Part III
Electrodynamics

Contents

1 Introduction

In this part we discuss the electric charge configurations and currents that are not constant in time. We start with a study of the effects of and electromagnetic force acting on charge. This study allows us to understand time dependent currents in electric circuits consisting of combinations of resistors, capacitors and inductors. Our main quest is the time dependent behavior of electric and magnetic fields, leading to an elegant description in term of the Maxwell equations.

2 Current and force

2.1 Why does current flow?

Current flows because something is pushing the charge carriers (electrons). And that something is an electric (or magnetic as we will see later) force. Let's analyze that. Consider a piece of wire made of copper with a current density $\vec{J}$ as illustrated in Fig. 1. To make the electrons that constitute

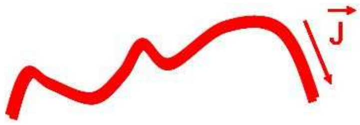

Figure 1: *A piece of wire made of copper with a current density $\vec{J}$.*

the current a force $\vec{f}$ is needed. The force $\vec{f}$ is defined as a force per unit charge. For normal everyday materials and current densities, the current is proportional to the force:

$$\vec{J} = \sigma \vec{f} \tag{1}$$

with σ the conductivity [1] depending on the material. Usually, in tables describing characteristics of materials, one gives the reciprocal value of this quantity, called the resistivity $\rho = 1/\sigma$.

The above expression looks reasonable at first sight, but when you think of it you may get confused about the following. When a force acts on the electrons in the wire, one would expect that the electrons acquire more and more speed with time. Consequently, the current would increase with time, which is not the case. Why not? Well, there appears to be a cancellation due

[1] Historically the symbol σ is used for conductivity, while we (and others) also use it for a surface charge density. So, don't get confused.

to 'collisions' of the free electrons and the nuclei in the material. This effect, that manifests itself as a constant friction, is illustrated in Fig. 2. Right after the current starts an equilibrium between the accelerating force $\vec{f}$ and the de-accelarating friction sets in.

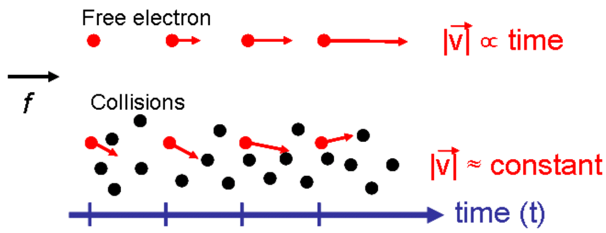

Figure 2: *Top: The free electron undergoes a force $\vec{f}$. Since it is really free its speed increases rapidly with time. Bottom: Another electron that also undergoes a force $\vec{f}$ is constrained in a material and collides with nuclei, such that it effectively senses a frictional force that cancels $\vec{f}$ eventually obtains a constant velocity.*

2.2 Ohm's law

So far, we discussed the relation between force $\vec{f}$ and the current density $\vec{J}$, without bothering about the origin of it. The origin of the force $\vec{f}$ for our purpose is an electric field caused by any device that establishes a potential difference, such as a battery, a van der Graaff generator or a dynamo. We know that in general the electric force is given by $\vec{F} = q\vec{E}$. The force $\vec{f}$ was defined per unit charge, thus we may write $\vec{f} = \vec{E}$. Now we consider a piece of wire with cross section A and length l. A potential difference V over the wire leads to a current I as shown in Fig. 3. According to Ohm's law, V and I are related by:

$$V = IR \tag{2}$$

with R the resistance of the wire. This law is the subject of many physics lectures at high school. Can we derive this law? Yes, in fact its derivation is rather straightforward!

For a perfect conductor, the conductivity $\sigma = \infty$ and thus $\vec{E} = \vec{J}/\sigma = 0$. For a real, everyday, conductor the electric field inside may be zero for stationary charges, but certainly not for the situation when current flows, like in our case. In our piece of wire the electric field is given by:

$$|\vec{E}| = \frac{V}{l} \tag{3}$$

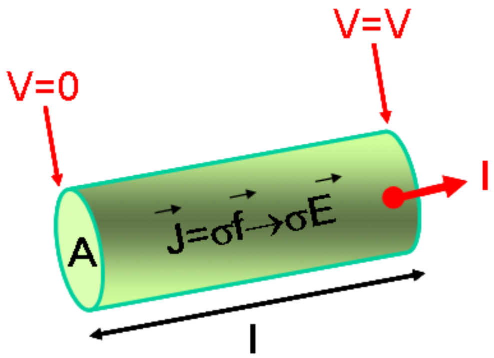

Figure 3: *A piece of wire with length l and cross section A. The potential difference V over its ends leads to a current $\vec{I}$.*

The current density in the wire can be written as:

$$|\vec{J}| = \sigma|\vec{f}| = \sigma|\vec{E}| = \sigma\frac{V}{l} \tag{4}$$

For $|\vec{J}|$ we write I/A and obtain:

$$|\vec{J}| = \frac{I}{A} = \sigma\frac{V}{l} \tag{5}$$

and thus:

$$V = \frac{Il}{\sigma A} \equiv IR \tag{6}$$

with $R \equiv \frac{l}{\sigma A}$. Hence, when we double the length l of the wire the resistance becomes twice as large. When we double the radius of the wire the resistance drops by a factor four. The conductivity of copper is about $\sigma = 6 \times 10^7$ $(\Omega\text{m})^{-1}$. A copper wire of 1 meter length and a cross section of 0.75 mm^2 has a resistance of about $R = 0.02$ Ω.

2.3 Electromotive force

Figure 4 shows an electric circuit with a current density $\vec{J}$. We know that a force $\vec{f}$ drives the current through the wire. In the wire this force is an electric field, an electrostatic field. The field through the wire is produced by the $+$ and $-$ side of a battery. We know from electrostatic theory that $\int_{circuit} \vec{E} \cdot d\vec{l} = 0$, and thus $\int_{wire} \vec{E} \cdot d\vec{l} = -\int_{battery} \vec{E} \cdot d\vec{l}$. Hence, the electric field in the battery points opposite to that in the wire. Anyway, you may ask now: if the field integrals cancel each other there is no net force, so there

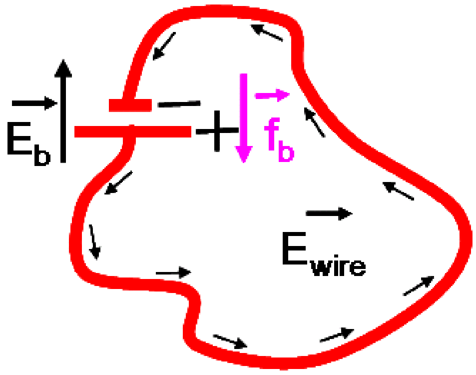

Figure 4: *An electric circuit with current density $\vec{J}$. The electric field and a battery ($\vec{f}_b$) are also indicated.*

can't be a current. Well, we forgot the force, f_b delivered by the battery itself, which is a chemical reaction that maintains the electric potential over the $+$ and $-$ side. The total force that drives a current to a circuit is thus:

$$\vec{f} = \vec{f}_b + \vec{E} \tag{7}$$

Note that $\vec{f}$ is not constant in the circuit: f_b is only present in the battery and $\vec{E}$ in the wire is different from $\vec{E}$ in the battery.

