Part I
Electrostatics

Contents

1 History and basics

The word electricity is derived from $\eta\lambda\epsilon\kappa\tau\rho o\nu$ (*elektron*), the Greek word for amber. The reference to amber is not a coincidence: when an amber rod is rubbed with fur it attracts certain objects, such as a piece of paper or a hair. This was documented first by Thales of Miletus (600 BC).

Amber is not unique in this sense. Many other materials, such as glass, rubber, PVC and Ebony can be electrified by rubbing it with for example fur, silk or wool. Many experiments in the 17^th and 18^th century were conducted to study this phenomenon. This work led to the discovery of two kinds of electricity (positive and negative charge) that attract, while electricity of the same kind repels. A simple experiment with a glass rod and a plastic ball as illustrated in Fig. 1 shows that like charges repel. Another

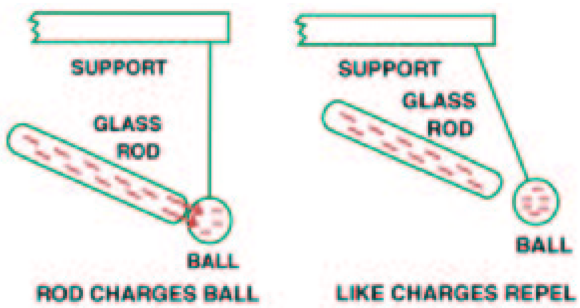

Figure 1: *A glass rod is electrically charged by rubbing it with fur. The rod is then used to charge a plastic ball. Now the two objects repel.*

experiment, using an additional rubber rod and ball with opposite electrical charge demonstrates that unlike charges attract, see Fig. 2. Well known is

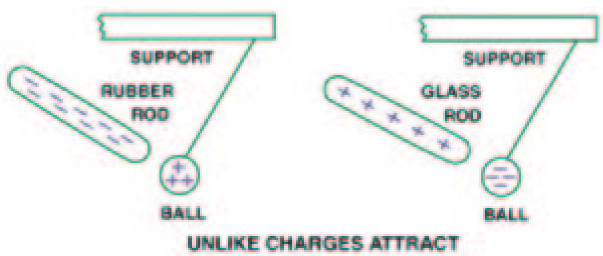

Figure 2: *The rod and ball have opposite electrical charge. The two objects attract.*

the glass rod that, rubbed with a silk cloth, obtains a 'positive' electrification.

Famous is the Ebony rod that after being rubbed with a cat's skin is 'negative' electrified.

2 Electrical Force and Field

2.1 Coulomb's law

Charles Augustin Coulomb (1736 - 1806) was a French physicist who studied the electrical forces in a quantitative manner utilizing a torsion balance. Through this experimentation, Coulomb found that the force between two (point) charges is given by:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r} \tag{1}$$

where $r = |\vec{r}|$ represents the distance between the test charge q and source charge Q. The vector $\vec{r}$ connects the two charges. The direction of the force $\vec{F}$ is given by $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, the unit vector pointing from Q to q, see also Fig.3. Equivalently, we can write for the Coulomb force:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \frac{\vec{r}}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^3} \vec{r} \tag{2}$$

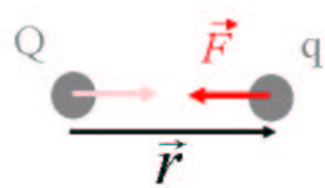

Figure 3: *The Coulomb force between a source charge Q and a test charge q.*

The SI unit of electrical charge is the Coulomb, which can be abbreviated to the unit C in equations. The factor $\frac{1}{4\pi\epsilon_0}$ is a constant term with ϵ_0 called the 'electrical permittivity'. The electrical permittivity has the numerical value:

$$\epsilon_0 = 8.85419 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2} \tag{3}$$

Often, Coulomb's law is also written as

$$\vec{F} = K\frac{qQ}{r^2}\hat{r} \tag{4}$$

with K the electrical constant:

$$K = 8.98755 \times 10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2} \tag{5}$$

Coulomb's law is about the electrical force between two point charges. If we have three, four or whatever number of charges at different positions, the total force on a test charge becomes:

$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} + ... \tag{6}$$

with $\vec{F_i}$ the electrical force by source charge Q_i on our test charge given by equation 1. Using this 'superposition principle' we can write this total force as:

$$\vec{F} = \sum_{i=1,N} \frac{1}{4\pi\epsilon_0}\frac{qQ_i}{|\vec{r_0} - \vec{r_i}|^2}\frac{\vec{r_0} - \vec{r_i}}{|\vec{r_0} - \vec{r_i}|} \tag{7}$$

with the position of the N charges Q_i denoted as $\vec{r_i}$ and the position of the test charge $\vec{r_0}$. The superposition principle is illustrated in Fig. 4. The vectors are defined with respect to some origin O. Note that the expression for the electrical force is independent on the choice of origin O.

