In Vino Veritas: A study of wineglass acoustics

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This paper describes an investigation of the natural resonant frequencies of vessels such as wineglasses. Measurements on a number of glasses are interpreted with the help of theoretical predictions based on the analysis of vibrating systems by means of the energy method. Results and analysis are given for empty glasses and for glasses containing different amounts of liquid. Evidence for vibrational modes above the lowest is presented.

I. INTRODUCTION

The pure tones emitted when one rubs a moistened finger around the rim of a wineglass must have been a subject of informal experiment at countless dinner tables. The quantitative study of this phenomenon contains some instructive physics. The present paper is based on observations and related analysis stemming from a high school science project undertaken by the author and his son about 15 years ago.

The theoretical analysis of the vibrational motion provides a nice illustration of the energy method for calculating the natural frequencies of a complex vibrating system.

II. BASIC MODE OF OSCILLATION OF A WINEGLASS

A wineglass typically has a strong, rigid supporting stem, and a bowl that is rather thick at the base and becomes progressively thinner toward the upper rim. Vibrations of this system occur most easily under conditions that leave the circumferential length of the rim unchanged, because glass is highly resistant to extension or compression. This condition is satisfied, to a high approximation, if the rim deforms from a circle into an ellipse, and back again through the circle into another ellipse with its major axis at right angles to the first [Fig. 1(a)]. All other horizontal sections of the wineglass will go through similar motions, but with amplitudes that decrease as one goes down from the level of the rim.

On a simplified view, one might represent the wineglass as a thin-walled cylinder, attached to a rigid circular base [Fig. 1(b)]. A vertical section through this cylinder looks rather like a tuning fork, and the complete glass is generated by rotating this section about the vertical axis of symmetry. In oscillation, the condition of constant rim perimeter forces a definite phase relationship, such that the "tuning fork" having maximum outward displacement of its prongs at a particular instant is 90° away in azimuth from the fork having maximum inward displacement.

Given the above form of the oscillation, its actual magnitude can be characterized by the amplitude of oscillation of any one chosen point. For convenience, let this be an antinodal point on the rim at the top of the glass. Its displacement at any instant can then be written

$$\Delta(t) = \Delta_0 \cos \omega t. \tag{1}$$

The displacement of any other point is related to this by a time-independent geometrical factor that depends on the azimuth θ and the vertical coordinate z. Assuming linear elastic restoring forces, the total elastic potential energy of the vibrating wineglass is proportional to Δ^2, and its total

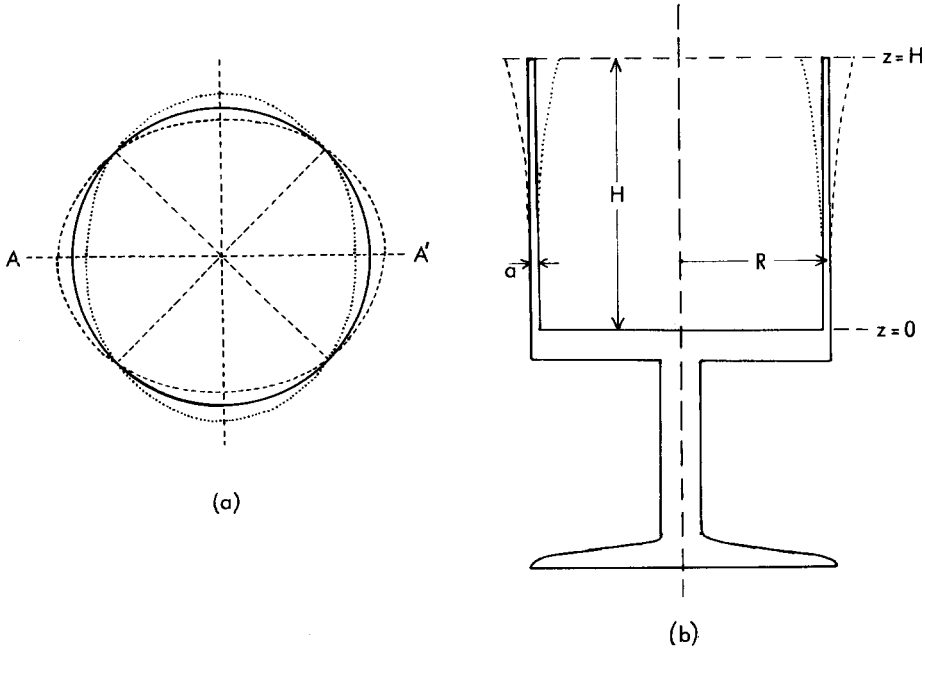

Fig. 1. (a) Elliptical deformations (greatly exaggerated) of wineglass rim in lowest vibrational mode. (b) Vertical section of vibrating wineglass through section AA' of (a).

