PHY3004: Nuclear and Particle Physics

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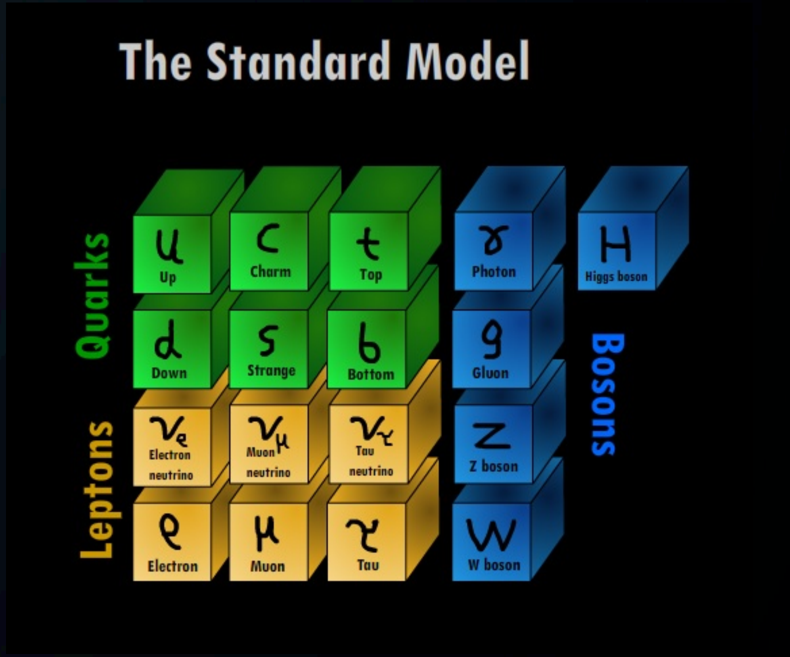
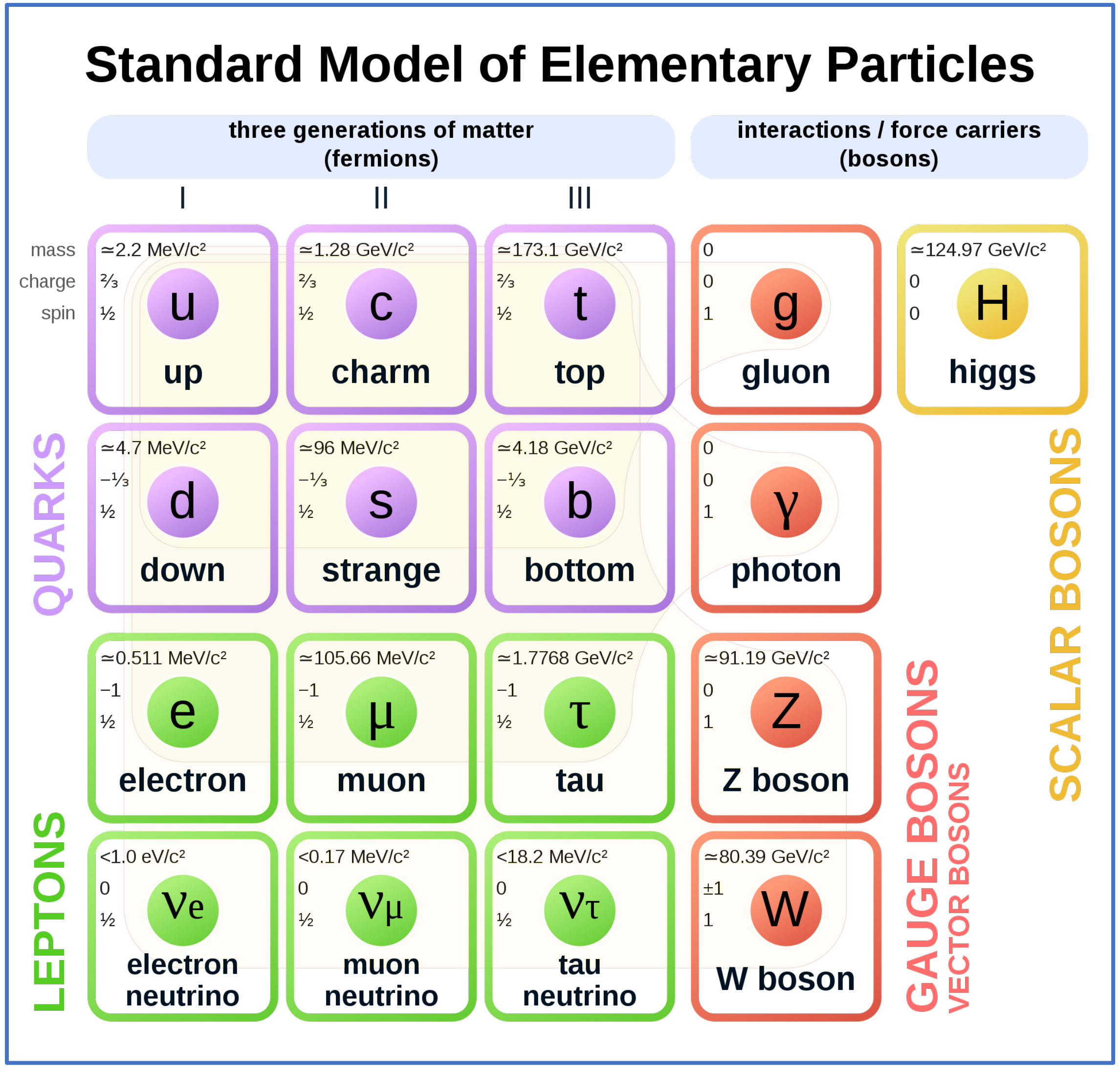