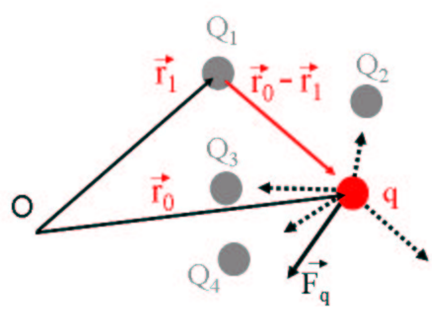

Figure 4: *The Coulomb force on a test charge q on position r_0 resulting from four source charges Q_1,..., Q_4. The connection vector, $\vec{r_0} - \vec{r_1}$ between charge Q_1 and the test charge q is also indicated. The vectors $\vec{r_0}$ and $\vec{r_1}$ are defined with respect to origin O.*

The physicist J.J. Thomson discovered in 1897 a new elementary particle: the electron. The electron carries a negative electrical charge and is responsible for most electrical currents. R.A. Millikan discovered in 1909 that all electrons carry a similar charge, $-e$, called the elementary charge. In SI units its value is

$$e = 1.6002 \times 10^{-19}\mathrm{C} \tag{8}$$

We know now that in atoms negative charge is carried by electrons, while protons are positively charged. When we go back to the old experiment, rubbing an ebony rod with fur or cat's skin, the electrons from the cat's skin get transferred to the ebony rod. The cat's skin now has a deficiency of electrons and so is positively charged. On the other hand, the ebony rod has an excess of electrons and hence is negatively charged. So, in everyday life, positive electrification is due to the deficiency of electrons and thus not by an excess of protons.

Protons are not so elementary as electrons. Protons consists of quarks and gluons. The quarks carry electrical charge of $-\frac{1}{3}e$ and $+\frac{2}{3}e$. In nature, the quarks and gluons are confined in other particles (like the proton). Scientific research in the field of 'particle physics' has led to a new 'table of elements', one with six quarks and six 'leptons' which interact by the exchange of force particles. The particle world is illustrated by the table in Fig. 5. All these

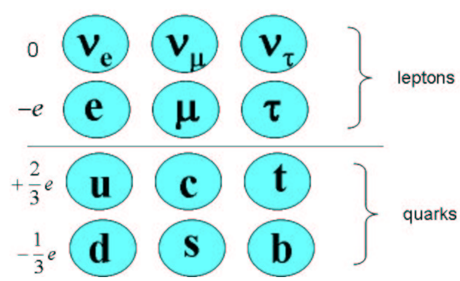

Figure 5: *The elementary particles. The leptons from left to right are: electron-neutrino, muon-neutrino, tau-neutrino, electron, muon, tauon. The quarks, in the same order, are: up, charm, top, down, strange, bottom. From left to right, the mass of the particles is increasing. For example, an up quark 'weighs' a few MeV while a top quarks weighs 175 GeV (for reference: a proton weighs 1 GeV). This mass difference between the particles is an open question, but the so called Higgs mechanism may be its origin.*

particles exist in nature, but one to remember is that our everyday world is made off only three particle: up and down quarks and electrons. Electrical charge in nature comes in discrete amounts: it is quantized, which has deep implications. The net charge before an interaction is equal to the charge after an interaction.

Elementary particles like quarks and gluons are studied in particle collisions at particle laboratories like CERN (Geneva) and Fermilab (Chicago) using large accelerators. Presently at CERN a new accellarator is being constructed, the Large Hadron Collider (LHC). The apparatus with a circumference of 27 km accellerates protons clockwise and counter-clockwise to energies of 7 TeV. At a few dedicated points, interaction points, the protons collide. The detectors to measure the products (i.c. new particles) of these collisions are large, typically 20 × 20 × 20 m^3. In the Netherlands, NIKHEF in Amsterdam, is the main institute where research in this field is being conducted. NIKHEF contributes to ATLAS, a detector for the LHC, as shown in Fig. 6. Although ATLAS is a multipurpose detector, the focus

