PHY3004: Nuclear and Particle Physics

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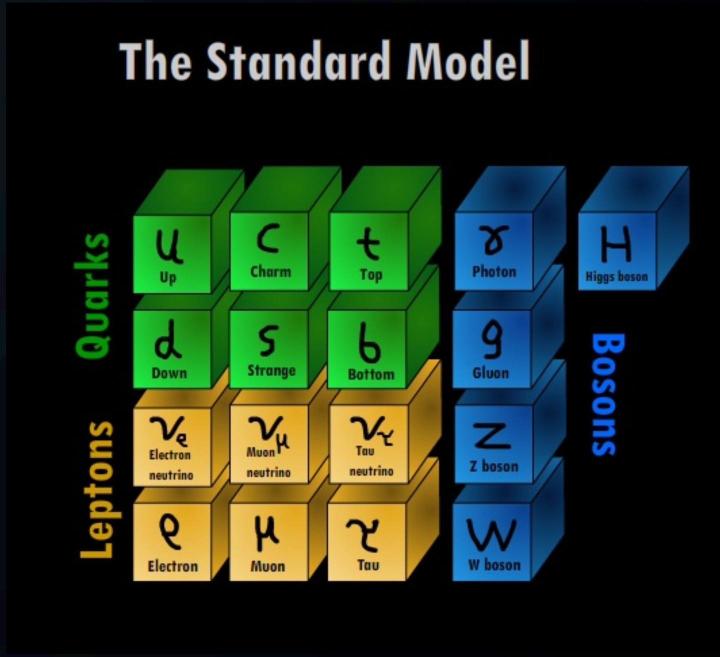


Today: Scattering Theory and Feynman Calculus

Griffiths Chapter 6

Part 1 : Decay and Cross Section

Part 2 : Perturbation Theory and the Golden Rule

Part 3 : Feynman Calculus

Part 1

Decay and Cross Section

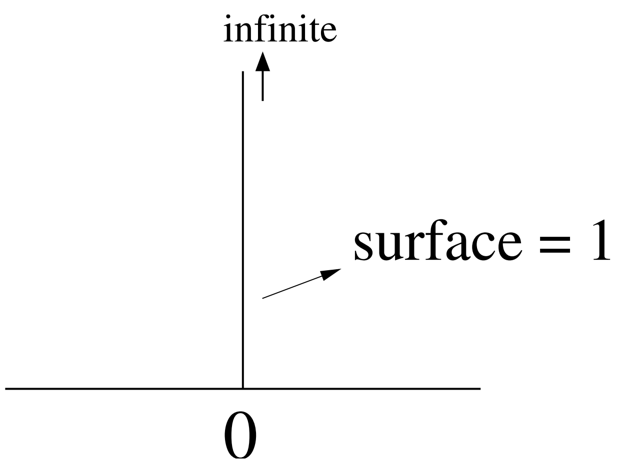
Griffiths §6.1

Exercise – 27: Dirac delta function (1)

See Griffiths Appendix A

- Consider a function defined by the following prescription:

$$\delta(x) = \lim_{\Delta \to 0} \begin{cases} 1/\Delta & \text{for } |x| < \Delta/2 \\ 0 & \text{otherwise} \end{cases}$$



- The integral of this function is normalized: $\int_{-\infty}^{\infty} \delta(x)\, \mathrm{d}x = 1$

- For a function $f(x)$ we have: $f(x)\delta(x) = f(0)\delta(x)$

...and therefore: $\int_{-\infty}^{\infty} f(x)\delta(x)\, \mathrm{d}x = f(0)$

- Exercise:
  a) Prove that: $\delta(kx) = \frac{1}{|k|}\delta(x)$
  b) Prove that: $\delta\big(g(x)\big) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x - x_i)$, where $g(x_i) = 0$ are the zero-points
     - Hint: make a Taylor expansion of g around the 0-points.

Exercise – 27: Dirac delta function (2)

- The delta function has many forms. One of them is: $\delta(x) = \lim_{\alpha\to\infty} \frac{1}{\pi}\frac{\sin^2 \alpha x}{\alpha x^2}$

c) Make this plausible by sketching the function $\sin^2(\alpha x)\,/(\pi\alpha x^2)$ for two relevant values of α

- Remember the Fourier transform:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)\, e^{ikx}\, \mathrm{d}k$$

$$g(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x$$

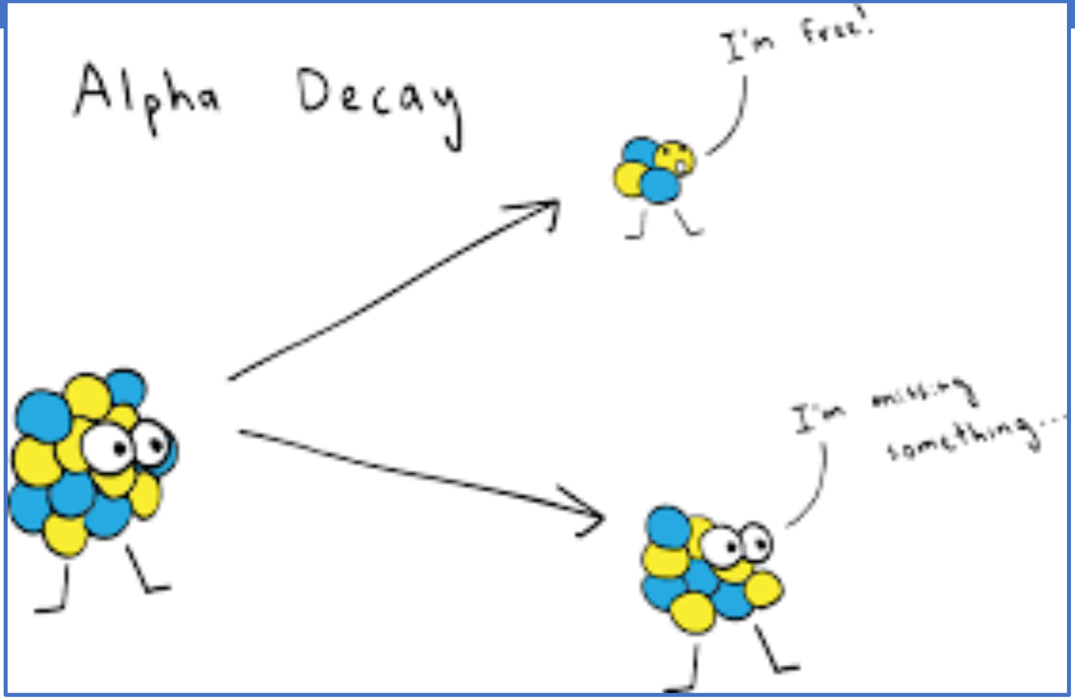
d) Use this to show that another (important!) representation of the Dirac delta-function is given by:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, \mathrm{d}k$$

🡐 We will use this later in the lecture!

Terminology: Decays

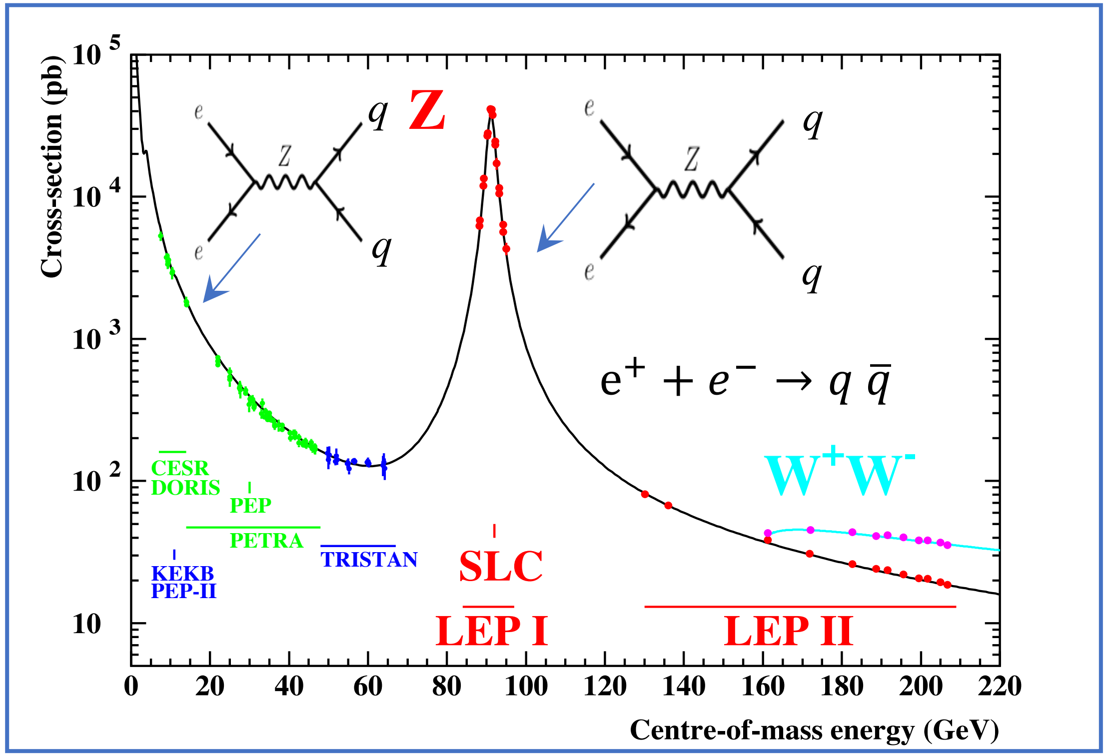
- A quantum particle decays with equal probability per unit time
  - $dN = -\Gamma N dt$ such that: $N = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau}$
  - $\Gamma \equiv$ decay rate $\tau \equiv$ mean lifetime



- Often particles can decay in many quantum ways; each with its own partial decay width Γ_i
  - ***Total decay rate*** $\Gamma_{tot} = \sum_i \Gamma_i$ and ***lifetime*** $\tau = \frac{1}{\Gamma_{tot}}$ and ***Branching Ratio*** $BR_i = \frac{\Gamma_i}{\Gamma_{tot}}$