The net effect of this force is the line integral over the circuit and is called the electromotive force:

$$\begin{aligned} EMF &= \int_{circuit} \vec{f} \cdot d\vec{l} = \int_{cicuit} (\vec{f}_b + \vec{E}) \cdot d\vec{l} \\ &= \int_{battery} \vec{f}_b \cdot d\vec{l} + \int_{cicuit} \vec{E} \cdot d\vec{l} \\ &= \int_{battery} \vec{f}_b \cdot d\vec{l} \end{aligned} \tag{8}$$

In an ideal battery there is no friction or net force on the charges and thus:

$$EMF = \int_{battery} \vec{f}_b \cdot d\vec{l} = -\int_{battery} \vec{E} \cdot d\vec{l} = V_b \left(= \int_{wire} \vec{E} \cdot d\vec{l} \right) \tag{9}$$

Hence, the electromotive force is just the potential (difference), V_b, of the battery.

2.4 Induced EMF

Consider the experiment of a circuit pulled out of an magnetic field with a speed $\vec{v}$ as illustrated in Fig. 5. The movement of the circuit out of the

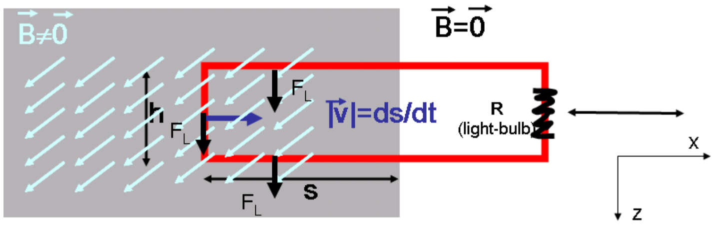

Figure 5: *An electric circuit with a light bulb with resistance R is pulled out of a magnetic field $\vec{B}$. The vertical height of the circuit is h. The length of the part of the circuit that is in the magnetic field is labeled s. The Lorentz force on the positive charge carriers is also indicated.*

magnetic field leads to an EMF. What is the source of the EMF? It is the Lorentz force on the (moving) charges in the wire. In the figure the Lorentz force on the positive charge carries is indicated, because that defines the direction of the current, while in reality the Lorentz force on the electrons is the relevant driving force. The Lorentz force in the horizontal pieces of the circuit point downward, perpendicular to the wire, as indicated. In the vertical piece of wire in the magnetic field the charge carriers that constitute a current will be pushed in the direction as indicated in the figure. The force per unit charge is $\vec{f} = \vec{v} \times \vec{B}$. For the EMF follows:

$$EMF = \int_{circuit} \vec{f} \cdot d\vec{l} = \int_0^h |\vec{f}| dl = \int_0^h |\vec{v} \times \vec{B}| dl = |\vec{v}||\vec{B}|h \qquad (10)$$

Note that the horizontal pieces do not contribute to the EMF. The EMF generated by the movement of the wire is called inductance. A potential difference over the light bulb is generated of size $V_{ind} = EMF$.

For reasons that become clear in a moment we study the flux of the magnetic field through the circuit. The magnetic flux is given by:

$$\Phi_B = \int_{surface} \vec{B} \cdot d\vec{o} = Bhs \qquad (11)$$

where s is the part of the circuit that lies in the magnetic field, which depends on the time when we pull the circuit out of the field. The time derivative of the magnetic flux is given by:

$$\frac{d\Phi_B}{dt} = \frac{d(Bhs)}{dt} = Bh\frac{ds}{dt} = -Bhv = -V_{ind} \qquad (12)$$

Hence, we found a relation between the EMF and the change of the magnetic flux:

$$EMF = -\frac{d\Phi_B}{dt} = -V_{ind} \qquad (13)$$

This principle to create electromotive force is exploited by electric generators (dynamo's).

2.5 Faraday's law

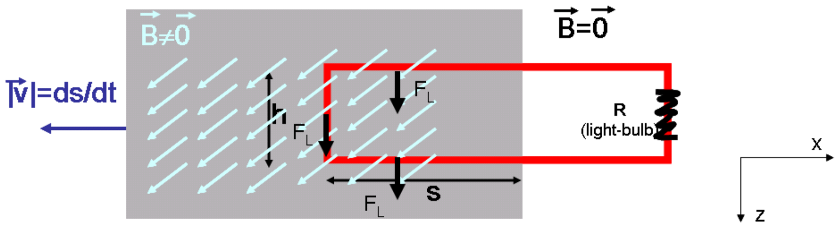

Figure 6: *An electric circuit with a light bulb with resistance R is located in an magnetic field $\vec{B}$. At time $t = 0$ the magnetic field is pulled to the left.*

Figure 6 shows one of the experiments Faraday conducted in 1831. In this experiment an electric circuit is first located in a magnetic field. Then, at time $t = 0$, the magnet is pulled away with a speed $\vec{v}$. The height of the wire loop is h and vertical part of the wire that is in the area of the magnetic field is s. An EMF leads to a current and the light bulb flashes, just like in the 'experiment' when the circuit was pulled away described in the previous section.

We ask the same question now: what is the source of the EMF? Well, based on relativity principle, this is physics-wise exactly the same experiment as we did before. So, to calculate the EMF we transform this experiment to the previous experiment and conclude that the EMF originates from the Lorentz force.