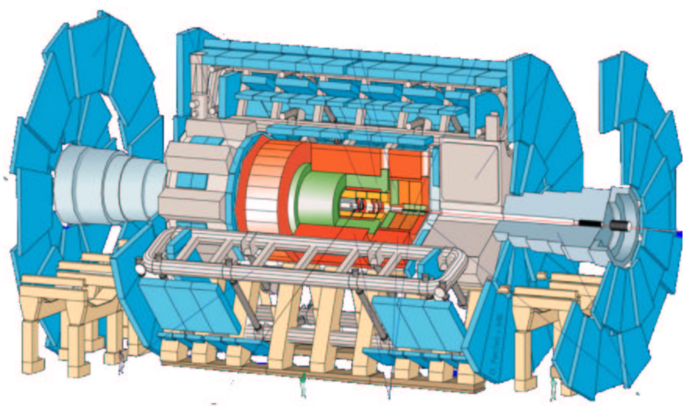

Figure 6: *An impression of the ATLAS detector which is being assembled. This detector is 44 m long! Particles that are produced by proton collisions are measured using several detection techniques. NIKHEF contributes to the silicon tracking detectors and muon drift tube chambers. See www.cern.ch for more information.*

of the research program is on the Higgs particle.

2.2 The electrical field

Let's have a closer look to equation 7 and rewrite it:

$$\vec{F} = q\left(\sum_{i=1,N} \frac{1}{4\pi\epsilon_0}\frac{Q_i}{|\vec{r_0}-\vec{r_i}|^2}\frac{\vec{r_0}-\vec{r_i}}{|\vec{r_0}-\vec{r_i}|}\right) \tag{9}$$

The electrical field $\vec{E}$ is defined by:

$$\vec{F} = q\vec{E}, \tag{10}$$

with:

$$\vec{E} = \sum_{i=1,N} \frac{1}{4\pi\epsilon_0} \frac{Q_i}{|\vec{r_0} - \vec{r_i}|^2} \frac{\vec{r_0} - \vec{r_i}}{|\vec{r_0} - \vec{r_i}|}. \tag{11}$$

In different textbooks you will find different notations. A common notation is:

$$\vec{E} = \sum_{i=1,N} \frac{1}{4\pi\epsilon_0} \frac{Q_i}{\vec{\mathrm{r_i}}^2} \hat{\mathrm{r_i}} \tag{12}$$

In this case the vector $\vec{\mathrm{r_i}}$ is defined as $\vec{\mathrm{r_i}} = \vec{r_0} - \vec{r_i}$. Be aware of this freedom of notation, which leads to many un-necessary mistakes.

A way to physcially interpret the above expression for the electrical field is that a charge q senses an electrical field and consequently undergoes a force $\vec{F} = q\vec{E}$. We have introduced the field as a relatively simple mathematical definition. However, the electrical field is a genuine physics quantity! In electrostatic theory, the field is present in space and you can calculate it using formula 11 when you know the (point) charge distribution. If the field is present in space, you may wonder, where is it made off? That is a bit harder to answer and beyond the scope of this course. In particle physics, fields consist of 'force particles' that are being exchanged when particles have an interaction. The electrical field consist of (virtual) photons that are exchanged between electrical charges.

Above, all equations are based on point charges. Since all charge are carried by individual particles this seems not unreasonable. However, in the classical theory charges can be continuously distributed. This is still reasonable when we describe physics at a scale much (much) larger than the size of and distances (10^{-10}m) between the particles. The classical theory describes nature in a macroscopic matter. We will discuss some examples of continuous charge distributions.

2.2.1 Line charge

A line charge can be described by a charge density $\lambda(\vec{r_l})$ with has the unit C/m. The position vector $\vec{r_l}$ is a coordinate defined with respect to some origin. To calculate the field in point P with coordinate $\vec{r}_P$, we integrate over the infinitesimal pieces of charge $\lambda(\vec{r_l})dl$ to find the electrical field:

$$\vec{E}(\vec{r}_P) = \frac{1}{4\pi\epsilon_0} \int_{line} \frac{\lambda(\vec{r_l})}{\vec{r}^2} \hat{r} dl \tag{13}$$

where $\vec{r}$ is the connection vector between a piece of charge and the point P with coordinate $\vec{r}_P$, thus $\vec{r} = \vec{r}_P - \vec{r_l}$. This is illustrated in Fig. 7.

