Exercise – 13: Dirac Algebra

- Dirac algebra:
  - Write the explicit form of the γ-matrices
  - Show that : $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
  - Show that : $(\gamma^0)^2 = \mathbb{1}_4$; $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{1}_4$
  - Use anti-commutation rules of α and β to show that: ${\gamma^\mu}^\dagger = \gamma^0\gamma^\mu\gamma^0$
  - Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and show: ${\gamma^5}^\dagger = \gamma^5$; $(\gamma^5)^2 = \mathbb{1}_4$; $\{\gamma^5, \gamma^\mu\} = 0$

Exercise – 14: Solutions of free Dirac equation

a) Show that the following plane waves are solutions to Dirac's equation

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \quad ; \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

$$\psi_3 = \begin{pmatrix} p_z/(E-m) \\ (p_x + ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)} \quad ; \quad \psi_4 = \begin{pmatrix} (p_x - ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x}-Et)}$$

b) Write the Dirac equation for particle in rest (choose $\vec{p} = 0$) and show that ψ_1 and ψ_2 are *positive energy* solutions: $E = +\left|\sqrt{p^2 + m^2}\right|$ whereas ψ_3 and ψ_4 are *negative energy* solutions: $E = -\left|\sqrt{p^2 + m^2}\right|$.

c) Consider the *helicity* operator $\vec{\sigma} \cdot \vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$ and show that ψ_1 corresponds to *positive helicity* solution and ψ_2 to *negative helicity.* Similarly for ψ_3 and ψ_4.

Spin and Helicity – hint for exercise 14c)

- For a given momentum p there still is a *two-fold degeneracy*: what differentiates solutions ψ_1 from ψ_2?
- Define the spin operator for Dirac spinors: $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, where $\vec{\sigma}$ are the three 2x2 Pauli spin matrices
- Define ***helicity*** λ as spin "up"/"down" wrt direction of motion of the particle

$$\lambda = \frac{1}{2}\vec{\Sigma}\cdot\hat{p} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} = \frac{1}{2|p|}(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)$$

- Split off the Energy and momentum part of Dirac's equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} m\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- Exercise: Try solutions ψ_1 and ψ_2 to see they are ***helicity eigenstates*** with $\lambda = +1/2$ and $\lambda = -1/2$
- Dirac wanted to solve negative energies and he found spin-½ fermions!

Exercise – 15: Electromagnetic Field

a) Show Maxwell equations can be summarized in covariant form:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$$

(Derive expressions for ρ and $\vec{j}$ and use: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$)

b) Field A^μ is just introduced as a mathematical tool

- Choose any A^μ as long as $\vec{E}$ and $\vec{B}$ fields don't change

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \lambda$$

$$V \rightarrow V' = V + \frac{\partial \lambda}{\partial t}$$

$$\vec{A} \rightarrow \overrightarrow{A'} = \vec{A} - \vec{\nabla}\lambda$$

- Show this explicitly!

c) Choose the Lorentz gauge condition: $\partial_\mu A^\mu = 0$

- Show that we can chose a gauge field such that this is possible

Exercise – 16 Antisymmetric tensor $F^{\mu\nu}$

- Maxwell's equation $\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$ can be further shortened by introducing the antisymmetric tensor: $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- Show that Maxwell's equations become: $\partial_\mu F^{\mu\nu} = j^\nu$

- Hint: derive the expressions for charge ($q = j^0$) and current ($\vec{I} = \vec{j}$) separately.

  Use the identity: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$. Remember the definitions:

  $A_\mu = (A_0, -\vec{A})$; $\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$; $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

Exercise – 17 : Lagrange Formalism classical

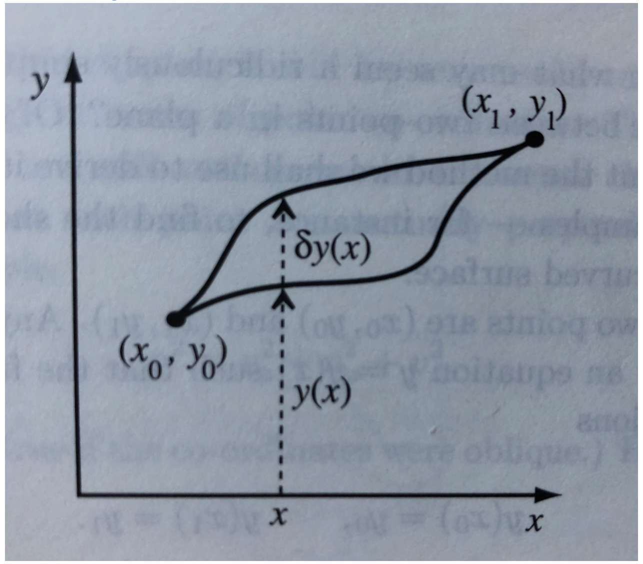
- Example of variational calculus and least action principle: what is the shortest path between two points in space?
  - Distance of two close points:

$$y = \sqrt{dx^2 + dy^2} = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 - y'^2}dx \quad \text{with } y' = dy/dx$$

  - Total length from (x_0, y_0) to (x_1, y_1) :

$$l = \int_{x_0}^{x_1} dl = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx = \int_{x_0}^{x_1} f(y, y')dx$$

  - Task is to find a function $y(x)$ for which l is minimal



- In general assume the path length is given by: $I = \int_{x_0}^{x_1} f(y, y')dx$
- Variational principle: shortest path is stationary: $\delta I = 0$

a) Write $\delta f(y, y') = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$ where $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

Show using partial integration that $\delta I = 0$ leads to the Hamilton Lagrange equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

b) Here for the shortest path we have $f(y') = l = \sqrt{1 + y'^2}$.

Then $\partial f/\partial y = 0$ and $\partial f/\partial y' = y'/\sqrt{1 + y'^2}$

Show that the variational principle leads to a straight line path: $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$ or that y' is a constant:

$dy/dx = a$; $y = ax + b$

Exercise – 18: Lagrangians and wave equations

Griffiths §10.2

- Scalar Field ("pion")
  a) Show that the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2$ results in the Klein-Gordon equation

- Dirac Field (Fermion)
  b) Show that the Euler-Lagrange equations for $\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$ results in the Dirac equation

- Electromagnetic field (photon)
  c) Show that $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$ results in Maxwell's equations

Exercise – 19: Covariant Derivative

Griffiths §10.3

- We insist that the Lagrangian does not change and invent a "covariant" derivative:
  - Replace in $i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$ the derivative by: $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu$
  - Require that the vector field A^μ transforms together with the particle wave ψ

$$\psi(x) \rightarrow \psi'(x) = \mathrm{e}^{iq\alpha(x)}\psi(x)$$
$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) - \frac{1}{q}\partial^\mu\alpha(x)$$

  - ➔ Exercise: check that the Lagrangian now is invariant!