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kinetic energy is proportional to $(d\Delta/dt)$.[2] Hence the total energy of the system can be written

$$E = A\left(\frac{d\Delta}{dt}\right)^2 + B\Delta^2, \tag{2}$$

where A and B are constants. Ignoring damping, this total energy is constant. If we substitute $\Delta = \Delta_0 \cos \omega t$ from Eq. (1), it can be seen that the requirement $E = \text{const}$ leads immediately to the result

$$\omega^2 = B/A. \tag{3}$$

Thus the calculation of the natural frequency of vibration reduces to a matter of evaluating the total kinetic and potential energies of the system in terms of the rim displacement Δ. This energy method is of course a well-known technique for calculating the frequencies of oscillating systems, and is far more powerful and tractable than the use of Newton's law when complicated systems are involved.

For the form of motion that we are assuming, the horizontal radial displacement x of any arbitrary point of the wall of the wineglass can be written

$$x(z,\theta,t) = \Delta_0 f(z) \cos 2\theta \cos \omega t, \tag{4}$$

where $f(z)$ rises from zero at the bottom of the wineglass to unity at the top rim. The $\cos 2\theta$ factor describes the elliptical distortion from the static circular shape, giving maximum amplitudes of oscillation along the perpendicular axes through $\theta = 0$ and $\pi/2$, and diagonal nodal lines through $\theta = \pi/4$ and $3\pi/4$, as indicated in Fig. 1(a). The form of this motion, with its nodes and antinodes, can be made visible if chalk dust or other fine powder is sprinkled on the liquid surface of a partially filled wineglass that is being excited into oscillation.

III. CALCULATION OF KINETIC AND POTENTIAL ENERGIES

Most of the basic elements of the detailed theory are to be found in Rayleigh's classic *Theory of Sound*.[1] However, the relevant results are scattered through the book, and the notation is not always perspicuous. It is probably useful, therefore, to develop the analysis here in a self-contained way.

A. Kinetic energy

Suppose for simplicity that the wineglass is modeled, as in Fig. 1(b), as a vertical cylinder of radius R and height H, with a rigid base and side walls of uniform thickness a. The mass of an element of the wall, lying between z and $z + dz$ in height, and between θ and $\theta + d\theta$ in azimuth, is equal to $\rho_g aR\, d\theta\, dz$, where ρ_g is the density of the glass. Its instantaneous radial velocity dx/dt is given, according to Eq. (4), by

$$\frac{dx}{dt} = -\omega\Delta_0 f(z) \cos 2\theta \sin \omega t.$$

In addition to this radial motion, however, the condition of fixed perimeter implies tangential displacements also. It is clear from symmetry that there is no such displacement for $\theta = 0$, but in general a point that initially has the coordinates R,θ will, when displaced, move to $R + x$, θ', where

$$\int_0^{\theta'} [R + x(\varphi)]\, d\varphi = R\theta.$$

The transverse displacement s is thus given by

$$s = R(\theta' - \theta) = -\tfrac{1}{2}\Delta_0 f(z) \sin 2\theta \cos \omega t$$

and so

$$\frac{ds}{dt} = \tfrac{1}{2}\omega\Delta_0 f(z) \sin 2\theta \sin \omega t.$$

Hence the total kinetic energy K of the vibrating glass at time t is given by

$$K = \tfrac{1}{2}\rho_g aR\omega^2\Delta_0^2 \sin^2 \omega t \times \int_{z=0}^{H}\int_{\theta=0}^{2\pi} [f(z)]^2(\cos^2 2\theta + \tfrac{1}{4}\sin^2 2\theta)\, d\theta\, dz,$$

which after the integration over θ simplifies to

$$K = \tfrac{5}{8}\rho_g aR\omega^2\Delta_0^2 \sin^2 \omega t \int_0^H [f(z)]^2\, dz. \tag{5}$$

Because $f(z)$ increases from zero at $z = 0$ to a maximum at $z = H$, the major contribution to K will come from the motion of the upper parts of the wineglass. This situation will be further enhanced in a real wineglass by the fact that the walls of the glass become thicker and less deformable lower down. In any case, the results of the calculation will be relatively insensitive to the details of the motion for $z \ll H$. We shall exploit this fact later.

B. Potential energy

The calculation of the potential energy is somewhat more complicated. It represents the energy of flexure of the glass in both horizontal and vertical planes. As a basis for the calculation, consider a curved segment of material which, when undeformed, has mean radius r_0 and radial thickness a [Fig. 2(a)]. Let its thickness perpendicular to the plane of the diagram be b. Suppose that the segment has its mean radius of curvature changed to r_c [Fig. 2(b)]. As in the usual analysis of bent beams, one can assume that the length l_0 of the center line of the segment remains unchanged. For a filament a distance y from this center line, we have

$$\text{initial length} = \frac{r_0 + y}{r_0} l_0,$$

$$\text{deformed length} = \frac{r_c + y}{r_c} l_0$$

$$\therefore \text{change of length } \delta l = l_0 y\left(\frac{1}{r_c} - \frac{1}{r_0}\right).$$

If the radial thickness of the filament is dy, and the Young's modulus is denoted Y, the force of tension or compression along the filament is given by