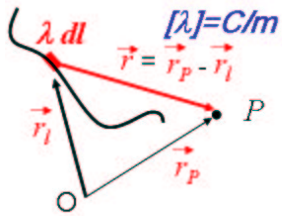

Figure 7: *Illustration of a line charge.*

In fact, we can still interpret the integral as the sum over point charges as we used to do in equation 11. A point charge is then just a piece of line charge $dq = \lambda(\vec{l})dl$ and so we obtain:

$$\begin{aligned} \vec{E}(\vec{r_P}) &= \frac{1}{4\pi\epsilon_0}\int_{charge}\frac{dq}{\vec{r}^2}\hat{r} \\ &= \frac{1}{4\pi\epsilon_0}\int_{charge}\frac{dq}{|\vec{r_P}-\vec{r_l}|^2}\frac{\vec{r_P}-\vec{r_l}}{|\vec{r_P}-\vec{r_l}|} \\ &\approx \sum_{pointcharges}\frac{1}{4\pi\epsilon_0}\frac{Q_i}{|\vec{r_P}-\vec{r_i}|^2}\frac{\vec{r_P}-\vec{r_i}}{|\vec{r_P}-\vec{r_i}|} \end{aligned} \quad (14)$$

Perhaps the interpretation of a continuous charge distribution as a collection of point charges helps you to make the math less abstract.

2.2.2 Example of a line charge

As an example we will calculate the electrical field in a point P with $z = z_P$ of a piece of wire with length L centered on the x axis. The wire carries an uniform charge density λ. The configuration is shown in Fig. 8.

When we consider the contribution, $d\vec{E}$, of a piece of charge λdx at position x to the electrical field in P we see that there are two components, an x and z component:

$$\begin{aligned} dE_x &= \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{z_P^2+x^2}sin(\alpha) \\ dE_z &= \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{z_P^2+x^2}cos(\alpha) \end{aligned} \quad (15)$$

Look at the symmetry of this problem. The components in the x direction cancel ($E_x = 0$) and thus we only have to integrate the z contributions. So, we need to know $cos(\alpha)$, which we can get from the figure (convince

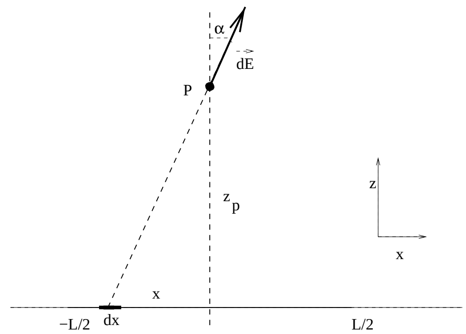

Figure 8: *A wire of length L with uniform charge distribution. Also indicated is the contribution to the electrical field in point P of a piece of line dx.*

yourself!): $cos(\alpha) = \frac{z_P}{\sqrt{z_P^2+x^2}}$. Now we can integrate all contributions:

$$\begin{aligned} E_z &= \int dE_z \\ &= \int_{-L/2}^{+L/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{z_P^2 + x^2} \frac{z_P}{\sqrt{z_P^2 + x^2}} \\ &= \int_{-L/2}^{+L/2} \frac{1}{4\pi\epsilon_0} \frac{z_P \lambda dx}{(z_P^2 + x^2)^{\frac{3}{2}}} \\ &= \frac{\lambda}{4\pi\epsilon_0} \frac{x}{z_P\sqrt{z_P^2 + x^2}} \Big|_{-L/2}^{+L/2} \\ &= \frac{\lambda}{4\pi\epsilon_0} \frac{L}{z_P\sqrt{z_P^2 + (L/2)^2}} \end{aligned} \tag{16}$$

Does the result make sense? Well, we know that the field of a point charge is linear with charge Q and drops quadratically with the distance. If we look from very large distance to the line charge, $z_P >> L$, so that all the charge on the line appears concentrated in a point, we find that

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z_P^2} = \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{z_P^2} \tag{17}$$

which is the field of a point charge $Q_{total} = \lambda L$, as expected.