Terminology: Cross Section

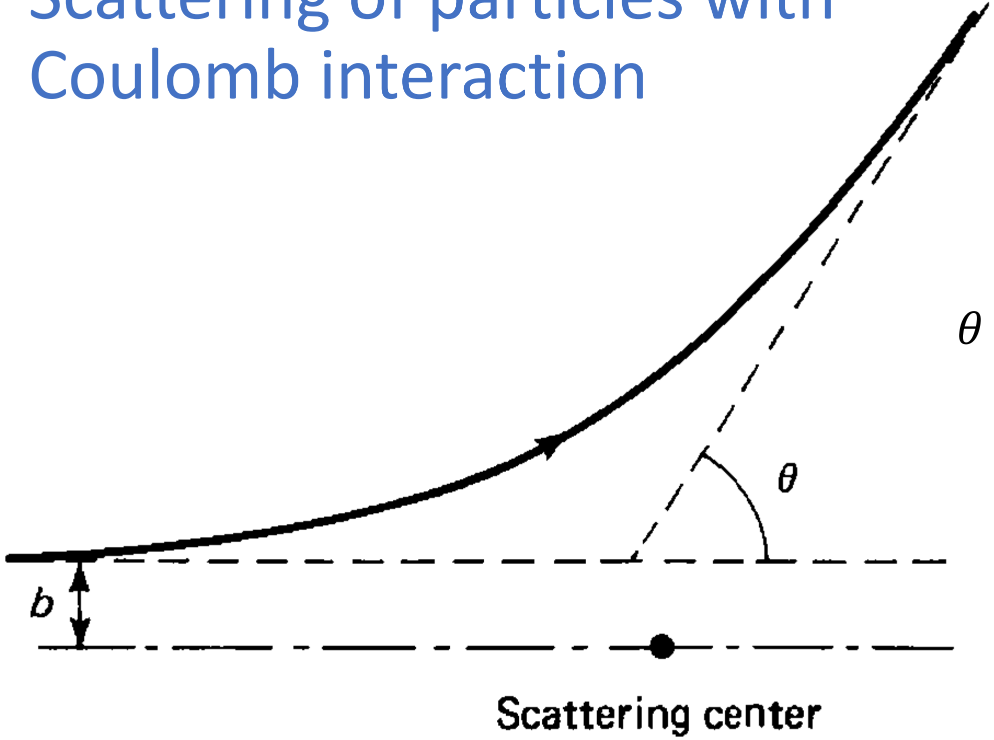
- A scattering process is measured using "cross section"; the ***effective surface*** seen by a particle colliding with a target. We use the same for collisions:
  - e.g. proton-proton colliders.
- For colliding protons many processes may happen:
  - ***Exclusive*** cross section σ_i : cross section for one specific process "i"
  - ***Inclusive*** cross section σ_{tot}: sum all possible exclusive cross sections: $\sigma_{tot} = \sum_i^N \sigma_i$
- The cross section can for example depend on the energy of the collision
- Look at the process $e^+e^- \rightarrow q\bar{q}$
  - There is a resonance at 91 GeV; the mass of the Z- boson
  - And there is a peak near 0 GeV; the photon resonance
  - "Electroweak process"



Scattering

Griffiths §6.1

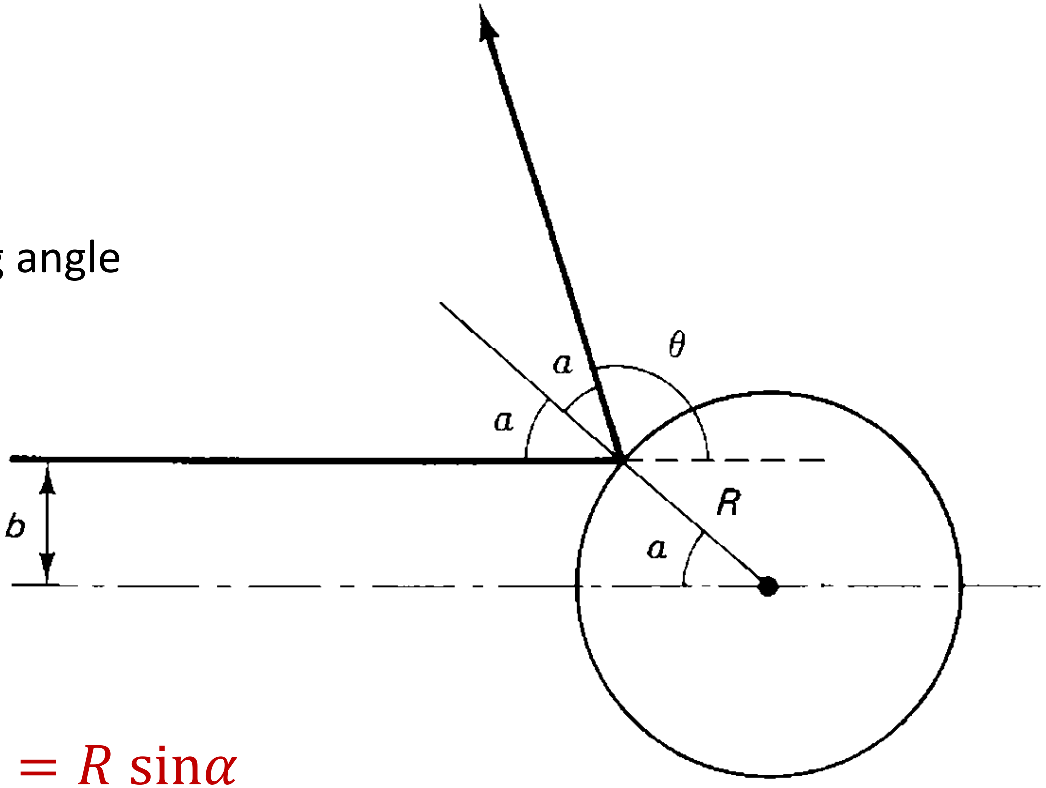
- Scattering of particles with Coulomb interaction



b = Impact Parameter
θ =Scattering Angle
1-to-1 relation b and θ

θ is scattering angle

- Scattering on a “hard sphere”



$b = R\sin\alpha$
$2\alpha + \theta = \pi$
$\sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$
$b = R\cos(\theta/2)$

Hard Sphere Scattering

Griffiths §6.1

- Calculation of hard sphere scattering

If particle goes through $d\sigma$ it will scatter through solid angle $d\Omega$:

$$\mathrm{d}\sigma = D(\theta)\mathrm{d}\Omega$$

$$\mathrm{d}\sigma = |b\,\mathrm{d}b\,\mathrm{d}\phi|$$

$$\mathrm{d}\Omega = |\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi|$$

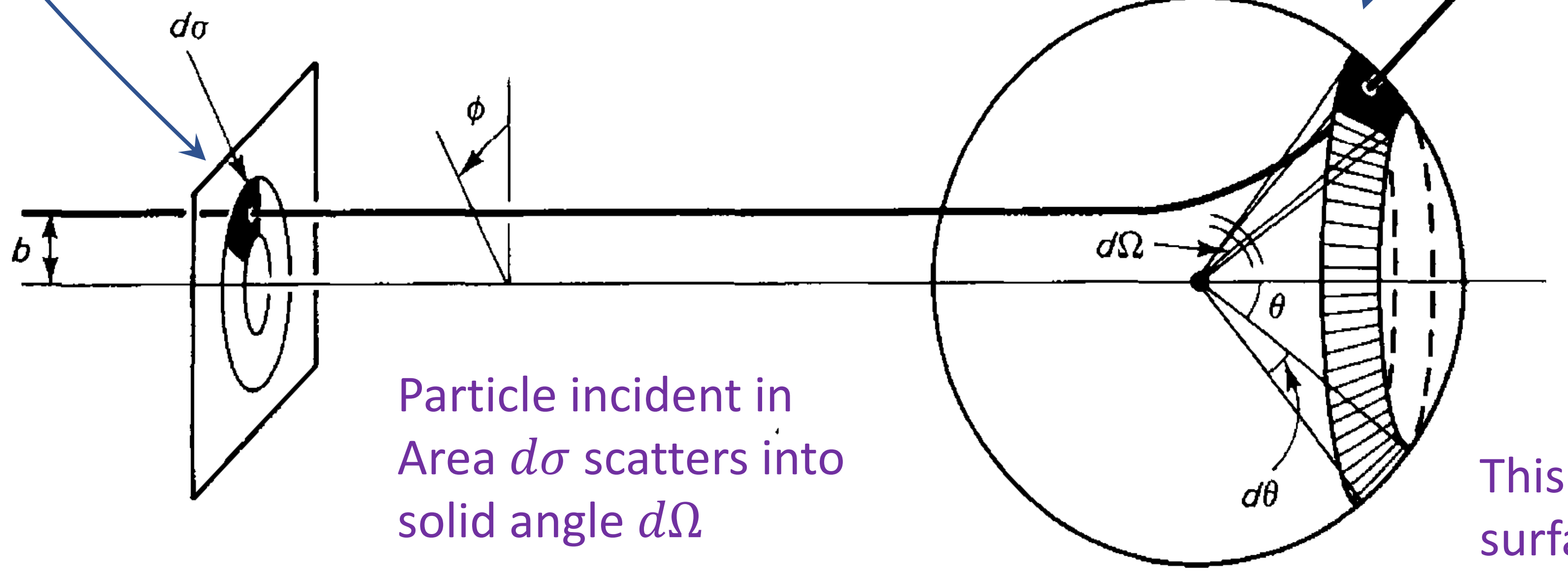
$$D(\theta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{b}{\sin\theta}\left(\frac{\mathrm{d}b}{\mathrm{d}\theta}\right)\right|$$

"Differential cross-section"