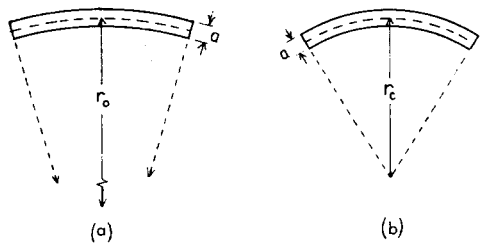

Fig. 2. (a) Curved section of material in stress-free state. (b) Same section under stress, with different curvature but same length along center line.

$$F = b\,dy\,Y\frac{\delta l}{l_0} = bYy\,dy\left(\frac{1}{r_c} - \frac{1}{r_0}\right)$$

and the stored energy $dU = \frac{1}{2}F\,\delta l$

$$\therefore dU = \tfrac{1}{2}l_0 bY\left(\frac{1}{r_c} - \frac{1}{r_0}\right)^2 y^2\,dy.$$

Integrating this between the limits $y = \pm a/2$ gives

$$\Delta U = \frac{l_0 Ya^3 b}{24}\left(\frac{1}{r_c} - \frac{1}{r_0}\right)^2. \tag{6}$$

Let us return now to the elastic deformation of the wineglass. For flexure in the horizontal plane of a curved segment lying between z and $z + dz$ and between θ and $\theta + d\theta$, we have

$$l_0 = R\,d\theta, \quad b = dz, \quad r_0 = R.$$

The radius of curvature r_c in the deformed state varies with the azimuthal angle θ:

$$\frac{1}{r_c} = \frac{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}{[r^2 + (dr/d\theta)^2]^{3/2}},$$

where $r = R + x$, with x as defined in Eq. (4). Since $x \ll R$, the above expression can be approximated by dropping all terms of higher than first order in x/R; the result is

$$\frac{1}{r_c} \approx \frac{1}{R}\left(1 - \frac{x}{R} - \frac{1}{R}\frac{d^2x}{d\theta^2}\right) = \frac{1}{R} + \frac{3x}{R^2}.$$

Substituting these values in Eq. (6), the elastic energy stored in a horizontal circular segment is given by

$$\Delta U_1 = \frac{Ya^3\,dz}{24}R\,d\theta\frac{9x^2}{R^4} = \frac{3Ya^3}{8R^3}\Delta_0^2\cos^2\omega t\,[f(z)]^2\cos^2 2\theta\,d\theta\,dz.$$

Integrating over θ and z then gives

$$U_1 = \frac{3\pi Ya^3}{8R^3}\Delta_0^2\cos^2\omega t\int_0^H [f(z)]^2\,dz. \tag{7a}$$

In the vertical plane, we can consider a segment of length $l_0 = dz$ and width $b = R\,d\theta$. Its radius of curvature in the deformed state is given by

$$\frac{1}{r_c} = \frac{d^2x/dz^2}{[1 + (dx/dz)^2]^{3/2}}.$$

Under the assumption of small deformations, this can be approximated simply as d^2x/dz^2. For these vertical segments we are assuming $r_0 = \infty$; hence we have

$$\frac{1}{r_c} - \frac{1}{r_0} \approx \Delta_0\cos\omega t\cos 2\theta f''(z)$$

and so, for the energy of flexure of this segment in the vertical plane,

$$\Delta U_2 = \frac{Ya^3\,dz}{24}R\,d\theta\,\Delta_0^2\cos^2\omega t\cos^2 2\theta\,[f''(z)]^2.$$

Integrating over θ and z then gives

$$U_2 = \frac{\pi Ya^3}{24}R\Delta_0^2\cos^2\omega t\int_0^H [f''(z)]^2\,dz. \tag{7b}$$

Combining Eqs. (7a) and (7b), we thus get the following expression for the total elastic potential energy at a given instant:

$$U = \frac{\pi Ya^3}{24R^3}\Delta_0^2\cos^2\omega t\left(9\int_0^H [f(z)]^2\,dz + R^4\int_0^H [f''(z)]^2\,dz\right). \tag{8}$$

If each vertical strip of the wineglass can be considered as a uniform bar, free at the top and clamped at the bottom end, then, as Rayleigh[2] has shown, the form of $f(z)$ is given by

$$f(z) = A(\cosh\beta z - \cos\beta z) + B(\sinh\beta z - \sin\beta z), \tag{9}$$

where A, B, and β are constants. However, as Rayleigh also shows,[3] it is a property of all of the flexural modes that the integral of $[f''(z)]^2$ over the whole length is just a multiple of the integral of $[f(z)]^2$:

$$\int_0^H [f''(z)]^2\,dz = \beta^4\int_0^H [f(z)]^2\,dz. \tag{10}$$

For the lowest "fixed/free" mode, $\beta \approx 1.875/H$. Since $(1.875)^4 \approx 12$, Eq. (8) can for this mode conveniently be rewritten, to a very good approximation, in the following simplified form:

$$U \approx \frac{3\pi Ya^3}{8R^3}\Delta_0^2\cos^2\omega t\left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right]\int_0^H [f(z)]^2\,dz. \tag{11}$$