2.2.3 Surface charge

An illustration of a surface charge is depicted in Fig. 9. A surface charge is described by a charge density $\sigma(\vec{r}_s)$ with has the unit C/m^2. When $\vec{r}_s$

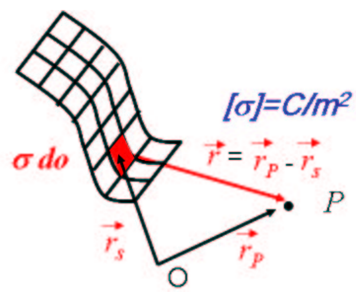

Figure 9: *Illustration of a surface charge.*

lies on the surface, $\sigma(\vec{r}_s)$ has some value that represents the charge density. When $\vec{r}_s$ lies outside the surface, $\sigma(\vec{r}_s) = 0$ C/m^2. To calculate the electrical field we integrate over the contribution of the infinitesimal pieces of charge $dq = \sigma(\vec{r}_s)do$. The electrical field in point P is given by:

$$\vec{E}(\vec{r_P}) = \frac{1}{4\pi\epsilon_0}\int_{surface}\frac{\sigma(\vec{r}_s)}{\vec{r}^2}\hat{r}do \tag{18}$$

Now $\vec{r}$ is defined as the connection vector between point $\vec{r}_P$ and the location of some infinitesimal piece of surface, thus $\vec{r} = \vec{r}_P - \vec{r}_s$.

Like the integration over a line, the integration over a surface is a difficult task. Only in special cases we can actually perform the integration. So, don't worry for the moment if you do not see how in general you could use the above expression. Finally we remark that we have been 'sloppy' with our notation: a surface is two dimensional, thus one would expect a double integral with two integration variables. Well, get used to it: in different textbooks you will find at least as many different notations. Be prepared and just 'bluf' through it.

2.2.4 Volume charge

An illustration of a volume charge is depicted in Fig. 10. The volume charge density $\rho(\vec{r}_v)$ is the most general continuous charge density and has the unit C/m^3. The contribution of the infinitesimal pieces of charge $dq = \rho(\vec{r}_v)dv$ lead to the following expression:

$$\vec{E}(\vec{r}_P) = \frac{1}{4\pi\epsilon_0}\int_{volume}\frac{\rho(\vec{r}_v)}{\vec{r}^2}\hat{r}dv \tag{19}$$

Now $\vec{r}$ is defined as the connection vector between point $\vec{r}_P$ and the location of some infinitesimal piece of volume dv at position $\vec{r}_v$. Applications of

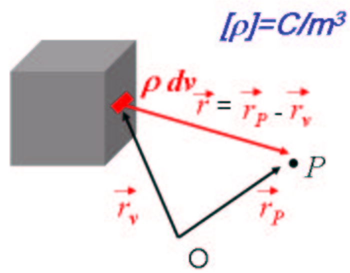

Figure 10: *Illustration of a volume charge.*

this expression come later. Note that we again have been sloppy with the notation. A volume (in this report) has three dimensions and thus three integration variables. For example when we inegrate a function f over a volume in cartesian coordinates:

$$\int_{volume} f dv = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{z=-\infty}^{\infty} f(x, y, z) dx dy dz \tag{20}$$

and in spherical coordinates:

$$\int_{volume} f dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} f(r, \phi, \theta) d\phi d\theta dr \tag{21}$$

If this is 'abacadabra' to you, first work through a textbook on mathematics[1].

2.3 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- Charge can be positive and negative. Electrons carry the elementary charge $-e$, with $e = 1.6002 \times 10^{-19}$ C. The total charge is conserved.
- The electrical or Coulomb force between two electrically charged objects is given by:

  $$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r} \tag{22}$$

  Make sure you understand the notation!

[1]Please, let the authors of this report know when integration using spherical coordinates are unknown to you at the time of the lectures. Don't hesitate! Additionally. in the slides, added as Appendix you can find more explanation and many examples.

- The electrical field of a point charge q is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{23}$$

- The electrical field of a volume charge is:

$$\vec{E}(\vec{r}_P) = \frac{1}{4\pi\epsilon_0}\int_{volume}\frac{\rho(\vec{r}_v)}{\vec{r}^2}\hat{r}dv \tag{24}$$

  You understand all the vectors in this notation and you can make a drawing that illustrates this expression. You can also write down the formulas for the electrical field of a line charge and a surface charge.

In addition, make the corresponding exercises of this section, which you can find in Appendix A.

3 Electrical flux and Gauss' Law

3.1 Electrical field lines and electrical flux

Figure 11 graphically displays the electrical field corresponding to the electrical field of a point charge. To make this drawing a grid was chosen and on each grid-point the electrical field is represented by an arrow. The length of the arrow corresponds to the magnitude of the field and, as can be seen in the figure, decreases quadratically as appropriate.