Standard Model: particles and forces

Classification of particles

- Lepton: fundamental particle
- Hadron: consist of quarks
  - Meson: 1 quark + 1 antiquark (π^+,B_s^0, …)
  - Baryon: 3 quarks (p ,n , Λ, …)
    - Anti-baryon: 3 anti-quarks
- ***Fermion***: particle with half-integer spin.
  - Antisymmetric wave function: obeys Pauli-exclusion principle and Pauli-Dirac statistics
  - All fundamental quarks and leptons are spin-½
  - Baryons (S=$^1/_2$, $^3/_2$)
- ***Boson***: particle with integer spin
  - Symmetric wave function: Bose-Einstein statistics
  - Mesons: (S=0, 1), Higgs (S=0)
  - Force carriers: γ, W, Z, g (S=1); graviton(S=2)



Wave Equations

Contents:

Griffiths chapter 7 and PP1 chapter 1

1. Wave equations
   a) Wave equations for spin-0 fields
      - Schrödinger (non relativistic), Klein-Gordon (relativistic)
   b) Wave equation for spin-½ fields
      - Dirac equation (relativistic)
      - Fundamental fermions
   c) Wave equations for spin-1 fields
      - Gauge boson fields; eg. electromagnetic field

Griffiths chapter 10 and PP1 chapter 1

2. Gauge field theory
   a) Variational Calculus and Lagrangians
   b) Local Gauge invariance
      i. QED
      ii. Yang-Mills Theory (Weak, Strong)

- Required Quantum Mechanics knowledge:
  - Angular momentum and spin: study Griffiths sections 4.2 ,4.3, In particular Pauli Matrices

Griffiths §7.1 – §7.3

Part 1
Wave Equations and Probability

1a) Spin-0

Schrödinger Equation and probability

- Quantization of classical non-relativistic theory:
  - Take $E = \frac{\vec{p}^2}{2m}$ and substitute energy and momentum by operators that operate on ψ:

$$E \to \hat{E} = i\hbar\frac{\partial}{\partial t} \qquad ; \qquad p \to \hat{p} = -i\hbar\vec{\nabla}$$

  - Result is Schrödinger's equation: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$
  - Plane wave solutions: $\psi = Ne^{i(\vec{p}\vec{x}-Et)/\hbar}$ with the kinematic relation $E = p^2/2m$

- Multiply both sides Schrödinger by ψ^* and add its complex conjugate

$$\psi^*\frac{\partial}{\partial t}\psi = \psi^*\left(\frac{i\hbar}{2m}\right)\nabla^2\psi$$

$$\psi\frac{\partial}{\partial t}\psi^* = \psi\left(\frac{-i\hbar}{2m}\right)\nabla^2\psi^*$$

+

$$\frac{\partial}{\partial t}\underbrace{(\psi^*\psi)}_{\rho} = -\vec{\nabla}\cdot\underbrace{\left[\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi\vec{\nabla}\psi\right)\right]}_{\vec{j}}$$

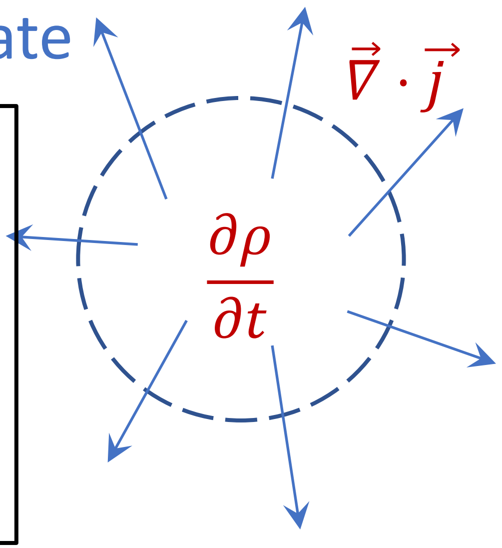
Recognize "continuity" equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

Law of conserved currents, with:

$$\rho \equiv \psi^*\psi = |N|^2$$

$$\vec{j} \equiv \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right) = \frac{|N|^2}{m}\vec{p}$$



*Use: $\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \psi^*\nabla^2\psi - \psi\nabla^2\psi^*$

- Interpret: probability waves!