IV. FREQUENCIES OF EMPTY WINEGLASSES

Using the values of total kinetic energy from Eq. (5) and total potential energy from Eq. (11) we deduce that the values of the constants A and B in Eq. (2) are

$$A = \frac{5\pi}{8}\rho_g aR\int_0^H [f(z)]^2\,dz,$$

$$B = \frac{3\pi Ya^3}{8R^3}\left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right]\int_0^H [f(z)]^2\,dz.$$

It follows from Eq. (3) that the natural angular frequency of oscillation in the assumed (lowest) mode of vibration is given by

$$\omega_0^2 = \frac{B}{A} = \frac{3}{5}\frac{Y}{\rho_g}\frac{a^2}{R^4}\left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right]. \tag{12}$$

The fundamental frequency $\nu_0 (= \omega_0/2\pi)$ in hertz is thus given by

$$\nu_0 = \frac{1}{2\pi}\left(\frac{3Y}{5\rho_g}\right)^{1/2}\frac{a}{R^2}\left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right]^{1/2}. \tag{13}$$

This relatively simple equation can then be tested against observed values. However, as mentioned earlier, there are complications arising from the fact that the wall thicknesses of real wineglasses increase downwards. To accommodate this fact, we shall take a in Eq. (13) to be the wall thickness near the upper rim, and use for H an equivalent height H^* for an idealized cylindrical wineglass of constant wall thickness. One way of inferring H^* involves observations on partially filled glasses and will be discussed later (Sec. VI). For the moment let us just verify that Eq. (13) gives reasonable values for the vibration frequencies.

Consider a typical wineglass of radius, say, 3 cm and depth 5 cm, with a wall thickness of 1.5 mm. Reasonable approximate values for Young's modulus and density are[4] $Y \approx 6\times 10^{11}$ dyn/cm^2, $\rho_g \approx 3$ g/cm^3. Substituting these values into Eq. (13) gives $\nu_0 \approx 1000$ Hz, which is certainly of the correct order.

V. SOME OBSERVATIONS WITH EMPTY GLASSES

The procedure for making measurements was simple, indeed rather primitive. A microphone connected to one input of a dual-trace oscilloscope was suspended close to the wineglass under investigation. An audio generator was connected both to the second input to the oscilloscope and to a small speaker placed near the microphone. A combination of aural and visual observation made it easy to judge when the frequency from the audio generator was equal to that from the wineglass when the latter was rubbed or tapped to stimulate it into vibration. Table I shows data for a mixed bag of glasses from the author's household. The observed values of frequency are compared with values ν_0' calculated from Eq. (13) but omitting the uncertain factor depending on the ratio R/H, i.e.,

$$\nu_0' = \frac{1}{2\pi}\left(\frac{3Y}{5\rho_g}\right)^{1/2}\frac{a}{R^2}. \tag{14}$$

Since the glasses were of different kinds, the actual values of the ratio Y/ρ_g probably vary from one to another; both Y and ρ_g can have widely different values according to the composition of the glass.[5] However, it can be seen that in all cases the quantity ν_0' is less, as it should be, than the observed frequency ν_0. From their ratio one can deduce the value of the ratio R/H in Eq. (13) and hence the effective height H^* of the wineglass modeled as a cylinder of uniform wall thickness. These values are listed in the next to last column of Table I; although no great significance should be attached to them they do appear quite reasonable in order of magnitude.

VI. FREQUENCIES OF PARTIALLY FILLED WINEGLASSES

It is easy to verify by direct observation that the pitch of the note from a filled wineglass is *lower* than that of the same glass when empty. This is readily understandable in very general terms. The added liquid is forced to participate in the vibrational motion, so the total kinetic energy for a given motion of the glass itself is increased. The potential energy of elastic deformation, however, remains unchanged. Thus, in Eq. (2) the constant A increases, the constant B remains the same, and the value of ω^2 in Eq. (3) consequently goes down.

To make the calculation quantitative, let us suppose that our idealized cylindrical wineglass is filled up to a height h from the bottom. An element of liquid, distance r from the axis, will in general undergo both radial and transverse displacements. A reasonable simplification is to assume that these displacements are proportional to r and can be written

$$\delta_r = \gamma_1(r/R)\Delta_0 f(z)\cos 2\theta\cos\omega t,$$

$$\delta_t = \gamma_2(r/R)\Delta_0 f(z)\sin 2\theta\cos\omega t,$$

where γ_1 and γ_2 are factors of the order of unity. Constancy of volume of an element of liquid would require $\gamma_2 = \gamma_1/2$, just as for the displacement of the material of the wineglass itself.