Good, but no cigar! Let's forget about the relativity principle and analyze this experiment based on what we know already about electrostatic, magnetostatic and electrodynamics. We do not transform to the previous experiment: physics-wise the results should be invariant anyway! In this experiment there is no moving charge, so there is no Lorentz force as source of the EMF. We can write:

$$EMF = \int_{circuit} \vec{f} \cdot d\vec{l} = \int_{circuit} \vec{E} \cdot d\vec{l} \neq 0 \tag{14}$$

How can $\int_{circuit} \vec{E} \cdot d\vec{l} \neq 0$? Well, this is electrodynamics, not electrostatics! So, this time the EMF is electric in nature. When we use Stokes Law, we obtain:

$$
\begin{aligned}
EMF &= \int_{circuit} \vec{E} \cdot d\vec{l} \\
&= \int_{surface-circuit} (\vec{\nabla} \times \vec{E}) \cdot d\vec{o}
\end{aligned} \tag{15}
$$

From the previous section we know that

$$
\begin{aligned}
EMF &= -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_{surface-circuit} \vec{B} \cdot d\vec{o} \\
&= -\int_{surface-circuit} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{o}
\end{aligned} \tag{16}
$$

When we combine these results we find Faraday's Law:

$$
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{17}
$$

Hence, a changing magnetic field induces an electric field.

In the first experiment we calculated the Lorentz force and therefore knew the direction of the current. In the second experiment it is already much harder to figure out what the direction of the current will be. And, you could get really lost when you have to predict the direction of the current in case of a time dependent magnetic field! Fortunately, there is a handy trick to do this, called Lenz's Law. It states that the:

'*induced current attempts to compensate the change of magnetic flux*'.

Hence, when the magnetic flux decreases (when the circuit is pulled out of the field) the induced current generates a magnetic field in the original field direction (think of the circuit as a solenoid with one winding). Use your right hand to deduce that the current flows counter clock wise in the experiments we just discussed.

2.6 Knowledge and skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- *You can derive Ohm's Law* $V = IR$ *from* $\vec{J} = \sigma\vec{f}$.
- *You can calculate the* EMF *of an electric circuit.*
- *You understand the following relation:*

$$
EMF = V_{induced} = -\frac{d\Phi_B}{dt} \tag{18}
$$

and that a changing magnetic field induces an electric field according to the relation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{19}$$

- *You know how to predict the induced current in an electric circuit using Lenz's Law.*

3 Electromagnetic inductance and circuits

3.1 Self-inductance

Figure 7 depicts an illustration of a current loop and its magnetic field. From

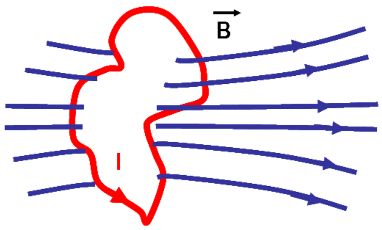

Figure 7: *An illustration of a current loop and its magnetic field.*

Biot-Savart's Law we know that whatever the exact shape of the current loop, its magnetic field is proportional to the current: $|\vec{B}| \propto I$. *When we consider the magnetic flux through the loop, originating from its own magnetic field we can write:*

$$\begin{aligned} \Phi_B &= \int_{loop} \vec{B} \cdot d\vec{\sigma} \propto I \\ &\equiv LI \end{aligned} \tag{20}$$

with L *a factor of proportionality called self-inductance, with unit Henry, 1 H=1 Vs/A. The self-inductance only depends on the geometry and size of the loop.*

When the wire loop is placed in a magnetic field, it requires an electromotive force to change its current:

$$EMF_{loop} = V_{loop} = -\frac{d\Phi_B}{dt} = L\frac{dI}{dt} \tag{21}$$

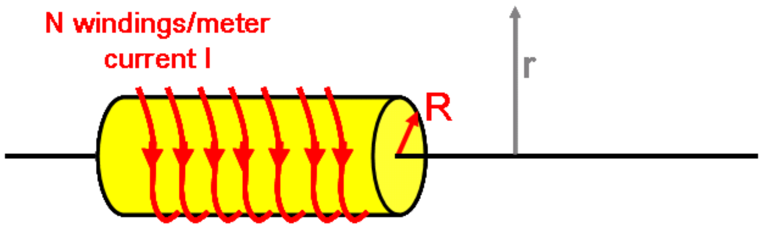

Figure 8: *A solenoid with N windings per meter, radius R and length l.*

3.1.1 Self-inductance of a solenoid

Figure 8 shows a solenoid with radius R, length l and N windings per meter. To determine the self-inductance of this solenoid we calculate the flux through all windings from its own magnetic field. Using Ampère's law we know the field in the solenoid: $\vec{B} = \mu_0 N I \hat{z}$. The magnetic flux though a single winding is:

$$\Phi_B^1 = \int_{winding-surface} \vec{B} \cdot d\vec{o} \tag{22}$$

The field points in the direction of normal of the surface, so

$$\begin{aligned} \Phi_B^1 &= \int_{winding-surface} |\vec{B}| do \\ &= \int_{\phi=0}^{2\pi} \int_{r=0}^{R} \mu_0 N I r dr d\phi \\ &= \mu_0 N I \pi R^2 \end{aligned} \tag{23}$$

The number of windings of the solenoid is Nl, which leads to a total flux through the solenoid of:

$$\Phi_B = Nl\Phi_B^1 = \mu_0 N^2 I \pi R^2 l \tag{24}$$

As predicted, this flux is proportional to the current I. Hence, the self-inductance, L, of this solenoid is:

$$L = \Phi_B / I = \mu_0 N^2 l \pi R^2 \tag{25}$$

3.1.2 Self-inductance of a coaxial cable

Figure 9 shows a coaxial cable with inner radius a and outer radius b and length l. A current I flows though the inner core in one direction and runs in the opposite direction in the outer core.

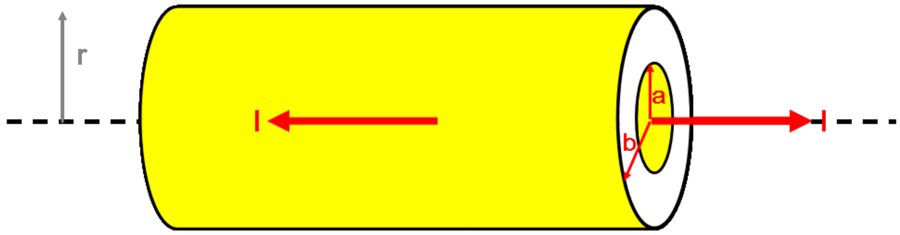

Figure 9: *A coaxial cable with inner radius a and outer radius b and length l. In the inner and outer core, but in opposite direction, flows a current I.*

Inside the cable in the region $a < r < b$, there is a magnetic field in the ϕ direction. Using Ampère's Law we determine the magnitude of this field:

$$\int_{loop} \vec{B} \cdot d\vec{l} = \int_0^{2\pi} B_\phi d\phi = B_\phi 2\pi r = \mu_0 I \tag{26}$$

and thus $B_\phi = (\mu_0 I)/(2\pi r)$.

