Lecture 8

Master Particle Physics 1

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Lectures PP1

L0 : Introduction

L1 : Particles and Fields (historical overview)
L2 : Wave Equations and Antiparticles
L3 : The Electromagnetic Field
L4 : Perturbation Theory and Fermi's Golden Rule
L5 : Electromagnetic Scattering of Spinless Particles

QED of Spinless Particles: Scattering Theory and Cross Sections

L6 : The Dirac Equation
L7 : Solutions of the Dirac Equation
L8 : Spin 1/2 Electrodynamics

QED for Fundamental Fermions

L9 : The Weak Interaction
(Fermi 4-point scattering: an analogy with QED)
L10: Local Gauge Invariance
(the role of symmetries in interactions)
L11: Electroweak Theory
L12: The Process: $e^+e^- \rightarrow \gamma,Z \rightarrow \mu^+\mu^-$

The Standard Model for massless particles $SU(2)_L \times U(1)_Y$

Solutions to the Dirac Equation

$$(i\gamma^\mu \partial_\mu - m)\ \psi(x) = 0 \qquad \Rightarrow \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} U_A(p) \\ U_B(p) \end{pmatrix} e^{-ipx}$$

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$$\left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} m \right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p})\, U_B &= (E - m)\, U_A \\ (\vec{\sigma} \cdot \vec{p})\, U_A &= (E + m)\, U_B \end{aligned} \qquad U_A = \begin{pmatrix} * \\ * \end{pmatrix} \quad ; \quad U_B = \begin{pmatrix} * \\ * \end{pmatrix} \qquad \vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

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$\boxed{\vec{p} = 0}$

$$U^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad U^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad U^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad U^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \cdot\ e^{-ipx}$$

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$\boxed{\vec{p} \neq 0}$

1)choose :

$$U_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ U_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad U_B^{(1)} = \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \ ; \ U_B^{(2)} = \ldots \quad \text{etc}$$

2)choose :

$$U_B^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ U_B^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad U_A^{(3)} = \ldots. \quad ; \quad U_A^{(4)} = \ldots$$

Solutions to the Dirac Equation

$$(i\gamma^\mu \partial_\mu - m)\ \psi(x) = 0 \qquad \Rightarrow \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} U_A(p) \\ U_B(p) \end{pmatrix} e^{-ipx}$$

$$\left[\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} m \right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$\vec{p} = 0$

$$U^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \ U^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \ U^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \ U^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \cdot\ e^{-ipx}$$

$\vec{p} \neq 0$

1) choose : $U_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ U_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad U_B^{(1)} = \begin{pmatrix} p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} \ ; \ U_B^{(2)} = \dots \quad \text{etc}$

2) choose : $U_B^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ U_B^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad U_A^{(3)} = \dots . \ ; \ U_A^{(4)} = \dots$

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1) Solutions are orthogonal

2) Normalisation: $N = \sqrt{E+m}$

3) Adjoint:

$$(\not{p} - m)\, u = 0 \qquad \Rightarrow \qquad \overline{u}\,(\not{p} - m) = 0$$

$$(\not{p} + m)\, v = 0 \qquad \Rightarrow \qquad \overline{v}\,(\not{p} + m) = 0$$

4) Completeness: $\displaystyle\sum_{s=1,2} u^{(s)}(p)\, \overline{u}^{(s)}(p) = \not{p} + m$

5) Helicity: $\lambda = \frac{1}{2} \vec{\Sigma} \cdot \vec{p} \ ; \ \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

Quantum Electrodynamics

1) Free particle wave equations

Plane wave solutions:

Current: $\partial_\mu j^\mu = 0$

K.G.: $\left(\partial_\mu \partial^\mu + m^2\right)\phi(x) = 0$ $\quad \phi(x) = N e^{-ipx}$ $\quad j^\mu = i\left[\phi^*\left(\partial^\mu \phi\right) - \left(\partial^\mu \phi^*\right)\phi\right]$ $\Rightarrow$ $\rho = 2|N|^2 E$

Dirac: $\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0$ $\quad \psi(x) = u(p)e^{-ipx}$ $\quad j^\mu = \overline{\psi}\gamma^\mu\psi$ $\quad \rho = \psi^\dagger\psi \geq 0$

2) Electromagnetic field

Gauge freedom:

Lorentz condition:

Plane wave solutions:

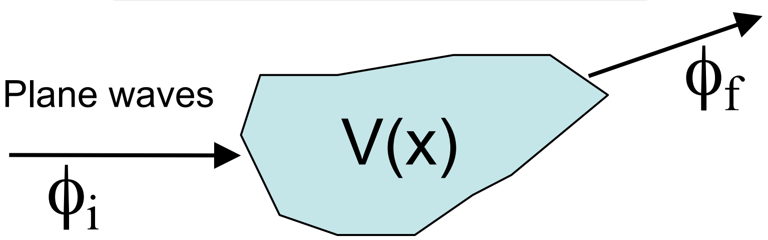
Maxwell $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ $\quad A_\mu \to A_\mu' = A_\mu + \partial_\mu \lambda$ $\quad \partial_\mu A^\mu = 0$ $\quad A^\mu = N\varepsilon^\mu(p)e^{-ipx}$

or $\partial_\mu F^{\mu\nu} = j^\nu$ $\quad$ with $\partial_\nu \partial^\nu \lambda = 0$ $\quad \partial_\mu \partial^\mu A^\nu = j^\nu$

(2 polarizations since m=0)

3) Scattering Perturbation Theory

A: non-relativistic derivation:



$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V(x,t)\right)\psi$$

$$\psi = \sum_{n=0}^{\infty} a_n(t)\phi_n(t)e^{-iEt}$$

$$T_{fi} = a_f(t\to\infty) = -i\int d^4x\, \phi_f^*(x)V(x)\phi_i(x)$$

1-st order:

$$W_{fi} = \lim_{T\to\infty}\frac{|T_{fi}|^2}{T} \quad \text{with}$$

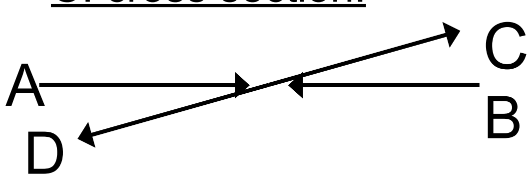
$$T_{fi} = -2\pi i\, V_{fi}\,\delta\left(E_f - E_i\right) \quad \text{with}$$

$$V_{fi} = \int d^3x\, \phi_f^*(\vec{x})V(\vec{x})\phi_i(\vec{x})$$

B: relativistic extension:

$$W_{fi} = \lim_{T,V\to\infty}\frac{|T_{fi}|^2}{TV} \quad \text{and} \quad T_{fi} = -iN_A N_B N_C^* N_D^*(2\pi)^4\delta^4\left(p_A + p_B - p_C - p_D\right)\mathcal{M}$$

C: cross section:



$$\frac{d\sigma}{d\Omega} = \frac{W_{fi}}{flux}d\Phi$$

$$flux = 4\sqrt{\left(p_A p_B\right)^2 - m_A^2 m_B^2}\,/V^2$$

$$d\Phi = \prod_i \frac{V}{(2\pi)^3}\frac{d^3p_i}{2E_i}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{s}\left|\frac{p_f}{p_i}\right||\mathcal{M}|^2$$

4) Electromagnetic Scattering

$$\partial^\mu \to \partial^\mu - ieA^\mu$$

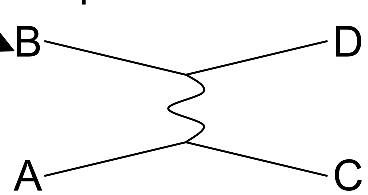
$$\left(\partial_\mu \partial^\mu + m^2\right)\phi(x) \to \left(\partial_\mu \partial^\mu + m^2 + V(x)\right)\phi(x) \;;\; V(x) = -ie\partial_\mu A^\mu + A_\mu \partial^\mu$$

$$T_{fi} = -i\int -ie\left[\phi_f^*\left(\partial_\mu \phi_i\right) - \left(\partial_\mu \phi_f^*\right)\phi_i\right]A^\mu\, d^4x$$

$$\left(i\gamma^\mu \partial_\mu - m\right)\phi(x) \to \left(i\gamma^\mu \partial_\mu - m + V(x)\right)\phi(x) \;;\; V(x) = -e\gamma^0\gamma_\mu A^\mu$$

$$T_{fi} = -i\int -e\left[\overline{\psi}_f\gamma_\mu\psi_i\right]A^\mu\, d^4x$$

Spin 0 case:



$$\phi_i = N_i e^{-ip_i x} \quad \phi_f^* = N_f^* e^{ip_f x} \quad \partial_\nu \partial^\nu A^\mu = -j_{BD}^\mu = -eN_B N_D^*\left(p_B^\mu + p_D^\mu\right)e^{i(p_D - p_B)} \quad A^\mu = -\frac{1}{q^2}j_{BD}^\mu$$

$$T_{fi} = -i\int j_\mu^{AC}A^\mu d^4x = -i\int j_{AC}^\mu \frac{-g_{\mu\nu}}{q^2}j_{BD}^\nu d^4x = -iN_A N_B N_C^* N_D^*(2\pi)^4\delta^4\left(p_A + p_B - p_C - p_D\right)\mathcal{M}$$

$$-i\mathcal{M} = ie\left(p_A + p_C\right)^\mu \frac{-ig_{\mu\nu}}{q^2} ie\left(p_B + p_D\right)^\nu$$

$\Rightarrow$ "Feynman rules"