Under these assumptions, the kinetic energy of an element of the liquid of volume $r\,dr\,d\theta\,dz$ and density ρ_l can be written

$$dK_l = \tfrac{1}{2}\rho_l(r\,dr\,d\theta\,dz)(r^2/R^2)\Delta_0^2[f(z)]^2 \times\omega^2\sin^2\omega t(\gamma_1^2\cos^2 2\theta + \gamma_2^2\sin^2 2\theta).$$

Integrating over r, θ, and z then gives the following equation for the total kinetic energy of the liquid:

$$K_l = \alpha\frac{\pi}{8}\rho_l R^2\omega^2\Delta_0^2\sin^2\omega t\int_0^h [f(z)]^2\,dz, \tag{15}$$

where $\alpha = \gamma_1^2 + \gamma_2^2$.

To this we add the kinetic energy of the glass itself, as given by Eq. (5). Hence the total kinetic energy of the wineglass when filled up to the level $z = h$ can be written

$$K = \omega^2\Delta_0^2\sin^2\omega t\left(\frac{5\pi}{8}\rho_g aR\int_0^H [f(z)]^2\,dz + \alpha\frac{\pi}{8}\rho_l R^2\int_0^h [f(z)]^2\,dz\right). \tag{16}$$

It then follows that the frequency ν_h of the partially filled wineglass should be related to its frequency ν_0 when empty through the relation

$$\left(\frac{\nu_0}{\nu_h}\right)^2 = 1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a}\int_0^h [f(z)]^2\,dz\Big/\int_0^H [f(z)]^2\,dz. \tag{17}$$

The form of $f(z)$ is in principle given by Eq. (9)—a complicated mixture of hyperbolic and trigonometric functions. However, for the lowest flexural mode, which we are assuming, $f(z)$ is a monotonically increasing function that happens to be excellently fitted by a simple power of the dimensionless variable z/H:

$$f(z) \approx (z/H)^{3/2}. \tag{18}$$

Figure 3 shows this function as a smooth curve, together with values of the exact form of $f(z)$ as given by Rayleigh.[2] For $z/H \gtrsim 0.4$ the fit is almost perfect; moreover, the integral of $[f(z)]^2$ according to Eq. (18) over all z from zero to H is equal to $H/4$, which is precisely the result of integrating $[f(z)]^2$ in its exact form.[6] Taking advantage of these simplifications, we can rewrite Eq. (17) as

$$\left(\frac{\nu_0}{\nu_h}\right)^2 \approx 1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a}\left(\frac{h}{H}\right)^4. \tag{19}$$

Table I. Frequencies of empty glasses.

No.	R (cm)	a(cm)	ν_0(Hz)	ν_0'(Hz)	R/H^*	H^*(cm)	H^*(Sec. VI)
1	2.0	0.14	4600	2200	1.26	1.6	2.6 cm
2	2.5	0.17	1700	1640	0.48	5.2	7.2 cm
3	3.4	0.12	1570	620	1.42	2.4	5.5 cm
4	4.0	0.13	1260	500	1.41	2.8	
5	4.0	0.16	1020	600	1.09	3.7	
6	4.7	0.22	1590	600	1.46	3.2	

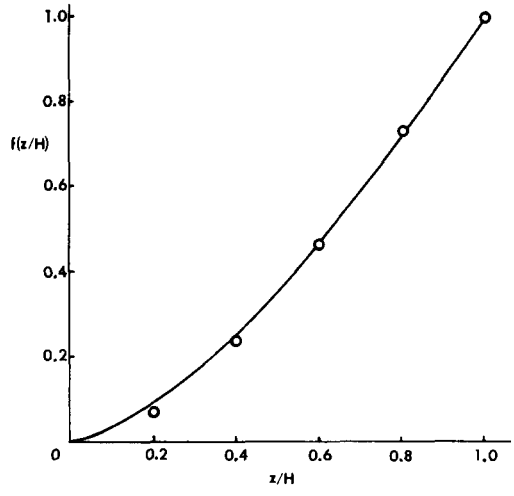

Fig. 3. Circles show relative transverse deformations as calculated by Rayleigh[2] for flexure of a bar fixed at one end and free at the other. Full line is the curve $f(z/H) = (z/H)^{1.5}$.

As with empty glasses, we run into the difficulty that the value of H is not really known. However, we can use observations on ν_h to deduce the effective value of H that we have called H^*. Instead of measuring the height of liquid from the bottom of the glass, we can use as variable the distance d of the liquid surface down from the *top* of the glass; this is a distance that can be unambiguously determined. Then, putting $h = H^* - d$, we can rewrite Eq. (19) as

$$\left(\frac{\nu_0}{\nu_d}\right)^2 \approx 1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a}\left(1 - \frac{d}{H^*}\right)^4. \tag{20}$$

For $d \ll H^*$, this can be written

$$\left(\frac{\nu_0}{\nu_d}\right)^2 \approx \left(1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a}\right) - \frac{4\alpha}{5}\frac{\rho_l R}{\rho_g a}\frac{d}{H^*}. \tag{21}$$