Relativistic: Klein-Gordon equation

- Quantization of relativistic theory
  - Start with $E^2 = p^2c^2 + m^2c^4$ and substitute again $E \rightarrow i\hbar\frac{\partial}{\partial t}$ and $p \rightarrow -i\hbar\vec{\nabla}$
  - Result is Klein-Gordon equation: $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi$

Use now: $\hbar = c = 1$

  - Plane wave solutions: $\psi = Ne^{i(\vec{p}\vec{x}-Et)/\hbar}$ with relativistic relation $E^2 = \vec{p}^2 + m^2$
- Use the covariant notation:

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \quad ; \quad \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$\partial_\mu\partial^\mu \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ (usually take $c = \hbar = 1$)

$p^0 = E$ and $x^0 = t$

$$p_\mu p^\mu = m^2$$

- Klein-Gordon in four-vector notation: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$
- Plane wave solutions: $\psi = Ne^{-i(p_\mu x^\mu)}$
- Time and space coordinates are now treated fully symmetric
  - This is needed in a relativistic theory where time and space for different observes are linear combinations of each other

Klein-Gordon conserved currents

- Similar to the Schrödinger case multiply both sides by $-i\phi^*$ from left and add the expression to its complex conjugate

$$-i\phi^*\left(-\frac{\partial^2\phi}{\partial t^2}\right) = -i\phi^*(-\nabla^2\phi + m^2\phi)$$

$$i\phi^*\left(-\frac{\partial^2\phi^*}{\partial t^2}\right) = i\phi(-\nabla^2\phi^* + m^2\phi^*)$$

$+$

$$\frac{\partial}{\partial t}\underbrace{i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)}_{\rho} = \vec{\nabla}\cdot\underbrace{\left[i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\right]}_{-\vec{j}}$$

Again recognize "continuity" equation, the law of conserved currents:

$$\partial_\mu j^\mu = 0$$

With now:

$$j^\mu = (\rho, \vec{j}) = i[\phi^*(\partial^\mu\phi) - \phi(\partial^\mu\phi^*)]$$

It gives for plane waves:

$$\rho = 2|N|^2E$$
$$\vec{j} = 2|N|^2\vec{p}$$

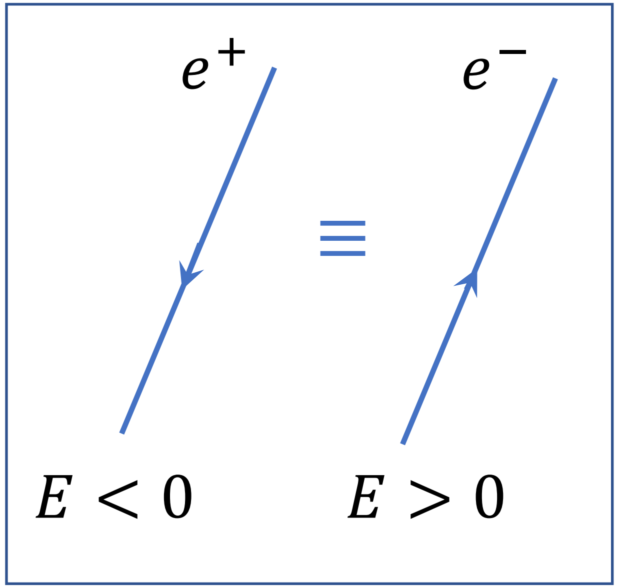
Or in 4-vector: $j^\mu = 2|N|^2p^\mu$

- The quadratic equation leads to *double solutions:* $E^2 = \cdots \Rightarrow E = \pm\cdots$
  - Positive and negative energy solutions
  - Negative solutions imply negative probability density ρ
  - This bothered Dirac and therefore he looked for an equation linear in E and p …

Antiparticles

- Feynman-Stückelberg interpretation
  - Charge current of an electron with momentum $\vec{p}$ and energy E
    $j^\mu(-e) = -2e|N|^2 p^\mu = -2e|N|^2(E, \vec{p})$
  - Charge current of a positron
    $j^\mu(+e) = +2e|N|^2 p^\mu = -2e|N|^2(-E, -\vec{p})$

  The positron current with energy $-E$ and momentum $-\vec{p}$ is the same as the electron current with E and $\vec{p}$



- The negative energy *particle* solutions going backward in time describe the positive-energy *antiparticle* solutions.
  - The wave function $\phi = Ne^{-ix_\mu p^\mu}$ stays invariant for negative energy and going backwards in time
  - Consider eg. $e^{-i(-E)(-t)} = e^{-iEt}$