To calculate the magnetic flux in the coaxial cable we have to determine 'amount' of magnetic field that runs through the region between $a < r < b$ and length l:

$$\begin{aligned} \Phi_B &= \int_0^l dl' \int_a^b dr \frac{\mu_0 I}{2\pi r} \\ &= l \int_a^b dr \frac{\mu_0 I}{2\pi r} \\ &= l \frac{\mu_0 I}{2\pi} ln(b/a) \end{aligned} \tag{27}$$

And again, the flux is proportional to I and the self-inductance becomes:

$$L = \frac{\Phi_B}{I} = l \frac{\mu_0}{2\pi} ln(b/a) \tag{28}$$

3.2 Mutual inductance

Consider two identical infinitesimal loops (we use square loops) as shown in Fig. 10, both with current $I = I_1 = I_2$. For the flux through loop 2 from loop 1 we can write $\Phi_2^{(1)} = MI_1 = MI$. With M a factor of proportionality, called the mutual inductance. Physically the situation is completely symmetric. Hence, for the flux through loop 1, we can write $\Phi_1^{(2)} = MI_2 = MI$, with the same factor M. If we change the orientation of the loops, the factor M changes accordingly, but the symmetry $\Phi_1^{(2)} = \Phi_2^{(1)}$ remains.

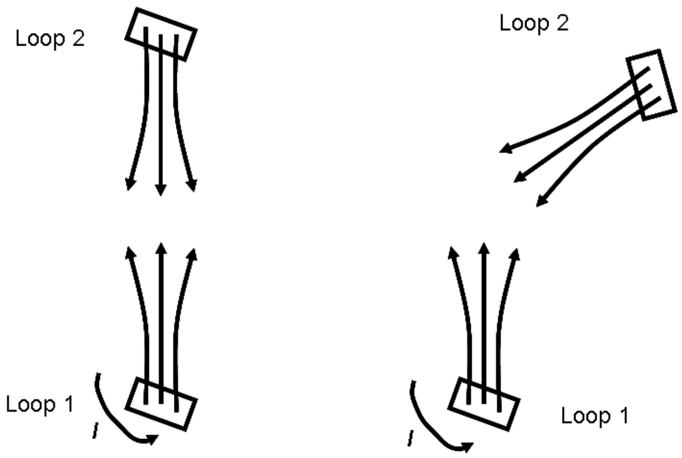

Figure 10: *Illustration of two infinitesimal loops (1 and 2) with current I that 'sense' each other's magnetic field, indicated by the arrows. The left and right drawing depict two different orientations of the loops.*

Is this relation also valid for macroscopic loops of any size and shape? We consider the infinitesimal loops again. Suppose we extend the second loop, 2a, with an additional infinitesimal loop, 2b, (with the same current) as depicted in Fig. 11. The extended loop 2=2a+2b now forms a single loop again, but just twice as large as the original. We can write:

$$\Phi_1^{(2)} = \Phi_1^{(a+b)} = \Phi_1^{(a)} + \Phi_1^{(b)} = M_a I + M_b I = M_{a+b} I \tag{29}$$

All the arguments before allow us to write for the individual contributions from a and b:

$$\begin{aligned} \Phi_1^{(a)} &= M_a I = \Phi_a^{(1)} \\ \Phi_1^{(b)} &= M_b I = \Phi_b^{(1)} \end{aligned}$$

and thus:

$$\Phi_2^{(1)} = \Phi_{(a+b)}^{(1)} = \Phi_a^{(1)} + \Phi_b^{(1)} = M_a I + M_b I = M_{a+b} I \tag{30}$$

We could extend the loop 2 again by adding another additional infinitesimal loop and would find the same relation again. In fact, we can make macroscopic loop of any size and shape of a collection of infinitesimal loops and thus we can conclude that

$$\begin{aligned} \Phi_1^{(2)} &= M I_2 \\ \Phi_2^{(1)} &= M I_1 \end{aligned} \tag{31}$$

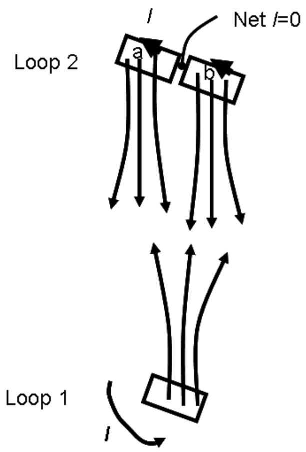

Figure 11: *Illustration of two loops (1 and 2) with current I that 'sense' each other's magnetic field, indicated by the arrows. The second loop consists of two infinitesimal loops, a and b.*

where M depends on the exact geometry of the configuration.

As an example we consider the configuration with a single current loop L, radius R_L, placed in a long solenoid S with radius R_S and N windings per meter as illustrated in Fig. 12. A current I runs through the winding(s) of L

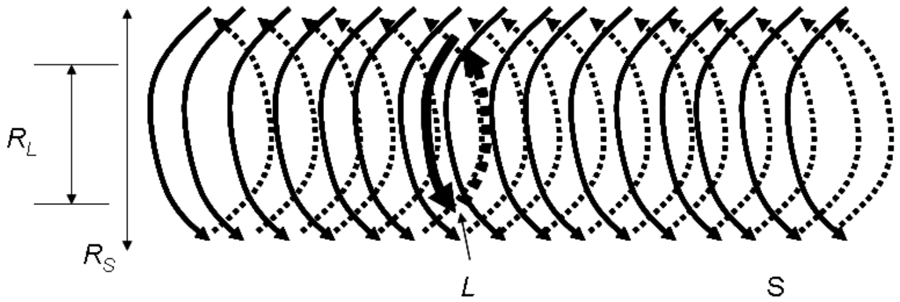

Figure 12: *Illustration of a single loop L in a long solenoid S, both with current I.*

and S. What is the flux through the solenoid S from L. You could calculate the dipole field of L and determine the flux that S sees by integration. Yes, you could, but you should not. Instead, you should write down: $\Phi_L^{(S)} = MI = \Phi_S^{(L)}$. Thus, when you calculate the flux through L from S, $\Phi_L^{(S)}$, you also have $\Phi_S^{(L)}$. The solenoid generates a magnetic field of $B = B_S = \mu_0 NI$

in the axial direction. The flux through L is:

$$\Phi_L^{(S)} = B\pi R_L^2 = \mu_0 \pi R_L^2 NI \tag{32}$$

We can read off that $M = \mu_0 \pi R_L^2 N$ *and thus the flux through the solenoid from the single loop is thus:*

$$\Phi_S^{(L)} = MI = \mu_0 \pi R_L^2 NI \tag{33}$$

3.3 Electric circuits

In the previous sections we discussed the relation between the current and the EMF *for a resistor and for an inductor. Another object is the capacitor which was introduced in the course on electrostatics:* $V_C = Q/C$. *Analogous to the self-inductance* L *for an inductor, the capacitance* C *depends only on the geometry of the configuration. The* EMF *to charge a capacitor equals* V_C *and a relation with the current can be easily derived:*

$$\frac{dV_C}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{C}I \tag{34}$$

