$

where we used that:

$$\frac{G_F}{\sqrt{2}}=\frac{g^2}{8M_W^2}=\frac{g^2}{8M_Z^2\cos^2\theta_w}\quad .$$

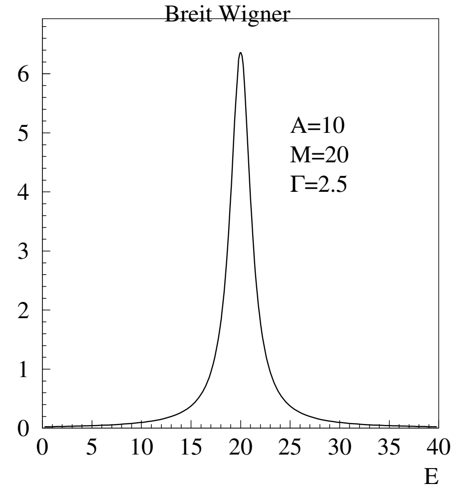
Similar expressions hold for the other two helicity configurations.

We note that there is a strange behaviour in the expression of the cross section of the Z-propagator. When $\sqrt{s}\to M_Z$ the cross section becomes ∞. In reality this does not happen (that would be unitarity violation) due to the fact that the Z-particle itself decays and has an intrinsic decay width Γ_Z. This means that the cross section has a Breit Wigner resonance shape. We are not going to derive it, but refer to the literature: Perkins[2].

The argument followed by H&M §2.10 goes as follows: The wave function for a non-stable massive particle state is:

$$\begin{aligned} |\psi(t)|^2 &= |\psi(0)|^2\,e^{-\Gamma t} \quad \text{with } \Gamma \text{ the lifetime.} \\ \psi(t) &\sim e^{-iMt}\,e^{-\frac{\Gamma}{2}t} \quad \text{with } M \text{ the mass.} \end{aligned}$$

[2]Perkins: Introduction to high energy Physics 3^{rd} ed. §4.8.



As function of the energy the state is described by the Fourier transform:

$$\chi(E) = \int \psi(t) e^{iEt} dt \sim \frac{1}{E - M + (i\Gamma/2)} \quad .$$

Such that experimentally we would observe:

$$|\chi(E)|^2 = \frac{A}{(E-M)^2 + (\Gamma/2)^2} \quad ,$$

the so-called Breit-Wigner resonance shape.

In the propagator for the z-boson we replace:

$$\frac{1}{s - M_Z^2} \to \frac{1}{s - \left(M_Z - i\frac{\Gamma_Z}{2}\right)^2} = \frac{1}{s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right) + iM_Z\Gamma_Z}$$

We observe two changes:

1. The maximum of the distribution shifts from $M_Z^2 \to M_Z^2 - \frac{\Gamma_Z^2}{4}$.
2. The expression will be finite because of the term $\propto M_Z\Gamma_Z$

For our expressions in the process $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$ it means that we only replace:

$$r = \frac{\sqrt{2}G_F M_Z^2}{e^2} \cdot \frac{s}{s - M_Z^2} \quad \text{by} \quad r = \frac{\sqrt{2}G_F M_Z^2}{e^2} \cdot \frac{s}{s - \left(M_Z - i\frac{\Gamma_Z}{2}\right)^2}$$

The total unpolarized cross section finally becomes the average over the four L, R helicity combinations. Inserting "lepton universality" $C_L^e = C_L^\mu$; $C_R^e = C_R^\mu$ and therefore also: $C_V^e = C_V^\mu$; $C_A^e = C_A^\mu$, the expression becomes (by writing it out):

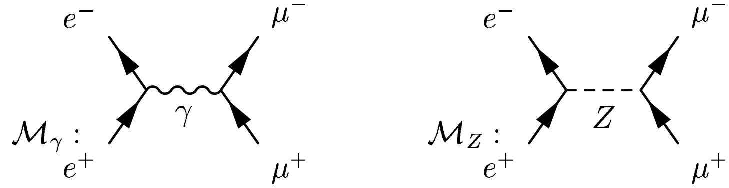
$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{4s}\left[A_0\left(1+\cos^2\theta\right) + A_1\ (\cos\theta)\right] \\ \text{with} \quad & A_0 = 1 + 2\,Re(r)\,C_V^2 + |r|^2\left(C_V^2 + C_A^2\right)^2 \\ & A_1 = 4\,Re(r)\,C_A^2 + 8|r|^2 C_V^2 C_A^2 \end{aligned}$$

In the Standard Model we have: $C_A = -\frac{1}{2}$ and $C_V = -\frac{1}{2} + 2\sin^2\theta$.