- A positron ***is*** an electron travelling backwards in time

Part 1
Wave Equations and Probability

1b) Spin-½

Dirac Equation

- Dirac did not like negative probabilities and looked for a wave equation of the form $E = i\frac{\partial}{\partial t}\psi = H\psi = (?\,)$, but relativistically correct.
- Try: $H = (\vec{\alpha}\cdot\vec{p} + \beta m)$ where $\vec{\alpha}\cdot\vec{p} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z$; $\vec{\alpha}?\ \ \beta?$
- We know that: $H^2\psi = E^2\psi = (\vec{p}^2 + m^2)\psi$
- Write it out: $H^2 = (\sum_i \alpha_i p_i + \beta m)(\sum_j \alpha_j p_j + \beta m)$

$$= \left(\sum_{i,j} \alpha_i\alpha_j p_i p_j + \sum_i \alpha_i \beta p_i m + \sum_i \beta\alpha_i p_i m + \beta^2 m^2\right)$$

$$= \left(\sum_i \alpha_i^2 p_i^2 + \underbrace{\sum_{i>j}(\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j + \sum_i(\alpha_i\beta + \beta\alpha_i)p_i m}_{=0} + \beta^2 m^2\right)$$

- This works out if:
  - $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$
  - $\alpha_i, \alpha_2, \alpha_3, \beta$ anti-commute: ie.: $\alpha_1\alpha_2 = -\alpha_2\alpha_1$ etc
- Anti-commutator: $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$; $\{\alpha_i, \beta\} = 0$; $\beta^2 = 1$
  - Using definition: $\{A, B\} = AB + BA$:

Dirac's idea

- Clearly α_i and β cannot be numbers. Let them be *matrices*!
  - In that case they operate on a wave function that is a column vector
  - The simplest case that allows the requirements are 4x4 matrices.
  - Dirac's equation becomes:

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix} = \left[-i\underbrace{\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix}}_{\vec{\alpha_i}}\cdot\vec{\nabla_i} + \underbrace{\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix}}_{\beta}\cdot m\right]\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}$$

  - It is possible making use of the Pauli spin matrices
    - $\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}$ and $\beta = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$ with $\sigma_1 = \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $\sigma_2 = \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$; $\sigma_3 = \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
    - Note that α and β are hermitian: $\alpha_i^\dagger = \alpha_i$ and $\beta^\dagger = \beta$ (Since Hamiltonian has real E eigenvalues.)

  - This is a very complicated equation!
    - What does it mean that the wave function ψ is now a 1-by-4 column vector?
    - ψ is ***not*** a 4-vector, since the indices do not represent kinematic variables, but matrices indices!

Covariant form of Dirac's equation

- Dirac equation: $i\frac{\partial}{\partial t}\psi = \left(-i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$

- Multiply Dirac's eq. from the left by β; then it becomes:
  - $\left(i\beta\frac{\partial}{\partial t}\psi + i\,\beta\vec{\alpha}\cdot\vec{\nabla} - m\right)\psi = 0$

- Introduce now the Dirac γ-matrices: $\gamma^{\mu} \equiv (\beta, \beta\vec{\alpha})$ (vector of 4 matrices!)
  - Covariant form of Dirac eq:

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0}$$

- Realise that Dirac's equation is a set of 4 coupled differential equations.

Dirac Gamma Matrices

- There is some freedom to implement: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ in 4x4 matrices.
  - We will use the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Or: $\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ and $\gamma^k = \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$

Note the indices:
(confusing!)

$\mu, \nu = 0{,}1{,}2{,}3$ are the
Lorentz indices in space-time:

Dirac matrix indices: 1,2,3,4
Have to do with the row and column indices of the matrix (and spinors)

- Note: although the gamma matrices indices are Lorentz-indices ("space-time", the gamma-matrices are not 4-vectors!

Exercise – 13: Dirac Algebra

- Dirac algebra:
  - Write the explicit form of the γ-matrices
  - Show that : $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$
  - Show that : $(\gamma^0)^2 = \mathbb{1}_4$; $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{1}_4$
  - Use anti-commutation rules of α and β to show that: $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$
  - Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and show: $\gamma^{5\dagger} = \gamma^5$; $(\gamma^5)^2 = \mathbb{1}_4$; $\{\gamma^5, \gamma^\mu\} = 0$

Exercise – 14: Solutions of free Dirac equation

See Griffith for a derivation of the solutions

a) Show that the following plane waves are solutions to Dirac's equation

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \; ; \; \psi_2 = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

$$\psi_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x + ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \; ; \; \psi_4 = \begin{pmatrix} (p_x - ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

b) Write the Dirac equation for particle in rest (choose $\vec{p} = 0$) and show that ψ_1 and ψ_2 are *positive energy* solutions: $E = +\left|\sqrt{p^2+m^2}\right|$ whereas ψ_3 and ψ_4 are *negative energy* solutions: $E = -\left|\sqrt{p^2+m^2}\right|$.

c) Consider the *helicity* operator $\vec{\sigma}\cdot\vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$ and show that ψ_1 corresponds to *positive helicity* solution and ψ_2 to *negative helicity.* Similarly for ψ_3 and ψ_4.

Spin and Helicity – hint for exercise 14c)

- For a given momentum p there still is a *two-fold degeneracy*: what differentiates solutions ψ_1 from ψ_2?
- Define the spin operator for Dirac spinors: $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, where $\vec{\sigma}$ are the three 2x2 Pauli spin matrices
- Define ***helicity*** λ as spin "up"/"down" wrt direction of motion of the particle

$$\lambda = \frac{1}{2}\vec{\Sigma}\cdot\hat{p} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} = \frac{1}{2|p|}(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)$$

- Split off the Energy and momentum part of Dirac's equation: $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- Exercise: Try solutions ψ_1 and ψ_2 to see they are ***helicity eigenstates*** with $\lambda = +1/2$ and $\lambda = -1/2$
- Dirac wanted to solve negative energies and he found spin-½ fermions!