Now we have three objects that we can combine to build electric circuits. Electric circuits that we consider consist of a battery that provides the potential difference V_0 *leading to current through a closed circuit that consist of, or combinations of, resistor(s), capacitor(s) and inductor(s). The voltage change going around the circuit in the direction of the current flow should be zero (Kirchoff's Voltage Law). Note that we wrote voltage change, we can have a voltage drop or a voltage rise, implying that we should keep track of* $+$ *and* $-$ *signs once again.*

For our three objects the voltage always drops when we follow the current I *in the positive direction. Remember that when we cross a battery (in the direction of positive current) the voltage increases by the battery potential* V_0. *We summarize these properties in Table 1.*

Consider the electric circuit consisting of a battery with potential V_0, *a self-inductance* L *and a resistor* R, *shown in Fig. 13. At time* $t = 0$ *the battery is just hooked on and the current through the circuit is zero:* $I(0) = 0$. *What is the current at any time? To solve this problem, we 'walk' around starting just before the resistor and add all voltage changes: We use Table 1 and obtain:*

$$\begin{aligned} -V_R - V_L + V_0 &= 0 \\ V_0 - V_L &= V_R \end{aligned}$$

Object	Potential change	relation with I
Resistor R	$-V_R$	$V_R = IR$
Inductor L	$-V_L$	$V_L = L\frac{dI}{dt}$
Capacitor C	$-V_C$	$\frac{dV_C}{dt} = \frac{1}{C}I$
Battery V_0	V_0	no resistance

Table 1: *The voltage change over possible elements in an electric circuit when we follow the direction of the positive current.*

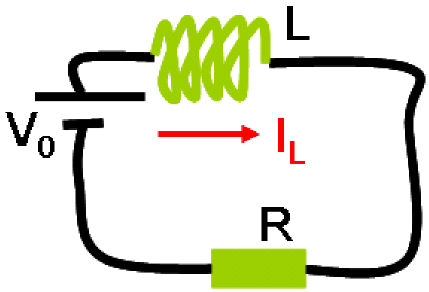

Figure 13: *An electric circuit with a battery, resistor and a self-inductance.*

$$V_0 - L\frac{dI}{dt} = I(t)R$$
$$\frac{dI}{dt} = -\frac{R}{L}I(t) - \frac{V_0}{L} \tag{35}$$

Physically we can interpret the last equation as the electromotive force $V_0 - L\frac{dI}{dt}$ of a battery and an inductor that establishes the potential $V_R = IR$ to drive a current through a resistor.

Mathematically, it is a first order differential equation with the structure $\frac{df}{dt} = af + b$. The general solution is $f(t) = ke^{at} - \frac{b}{a}$, with a, b and k constants. Returning to our LR circuit, we write:

$$I(t) = \frac{V_0}{R}(1 - e^{-(R/L)t}) \tag{36}$$

Figure 14 shows the current as function of time. At time intervals L/R, called the time constant, the current increases with fractions $1 - 1/e$.

Another first order circuit is shown in Fig. 15, which consists of a resistor, a capacitor and a battery. At time $t = 0$, the battery is just hooked on, the charge on the capacitor $Q_C(0) = 0$ (and $V_C(0) = 0$) and the current $I(0) = 0$. What is the potential over the capacitor at any time? We 'walk'

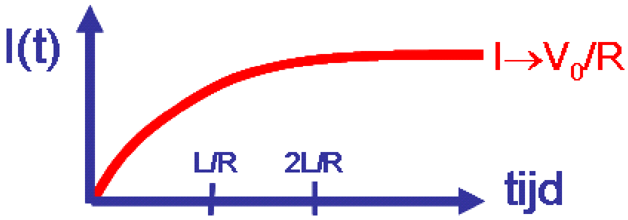

Figure 14: *The current of an LR circuit as function of time.*

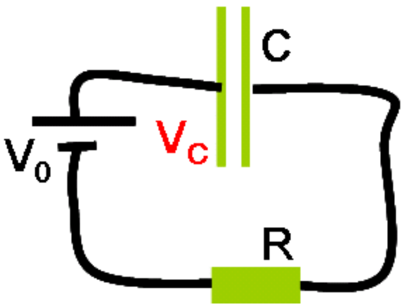

Figure 15: *An electric circuit with a battery, resistor and a capacitor.*

around starting just before the resistor and add all voltage changes: We use Table 1 and obtain:

$$\begin{aligned} -V_R - V_C + V_0 &= 0 \\ V_0 - V_C &= V_R \\ V_0 - V_C &= I(t)R \\ V_0 - V_C &= RC\frac{dV_C}{dt} \end{aligned} \tag{37}$$

with solution:

$$V_C(t) = V_0(1 - e^{-t/(RC)}) \tag{38}$$

The time constant of this circuit is RC as illustrated by Fig. 16.

3.4 Energy in electric circuits

3.4.1 Energy dissipation in a resistor

Consider the basic circuit of a battery and a resistor, shown in Fig. 17 below. Each second a charge $Q = I$ is pumped through the resistor, which requires

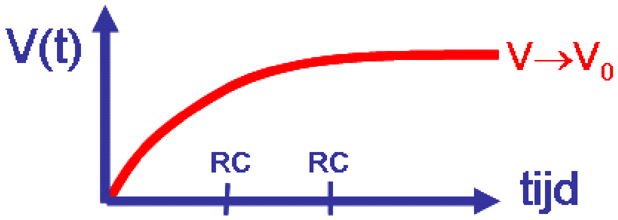

Figure 16: *The potential difference over a capacitor in an RC circuit as function of time.*

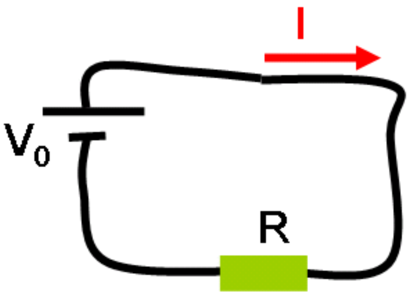

Figure 17: *An electric circuit consisting of a battery V_0 and a resistor R.*

work W. Some chemical reaction in the battery provides the necessary energy. The dissipated energy in the resistor is transformed into heat. Since the voltage change over the resistor is $V = V_0$, the energy needed to transport the charge equals $W = VQ$. The energy per second dW/dt, the power, dissipated by resistor is $P = VQ/1\mathrm{s} = VI$ with unit Watt. Using Ohm's Law we obtain the equivalent expressions $P = I^2R$ and $P = V^2/R$.