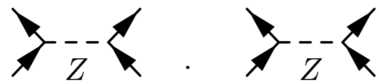
The general expression for $e^-e^+ \to \gamma, Z \to \mu^-\mu^+$ is (assuming seperate couplings for initial and final state):

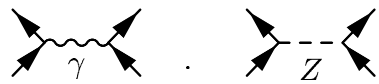
$$\begin{aligned} A_0 &= 1 + 2Re(r)\,C_V^e C_V^f + |r|^2\left({C_V^e}^2 + {C_A^e}^2\right)\left({C_V^f}^2 + {C_A^f}^2\right) \\ A_1 &= 4Re(r)\,C_A^e C_A^f + 8|r|^2\,C_V^e C_V^e C_A^f C_A^f \end{aligned}$$

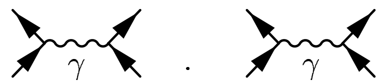
To summarize, on the *amplitude level* there are two diagrams that contribute:



Introducing the following notation:

$\frac{d\sigma}{d\Omega}\,[Z,Z] =$  $\propto |r|^2$

$\frac{d\sigma}{d\Omega}\,[\gamma Z] =$  $\propto Re\,(r)$

$\frac{d\sigma}{d\Omega}\,[\gamma,\gamma] =$  $\propto 1$

Explicitly, the expression is:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}[\gamma,\gamma] + \frac{d\sigma}{d\Omega}[Z,Z] + \frac{d\sigma}{d\Omega}[\gamma,Z]$$

with

$$\frac{d\sigma}{d\Omega}[\gamma,\gamma] = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

$$\frac{d\sigma}{d\Omega}[Z,Z] = \frac{\alpha^2}{4s}\,|r|^2\,\left[\left(C_V^{e\,2}+C_A^{e\,2}\right)\left(C_V^{f\,2}+C_A^{f\,2}\right)\left(1+\cos^2\theta\right) + 8C_V^eC_V^fC_A^eC_A^f\cos\theta\right]$$

$$\frac{d\sigma}{d\Omega}[\gamma,Z] = \frac{\alpha^2}{4s}\,Re|r|\,\left[C_V^eC_V^f\left(1+\cos^2\theta\right) + 2C_A^eC_A^f\cos\theta\right]$$

Let us take a look at the cross section close to the peak of the distribution:

$$r \propto \frac{s}{s-\left(M_z - i\frac{\Gamma_Z}{2}\right)^2} = \frac{s}{s-\left(M_z^2-\frac{\Gamma_Z^2}{4}\right)+i\mathcal{M}_Z\Gamma_Z}$$

The peak is located at $s_0 = M_Z^2 - \frac{\Gamma_Z^2}{4}$.

Exercise: *Show that:*

$$Re(r) = \left(1-\frac{s_0}{s}\right)|r|^2 \quad \text{with} \quad |r|^2 = \frac{s^2}{\left(s-\left(M_Z^2-\frac{\Gamma_Z^2}{4}\right)\right)^2 + M_Z^2\Gamma_Z^2}$$

This shows that the interference term is 0 at the peak.

In that case we have for the Z-cross section:

$$A_0 = |r|^2\left(C_V^{e\,2}+C_A^{e\,2}\right)\left(C_V^{f\,2}+C_A^{f\,2}\right)$$

$$A_1 = 8|r|^2\left(C_V^e\,C_A^e\,C_V^f\,C_A^f\right)$$

The total cross section (integrated over $d\Omega$) is then:

$$\sigma(s) = \frac{G_F^2 M_Z^4}{\left(s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right)\right)^2 + M_Z^2\Gamma_Z^2} \cdot \frac{s}{6\pi} \left({C_V^e}^2 + {C_A^e}^2\right)\left({C_V^f}^2 + {C_A^f}^2\right) \quad .$$