Hard scattering:

$$b = R\cos(\theta/2)$$

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$



Particle incident in Area $d\sigma$ scatters into solid angle $d\Omega$

Then:

$$D(\theta) = \frac{Rb\ \sin(\theta/2)}{2\sin\theta}$$

$$= \frac{R^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta}$$

$$= \frac{R^2}{4} \qquad \left(\cos\alpha\sin\alpha = \tfrac{1}{2}(\sin 2\alpha)\right)$$

$$\int \mathrm{d}\Omega = \int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$$

$$\sigma = \int \mathrm{d}\sigma = \int D(\theta)\,\mathrm{d}\Omega$$

$$= \int \frac{R^2}{4}\mathrm{d}\Omega = \pi R^2$$

This corresponds to the projected surface of the circle seen by the particle

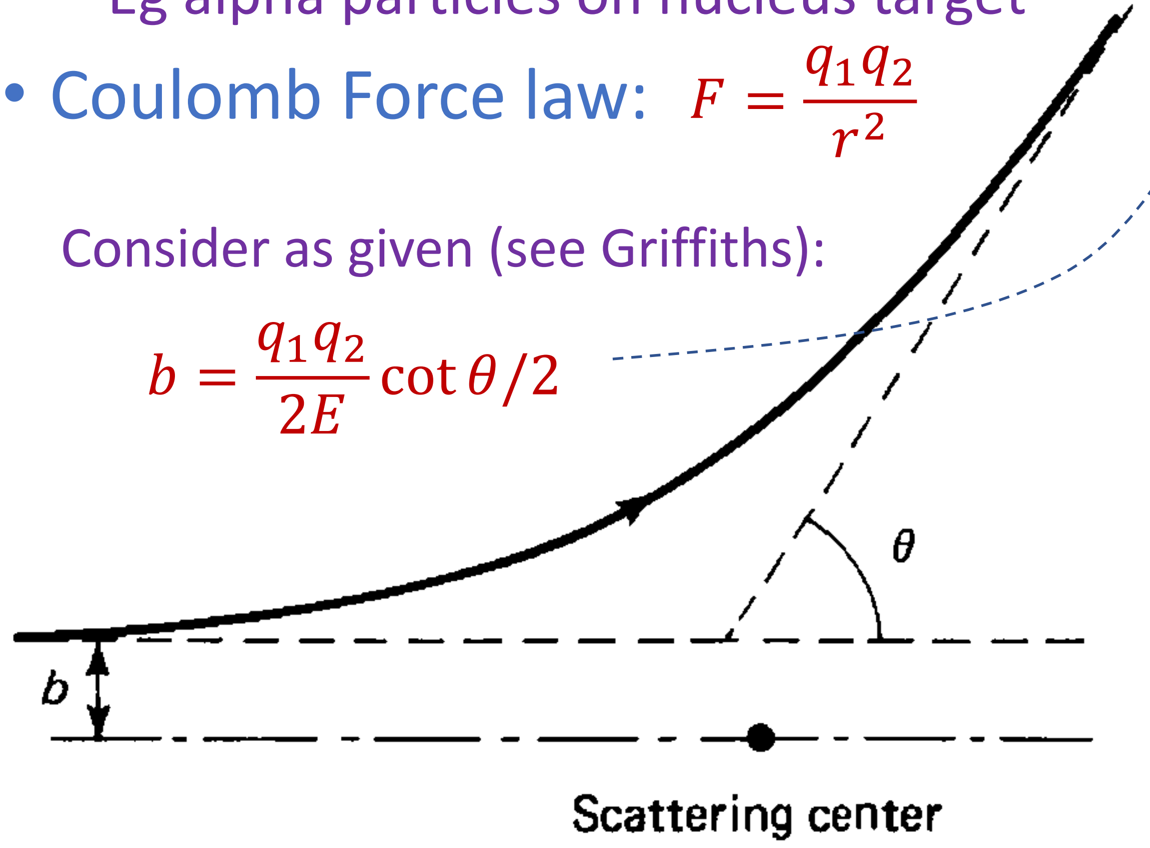
Rutherford Scattering

Griffiths §6.1

- Scattering of charged particles with Coulomb force
  - Eg alpha particles on nucleus target
- Coulomb Force law: $F = \frac{q_1 q_2}{r^2}$

Consider as given (see Griffiths):

$$b = \frac{q_1 q_2}{2E} \cot \theta/2$$



$$D(\theta) = \frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \left(\frac{db}{d\theta} \right) \right|$$

$$D(\theta) = \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2$$

$$\sigma = \int d\sigma = \int D(\theta)\, d\Omega$$

$$\sigma = 2\pi \left(\frac{q_1 q_2}{4E} \right)^2 \int_0^\pi \frac{1}{\sin^4(\theta/2)} \sin\theta\, d\theta$$

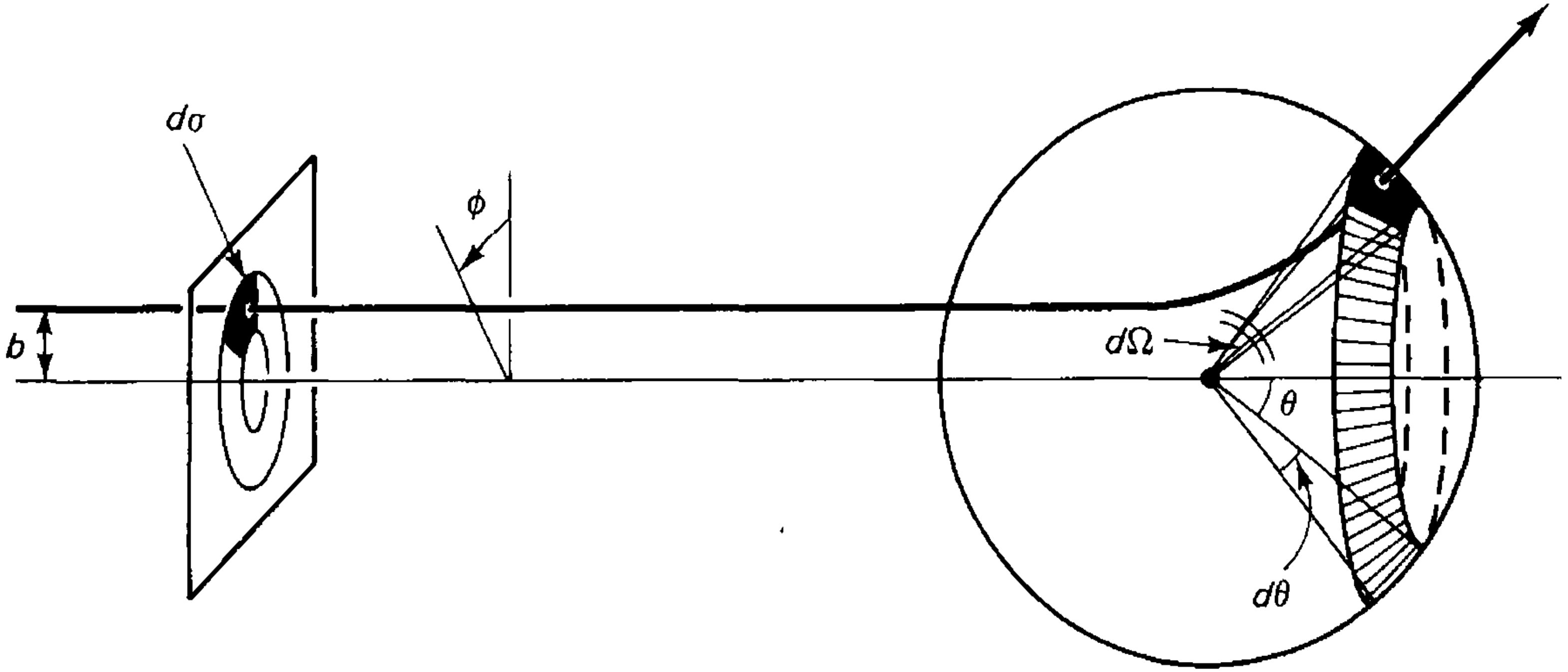
The integral turns out to be infinite!
Particle sees an infinitely large scattering surface?
Reason is that Coulomb force has infinite range.

Most “collisions” will happen at large distance, which is what Rutherford observed.

Luminosity

Griffiths §6.1

- Consider beam of particles on a target
  - Luminosity $\mathcal{L}$ is number of particles per unit time, per unit area.
  - Number of particles passing through area $\mathrm{d}\sigma$: $dN = \mathcal{L}\,\mathrm{d}\sigma$
  - Same number of particles scattering into solid angle $\mathrm{d}\Omega$: $dN = \mathcal{L}\,\mathrm{d}\sigma = \mathcal{L}\,D(\theta)\,\mathrm{d}\Omega$
  - By counting one can measure the *differential cross section*: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = D(\theta) = \frac{\mathrm{d}N}{\mathcal{L}\,\mathrm{d}\Omega}$



These aspects are needed when you Compare theory with experiments.