From a graph of $(\nu_0/\nu_d)^2$ against d, we can read off the intercept y_0 and the initial slope y_0':

$$y_0 = 1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a},$$

$$y_0' = -\frac{4\alpha}{5}\frac{\rho_l R}{\rho_g a}\frac{1}{H^*}.$$

From these we can get $H^* = -4(y_0 - 1)/y_0'$. Using this value of H^* we can then test the complete relation, Eq.(20), over all values of d/H^*. Figures 4(a) and 4(b) show how remarkably well this analysis works for one particular wineglass for which the relevant data are tabulated in Table II; the resulting graph of $(\nu_0/\nu_d)^2$ against $(1 - d/H^*)^4$ is a good straight line. Similarly good results were obtained for several different types of wineglass. The value of the parameter α was found to be about 1.4 in each case. Values of H^* deduced by this method are shown in the last column of Table I for comparison. Since they do not depend on assumed values of Y and ρ_g for the glass, they should be more meaningful than the values listed in the preceding column; the differences do, however, seem rather large.

VII. EXCITATION OF HIGHER MODES

We return now to the problem of the empty wineglass. Our discussion so far has been limited to the mode that involves the lowest-order deformations in both vertical and horizontal planes. One can of course envisage higher modes. These can in principle arise in two ways. The first is that the flexure of a vertical section could have one or more nodes between the fixed bottom and the free top rim. The form of $f(z)$ is still given by Eq. (9), but the quantity β takes on higher values, which can be written $\beta_m = m/H$, where m takes on the successive values 1.875 (lowest mode), 4.694, 7.855, ... (Rayleigh[2]). The other way of obtaining higher modes is for the shape of a horizontal section of the vibrating glass to be modified by radial displacements proportional, not to cos 2θ, but to cos $n\theta$ with $n > 2$. It is not

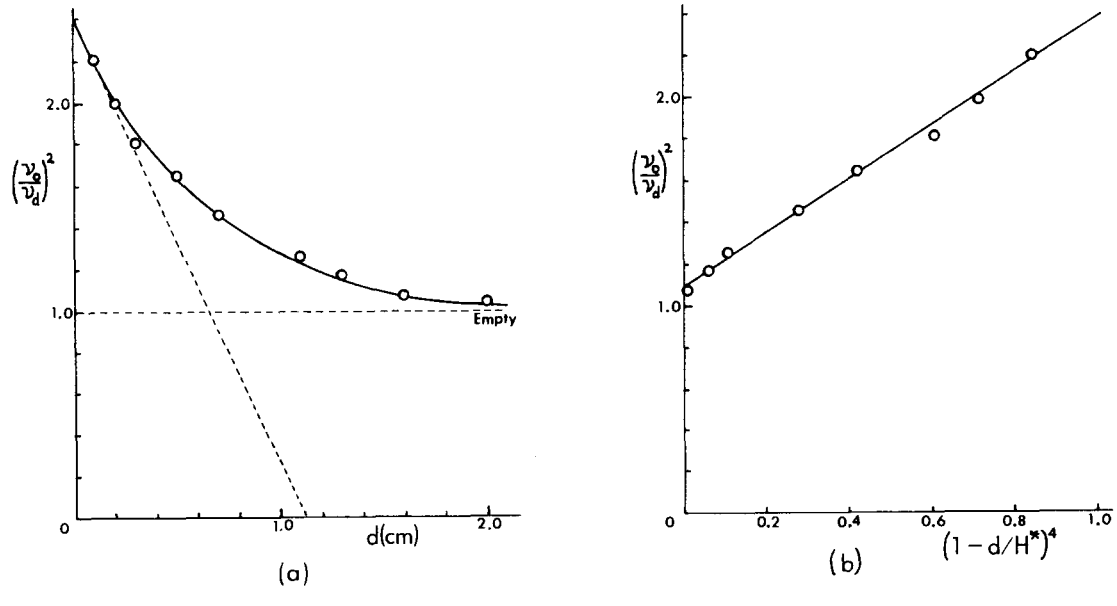

Fig. 4. (a) Data on dependence of frequency of a particular wineglass on distance d of liquid surface below top rim. The ordinate is $(\nu_0/\nu_d)^2$, where ν_0 is the frequency when empty. (b) Same data—values of $(\nu_0/\nu_d)^2$—plotted against $(1 - d/H^*)^4$, for which a linear relation is predicted.

difficult to verify that Eqs. (5) and (11) for the total kinetic and potential energies are replaced by the following more general expressions:

$$K_{m,n} = \frac{\pi}{2}\rho_g aR\omega^2\left(1+\frac{1}{n^2}\right)\Delta_0^2 \sin^2\omega t \int_0^H [f_m(z)]^2\,dz, \quad (22)$$

$$U_{m,n} = \frac{\pi Ya^3}{24R^3}\left[(n^2-1)^2 + \left(\frac{mR}{H}\right)^4\right]\Delta_0^2 \cos^2\omega t \times \int_0^H [f_m(z)]^2\,dz. \quad (23)$$

From these we would deduce a frequency spectrum given by

$$\nu_{m,n} = \frac{1}{12\pi}\left(\frac{3Y}{\rho_g}\right)^{1/2}\frac{a}{R^2}\left[\frac{(n^2-1)^2+(mR/H)^4}{1+1/n^2}\right]^{1/2}. \quad (24)$$

If the ratio R/H is of the order of unity, the increase in frequency in going from the vertical mode of lowest m to the next higher mode is likely to be far greater than in going from the lowest azimuthal mode ($n = 2$) to $n = 3$, for $(n^2-1)^2$ changes from 9 to 64, but m^4 jumps from about 12 to 484. Thus one would in general expect that it might be hard to excite any mode having m greater than the lowest. This does not, however, seem inevitably to be the case, as illustrated by the following observations.

