Exercise 4. Weight 3 points. Due Monday Feb. 20, 2017.

A heart-shaped volume can be created with boundary

$$x = r\cos(a)$$
$$y = r\sin(a)$$
$$r = 1.6\sin^3(t)\ ;\ t\,(\text{or }\pi - t) = \arcsin\left(\left(\frac{r}{1.6}\right)^{1/3}\right)$$
$$z = 1.3\cos(t) - 0.5\cos(2t) - 0.2\cos(3t) - 0.1\cos(4t)$$
$$0 \le a \le 2\pi,\quad 0 \le t \le \pi$$

Integrate the function $f(x, y, z) = (x^2 + y^2)^{0.5z} = r^z$ over the volume enclosed by the boundary surface given above. One can use the radial symmetry to obtain

$$\oiiint_V f(x, y, z)\,dx\,dy\,dz = 2\pi \iint_V f(r, z)\,r\,dr\,dz \quad .$$

Furthermore, the boundary is described as a function of the single parameter *t*. Verify that using

$$r(t) = r(\pi - t)\ ,\ dr/dt > 0 \quad for \quad 0 < t < \pi/2$$

one can write the integral as:

$$2\pi \int_0^{\pi/2} \left[\frac{dr}{dt} \int_{z(\pi-t)}^{z(t)} r\,f(r, z)\,dz\right] dt$$

Determine the value of the integral with Gauss-Legendre quadrature, using 40 and 50 abscissa in z, and 50, 100, 200, 300, 400, 500, and 600 abscissa in t or in r. Give the results in 15 digits precision. You may choose whether you perform the integral using dr or dt.

Determine the value of the integral with Simpson's rule using EPS=1e-8. This should yield 8 digits of precision. Give the number of function calculations (of f(r,z)) needed. Since the function at r=0 is singular (for negative z) but integrable (over r dr, not over r), one can start from e.g. $t=10^{-10}$ or $r=10^{-30}$. (To avoid exceptions, it may be necessary to increase JMAX in the include files).

Determine the value of the integral with Romberg's algorithm with EPS = 1e-9 and 1e-12. Start again at $t=10^{-10}$ or $r=10^{-30}$. Determine the number of function calculations needed. (Also here, you might need to increase JMAX in the include files).

(Optionally, you may also integrate the inner integral analytically; then in the case of Simpson’s rule or Romberg’s rule you typically gain a factor of a million in speed for the same accuracy. Gaussian quadrature is already quite fast.)