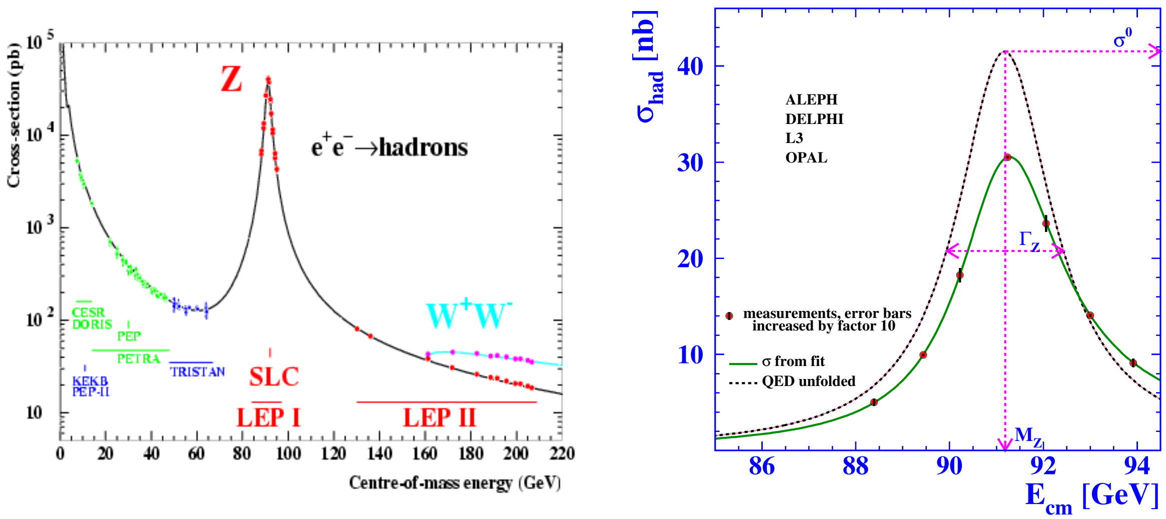
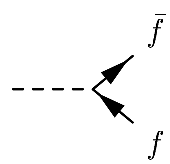


Figure 3.4: *left:* The Z-lineshape as a function of $\sqrt{s}$. *right:* The Lineshape parameters for the lowest order calculations and including higher order corrections.

3.2 Decay Widths

We can also calculate the decay width:



$$\Gamma\left(Z \to f\bar{f}\right)$$

which is according Fermi's golden rule:

$$\begin{aligned}
\Gamma\left(Z \to f\bar{f}\right) &= \frac{1}{16\pi}\frac{1}{M_Z}\left|\overline{\mathcal{M}}\right|^2 \\
&= \frac{g^2}{48\pi}\frac{M_z}{\cos^2\theta_w}\left({C_V^f}^2 + {C_A^f}^2\right) \\
&= \frac{G_F}{6\sqrt{2}}\frac{M_Z^3}{\pi}\left({C_V^f}^2 + {C_A^f}^2\right)
\end{aligned}$$

Using this expression for $\Gamma_e \equiv \Gamma(Z \to e^+e^-)$ and $\Gamma_f \equiv \Gamma(Z \to f\bar{f})$ we can re-write:

$$\sigma(s) = \frac{12\pi}{M_Z^2} \cdot \frac{s}{\left(s - \left(M_Z^2 - \frac{\Gamma_Z^2}{4}\right)\right)^2 + M_Z^2\Gamma_Z^2} \cdot \Gamma_e\Gamma_f \quad .$$

Close to the peak we then find:

$$\sigma_{peak} \approx \frac{12\pi}{M_Z^2}\frac{\Gamma_e \Gamma_f}{\Gamma_Z^2} = \frac{12\pi}{M_Z^2} BR(Z \to ee) \cdot BR(Z \to ff)$$

Let us now finally consider the case when $f = q$ (a quark). Due to the fact that quarks can be produced in 3 color-states the decay width is:

$$\Gamma(Z \to \bar{q}q) = \frac{G_F}{6\sqrt{2}}\frac{M_Z^3}{\pi}\left(C_V^{f\,2} + C_A^{f\,2}\right) \cdot N_C$$

with the colorfactor $N_C = 3$. The ratio between the hadronic and leptonic width: $R_l = \Gamma_{had}/\Gamma_{lep}$ can be defined. This ratio can be used to test the consistency of the standard model by comparing the calculated value with the observed one.

3.3 Forward Backward Asymmetry

The forward-backward asymmetry can be defined using the polar angle distribution:

$$\frac{d\sigma}{d\cos\theta} \propto 1 + \cos^2\theta + \frac{8}{3}A_{fb}\cos\theta$$

This defines the forward-backward asymmetry with:

$$A_{FB}^{0,f} = \frac{3}{4}A_e A_f \quad \text{where} \quad A_f = \frac{2C_V^f C_A^f}{C_V^2 + C_A^2}$$

The precise measurements of the forward-backward asymmetry can be used to determine the couplings C_V and C_A.