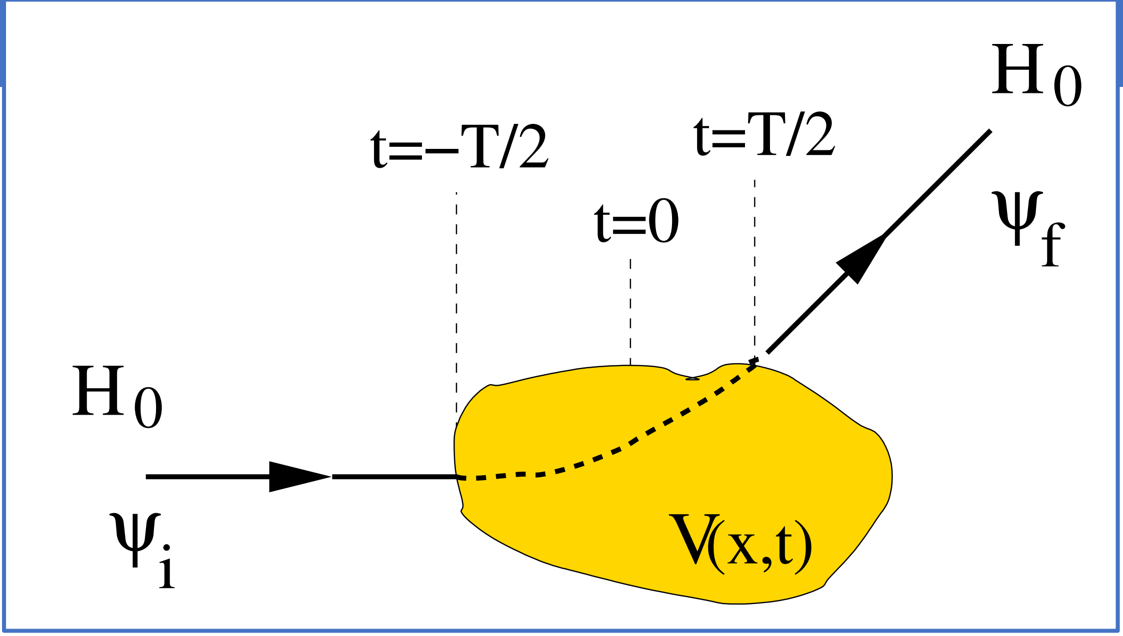
Griffiths §6.2 and PP1 Chapter 2

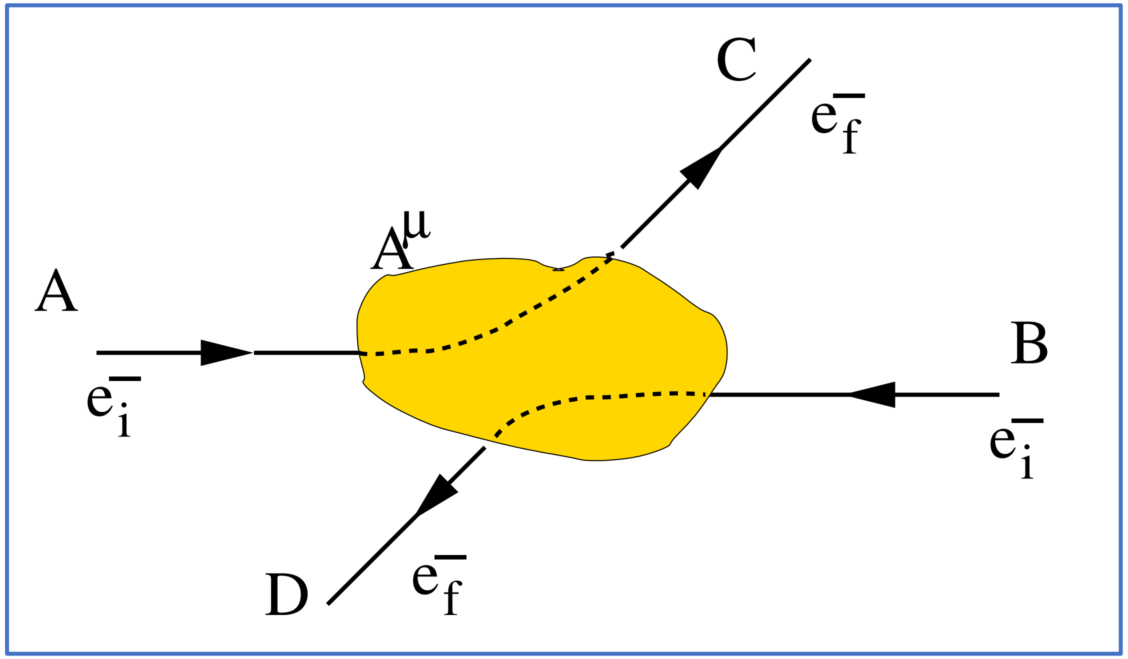
Part 2

Perturbation Theory and the Golden Rule

Scattering with waves

- An incoming particle is represented by a wave packet of incoming plane waves: $\psi(x) = Ne^{-ipx}$
- Example 1:
  - Calculate the scattering of these waves in an external potential
- Example 2:
  - For collisions the scattering potential A^{μ} of particle A is determined by the field of particle B and vice versa.





Fermi’s Golden Rule

- To calculate decay rates and cross section in relativistic scattering we use a general formula that we cannot fully derive within the scope of these lectures
  - For a fully relativistic derivation: quantum field theory
  - We will ”make it plausible” using non-relativistic single particle theory
    - see also Nikhef PP1 lecture notes chapter 2, or the book of Thomsom §2.3.6 and chapter 3
- Here is the end-result:
  - Golden Rule for decays: $1 \rightarrow 2 + 3 + 4 + \cdots n$

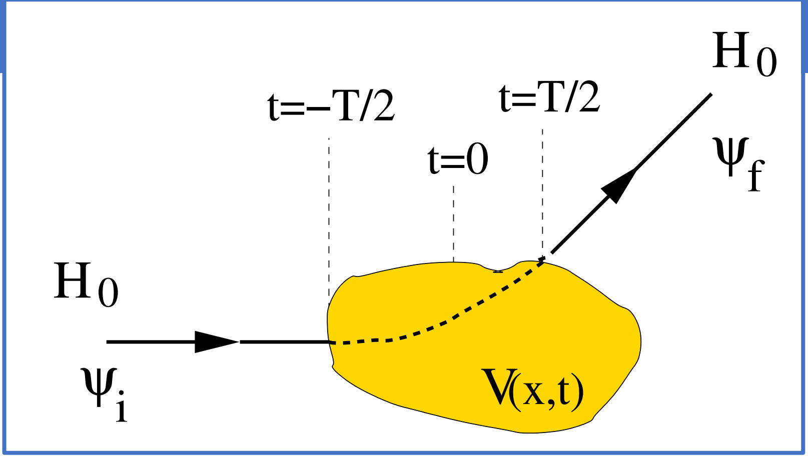
$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right) \, \frac{d^4 p_j}{(2\pi)^4}$$

  - Golden Rule for scattering: $1 + 2 \rightarrow 3 + 4 + 5 + \cdots n$

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right) \, \frac{d^4 p_j}{(2\pi)^4}$$

OK, let’s go….

Perturbation Theory



- Consider the free Schrödinger equation $i\frac{\partial\psi}{\partial t} = H_0\psi$
  - H_0 is time-independent Hamiltonian $H_0\phi_m(\vec{x}) = E_m\phi_m(\vec{x})$
  - Eigenstates equation: $\int \phi_m^*(\vec{x})\phi_n(\vec{x})d^3x = \delta_{mn}$
  - where $\phi_m(\vec{x})$ form orthonormal basis for any solution $\psi_m(\vec{x},t) = \phi_m(\vec{x})e^{-iE_mt}$

- Hamiltonian with time-dependent perturbation $i\frac{\partial\psi}{\partial t} = \left(H_0 + V(\vec{x},t)\right)\psi$
  - Solutions are of the form can be written as $\psi = \sum_{n=0}^{\infty} a_n(t)\phi_n(\vec{x})e^{-iE_nt}$
  - Substituting gives:

$$i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\phi_n(\vec{x})e^{-iE_nt} = \sum_{n=0}^{\infty}V(\vec{x},t)a_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

Perturbation Theory

- Multiply the equation: $i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\phi_n(\vec{x})e^{-iE_nt} = \sum_{n=0}^{\infty}V(\vec{x},t)a_n(t)\phi_n(\vec{x})e^{-iE_nt}$

… from the left by $\int \psi_f^* d^3x$ with $\psi_f^* = \phi_f^* e^{iE_ft}$

to find:

$$i\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}\underbrace{\int \mathrm{d}^3x\phi_f^*(\vec{x})\phi_n(\vec{x})}_{\delta_{fn}}\ e^{-i(E_n-E_f)t} = \sum_{n=0}^{\infty}a_n(t)\int \mathrm{d}^3x\ \phi_f^*(\vec{x})V(\vec{x},t)\phi_n(\vec{x})e^{-i(E_n-E_f)t}$$

- Using orthonormality gives: $\frac{da_f(t)}{dt} = -i\sum_{n=0}^{\infty}a_n(t)\int \mathrm{d}^3x\ \phi_f^*(\vec{x})V(\vec{x},t)\phi_n(\vec{x})e^{-i(E_n-E_f)t}$

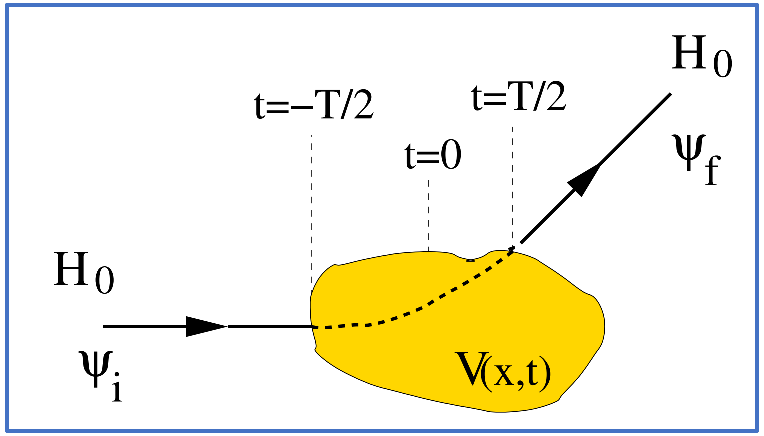
or in short: $i\frac{\mathrm{d}a_f(t)}{\mathrm{d}t} = \sum_{n=0}^{\infty}a_n(t)V_{fn}\ e^{i\omega_{fn}t}$

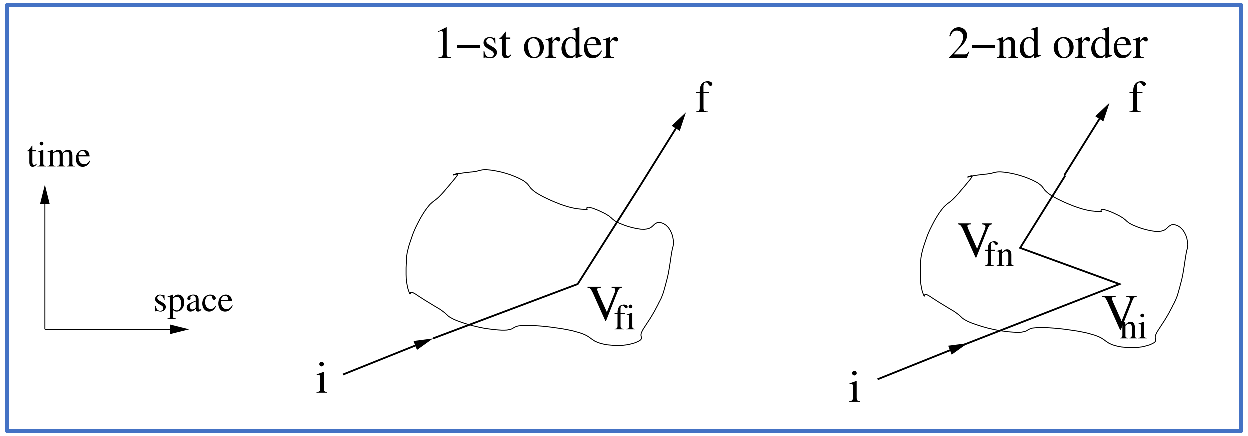
with $\omega_{fn} = (E_f - E_n)$

and the transition matrix element $V_{fn} = \int \mathrm{d}^3x\ \phi_f^*\ V(\vec{x},t)\ \phi_n(\vec{x})$