Antiparticles

- Dirac spinor solutions $\psi_i(x^\mu) = \psi_i(t,\vec{x}) = u_i(E,\vec{p})e^{i(\vec{p}\vec{x}-Et)} = u_i(p^\mu)e^{-ip_\mu x^\mu}$ with $i = 1,2,3,4$

- Since we work with antiparticles, instead of *negative energy particles* travelling backwards instead in time, *antiparticle solutions* are defined

$$u_3(-E,-\vec{p})e^{i((-\vec{p})\vec{x}-(-E)t)} = v_2(E,\vec{p})e^{-i(\vec{p}\vec{x}-Et)} = v_2(p^\mu)e^{ip_\mu x^\mu}$$

$$u_4(-E,-\vec{p})e^{i((-\vec{p})\vec{x}-(-E)t)} = v_1(E,\vec{p})e^{-i(\vec{p}\vec{x}-Et)} = v_1(p^\mu)e^{ip_\mu x^\mu}$$

  - Where now the energy of the antiparticle solutions v_1 and v_2 is positive: $E > 0$

- Explicit: $v_1 = \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} p_z/(E+m) \\ (p_x + ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}$
  - Where E and $\vec{p}$ are now the energy and momentum of the antiparticle

Adjoint spinors

- Adjoint spinors
  - Solutions of the Dirac equation are called *spinors*
  - Current density and continuity equation require *adjoints* instead of *complex conjugates*

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i \sum_{k=1,2,3} \gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$$

$$-i \frac{\partial \psi^\dagger}{\partial t} \gamma^0 - i \sum_{k=1,2,3} \frac{\partial \psi^\dagger}{\partial x^k} (-\gamma^k) - m\psi^\dagger = 0$$

$$\boxed{\gamma^{0\dagger} = \gamma^0 \quad ; \quad \gamma^{k\dagger} = -\gamma^k}$$

- The minus sign in $(-\gamma^k)$ disturbs the Lorentz invariant form
- Restore by defining adjoint spinor:

$$\bar{\psi} = \psi^\dagger \gamma^0$$

- Dirac spinor: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$, adjoint Dirac spinor: $\bar{\psi} = (\overline{\psi_1}, \overline{\psi_2}, \overline{\psi_3}, \overline{\psi_4})$
- Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$; adjoint Dirac equation: $i\partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi} = 0$

Dirac Current density and conserved current

- Apply a similar trick as before:
  - Multiply adjoint Dirac eq from from right by ψ and multiply Dirac eq. from left by $\bar{\psi}$

$$\left(i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}\right)\quad\psi = 0$$

$$\bar{\psi}\quad\left(i\partial_\mu\psi\gamma^\mu - m\psi\right)\qquad = 0$$

+

$$\bar{\psi}\left(\partial_\mu\gamma^\mu\psi\right) + \left(\partial_\mu\bar{\psi}\gamma^\mu\right)\psi = 0$$

→

Define the 4-vec current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

Satisfies the continuity equation:

$$\partial_\mu j^\mu = 0$$

- Probability: Zero-th component of the current:

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2$$

- This always gives a positive probability, which was the motivation of Dirac.

Dirac in summary

- Dirac was looking for an explanation for positive and negative energy solutions by linearising Klein-Gordon equation
  - He found that his solutions described spin-½ particles
  - He predicted, based on symmetry, that for each particle there should exist an antiparticle (the negative energy solution).

- We had relativistic fields:
  - Spin-0: Klein-Gordon: e.g. pion particles
  - Spin-1/2: Dirac : e.g. quarks and leptons
  - How about forces? Spin=1

Part 1
Wave Equations and Probability

1c) Spin-1

The Electromagnetic Field – including exercise 15

Griffiths §7.4

- Maxwell equations describe electric and magnetic fields induced by charges and currents: (used Heavyside-Lorentz units: $c = 1, \epsilon_0 = 1, \mu_0 = 1$)

1. Gauss' law: $\vec{\nabla} \cdot \vec{E} = \rho$
2. No magnetic charges: $\vec{\nabla} \cdot \vec{B} = 0$
3. Faraday's law of induction: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
4. Modified Ampère's law: $\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$

From 1. and 4. derive continuity

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

➔ charge conservation

This was the motivation for Maxwell to modify Ampère's law

- Define a Lorentz covariant 4-vector field $A^\mu = (V, \vec{A})$ as follows:

$\vec{B} = \vec{\nabla} \times \vec{A}$ (then automatically 2. follows)

$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$ with $V = A^0$ (then automatically 3. follows)

a) Show Maxwell equations can be summarized in covariant form:

$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ (Derive expressions for ρ and $\vec{j}$ and use: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$)