3.4.2 Energy stored in a capacitor

In the course on electrostatics we have derived that the energy in a capacitor is given by $W = \frac{1}{2}CV^2$. In this section we study the energy stored in a capacitor in an electric circuit and obviously expect to find the same result. Consider an RC circuit as discussed in section 3.3. From the calculation in this section we conclude that the capacitor gets charged (with time constant RC). For the power required to charge the capacitor we write:

$$P = \frac{dW_C}{dt} = V_C(t)I(t) = \frac{Q_C}{C}\frac{dQ_C}{dt} \tag{39}$$

The total energy in a charged capacitor ($Q_C = Q$) is obtained by integration over time:

$$\begin{aligned} W &= \int_{t=0}^{\infty} \frac{dW}{dt} dt = \int_{t=0}^{\infty} \frac{Q_C}{C} \frac{dQ_C}{dt} dt \\ &= \int_{t=0}^{\infty} \frac{Q_C}{C} \frac{dQ_C}{dt} dt \\ &= \int_{Q_C=0}^{Q} \frac{Q_C}{C} dQ_C = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2 \end{aligned} \tag{40}$$

which is no surprise.

An alternative expression for the energy in the electric field, also derived in the Chapter on electrostatics, is:

$$W_C = \frac{\epsilon_0}{2} \int_{volume} E^2 dv \tag{41}$$

Let's check this expression for a parallel plate capacitor. The distance between the plates is d and the plates have a surface area A, leading to a capacity $C = \frac{\epsilon_0 A}{d}$. In the ideal case, the electric field outside the plates is zero, while in between the plates the field is given by $|\vec{E}| = V/d$. For the energy we find

$$\begin{aligned} W_C &= \frac{\epsilon_0}{2} \int_{volume} \frac{V^2}{d^2} dv \\ &= \frac{\epsilon_0}{2} \frac{V^2}{d^2} A d = \frac{\epsilon_0 A}{2d} V^2 = \frac{1}{2} C V^2 \end{aligned} \tag{42}$$

as expected.

3.4.3 Energy stored in a self-inductance

To determine the energy stored in a self-inductance we can follow the same strategy as above. For the power required to reach a current $I_L = I$ in a self-inductance L we write:

$$P = \frac{dW_L}{dt} = I_L(t) V_{induced}(t) = I_L(t) L \frac{dI_L}{dt} \tag{43}$$

The total energy in the self-inductance is obtained by integration over time:

$$\begin{aligned} W_L &= \int_{t=0}^{\infty} \frac{dW}{dt} dt = \int_{t=0}^{\infty} I_L(t) L \frac{dI_L}{dt} dt \\ &= \int_{t=0}^{\infty} I_L(t) L dI_L \\ &= \frac{1}{2} L I^2 \end{aligned} \tag{44}$$

This formula has the same structure as the expression for a capacitor. Is there also an alternative expression for energy in a self-inductance in terms of the magnetic field? Well, we could just try:

$$W_L = \frac{1}{2\mu_0} \int_{volume} B^2 dv \tag{45}$$

Let's check this expression for a solenoid. For a solenoid $L = \mu_0 l N^2 \pi R^2$ *and thus (using equation 44)* $W_L = \frac{\mu_0}{2} l N^2 \pi R^2 I^2$. *Now we use the trial equation 45:*

$$\begin{aligned} W_L &= \frac{1}{2\mu_0} \int_{volume} (\mu_0 N I)^2 dv \\ &= \frac{1}{2\mu_0} (\mu_0 N I)^2 l \pi R^2 \\ &= \frac{\mu_0}{2} l N^2 \pi R^2 I^2 \end{aligned} \tag{46}$$

which leads to the same result! Equation 45 is indeed the correct expression for the energy of the magnetic field. By the way, we can also use this expression for the energy to calculate the self-inductance of an object: $L = 2W_L/I^2$.

3.5 Knowledge and skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- *You understand the meaning of self-inductance and mutual inductance.*
- *You can calculate the self-inductance of the solenoid and a coaxial cable.*
- *You can calculate the time dependent current in RC and LR circuits.*
- *You can calculate the energy in a self-inductance and in the magnetic field.*

4 Maxwell equations

In this section we complete the field equations and derive the existence of electromagnetic waves: light! First we summarize the field equations, we encountered so far. Look at the structure of these equations in Table 2. There is an asymmetry between the electric and magnetic field, due to the

Comment	Integral
Gauss	$\int_{surface} \vec{E} \cdot d\vec{o} = Q_{enclosed}/\epsilon_0 = \int_{volume} \frac{\rho}{\epsilon_0} dv$
No magnetic monopoles	$\int_{surface} \vec{B} \cdot d\vec{o} = 0$
Faraday	$\int_{loop} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampère	$\int_{loop} \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} = \int_{surface} \vec{J} \cdot d\vec{o}$
	Differential
Gauss	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
No magnetic monopoles	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Table 2: *The (incomplete) field equations based on electrostatics, magnetostatics and Faraday's Law in integral and differential form.*

fact that there exist no magnetic monopoles. In addition, there is nowhere a term $\frac{\partial \vec{E}}{\partial t}$. Did we miss something? How can we find that term, called the Maxwell term?

4.1 The Maxwell term from a gedanken experiment

At time $t = 0$ an empty capacitor is charged in a circuit as depicted in Fig. 18. Around the wire at position a a magnetic field is generated by the current I such that Ampère's law is fulfilled:

$$\int_{loop\,a} \vec{B} \cdot d\vec{l} = \mu_0 I \tag{47}$$

In this case, the loop a spans a surface, a disk, penetrated by the the wire, thus the enclosed current is I.