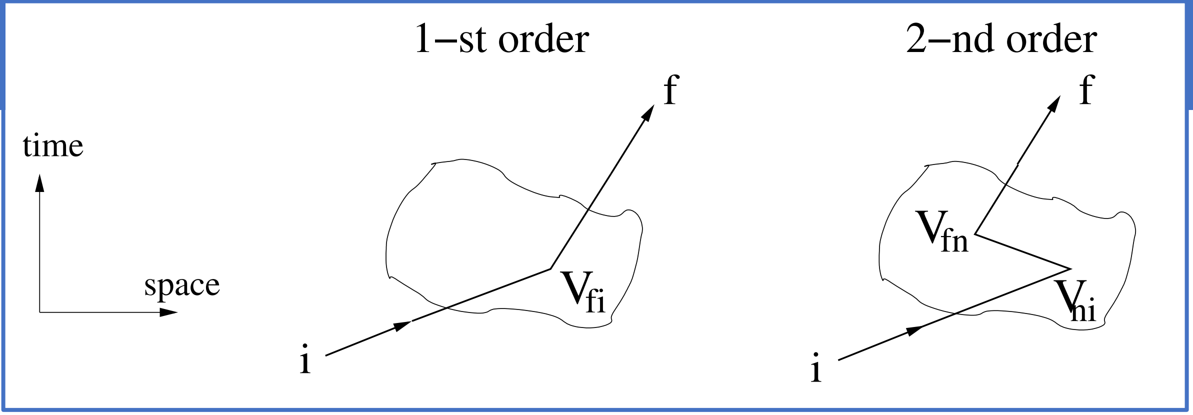
Perturbation Theory

- Solving differential equation: $$i\frac{\mathrm{d}a_f(t)}{\mathrm{d}t}=\sum_{n=0}^{\infty}a_n(t)V_{fn}\,e^{i\omega_{fn}t}$$
- Start with some assumption of zero-th order for a_n and then for each order o: $$i\frac{\mathrm{d}a_f^{(o+1)}(t)}{\mathrm{d}t}=\sum_{n=0}^{\infty}a_n^{(o)}(t)V_{fn}\,e^{i\omega_{fn}t}$$
- First order: assume one step interaction: $a_i(-\infty)=1$ and $a_f(-\infty)=0$ "during" interaction: $a_f^{(0)}(t)=\delta_{fi}$: $$i\frac{\mathrm{d}a_f^{(1)}(t)}{\mathrm{d}t}=V_{fi}(t)e^{i\omega_{fi}t}$$
- Perturbation theory: $$a_f^{(1)}(t)=\int_{-\infty}^{t}\frac{\mathrm{d}a_f(t')}{\mathrm{d}t}\mathrm{d}t'=-i\int_{-\infty}^{t}V_{fi}(t')e^{i\omega_{fi}t'}\mathrm{d}t'$$ for $f\neq i$





Perturbation Theory



- First order perturbation:

$$a_f^{(1)}(t) = \int_{-\infty}^{t} \frac{\mathrm{d}a_f(t')}{\mathrm{d}t}\mathrm{d}t' = -i\int_{-\infty}^{t} V_{fi}(t')e^{i\omega_{fi}t'}\mathrm{d}t'$$

$$\left(V_{fi} = \int \mathrm{d}^3x\, \phi_f^*\, V(\vec{x},t)\, \phi_i(\vec{x}) \right)$$

Results in (“Born approximation”) *“transition amplitude”* T_{fi}:

$$T_{fi} \equiv a_f(t\to\infty) = -i\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3x\psi_f^*(\vec{x},t)V(\vec{x},t)\psi_i(\vec{x},t) = -i\int \mathrm{d}^4x\,\psi_f^*(x)V(x)\psi_i(x)$$

Note: co-variant form!

- If the potential is time independent (“static”) we find:

$$T_{fi} \equiv a_f(t\to\infty) = -i\,V_{fi}\int_{-\infty}^{\infty} e^{i\omega_{fi}t}\mathrm{d}t = -2\pi V_{fi}\delta\left(E_f - E_i\right)$$

- Where we have used an implementation of the Dirac delta function:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}k$$

Golden Rule – non-relativistic

- *Transition rate* is defined as: $W_{fi} \equiv \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$ $\quad T_{fi} = -2\pi V_{fi}\delta(E_f - E_i)$
- After squaring of the delta function (not trivial, see PP1) results in transition probability per unit time: $W_{fi} = 2\pi\, |V_{fi}|^2 \delta(E_f - E_i)$
  - Where the delta function takes care of *energy conservation*
  - The name V_{fi} was used here since it relates to the potential V
  - We adapt the more common name for the matrix element $\mathcal{M}$
- The differential cross section is: $\mathrm{d}\sigma = \frac{W_{fi}}{\mathrm{flux}}\mathrm{d}\Phi$
  - Where flux represents the "density" of the number of incoming states per particle: states →particle
  - The phase space factor $\mathrm{d}\Phi$ (also 'dLIPS') is the density of outgoing states (final state "realisation possibilities")

For more, see Chapter 2 of the Nikhef PP1 Lectures

- Next extend it to relativistic scattering using the matrix element $\mathcal{M}$

Fermi's Golden Rule - Relativistic

Griffiths §6.2

- In Griffiths the relativistic Golden Rule for decay and scattering are just stated.
  - Try to gain understanding by considering the terms in comparison to the non-relativistic case that we derived.
- Golden Rule for decays: $1 \rightarrow 2+3+4+\cdots n$

$$\Gamma = \frac{S}{2m_1}\int |\mathcal{M}|^2\,(2\pi)^4\delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right)\,\frac{d^4p_j}{(2\pi)^4}$$

- Golden Rule for scattering: $1+2 \rightarrow 3+4+5+\cdots n$

$$\sigma = \frac{S}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2)^2}}\int |\mathcal{M}|^2\,(2\pi)^4\delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right)\,\frac{d^4p_j}{(2\pi)^4}$$

- We will discuss them in turn…

Decay

- Golden Rule for Decay:

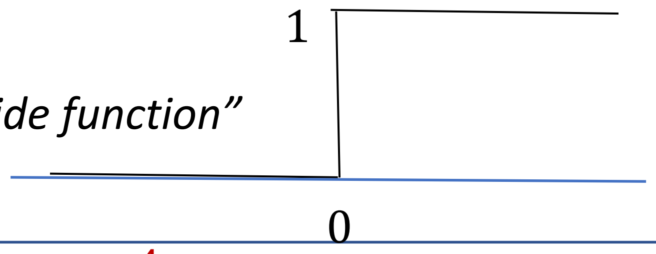
$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0) \frac{d^4 p_j}{(2\pi)^4}$$

- S is a quantum factor to prevent double counting for identical particles
  - Each species with s particles in *the final state* gives factor $1/s!$
  - Eg.: decay $a \to b + b + c + c + c$ gets factor $(1/2!)\times(1/3!) = 1/12$
- $2m_1 = 2E$: density of incoming states (see lecture 3: $\rho = 2|N|^2 E$).
- $\mathcal{M}$ is the Matrix Element: contains the *dynamics* (the interesting particle physics). It is given by the Feynman rules.
- $\int$ implements the integral over *all realization possibilities* to obtain the final state.
- δ is the Dirac delta function. δ^4 implements energy-momentum conservation and $\prod \delta$ assures produced particles are *on-mass shell*: $p^2 = m^2 \to E^2 - \vec{p}^2 = m^2$.
- θ step function so that only $E > 0$.
- Each δ-function comes with a factor 2π (ex. 27) and each d^4p with $1/(2\pi)$.

underlying reason: $\hbar = h/2\pi$

Decay

$\theta(p_j^0) =$
"Heavyside function"



- Decay:

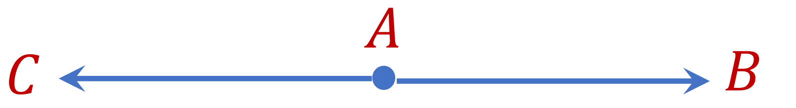
$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 \,(2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\, \frac{d^4 p_j}{(2\pi)^4}$$

- Using the mathematical characteristics of Dirac δ functions (optional exercise, Griffiths page 205: "the $\theta(p_j^0)$-function kills the $\delta(p_j^0)$"), the second part can be shortened into:

$$\Gamma = \frac{S}{2m_1} \int |\mathcal{M}|^2 \,(2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} \frac{1}{2E_j} \frac{d^3 \vec{p}_j}{(2\pi)^3}$$

Eg.: $K \rightarrow \pi^+ + \pi^-$

- Consider the example $A \rightarrow B + C$



$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \,\delta^4(p_1 - p_2 - p_3)\, \frac{d^3 \vec{p}_2}{E_2} \frac{d^3 \vec{p}_3}{E_3}$$

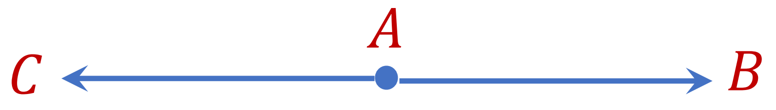
A: $p_1^\mu = (m_1, 0, 0, 0)$

B: $p_2^\mu = (E_2, p_2, 0, 0)$

C: $p_3^\mu = (E_3\ p_3, 0, 0)$

- Kinematics for the two-particle case……

Two-particle decay … calculate…



A: $p_1^\mu = (m_1, 0, 0, 0)$

B: $p_2^\mu = (E_2, p_2, 0, 0)$

C: $p_3^\mu = (E_3\ p_3, 0, 0)$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2\, \delta^4(p_1 - p_2 - p_3)\, \frac{d^3\vec{p}_2}{E_2}\, \frac{d^3\vec{p}_3}{E_3}$$

$$\delta^4(p_1 - p_2 - p_3) = \delta(p_1^0 - p_2^0 - p_3^0)\delta^3(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)$$

Now: $p_1^0 = m_1$ and $\vec{p}_1 = 0$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2\, \frac{\delta\left(m_1 - \sqrt{\vec{p}_2^{\,2} + m_2^2} - \sqrt{\vec{p}_3^{\,2} + m_3^2}\right)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_3^{\,2} + m_3^2}}\, \delta^3(\vec{p}_2 + \vec{p}_3) d^3\vec{p}_2 d^3\vec{p}_3$$

Next: use $\vec{p}_3 = -\vec{p}_2$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2\, \frac{\delta\left(m_1 - \sqrt{\vec{p}_2^{\,2} + m_2^2} - \sqrt{\vec{p}_2^{\,2} + m_3^2}\right)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_2^{\,2} + m_3^2}}\, d^3\vec{p}_2$$