Gauge Invariance (including exercise 15)

b) Field A^μ is just introduced as a mathematical tool

- Choose any A^μ as long as $\vec{E}$ and $\vec{B}$ fields don't change

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda$$

$$V \rightarrow V' = V + \frac{\partial \lambda}{\partial t}$$

$$\vec{A} \rightarrow \overrightarrow{A'} = \vec{A} - \vec{\nabla} \lambda$$

- Exercise: show this explicitly!

c) Choose the Lorentz gauge condition: $\partial_\mu A^\mu = 0$

- Exercise: show that we can chose a gauge field such that this is possible

- Maxwell equation in Lorentz gauge becomes: $\partial_\mu \partial^\mu A^\nu = j^\nu$ also: $A^\nu = j^\nu$
  - Very similar to Klein-Gordon equation $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$
    - But now mass of the photon = 0.
    - Also now 4-equations → polarizations states of the photon field

- Photon field solutions: $A^\mu(x) = N \varepsilon^\mu(p) e^{-ip_\nu x^\nu}$
  - A gauge transformation implies: $\varepsilon^\mu \rightarrow \varepsilon'^\mu = \varepsilon^\mu + a p^\mu$
  - Different polarization vectors which differ by multiple of p^μ describe same photon

Exercise – 16 Antisymmetric tensor $F^{\mu\nu}$

- Maxwell's equation $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ can be further shortened by introducing the antisymmetric tensor: $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- Show that Maxwell's equations become: $\partial_\mu F^{\mu\nu} = j^\nu$

- Hint: derive the expressions for charge ($q = j^0$) and current ($\vec{I} = \vec{j}$) separately.

Use the identity: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$. Remember the definitions:

$A_\mu = (A_0, -\vec{A})$; $\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$; $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Part 2
Gauge Theory

2a) Variational Calculus and Lagrangians

Lagrange Formalism classical

Griffiths §10.1

- Classical Mechanics: The Lagrangian leads to equations of motion
  - $L(q_i, \dot{q}_i) = T - V$ where q_i and $\dot{q}_i$ are the generalized coordinates and velocities.
  - The path of a particle is found from Hamilton's principle of least action

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}) = 0 \qquad \delta S = 0$$

From this the Euler Lagrange equations follow and provide the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

See: https://en.wikipedia.org/wiki/Lagrangian_mechanics

- Example: Ball falls from height $y = h$: $q = y$, $\dot{q} = dy/dt = v_y$
  - $E_{pot} = T = mgq$
  - $E_{kin} = \frac{1}{2} m\dot{q}^2$
- Euler Lagrange: $dL/dq = mg$; $dL/d\dot{q} = m\dot{q}$
  - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ gives $m\ddot{q} = mg$ → $\dot{q} = gt + v_0$ → $q = y = \frac{1}{2} gt^2 + v_0 t + y_0$

Exercise – 17 : Lagrange Formalism classical

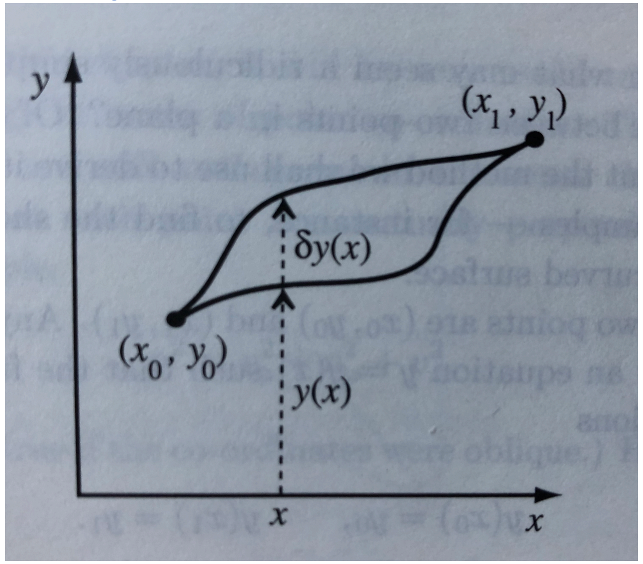
- Example of variational calculus and least action principle: what is the shortest path between two points in space?
  - Distance of two close points:

$$y = \sqrt{dx^2 + dy^2} = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 - y'^2}dx \quad \text{with } y' = dy/dx$$

  - Total length from (x_0, y_0) to (x_1, y_1) :

$$l = \int_{x_0}^{x_1} dl = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx = \int_{x_0}^{x_1} f(y, y')dx$$

  - Task is to find a function $y(x)$ for which l is minimal



- In general assume the path length is given by: $I = \int_{x_0}^{x_1} f(y, y')dx$
- Variational principle: shortest path is stationary: $\delta I = 0$

a) Write $\delta f(y, y') = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$ where $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

Show using partial integration that $\delta I = 0$ leads to the Hamilton Lagrange equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

b) Here for the shortest path we have $f(y') = l = \sqrt{1 + y'^2}$.