One set of glasses in the author's possession, having unusually thin walls, appeared to vibrate readily in the second vertical mode, and to exhibit resonances for several different azimuthal vibrations associated with this mode. For any reasonable assumption about the value of the ratio R/H^* the frequency of the lowest mode should have been about 400 Hz. No such frequency was present in any perceptible strength. Every glass in the set had an apparently lowest frequency of between about 1400 and 2000 Hz, and a well-defined second frequency usually between about a semitone and a full tone higher. More detailed observations on one glass arbitrarily chosen from the set showed strong resonances at about 1950, 2340, 3850, and 4850 Hz.

According to Eq. (24), a graph of $(1+1/n^2)\nu^2$ against $(n^2-1)^2$ for a particular value of m should be a straight line. The intercept on the vertical axis, divided by the slope, should be equal to $(mR/H^*)^4$. Figure 5 shows such a graph, based on the assumption that the lowest observed frequency belongs to $n = 2$ or 3, and that the others belong successively to $n = 4$, 6, and 7 ($n = 5$ was apparently missed). This identification can be only tentative, but it has a measure of plausibility. The ratio of intercept to slope is about 530, giving $R/H^* \approx 1.02$ (which is reasonable) if we assume $m = 4.694$ (second vertical flexural mode). Furthermore, the slope itself is about 8.7×10^3. According to Eq. (24), we

Table II. Frequency variation for partially filled glass (#1).

d (cm)	ν_d (Hz)	$(\nu_0/\nu_d)^2$	d/H^*	$(1-d/H^*)$	$(1-d/H^*)^4$
0.1	3100	2.20	0.038	0.962	0.844
0.2	3260	1.99	0.077	0.923	0.718
0.3	3420	1.81	0.115	0.885	0.607
0.5	3575	1.65	0.192	0.808	0.421
0.7	3800	1.46	0.270	0.730	0.282
1.1	4100	1.26	0.424	0.576	0.110
1.3	4250	1.17	0.500	0.500	0.062
1.6	4450	1.07	0.616	0.384	0.022
2.0	4500	1.04	0.770	0.230	0.003

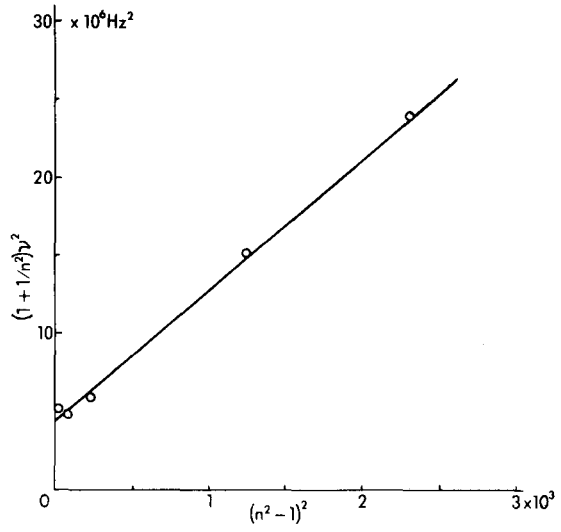

Fig. 5. Higher resonances of a particular wineglass. Azimuthal modes belonging to $n = 2$ (or 3), 4, 6, and 7 are indicated (all for the second mode of vertical flexure). The lowest two points belong to the same measured ν, but plotted for both of the possibilities $n = 2$ and 3.

should have

$$\text{slope} = \frac{1}{48\pi^2}\left(\frac{Y}{\rho_g}\right)\left(\frac{a}{R^2}\right)^2.$$

For this glass, direct measurement gave $a \approx 0.07$ cm, $R = 3.8$ cm. If we put $Y \approx 6\times10^{11}$ dyn/cm^2, $\rho_g \approx 3$ g/cm^3, this theoretical slope would be equal to 9.9×10^3; the agreement can be considered remarkably good in view of the many uncertainties in the analysis. And it seems possible that the upper of the pair of fairly close frequencies for every glass in the set could be attributed to $n = 3$ or 4, and the lower to $n = 2$ or 3, or some mixture thereof.

