Lecture 4

Master Particle Physics 1

Marcel Merk

September 14, 2011

Lectures PP1

L0 : Introduction

L1 : Particles and Fields (historical overview)
L2 : Wave Equations and Antiparticles
L3 : The Electromagnetic Field
L4 : Perturbation Theory and Fermi's Golden Rule
L5 : Electromagnetic Scattering of Spinless Particles

QED of Spinless Particles: Scattering Theory and Cross Sections

L6 : The Dirac Equation
L7 : Solutions of the Dirac Equation
L8 : Spin 1/2 Electrodynamics

QED for Fundamental Fermions

L9 : The Weak Interaction
(Fermi 4-point scattering: an analogy with QED)
L10: Local Gauge Invariance
(the role of symmetries in interactions)
L11: Electroweak Theory
L12: The Process: $e^+e^- \rightarrow \gamma, Z \rightarrow \mu^+\mu^-$

The Standard Model for massless particles $SU(2)_L \times U(1)_Y$

Matter Waves and EM Field

Matter Waves

$$E \rightarrow i\frac{\partial}{\partial t} \quad ; \quad \vec{p} \rightarrow -i\vec{\nabla}$$

Non-relativistic:

$$E = \frac{\vec{p}^2}{2m}$$

$$i\frac{\partial}{\partial t}\psi = \frac{-1}{2m}\nabla^2\psi$$

Relativistic:

$$E^2 = \vec{p}^2 + m^2$$

$$-\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + m^2\phi$$

Continuity: $\partial_\mu j^\mu = 0$ with: $j^\mu = i\left(\phi^* \, \partial^\mu \phi - \phi \, \partial^\mu \phi^*\right)$

EM Field

Maxwell Equations:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \rho \\
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}
\end{aligned}$$

Maxwell 1+4
-> continuity

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} \\
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V \\
A^\mu &= \left(V, \vec{A}\right) \\
j^\nu &= \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu
\end{aligned}$$

Gauge Invariance: $A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda$

Lorentz Condition: $\partial_\mu A^\mu = 0$

Coulomb Condition: $A^0 = 0 \quad ; \quad \vec{\nabla}\cdot\vec{A} = 0$

Photon has 2-polarizations!

Bohm - Aharanov Experiment!

Perturbation Equations (I)

general solution:

$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V\left(\vec{x},t\right)\right)\ \psi \quad , \quad \psi = \sum_{n=0}^{\infty} a_n(t)\ \phi_n(\vec{x})\ e^{-iE_n t}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

Perturbation Equations (1)

$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V\left(\vec{x}, t\right)\right)\ \psi$$

general solution:

$$,\quad \psi = \sum_{n=0}^{\infty} a_n(t)\ \phi_n(\vec{x})\ e^{-iE_n t}$$

$$i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\ \phi_n(\vec{x})\ e^{-iE_n t} + i\sum_{n=0}^{\infty}(-i)\ E_n\ a_n(t)\ \phi_n(\vec{x})\ e^{-iE_n t} =$$

$$\sum_{n=0}^{\infty} E_n\ a_n(t)\ \phi_n(\vec{x})\ e^{-iE_n t} + \sum_{n=0}^{\infty} V(\vec{x}, t)\ a_n(t)\ \phi_n(\vec{x})\ e^{-iE_n t}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

Perturbation Equations (1)

$$i\,\frac{\partial \psi}{\partial t} = \left(H_0 + V\left(\vec{x},t\right)\right)\,\psi \quad , \quad \psi = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}$$

general solution:

$$i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\,\phi_n(\vec{x})\,e^{-iE_n t} + i\sum_{n=0}^{\infty}(-i)\,\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})}\,e^{-iE_n t} =$$

$$\sum_{n=0}^{\infty}E_n\,\cancel{a_n(t)\,\phi_n(\vec{x})}\,e^{-iE_n t} + \sum_{n=0}^{\infty}V(\vec{x},t)\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

Perturbation Equations (I)

$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V\left(\vec{x},t\right)\right)\,\psi$$

general solution:

$$\psi = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$

$$\int \psi_f^* d^3x \quad \times \quad i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\,\phi_n(\vec{x})\,e^{-iE_nt} + i\sum_{n=0}^{\infty}(-i)\,\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}} =$$

$$= \left(\phi_f^* e^{-iE_nt}\right)\sum_{n=0}^{\infty} E_n\,\cancel{a_n(t)\,\phi_n(\vec{x})}\,e^{-iE_nt} + \sum_{n=0}^{\infty} V(\vec{x},t)\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