Then $\partial f/\partial y = 0$ and $\partial f/\partial y' = y'/\sqrt{1 + y'^2}$

Show that the variational principle leads to a straight line path: $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$ or that y' is a constant:

$dy/dx = a$; $y = ax + b$

Lagrange Formalism in field theory

Griffiths §10.2

- Relativistic Field theory: fields replace the generalized coordinates
  - Also time and space will be treated symmetric
  - Replace $L(q, \dot{q})$ by a Lagrange density $\mathcal{L}\big(\phi(x), \partial\phi(x)\big)$ in terms of fields and gradients such that $L \equiv \int d^3x \mathcal{L}(\phi, \partial\phi)$
- Principle of least actions becomes:

$$S = \int_{t_1}^{t_2} d^4x \, \mathcal{L}\Big(\phi(x), \partial\phi(x)\Big) \qquad \text{and again} \qquad \delta S = 0$$

t_1, t_2 are endpoints of the path

- Euler Lagrange Equations of motion becomes:

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} = \partial_\mu \frac{\partial \mathcal{L}}{\partial\Big(\partial_\mu \phi(x)\Big)}$$

Exercise – 18: Lagrangians and wave equations

Griffiths §10.2

- Scalar Field ("pion")
  a) Show that the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2$ results in the Klein-Gordon equation

- Dirac Field (Fermion)
  b) Show that the Euler-Lagrange equations for $\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$ results in the Dirac equation

- Electromagnetic field (photon)
  c) Show that $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$ results in Maxwell's equations

The Gauge Principle: Interactions

Griffiths §10.3

- *global* gauge invariance: the phase of the wave function is not observable: Changing the wave function $\psi(x) \rightarrow \psi'(x) = e^{i\alpha}\psi(x)$ should not change the Lagrangian for an electron
  - Look at Dirac Lagrangian: $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$
  - It should not change for $\psi \rightarrow \psi'$ and $\bar{\psi} \rightarrow \overline{\psi'} = \psi'^\dagger\gamma^0$; $\overline{\psi'} = e^{+i\alpha}\bar{\psi}$ ➔ OK.

- *local* gauge invariance: invariance under chaging phases in space and time
  - An electron wave function can have a different phases at different places and times
    - $\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$ and $\bar{\psi}(x) \rightarrow \overline{\psi'}(x) = e^{-i\alpha(x)}\bar{\psi}(x)$
  - Check this for the Dirac Lagrangian

  Problem in the term: $\partial_\mu\psi(x) \rightarrow \partial_\mu\psi'^{(x)} = e^{i\alpha(x)}\left(\partial_\mu\psi(x) + i\partial_\mu\alpha(x)\psi(x)\right)$

- It seems that the Lagrangian will change, but this is not allowed!

Part 2
Gauge Theory

2b) Local Gauge Invariance

i) QED

Exercise – 18: Covariant Derivative

Griffiths §10.3

- We insist that the Lagrangian does not change and invent a “covariant” derivative:
  - Replace in $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ the derivative by: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$
  - Require that the vector field A^μ transforms together with the particle wave ψ

$$\psi(x) \rightarrow \psi'(x) = \mathrm{e}^{iq\alpha(x)}\psi(x)$$

$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \frac{1}{q}\partial^\mu\alpha(x)$$

  - ➔ Exercise: check that the Lagrangian now is invariant!

- What have we done?
  - We ***insist*** the electron can have a local phase factor $\alpha(x)$ without changing the physics
  - We then ***must*** at the same time introduce a photon, which couples to charge!

➔ ***Gauge invariance implies interactions!***

- Remember gauge transformations EM field: $A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu\lambda$
  - λ is coupled to the phase of the wave function of the electrons

- The same principle can also be used for weak and strong interactions: implement other symmetries

Quantum Electrodynamics (QED)

- The free Dirac Lagrangian is: $\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$

- Introducing electromagnetism implies: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$

- Resulting in: $\mathcal{L} = i\bar{\psi}\gamma_\mu\, D^\mu\psi - m\bar{\psi}\psi$

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\, \partial^\mu\psi - m\bar{\psi}\psi\; - q\bar{\psi}\gamma_\mu A^\mu\psi$$

$\mathcal{L} = \mathcal{L}_{\text{free}} - \mathcal{L}_{\text{int}}$ with $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$ and $J_\mu = q\bar{\psi}\gamma_\mu\psi$

- Remember that the probability current was $\bar{\psi}\gamma_\mu\psi$ such that we now have a charge current: $J_\mu = q\bar{\psi}\gamma_\mu\psi$

- The system is described as free Lagrangian plus an interaction Lagrangian of the form: "current × field" $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$

Part 2
Gauge Theory

2b) Local Gauge Invariance

ii) Yang-Mills theories* (Weak, Strong)

* Note: this is a more technical part: focus on the concept involved; the precise mathematics is less important for now

Yang Mills Theories

- QED is called a U(1) symmetry. It means that a 1-dimensional unitary transformation (the phase factor) does not change the physics.
  - The unitary symmetry couples to the charge quantum number