Next, go to spherical coordinates:
$\vec{p}_2 \to (p, \theta, \phi)$

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2\, \frac{\delta\left(m_1 - \sqrt{p^2 + m_2^2} - \sqrt{p^2 + m_3^2}\right)}{\sqrt{p^2 + m_2^2}\sqrt{p^2 + m_3^2}}\, p^2 dp \underbrace{\int \sin\theta\, d\theta\, d\phi}_{4\pi}$$

Two-particle decay … calculate…

The integral over dp is not easy to calculate. Make the substitution: $u \equiv \sqrt{p^2+m_2^2} + \sqrt{p^2+m_3^2}$

Then: $\frac{du}{dp} = \frac{up}{\sqrt{p^2+m_2^2}\,\sqrt{p^2+m_3^2}}$; $pdp = \frac{du}{u}\sqrt{p^2+m_2^2}\sqrt{p^2+m_3^2}$

Such that we recognize:

$$\Gamma = \frac{S}{8\pi m_1}\int_{(m_2+m_3)}^{\infty} |\mathcal{M}|^2\,\delta(m_1-u)\,\frac{p}{u}\,du$$

which only has a contribution for $u = m_1$ (δ-function):

Note: $m_1 > m_2 + m_3$

Inverting the equation for u and p and putting $u = m_1$ gives (*exercise*):

$$p = |\vec{p}| = \frac{1}{2m_1}\sqrt{m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2}$$

and putting $u = m_1$ gives finally : $\Gamma = \frac{S|\vec{p}|}{8\pi m_1^2}|\mathcal{M}|^2$

Note that the δ-functions were enough to do all the integrals and put the required kinematic value for $\vec{p}$

Exercise – 28: Kinematics relation

- Show explicitly that by inverting the equation:

$$m_1 = \sqrt{p^2 + m_2^2} + \sqrt{p^2 + m_3^2}$$

it follows that:

$$p = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

Cross Section

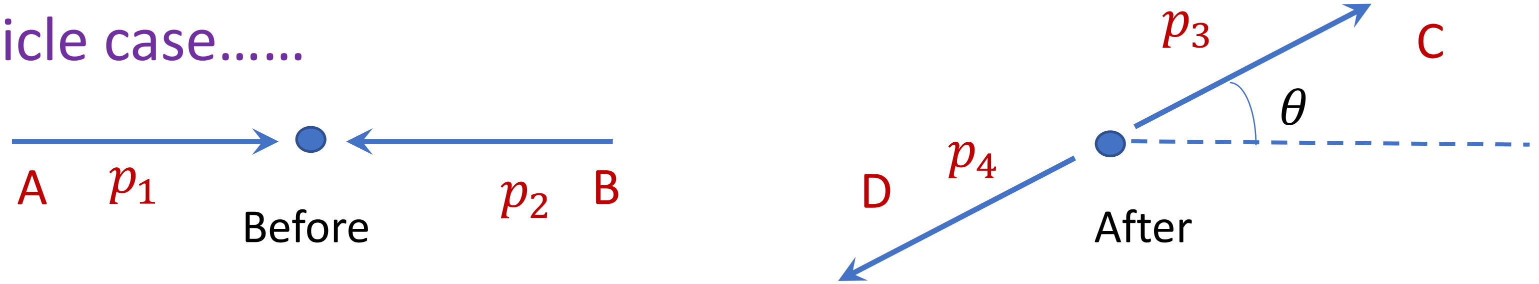
- Golden rule for cross section:

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2\right)\theta\left(p_j^0\right) \frac{d^4 p_j}{(2\pi)^4}$$

- Can be shortened, just as for decay, by doing the integrals over p^0 requiring on-mass relation: $p_j^0 = \sqrt{p_j^2 + m_j^2} = E_j$ to find:

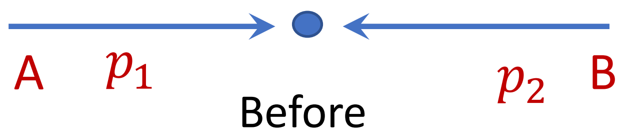
$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \times \prod_{j=3}^{n} \frac{1}{2E_j} \frac{d^3 \vec{p}_j}{(2\pi)^3}$$

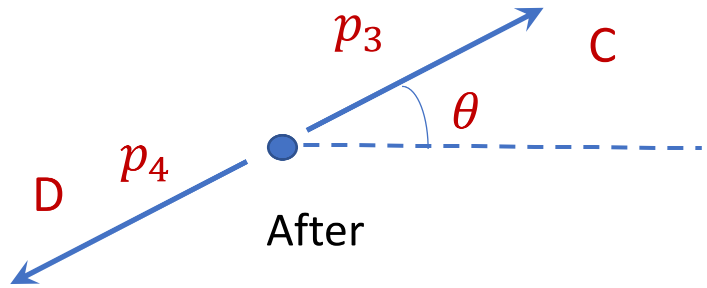
- Consider the ("2-to-2") example $A + B \rightarrow C + D$
  - Kinematics for the two-particle case……



Two-to-Two cross section …

- $A + B \rightarrow C + D$





Kinematics for the two-particle case in Center-of-Mass: $\vec{p}_2 = -\vec{p}_1$

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times \prod_{j=3}^{n} \frac{1}{2E_j} \frac{d^3\vec{p}_j}{(2\pi)^3}$$

Use: $p_1^\mu = (E_1, |\vec{p}_1|, 0,0)$ and $p_2^\mu = (E_2, -|\vec{p}_1|, 0,0)$ to see that, after kinematic calculation's – see Griffiths…

$$\sigma = \frac{S}{64\pi^2(E_1 + E_2)|\vec{p}_1|} \int |\mathcal{M}|^2 \, (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \, \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4}$$

Again, split up the $\delta^4(p_1 + p_2 - p_3 - p_4)$ into $\delta(E_1 + E_2 - E_3 - E_4) \times \delta^3(\vec{p}_3 + \vec{p}_4)$ …etc… similar as decay.

Complication: there is an angle θ in the game and we cannot carry out the integral , since $\mathcal{M}$ can depend on it. (Q: Why was there no θ in the case of decay?).

Determine the angle dependent cross section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

Part 3

Feynman Calculus

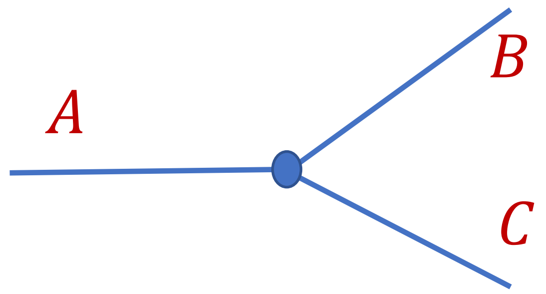
Griffiths §6.3

Or: how to find the matrix element $\mathcal{M}$

Feynman Rules: ABC Toy theory

- All the "real" particle physics is in the calculation of the matrix $\mathcal{M}$.

- A full derivation of QED is not in the scope of the lectures. We give a "recipe".



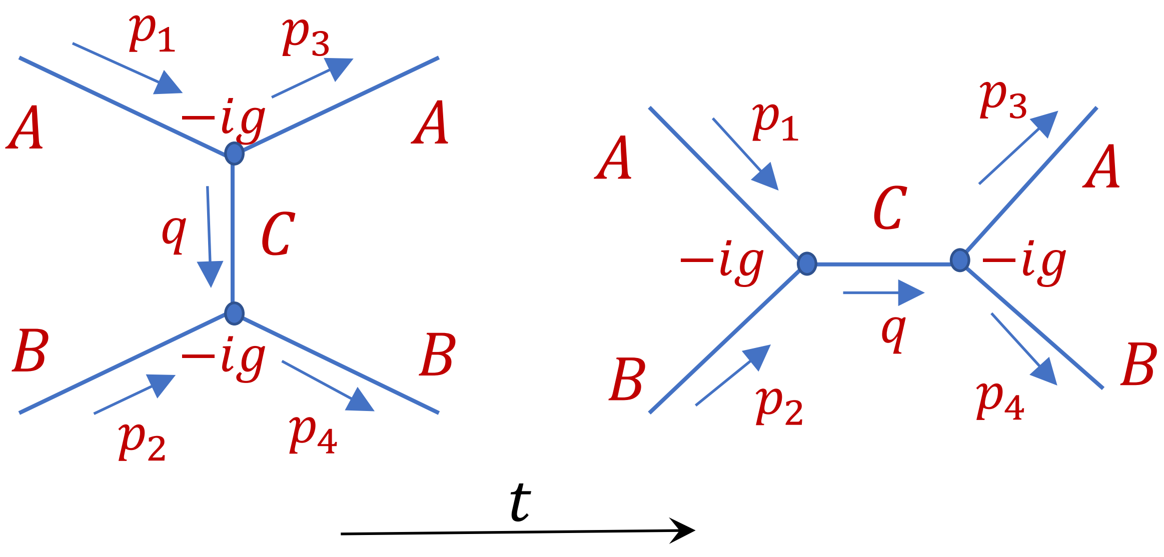
Only one fundamental vertex

- Consider ABC example theory
  - ABC model is simplest possible "theory".
  - Particles have no spin: no "arrows" needed
    - Think of π^0, K^0, η particles etc
  - No real forces, just "particles"

- For the following recipe keep perturbation theory and the golden rule in mind.

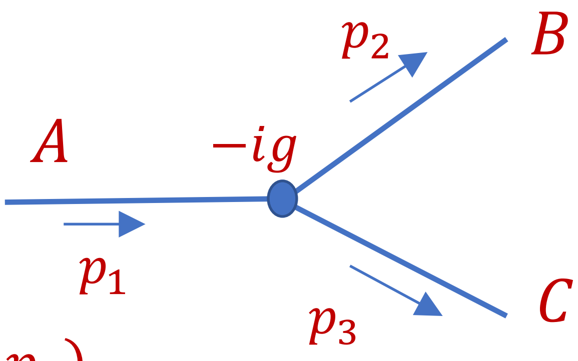
Feynman rules: ABC Toy Theory

- Recipe to find $\mathcal{M}$:
  1. Draw all the possible diagrams
  2. Label the external 4-momenta p_i and put an arrow for the direction forward in time
  3. For each vertex write a factor $-ig$
  4. Propagators: for each internal line write: $\frac{i}{q_j^2 - m_j^2}$
     - Note that for an internal line: $q_j^2 \neq m_j^2$



5. Conservation of energy and momentum:
   - For each vertex write a δ-function of the form: $(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)$, with a positive sign for momenta going into the vertex. This δ makes sure that no momentum is "disappearing into a vertex"
6. Integrate over all internal momenta. For each internal line write a factor $\frac{1}{(2\pi)^4} d^4 q_i$
7. Delta function: Result will include a delta function: $(2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n)$ reflecting overall energy and momentum conservation. Erase this factor and multiply by i

Decay: Lifetime of $A \rightarrow B + C$



Feynman rules:

1. Diagrams: see sketch
2. Labels: see sketch
3. One vertex: $-ig$
4. Propagators: no internal lines
5. Conservation of energy and momentum: $(2\pi)^4\,\delta^4(p_1 - p_2 - p_3)$
6. Integrate: no internal momenta
7. Discard delta-function and multiply by i.