Now we modify the surface a little bit and stretch it in between the capacitor plates, such that it acquires the shape of a balloon b. In fact, no wire is pointing through the surface and the the enclosed current is zero:

$$\int_{loop\,b} \vec{B} \cdot d\vec{l} = 0 \tag{48}$$

which suggests that the magnetic field has vanished in conflict with equation 47. Something must be wrong with Ampère's law! How can we fix that? Well, the electric field between the plates of the capacitors is $|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q(t)}{A\epsilon_0}$. The time derivative is $|\frac{\partial \vec{E}}{\partial t}| = \frac{1}{A\epsilon_0}\frac{dQ}{dt} = \frac{I}{A\epsilon_0}$. A changing electric field is related

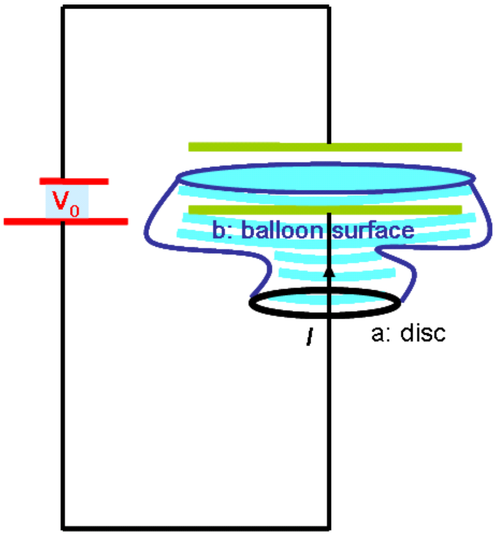

Figure 18: *An electric circuit with a parallel plate capacitor. An Amperian loop is indicated. Also indicated are two surfaces (a and b) that are enclosed by the Amperian loop. Surface a has the 'usual' shape of a disk. Surface b has the shape of a balloon and is stretched in between the plates of the capacitor.*

to a current, which fixes Ampère's Law:

$$\int_{loop} \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o} \tag{49}$$

and we conclude that a changing electric field induced a magnetic field!

This calculation may look not so scientific to you or perhaps it even looks like a hat-trick. Right, but nevertheless the result correctly describes the original gedanken experiment!

4.2 The Maxwell term from a controlled charge explosion

Another example that yields the Maxwell term is a slowly exploding charge as illustrated in Fig. 19. A large collection of charge at the origin slowly explodes. In the figure, an imaginary sphere is also shown. The charge that emerges from the explosion uniformly traverses the sphere. You could compare this situation with a radioactive decay. However, in radioactive decays usually, besides electrons, also photons and neutrinos are produced. Anyway, charge-wise, the explosion is comparable to radioactive decays. When

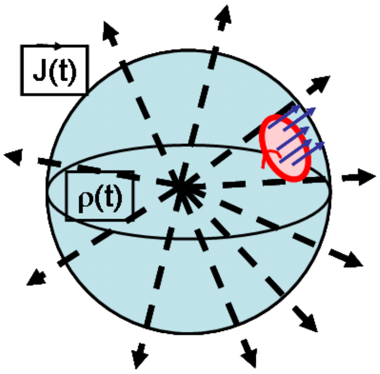

Figure 19: *A large amount of charge at the origin slowly 'explodes'. The charge that emerges from the explosion traverses an imaginary sphere. On the surface of this sphere an infinitesimal Amperian loop is also indicated*

we start with N_0 nuclei with charge q and lifetime τ, then the number of nuclei at a given time is $N(t) = N_0 e^{-t/\tau}$ and the total charge is $Q(t) = qN(t)$. The electricfield of the charged nuclei is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{qN_0 e^{-t/\tau}}{r^2} \hat{r} \tag{50}$$

Each decay produces a charge q which leads to a 'shower of charge' that escapes: with corresponding current:

$$I = -\frac{dQ(t)}{dt} = \frac{qN_0}{\tau} e^{-t/\tau} \tag{51}$$

For the current density through a spherical surface we find:

$$\vec{J} = \frac{qN_0}{\tau} \frac{e^{-t/\tau}}{4\pi r^2} \hat{r} \tag{52}$$

where r represents the radius of the surface.

Now we investigate the integral $\int_{loop} \vec{B}(r) \cdot d\vec{l}$ of the magnetic field over a small loop on the surface of the sphere, also indicated in Fig. 19. The dot-product filters out the component of the magnetic field along the surface of the sphere, $B_{//}$. What is the magnitude of this component? Well, look at the symmetry of the configuration. The can not be a component of the magnetic field along the surface, and thus:

$$\int_{loop} \vec{B}(r) \cdot d\vec{l} = \int_{loop} B_{//}(r) dl = 0 \tag{53}$$

According to Ampère's law, this loop integral should be proportional with the enclosed current: $\mu_0 I_{enclosed} = \mu_0 \vec{J} \cdot d\vec{o}$. Is the enclosed current zero as equation 53 demands? No, because there is certainly charge showering through the surface enclosed by the loop. So, Ampère's Law fails again. Let's see whether the changing electric field completes the equation:

$$\begin{aligned}\int_{loop} \vec{B} \cdot d\vec{l} &= \mu_0 I_{enclosed} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o} \\ &= \mu_0 \int_{surface} \vec{J} \cdot d\vec{o} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o}\end{aligned} \tag{54}$$

Now we substitute the expressions we derived for the current density (equation 52) and the electric field (equation 50) and obtain:

$$\begin{aligned}\int_{loop} \vec{B} \cdot d\vec{l} &= \mu_0 \int_{surface} N_0 \frac{Q}{\tau} \frac{e^{-t/\tau}}{4\pi r^2} do + \mu_0 \epsilon_0 \int_{surface} \frac{N_0 Q}{4\pi \epsilon_0 r^2} \frac{\partial e^{-t/\tau}}{\partial t} do \\ &= \mu_0 \int_{surface} N_0 \frac{Q}{\tau} \frac{e^{-t/\tau}}{4\pi r^2} do - \mu_0 \int_{surface} N_0 \frac{Q}{\tau} \frac{e^{-t/\tau}}{4\pi r^2} do \\ &= 0\end{aligned} \tag{55}$$

So, the Maxwell term $\mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o}$, saves the day again!

4.3 Continuity equation

In the previous section we used the fact that charge is conserved. A current through a (closed) surface was the result of a changing charge in the enclosed volume:

$$\begin{aligned}I_{surface} &= -\frac{dQ_{volume}}{dt} \\ \int_{surface} \vec{J} \cdot d\vec{o} &= -\frac{d}{dt} \int_{volume} \rho dv \\ \int_{surface} \vec{J} \cdot d\vec{o} + \frac{d}{dt} \int_{volume} \rho dv &= 0\end{aligned} \tag{56}$$

Using Gauss's rule we can write:

$$\int_{volume} \vec{\nabla} \cdot \vec{J} dv + \int_{volume} \frac{d\rho}{dt} dv = 0 \tag{57}$$

and thus:

$$\vec{\nabla} \cdot \vec{J} + \frac{d\rho}{dt} = 0 \tag{58}$$

This expression is known as the continuity equation.