Perturbation Equations (I)

general solution:

$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V(\vec{x},t)\right)\psi \quad , \quad \psi = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}$$

$$\int \psi_f^* d^3x \quad \times \quad i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\phi_n(\vec{x})\,e^{-iE_n t} + i\sum_{n=0}^{\infty}(-i)\,\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}} =$$

$$= \left(\phi_f^* e^{-iE_n t}\right)\sum_{n=0}^{\infty}\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}} + \sum_{n=0}^{\infty} V(\vec{x},t)\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

$$i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\underbrace{\int d^3x\,\phi_f^*(\vec{x})\,\phi_n(\vec{x})}_{\delta_{fn}}\,e^{-i(E_n-E_f)t} =$$

$$\sum_{n=0}^{\infty} a_n(t)\int d^3x\,\phi_f^*(\vec{x})\,V(\vec{x},t)\,\phi_n(\vec{x})\,e^{-i(E_n-E_f)t}$$

Perturbation Equations (I)

$$i\frac{\partial\psi}{\partial t} = \left(H_0 + V\left(\vec{x},t\right)\right)\,\psi$$

general solution:

$$\psi = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$

$$\int \psi_f^* d^3x \quad \times \quad i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\,\phi_n(\vec{x})\,e^{-iE_nt} + i\sum_{n=0}^{\infty}(-i)\,\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}} =$$

$$= \left(\phi_f^* e^{-iE_nt}\right)\sum_{n=0}^{\infty}\cancel{E_n\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}} + \sum_{n=0}^{\infty} V(\vec{x},t)\,a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$

multiply with Psi_f^* = phi_f^* exp (iE_f t) and integrate over d3x

$$i\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}\underbrace{\int d^3x\,\phi_f^*(\vec{x})\,\phi_n(\vec{x})}_{\delta_{fn}}\;e^{-i(E_n-E_f)t} =$$

$$\sum_{n=0}^{\infty} a_n(t)\int d^3x\,\phi_f^*(\vec{x})\,V(\vec{x},t)\,\phi_n(\vec{x})\,e^{-i(E_n-E_f)t}$$

$$\frac{da_f(t)}{dt} = -i\sum_{n=0}^{\infty} a_n(t)\int d^3x\,\phi_f^*(\vec{x})\,V(\vec{x},t)\,\phi_n(\vec{x})\,e^{-i(E_n-E_f)t}$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}, t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}, t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

$$\frac{da_f(t)}{dt} = -i \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}, t) \, \phi_i(\vec{x}) \, e^{-i(E_i - E_f)t}$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}, t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

$$\frac{da_f(t)}{dt} = -i \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}, t) \, \phi_i(\vec{x}) \, e^{-i(E_i - E_f)t}$$

$$\begin{aligned} a_f(t') &= \int_{-T/2}^{t'} \frac{da_f(t)}{dt} \, dt \\ &= -i \int_{-T/2}^{t'} dt \int d^3x \, \left[\phi_f(\vec{x}) \, e^{-iE_f t}\right]^* \, V(\vec{x}, t) \, \left[\phi_i(\vec{x}) \, e^{-iE_i t}\right] \end{aligned}$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

$$\frac{da_f(t)}{dt} = -i \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_i(\vec{x}) \, e^{-i(E_i - E_f)t}$$

$$\begin{aligned} a_f(t') &= \int_{-T/2}^{t'} \frac{da_f(t)}{dt} \, dt \\ &= -i \int_{-T/2}^{t'} dt \int d^3x \, \left[\phi_f(\vec{x}) \, e^{-iE_f t}\right]^* \, V(\vec{x},t) \, \left[\phi_i(\vec{x}) \, e^{-iE_i t}\right] \end{aligned}$$

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{T/2} dt \int d^3x \, \phi_f^*(\vec{x},t) \, V(\vec{x},t) \, \phi_i(\vec{x},t)$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

$$\frac{da_f(t)}{dt} = -i \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_i(\vec{x}) \, e^{-i(E_i - E_f)t}$$

$$\begin{aligned} a_f(t') &= \int_{-T/2}^{t'} \frac{da_f(t)}{dt} \, dt \\ &= -i \int_{-T/2}^{t'} dt \int d^3x \, \left[\phi_f(\vec{x}) \, e^{-iE_f t}\right]^* \, V(\vec{x},t) \, \left[\phi_i(\vec{x}) \, e^{-iE_i t}\right] \end{aligned}$$