Result for the amplitude: $\mathcal{M} = g$

We obtain: $\Gamma = \frac{S|\vec{p}|}{8\pi m_A^2}\,|\mathcal{M}|^2 = \frac{g^2|\vec{p}|}{8\pi m_A^2}$ (no identical particles: $S = 1$)

where $|\vec{p}| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2m_B^2 - 2m_A^2m_C^2 - 2m_B^2m_C^2}$

So that the lifetime is: $\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2|\vec{p}|}$

Exercise – 29: Pion Decay

Griffiths exercise 6.6

Calculate the lifetime of the neutral pion π^0

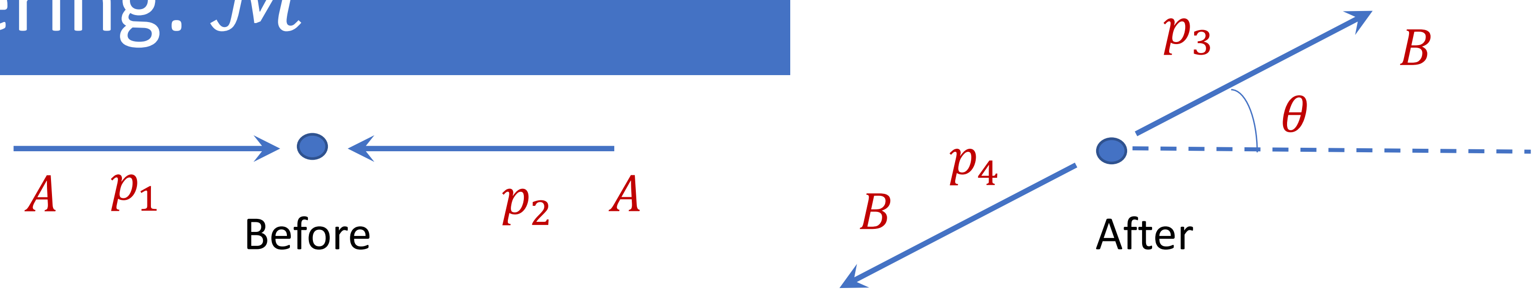
The neutral pion decays mainly via: $\pi^0 \rightarrow \gamma\gamma$. Assume that the amplitude has dimensions $[\text{mass}] \times [\text{velocity}]$.

a) Motivate the reason that the amplitude should be proportional to the coupling constant α. Try to sketch a diagram of the decay.

b) Use Fermi's golden rule for two-body decays to estimate the lifetime of the pion.

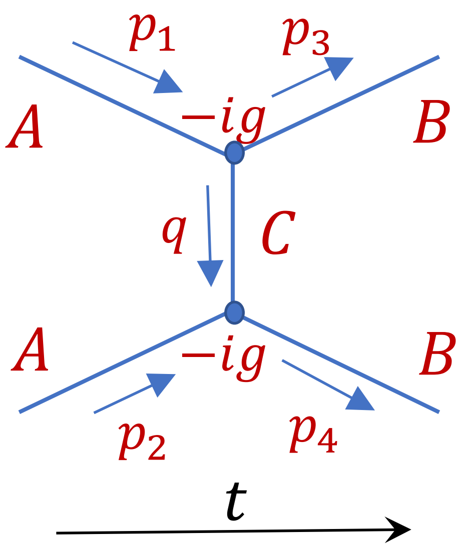
c) Compare it with the experimental value. What do you think?

$A + A \rightarrow B + B$ Scattering: $\mathcal{M}$



Feynman rules:

1. Diagram: see sketch
2. Labels: see sketch
3. Two vertices: $(-ig)^2 = -g^2$
4. Propagators: one internal line: $\frac{i}{q^2 - m_C^2}$
5. Conservation of energy and momentum twice:
$(2\pi)^4\ \delta^4(p_1 - p_3 - q)$ and $(2\pi)^4\ \delta^4(p_2 + q - p_4)$
6. Integrate: one integral: $\frac{1}{(2\pi)^4} d^4q$



Result so far:

$$-(2\pi)^4 g^2 \int \frac{i}{q^2 - m_C^2} \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4) d^4q$$

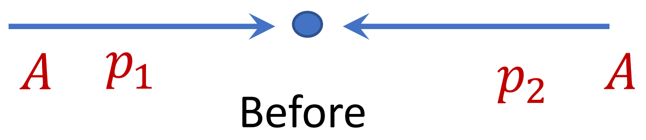
Doing integral over 2nd δ^4 sets $q = p_4 - p_2$ into first δ^4 to find

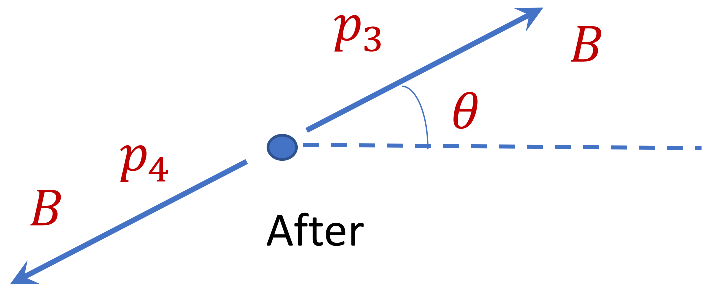
$$-g^2 \frac{i}{q^2 - m_C^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

7. Erase delta-function and multiply by i to find:

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

$A + A \to B + B$ Scattering: $\mathcal{M}$

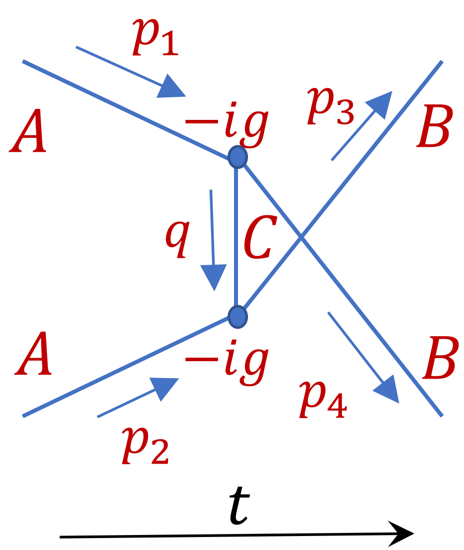




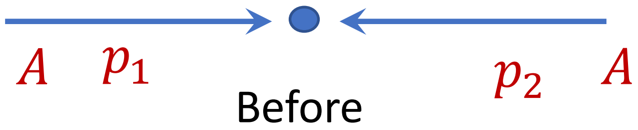
- There is a second diagram: see sketch
- Repeat the computation?
  - No, just replace: $p_3 \leftrightarrow p_4$ and fill in the end result:

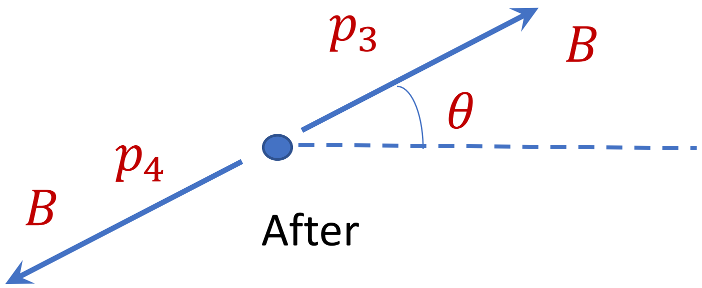
$$\mathcal{M} = \frac{g^2}{(p_3 - p_2)^2 - m_C^2} + \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

  - Note: $\mathcal{M}$ does *not* depend on Lorentz frame: it is a Lorentz invariant (scalar) quantity.



$A + A \to B + B$ Scattering: $d\sigma/d\Omega$





- Look at the matrix element and assume that $m_A = m_B = m$ and $m_C = 0$ (eg. a photon): $\mathcal{M} = \dfrac{g^2}{(p_3 - p_2)^2} + \dfrac{g^2}{(p_4 - p_2)^2}$

$$
\begin{aligned}
(p_4 - p_2)^2 - m_C^2 &= p_4^2 + p_2^2 - 2p_2 \cdot p_4 \\
&= m_4^2 + m_2^2 - 2p_2 \cdot p_4 \\
&= 2m^2 - 2E_2E_4 + 2(\vec{p}_2 \cdot \vec{p}_4) \\
&= 2m^2 - 2\left(\sqrt{m^2 + \vec{p}^2}\right)\left(\sqrt{m^2 + \vec{p}^2}\right) + 2\vec{p}^2 \cos\theta \\
&= -2\vec{p}^2(1 - \cos\theta)
\end{aligned}
$$

$$(p_3 - p_2)^2 - m_C^2 = -2\vec{p}^2(1 + \cos\theta)$$

Note that for 4-vectors:
$p_i \cdot p_j = p_{i_\mu} p_j^\mu = E_iE_j - \vec{p}_i \cdot \vec{p}_j$
and that $p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$

$$\mathcal{M} = \frac{g^2}{-2\vec{p}^2(1 - \cos\theta)} + \frac{g^2}{-2\vec{p}^2(1 + \cos\theta)} = -\frac{g^2}{2\vec{p}^2 \sin^2\theta}$$

- Plug in: ($S = 1/2$)

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi E \vec{p}^2 \sin^2\theta}\right)^2$$