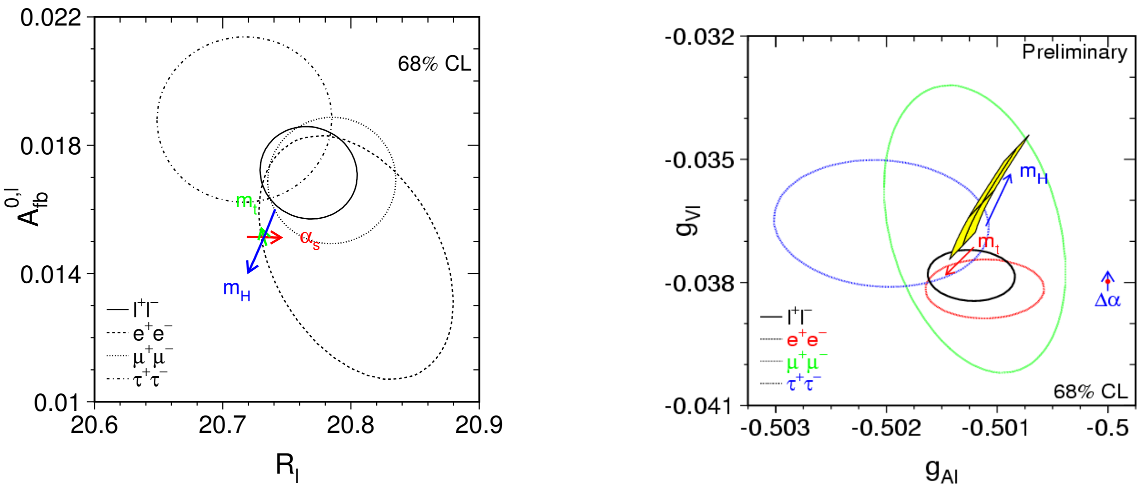


Figure 3.5: *left:* Test of lepton-universality. The leptonic A_{fb} vs. R_l. The contours show the measurements while the arrows show the dependency on Standard Model parameters. *right:* Determination of the vector and axial vector couplings.

3.4 The Number of Light Neutrino Generations

Since the total decay width of the Z must be equal to the sum of all partial widths the following relation holds:

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + 3\Gamma_{uu} + 3\Gamma_{dd} + 3\Gamma_{ss} + 3\Gamma_{cc} + 3\Gamma_{bb} + N_\nu \cdot \Gamma_{\nu\nu}$$

From a scan of the Z-cross section as function of the center of mass energy we find:

$$\begin{aligned}
\Gamma_Z &\approx 2490 \text{ MeV} \\
\Gamma_{ee} &\approx \Gamma_{\mu\mu} \approx \Gamma_{\tau\tau} = 84 \text{ MeV} && C_V \approx 0 && C_A = -\frac{1}{2} \\
\Gamma_{\nu\nu} &= 167 \text{ MeV} && C_V = \frac{1}{2} && C_A = \frac{1}{2} \\
\Gamma_{uu} &\approx \Gamma_{cc} = 276 \text{ MeV} && C_V \approx 0.19 && C_A = \frac{1}{2} \\
\Gamma_{dd} &\approx \Gamma_{ss} \approx \Gamma_{bb} = 360 \text{ MeV} && C_V \approx -0.35 && C_A = -\frac{1}{2}
\end{aligned}$$

(Of course $\Gamma_{tt} = 0$ since the top quark is heavier than the Z.)

$$N_\nu = \frac{\Gamma_Z - 3\Gamma_l - \Gamma_{had}}{\Gamma_{\nu\nu}} = 2.984 \pm 0.008 \quad .$$