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{T/2} dt \int d^3x \, \phi_f^*(\vec{x},t) \, V(\vec{x},t) \, \phi_i(\vec{x},t)$$

$$\boxed{T_{fi} = -i \int d^4x \, \phi_f^*(x) \, V(x) \, \phi_i(x)}$$

Perturbation Equations (2)

$$\frac{da_f(t)}{dt} = -i \sum_{n=0}^{\infty} a_n(t) \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_n(\vec{x}) \, e^{-i(E_n - E_f)t}$$

$$\frac{da_f(t)}{dt} = -i \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x},t) \, \phi_i(\vec{x}) \, e^{-i(E_i - E_f)t}$$

$$\boxed{V_{fi} \equiv \int d^3x \, \phi_f^*(\vec{x}) \, V(\vec{x}) \, \phi_i(\vec{x})}$$

$$\begin{aligned} a_f(t') &= \int_{-T/2}^{t'} \frac{da_f(t)}{dt} \, dt \\ &= -i \int_{-T/2}^{t'} dt \int d^3x \, \left[\phi_f(\vec{x}) \, e^{-iE_f t}\right]^* \, V(\vec{x},t) \, \left[\phi_i(\vec{x}) \, e^{-iE_i t}\right] \end{aligned}$$

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{T/2} dt \int d^3x \, \phi_f^*(\vec{x},t) \, V(\vec{x},t) \, \phi_i(\vec{x},t)$$

$$\boxed{T_{fi} = -i \int d^4x \, \phi_f^*(x) \, V(x) \, \phi_i(x)}$$

Relativistic Scattering

$$T_{fi} = -i\, N_A N_B N_C N_D\, (2\pi)^4\, \delta(p_A + p_B - p_C - p_D)\, \mathcal{M}$$

Relativistic Scattering

$$T_{fi} = -i\, N_A N_B N_C N_D\, (2\pi)^4\, \delta(p_A + p_B - p_C - p_D)\, \mathcal{M}$$

$$\begin{aligned}
|T_{fi}|^2 &= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2 \int d^4x\, e^{-i(p_A+p_B-p_C-p_D)x} \times \int d^4x'\, e^{-i(p_A+p_B-p_C-p_D)x'} \\
&= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} \int_{TV} d^4x \\
&= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} TV
\end{aligned}$$

Relativistic Scattering

$$T_{fi} = -i\, N_A N_B N_C N_D\, (2\pi)^4\, \delta(p_A + p_B - p_C - p_D)\, \mathcal{M}$$

$$\begin{aligned}
|T_{fi}|^2 &= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2 \int d^4x\, e^{-i(p_A+p_B-p_C-p_D)x} \times \int d^4x'\, e^{-i(p_A+p_B-p_C-p_D)x'} \\
&= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} \int_{TV} d^4x \\
&= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2\, (2\pi)^4\, \delta^4(p_A + p_B - p_C - p_D) \times \lim_{T,V\to\infty} TV
\end{aligned}$$

$$\begin{aligned}
W_{fi} &= \lim_{T,V\to\infty} \frac{|T_{fi}|^2}{TV} \\
&= |N_A N_B N_C N_D|^2\, |\mathcal{M}|^2\, (2\pi(^4\, \delta\,)p_A + p_B - p_C - p_D)
\end{aligned}$$

Cross Section: $A + B \rightarrow C + D$

$$\boxed{d\sigma = \frac{(2\pi)^4 \ \delta^4\left(p_A + p_B - p_C - p_D\right)}{4\sqrt{\left(p_A \cdot p_B\right)^2 - m_A^2 m_B^2}} \cdot |\mathcal{M}|^2 \cdot \frac{d^3 p_C}{(2\pi)^3 \ 2E_C} \frac{d^3 p_D}{(2\pi)^3 \ 2E_D}}$$

Decay Rate: $A \rightarrow B + C$

$$\boxed{d\Gamma = \frac{(2\pi)^4 \ \delta^4\left(p_A - p_C - p_D\right)}{2E_A} \cdot |\mathcal{M}|^2 \cdot \frac{d^3 p_C}{(2\pi)^3 \ 2E_C} \frac{d^3 p_D}{(2\pi)^3 \ 2E_D}}$$