- Let us require that the weak interaction can not differentiate between an up and a down quark
  - $\mathcal{L} = \bar{u}(i\gamma^\mu \partial_\mu - m)u + \bar{d}(i\gamma^\mu \partial_\mu - m)d$ where u and d are spinor waves

- Rewrite it as $\mathcal{L} = \bar{\psi}(i\gamma^\mu \, I \, \partial_\mu - I \, m)\psi$ with $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - We think of the “up” and “down” directions in weak isospin space

SU2 Gauge Invariance

- We require gauge invariance: $\psi(x) \to \psi'(x) = G(x)\psi(x)$ with $G(x) = \exp\left(\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)\right)$
  - $\vec{\tau} = \tau_1, \tau_2, \tau_3$ are the Pauli Matrices
  - This is now a rotation in isospin space generated by 2x2 Pauli matrices!

- Just like QED there is the problem that the Lagrangian does not automatically stay invariant (just write it out), because: $\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = G(x)(\partial_\mu\psi) + (\partial_\mu G)\psi$

- To solve a corresponding covariant derivative must be introduced to keep the Lagrangian invariant: $I\partial_\mu \to D_\mu = I\partial_\mu + igB_\mu$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  - g is the coupling constant that replaces charge q in QED and B_μ is now e a new vector force field that replaces A_μ of QED.
  - The object B_μ is now a 2x2 matrix: $B_\mu = \frac{1}{2}\vec{\tau}\cdot\vec{b}_\mu = \frac{1}{2}\tau_1^a b_\mu^a = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

    $\vec{b}_\mu = (b_1, b_2, b_3)$ are now three new gauge fields
  - We need 3 instead of one, because there are three generators of 2x2 rotations

- We now get the desired behaviour if : $D_\mu\psi(x) \to D'_\mu\psi'(x) = G(x)(D_\mu\psi)$

Gauge transformation for B_μ field – (for experts)

- We get the desired behaviour if: $D_\mu\psi(x) \to D'_\mu\psi'(x) = G(x)(D_\mu\psi)$
- The left side of this equation is: $$D'_\mu\psi'(x) = (\partial_\mu + igB'_\mu)\psi'$$ $$= G(\partial_\mu\psi) + (\partial_\mu G)\psi + ig\, B'_\mu\, (G\psi)$$
- While the right hand side is: $G(D_\mu\psi) = G(\partial_\mu\psi) + \, ig\; G\; B_\mu\; \psi$
- So the required transformation of the field is: $igB'_\mu(G\psi) = igG(B_\mu\psi) - \, (\partial_\mu G)\psi$
- Multiply the equation by G^{-1} on the right (and omitting ψ): $B'_\mu = GB_\mu G^{-1} + \frac{i}{g}\,(\partial_\mu G)G^{-1}$
- Compare this to the case of electromagnetism where $G_{em} = e^{i\alpha(x)}$gives:

$$A'_\mu = G_{em}AG_{em}^{-1} + \frac{i}{g}\,(\partial_\mu G_{em})G_{em}^{-1} = A_\mu - \frac{1}{q}\partial_\mu\alpha$$

… which is exactly what we had before.

Interpretation

- We try to describe an interaction with a symmetry between two states:
  - "up" and "down" states with invariance under SU2 rotations

- To do this requires the existence of three force fields, related to the gauge field: $\vec{B}_\mu$
  - What are they?
  - They must be three massless bosons, similar to the photon, that couple to "up" and "own" states.
    - They are the W^-, Z^0, W^+ bosons.
    - How come they have a mass (unlike the photon? → Higgs mechanism

- Again the interaction Lagrangian will be of the form "current x field:" $\vec{J}_\mu \vec{b}^\mu$, where the current is now: $J_\mu = \frac{g}{2} \bar{\psi} \gamma_\mu \vec{\tau} \psi$

- The "up" and "down" states are $\psi = \binom{u}{d}$ and $\psi = \binom{\nu}{e}$ and we describe the *weak interaction.*

- How about the *strong interaction*?

The strong interaction

- The "charge" of the strong interaction is "colour"
- The wave function of a quark has three components:
  - $\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$; Require a symmetry generated by 3x3 rotations in 3-dim color space: SU(3)
- There are 8 generator matrices λ_i and as a consequence there are 8 vector fields needed to keep the Lagrangian invariant
  - There exist 8 gluons, related to:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The Standard Model

- The Standard Model applies gauge invariance at the same time to
  - Electromagnetism (U(1) symmetry transformations) → 1 photon
  - Weak interaction (SU(2) symmetry transformations) → 3 weak bosons
  - Strong interaction (SU(3) symmetry transformations) → 8 gluons

- The SM gauge group is $SU(3) \otimes SU(2) \otimes U(1)$

- For an exact symmetry the force particles should be massless for
  - $SU(3)$ is exact.
  - $SU(2) \otimes U(1)$ is an approximate (ie “broken”) symmetry.
    - It is broken in the Higgs mechanism such that there remains one massless boson and three massive particles.

Standard Model