Exercise – 30: Two-to-Two Scattering

Griffiths exercise 6.15

Consider the process: $A + B \rightarrow A + B$ in the ABC theory

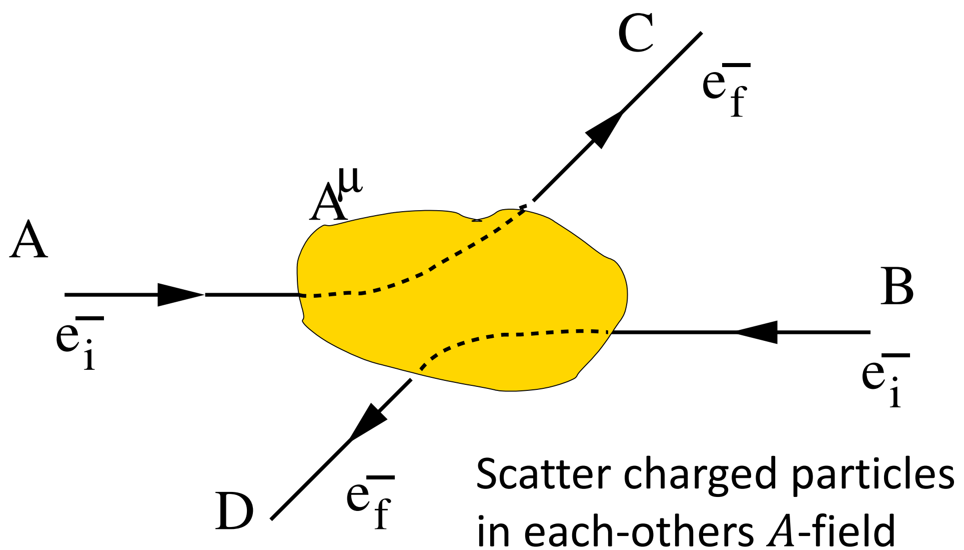
a) Draw the (two) lowest-order Feynman diagrams, and calculate the amplitudes
b) Find the differential cross-section in the CM frame, assuming $m_A = m_B = m\ , m_C = 0$, in terms of the (incoming) energy E and the scattering angle θ.
c) Assuming next that B is much heavier than A, calculate the differential cross-section in the lab frame.
d) For case c), find the total cross-section.

From ABC to QED

- ABC does not describe electrodynamics in the real world
  - We have charged particles and photons
  - Fermions have spin=½, forces have spin=1
    - Spin is a complication, we will leave that for a course at master level

- How to get the matrix element and Feynman rules for QED scattering?
  - This section will be quite “dense” but try to get the gist of it…

A taste of Relativistic QED Scattering



- Dirac equation in QED: $(\gamma_\mu p^\mu - m)\psi + e\gamma_\mu A^\mu = 0$

  $\partial^\mu \rightarrow \partial^\mu - ieA^\mu$ $\qquad$ $p^\mu \rightarrow p^\mu + eA^\mu$

- Perturbation theory: $(H_0 + V)\psi = E\psi$

  $H_0 = \vec{\alpha}\cdot\vec{p} + \beta m = \gamma^0\gamma^k p^k + \gamma^0 m$

  ➔ $V = -e\gamma^0\gamma_\mu A^\mu$

- Transition amplitude: no spin (see before): $T_{fi} = -i\int d^4x\, \psi_f^*(x)V(x)\psi_i(x)$

  spin ½ (Dirac): $T_{fi} = -i\int d^4x\, \psi_f^\dagger(x)V(x)\psi_i(x)$

  $$= -i\int d^4x\, \bar{\psi}_f(x)(-e)\gamma_\mu A^\mu(x)\psi_i(x) = -i\int j_\mu^{fi} A^\mu d^4x$$

  $$= -2\pi\delta^4(p_A + p_B - p_C - p_D)\,\mathcal{M}$$

  j_μ^{fi} = "transition current"

- To determine $\mathcal{M}$ insert the electromagnetic field that one particle A observes from the other particle B and vice versa. Remember also lecture 3:

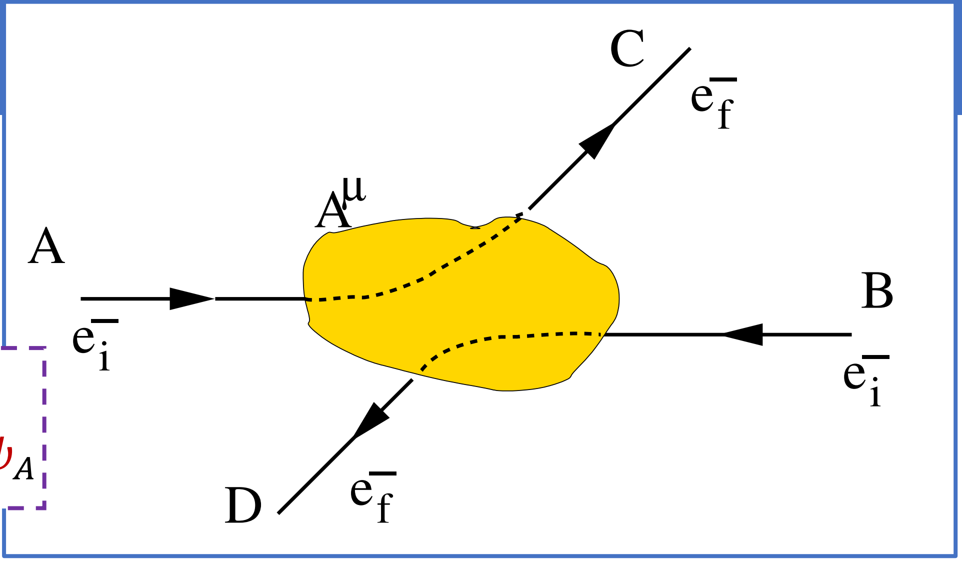
  $\mathcal{L}_{QED} = \mathcal{L}_{\text{free}} - \mathcal{L}_{\text{int}} = \mathcal{L}_{\text{Dirac}} - q\bar{\psi}\gamma_\mu A^\mu\psi \quad\rightarrow\quad \mathcal{L}_{\text{int}} = -J_\mu A^\mu$ with $J_\mu = q\bar{\psi}\gamma_\mu\psi$

A sketch of QED scattering

- Particle BD scatters in the field A^μ of particle AC
- The field A^μ is obtained from Maxwell: $\partial_\nu \partial^\nu A^\mu = j^\mu_{AC}$
- Solution: $A^\mu = -\frac{1}{q^2} j^\mu_{AC}$

Remember: $j^\mu = -e\bar{\psi}\gamma^\mu\psi$

Transition current: $j^\mu_{AC} = -e\bar{\psi}_C\gamma^\mu\psi_A$



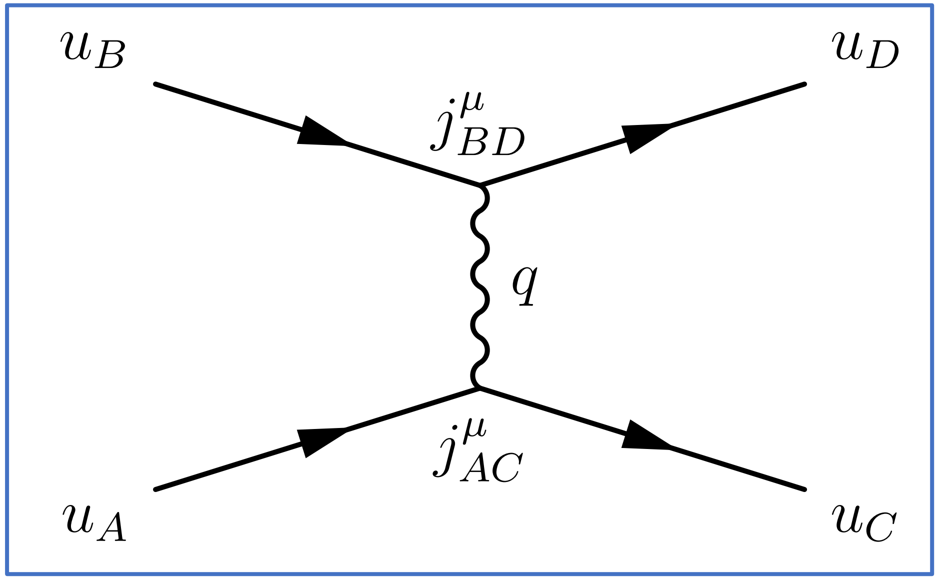
- Transition amplitude becomes:

$$T_{fi} = -i\int j^\mu_{fi} A_\mu \, d^4x = -i\int j^{(BD)}_\mu \frac{-1}{q^2} j^\mu_{(AC)} = -i\int j^{(BD)}_\mu \frac{-1}{q^2} j^\mu_{(AC)}$$

- Inserting plane wave solutions: $\psi(x) = u(p)e^{-ipx}$

  into the current gives: $j^\mu_{(AC)} = -e\bar{u}_C\gamma^\mu u_A \, e^{i(p_C - p_A)x}$

  and: $j^\mu_{(BD)} = -e\bar{u}_D\gamma^\mu u_B \, e^{i(p_D - p_B)x}$



- Hence: $T_{fi} = -i(2\pi)^4\delta^4(p_D + p_C - p_B - p_A)\,\mathcal{M}$

  with: $-i\mathcal{M} = \underbrace{ie\,(\bar{u}_C\gamma^\mu u_A)}_{\text{vertex}} \; \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{\text{propagator}} \; \underbrace{ie(\bar{u}_D\gamma^\nu u_B)}_{\text{vertex}}$

"Matrix element"

A taste of Relativistic QED Scattering

- Note that the current is of the form:

$$j_{fi}^{\mu} = \left(\quad \bar{\psi}_f \quad \right) \begin{pmatrix} & \cdots & \\ \vdots & \gamma^{\mu} & \vdots \\ & \cdots & \end{pmatrix} \begin{pmatrix} \psi_i \end{pmatrix}$$

It is a 4-*vector* in Lorentz space (t, x, y, z) ; $\mu = 0,1,2,3$
It is a *scalar* in Dirac space (1, 2, 3, 4)

- Finally:
  - Feynman rules for QED are given in Griffiths section 7.5
  - To calculate cross sections with spin-½ particles is mathematically involved; it requires taking the square of the matrix element and summation over spin states of spinor objects.
  - We leave this fun for a topic of a master level course
  - Section 7.6, 7.7 and 7.8 of Griffiths give you an idea
    - It goes beyond the scope of this course

- Next week: “Detectors”, measuring the particle processes