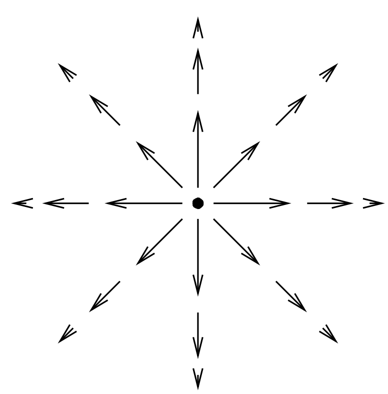

Figure 11: *The electrical field of a point charge depicted by vectors.*

Traditionally, the electrical field is depicted by field lines as shown in Fig. 12. The density of the lines represent the magnitude of the field. Note

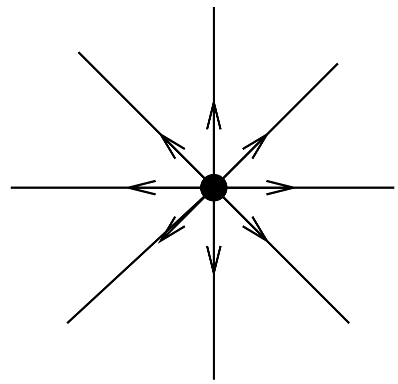

Figure 12: *The electrical field of a point charge depicted by field lines.*

that such drawings are two dimensional projections and thus that the density of the lines drops with the circumference of a circle, $2\pi r$. In three dimensions the line-density drops with the surface of a sphere, $4\pi r^2$

Have another look to the field lines in Fig. 12 and imagine a spherical surface around a point charge. The number of field lines through the surface is constant, thus independent on the size (or better radius) of the sphere. You find an animation of this phenomena on the web-page.

The number of field lines can be expressed by the electrical flux:

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{o} \tag{25}$$

This the flux through a surface. The infinitesimal surface element $d\vec{o}$ is a vector with magnitude do and its direction perpendicular to the surface. We can write

$$\vec{E} \cdot d\vec{o} = \vec{E} \cdot \hat{n} do = |\vec{E}| cos(\phi) do \tag{26}$$

with $\hat{n}$ the normal on the surface[2] and ϕ represents the angle between the electrical field and normal direction of the surface.

Figure 13 shows three examples of the electrical flux through a surface for different angles between the field and surface. With the electrical field being constant, the flux through each surface S in Fig. 13a-c is:

$$\begin{aligned} \mathrm{a}: \Phi_\mathrm{S} &= |\vec{E}| cos(0) S = |\vec{E}| S \\ \mathrm{b}: \Phi_\mathrm{S} &= |\vec{E}| cos(\pi/2) S = 0 \\ \mathrm{c}: \Phi_\mathrm{S} &= |\vec{E}| cos(\phi) S \end{aligned} \tag{27}$$

[2] In textbooks several notations are used for the normal vector, a few common notations are: $\hat{n} = \vec{n} = \vec{e}_n = \hat{e}_n$.

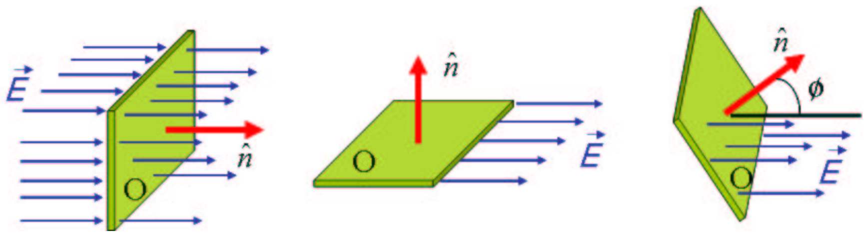

Figure 13: *The electrical flux through a surface S for different angles between the field and surface. The normal $\hat{n}$ is also indicated.*

3.2 Gauss' Law

Gauss' Law states that the electrical flux through a closed surface, independent of its shape, equals the total charge enclosed by this surface multiplied by the factor $\frac{1}{\epsilon_0}$. Thus:

$$\Phi = \int_{closed-surface} \vec{E} \cdot d\vec{o} = \sum_{charges-enclosed} \frac{Q_i}{\epsilon_0} \tag{28}$$

This relation allows us to calculate the electrical field in many cases in an remarkable elegant way.