It may seem surprising that no mention has been made throughout this paper of the possibility of simple torsional oscillation of the wineglass about its vertical axis. The process of rubbing around the rim of the glass to excite its oscillations would be expected to be particularly favorable to the excitation of this type of vibration. The frequency of such a torsional vibration would, however, be far higher than the modes we have discussed. If a wineglass is again modeled as a thin-walled cylinder of radius R and height H, rigidly fixed at its base, it is easy to verify that its natural frequency of torsional vibration would be given by

$$\nu_{\text{tors}} = \frac{1}{2\pi}\left(\frac{3E_s}{\rho_g}\right)^{1/2}\frac{1}{H}$$

(independent of wall thickness a), where E_s is the shear modulus of the glass. This frequency will be greater, by a factor of the order of R/a, than the basic elliptical mode frequency of Eq. (13). This means in practice a factor of at least 10 in frequency, i.e., into the range of about 10 kHz or more, toward the upper limit of audible vibrations.

VII. CONCLUDING REMARKS

Obviously the observations and analysis described in this paper cannot be regarded as very fundamental physics. The subject of classical mechanical vibrations has no mysteries, but even so a serious test of its predictions would call for better defined physical systems than a random set of ordinary wineglasses. Nonetheless, it is satisfying to see how

physical analysis can be applied, with some success, to real wineglasses and other such vessels that can be found in any household. The analysis is, to be sure, rather unreasonably heavy in relation to the importance (or lack of it) of the specific topic, but it does exemplify the power of the energy method for the analysis of relatively complex vibrating systems.

One interesting feature is the way in which seemingly identical glasses (such as the set mentioned in Sec. VI) have distinctive and widely different frequencies of vibration. The natural frequency could well be used as an identifying label or "signature" for a glass or other vessel in cases where the original was rare and valuable and one wanted a simple noninvasive test to distinguish it from copies.

ACKNOWLEDGMENT

I wish to thank my son, Martin C. French, for helping to stimulate this investigation and for his partnership in all the observations.

[1]Lord Rayleigh, *The Theory of Sound, Vol. I* (Macmillan, London, 1894; Dover, New York, 1945).
[2]Rayleigh, Ref. 1, Sec. 173-4.
[3]Rayleigh, Ref. 1, Sec. 165.
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An electrostatic example to illustrate dimensional regularization and renormalization group technique

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An easy and intuitive introduction to the regularization and renormalization techniques in elementary particle physics is given. It is based on the use of a simple electrostatic problem.

I. INTRODUCTION

High-energy physics needs by its very nature an elaborate relativistic quantum field theory. In quantum electrodynamics the successes of this theory are remarkable. However, in all its perturbative aspects it has heavy divergence problems. We would like to show schematically what these problems are and how to solve them. We illustrate this last point precisely with an electrostatic example which shows the same type of divergence. The method used is dimensional regularization.[1] Since 1973, it is most used in applications. This example allows us to show the simplicity and the elegance of dimensional regularization. Moreover, it illustrates the crucial role played by dimensional analysis especially for any problem in which there is no fixed length (or energy) scale. We show that renormalization implies the introduction of a scale and therefore breaks the so-called scale invariance. We study in Sec. V the behavior of renormalized quantities such as an electrostatic potential and a dimensionless quantity of high-energy physics called R. By doing this, we illustrate a few aspects of the renormalization group techniques, which are commonly used in elementary particle physics.

II. DIVERGENCES IN QUANTUM FIELD THEORY

Every quantum field theory is basically of perturbative character. Without going into more details, we can say that it is most useful whenever we deal with a physical quantity which can be expressed as a truncated series. Every cross section and every decay width are of that type. The development parameter of this series is the coupling constant. It measures the intensity of the interaction under study. In quantum electrodynamics, this constant is[2]

$$\alpha = e^2/4\pi = 1/137$$

(the fine structure constant).

To be specific, let us consider the scattering between two particles interacting electromagnetically such as Compton scattering ($e^-\gamma \to e^-\gamma$), electron scattering ($e^-e^- \to e^-e^-$) or the annihilation process $e^-e^+ \to 2\gamma$. The cross section is proportional to the square of the scattering amplitude. It is this last quantity that quantum electrodynamics determines as a development in powers of α. Each term of this development can be calculated directly, but it is more convenient to write it as a sum of algebraic quantities represented by the so-called Feynman diagrams. These diagrams have a much more intuitive interpretation than the corresponding algebraic quantities. In the case of e^-e^- scattering the first nontrivial order (Born approximation) is represented by the very simple diagram of Fig. 1. With this diagram we can calculated its cross section. (Analogously for $e^-\gamma \to e^-\gamma$ and $e^-e^+ \to 2\gamma$.) The agreement with experiment is very good.

However, experimental data have sometimes such an accuracy that we must calculate the next-order corrections. These correspond to more complicated diagrams [with

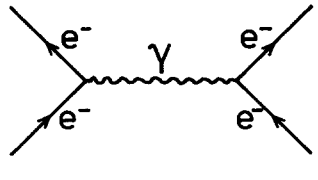

Fig. 1. Two electrons interact by exchanging a virtual (nonphysical, nonobservable) photon.

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