Is this consistent with our renewed Ampère's Law? We start with:

$$\int_{loop} \vec{B} \cdot d\vec{l} = \mu_0 \int_{surface} \vec{J} \cdot d\vec{o} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o} \tag{59}$$

and use Stoke's Law to write:

$$\int_{surface} \vec{\nabla} \times \vec{B} \cdot d\vec{o} = \mu_0 \int_{surface} \vec{J} \cdot d\vec{o} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{o} \tag{60}$$

Note that the surface is arbitrary, thus the integrands must be equal:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{61}$$

(which by the way is the renewed Ampère's Law in differential form!) Now take the divergence of this equation and realize that the divergence of a rotation is zero by construction:

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = 0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \tag{62}$$

or equivalently

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = 0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \frac{\partial \rho}{\partial t} \tag{63}$$

from which we can read off the continuity equation. As a matter of fact we can conclude that Ampère's Law extended with Maxwell's term leads to charge conservation.

4.4 The complete set of Maxwell equations

The time has come to write down the complete set of field equations which we derived during our tour through electrostatics, magnetostatics and electrodynamics leading to Faraday's Law and the Maxwell term. The equations are know as the Maxwell Equations and are listed in table 3. The physical behavior of electric and magnetic field are described by these equations. In the following section we use the Maxwell Equations to derive the existence of electromagnetic waves of which light is a specific example.

4.5 Electromagnetic waves

The Maxwell equations, based on empirical studies, can now be used to further investigate the physics of the electric and magnetic field. In this

Comment	Integral
Gauss	$\int_{surface} \vec{E} \cdot d\vec{\sigma} = Q_{enclosed}/\epsilon_0 = \int_{volume} \frac{\rho}{\epsilon_0} dv$
No magnetic monopoles	$\int_{surface} \vec{B} \cdot d\vec{\sigma} = 0$
Faraday	$\int_{loop} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
Ampère+Maxwell term	$\int_{loop} \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \epsilon_0 \int_{surface} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{\sigma}$
	Differential
Gauss	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
No magnetic monopoles	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday	$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$
Ampère+Maxwell term	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Table 3: *The complete set of field equations based on electrostatics, magnetostatics and electrodynamics, called the Maxwell Equations, in integral and differential form.*

section we study the fields in free space, which requires some mathematics, but the result will be worth it.

In vacuum, away from electric charges and currents the Maxwell Equations (in differential form) simplify to:

$$\begin{aligned} (a) \quad & \vec{\nabla} \cdot \vec{E} = 0 \\ (b) \quad & \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ (c) \quad & \vec{\nabla} \cdot \vec{B} = 0 \\ (d) \quad & \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{64}$$

Now take the rotation of 64b and write:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} \tag{65}$$

First, we concentrate on the left hand side. Using basic calculus we obtain:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} \tag{66}$$

Remember that we are in vacuum and use equation 64a to write:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla}^2 \vec{E} \tag{67}$$

Now, we proceed with the right hand side:

$$-\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t} \tag{68}$$

and use 64d to write:

$$-\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{69}$$

When the results are combined we obtain:

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{70}$$

Starting with the rotation of 64d you find a similar relation for the magnetic field:

$$\vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{71}$$

Fine, so what? Well, remember the theory of waves. The classical wave equation for a wave in the z direction with speed v is:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial z^2} \tag{72}$$

with typical solution $A\,cos(kz - \omega t)$ where $\omega^2 = k^2 v^2$. Hence, the equations 70 and 71 imply the existence of electromagnetic waves with speed $v = \sqrt{\frac{1}{\mu_0 \epsilon_0}} = c$, the speed of light!

4.5.1 Monochromatic waves in one dimension

In this section we study a typical solution of the electromagnetic wave equation. To simplify the math, we assume:

- *The electric and magnetic field only depend on z and t;*
- *We stay in the vacuum which is infinitely large.*

This leads to the following wave equations:

$$\begin{aligned} \frac{\partial^2 \vec{E}}{\partial t^2} &= c^2 \frac{\partial^2 \vec{E}}{\partial z^2} \\ \frac{\partial^2 \vec{B}}{\partial t^2} &= c^2 \frac{\partial^2 \vec{B}}{\partial z^2} \end{aligned}$$

with solutions of the form:

$$\vec{E} = \vec{E}^0 cos(kz - \omega t)$$
$$\vec{B} = \vec{B}^0 cos(kz - \omega t) \tag{73}$$

with $\omega = kc$. *The constants* $\vec{E}^0$ *and* $\vec{B}^0$ *can be determined by applying Maxwell Equations in vacuum (again):*

- $\vec{\nabla} \cdot \vec{E} = 0 = -E_z^0 k sin(kz - \omega t)$ *implies that* $E_z^0 = 0$.
- $\vec{\nabla} \cdot \vec{B} = 0 = -B_z^0 k sin(kz - \omega t)$ *implies that* $B_z^0 = 0$.
- $\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ *leads to:*

$$+cB_y^0 = E_x^0$$
$$-cB_x^0 = E_y^0 \tag{74}$$

which can be written as:

$$\vec{B}^0 = \frac{1}{c}\hat{z} \times \vec{E}^0 \tag{75}$$

The equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ *implies the same and adds no information.*

We summarize our findings by saying that the Maxwell Equations have solutions that can be interpreted as electromagnetic waves with speed c, the speed of light. These waves are transverse: the electric and magnetic fields have no components in the direction of propagation ($E_z = B_z = 0$). *In addition the fields are also mutual transverse and in phase. Figure 20 shows an illustration of an electromagnetic wave.*

4.6 Knowledge and skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- *You understand the problem in Ampère's Law.*
- *You can fix Ampère's Law.*
- *You can write down the Maxwell Equations.*
- *You can derive the continuity equation from these equations.*
- *From the Maxwell Equations in vacuum you can proof the existence of light and its properties.*

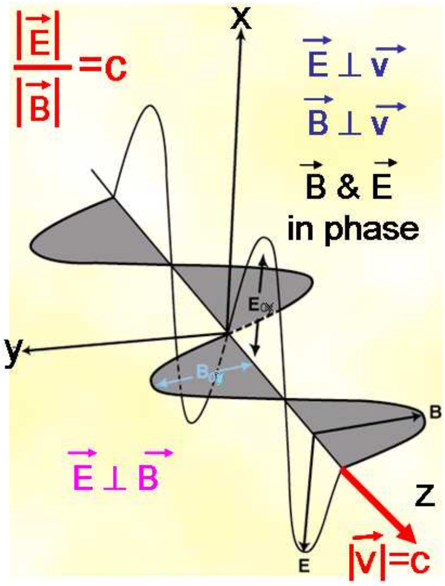

Figure 20: *A schematic view of an electromagnetic wave.*