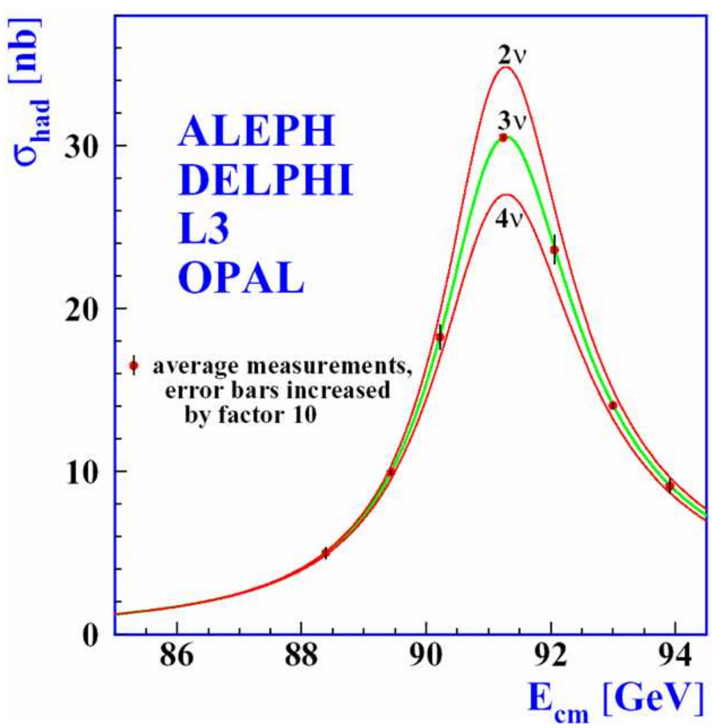


Figure 3.6: The Z-lineshape for resp. $N_\nu = 2,\ 3,\ 4$.

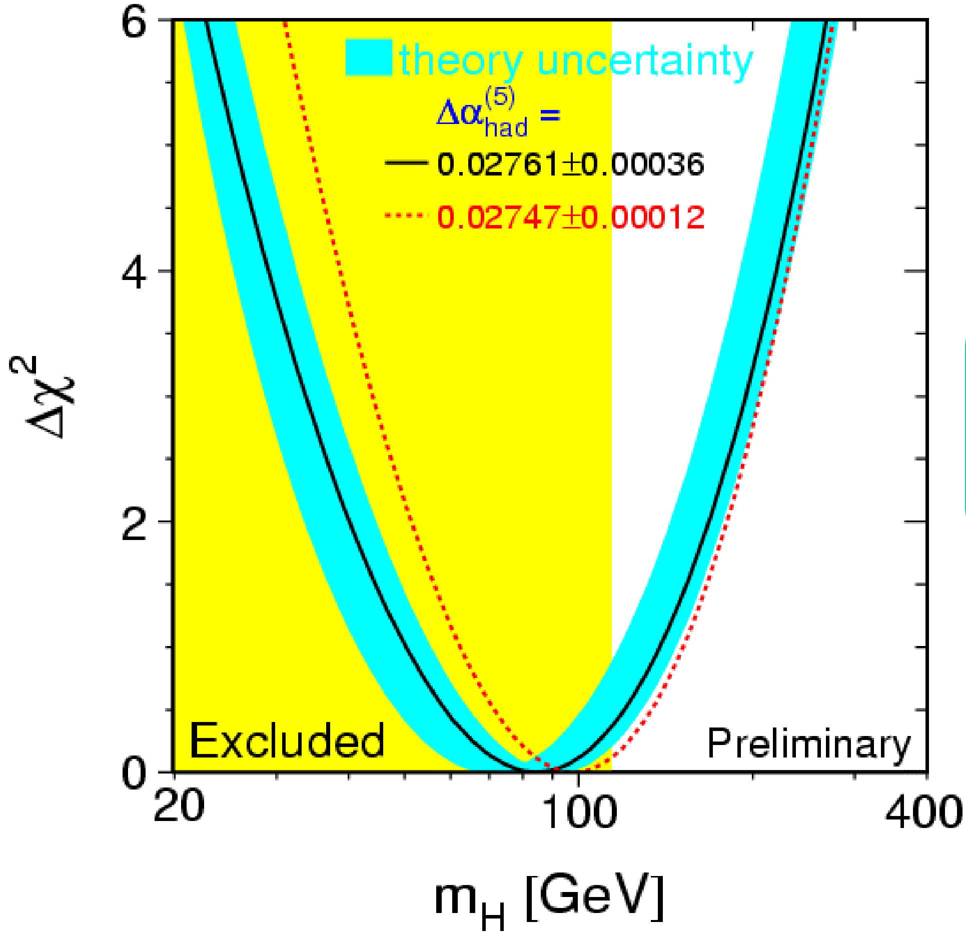


Figure 3.7: Standard Model fit of the predicted value of the Higgs boson.