But first we will deduce, or better, verify the validity of Gauss' Law for a simple case. Therefore we consider a point charge, Q, and calculate the electrical flux through a spherical surface with radius R. This configuration is illustrated in Fig. 14. The flux is given by:

$$\Phi = \int_{surface} \vec{E} \cdot d\vec{o} = \int_{surface} \vec{E} \cdot \hat{n} do \tag{29}$$

where $\hat{n}$ is the normal vector on the surface. Now we use the argument that both $\vec{E}$ and $\hat{n}$ point in the radial direction, hence

$$\Phi = \int_{surface} |\vec{E}| cos(\alpha_{\vec{E},\hat{n}}) do = \int_{surface} |\vec{E}| do \tag{30}$$

The electrical field at the surface follows from equation 11, $|\vec{E}| = E_r(R) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$ (independent of the location on the surface). We find

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\int_{surface} do \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} 4\pi R^2 \\ &= \frac{Q}{\epsilon_0} \end{aligned} \tag{31}$$

in accordance with Gauss' Law.

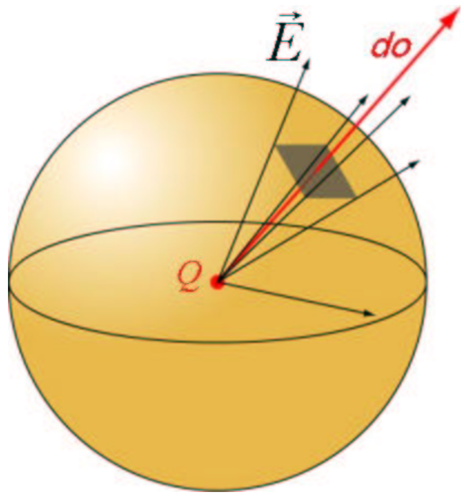

Figure 14: *Illustration of the electrical flux from a point charge Q through a spherical surface S. Some infinitesimal piece of surface do on the sphere is also indicated.*

3.2.1 Intermezzo: a more mathematical approach

To evaluate (i.e. to get rid off) the dot-product $\vec{E} \cdot d\vec{o}$ we used the argument that both vectors point in the radial direction. Perhaps this step is hard to digest and you want to see what is behind it. Well, let's give it a try. If we had started using Cartesian coordinates we would have written:

$$\begin{aligned} \Phi &= \int_{surface} \vec{E} \cdot d\vec{o} \\ &= \int_{surface} (E_x\hat{x} + E_y\hat{y} + E_z\hat{z}) \cdot (n_x\hat{x} + n_y\hat{y} + n_z\hat{z}) do \end{aligned} \tag{32}$$

with $\hat{n}$ the normal vector on the spherical surface. The unit vectors in the x, y and z direction are represented as $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively. Another common notation for these unit vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$.

The expression in cartesian coordinates is for our case just terrible because the physical symmetry of the original problem is obscured. Obviously we need to work in spherical coordinates. Then, formally we obtain:

$$\begin{aligned} \Phi &= \int_{surface} \vec{E} \cdot d\vec{o} \\ &= \int_{surface} (E_\phi\hat{\phi} + E_\theta\hat{\theta} + E_r\hat{r}) \cdot (n_\phi\hat{\phi} + n_\theta\hat{\theta} + n_r\hat{r}) do \\ &= \int_{surface} (E_\phi n_\phi\hat{\phi} + E_\theta n_\theta\hat{\theta} + E_r n_r\hat{r}) do \end{aligned} \tag{33}$$

With $\hat{\phi}$, $\hat{\theta}$ and $\hat{r}$ the unit direction vectors in the ϕ, θ and r direction respectively. It looks abstract, (it is), but just try to see through the notation and realize that the vectors are written out in components.

We then use $\vec{E} = (0\hat{\phi}, 0\hat{\theta}, E_r\hat{r})$ with $E_r = \frac{Q}{4\pi\epsilon_0 r^2}$. In fact E_r is a function and depends on $\vec{r}$ or in this case for a point charge just on r. Hence, on the surface $E_r = E_r(r = R)$. The normal vector $\hat{n}$ on the surface points in the radial direction and has the unit length: $\hat{n} = (0, 0, \hat{r})$. When we substitute these expressions and reshuffle, we acquire:

$$
\begin{aligned}
\Phi &= \int_{surface} \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} do \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^2 sin(\theta) d\phi d\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\int_{\theta=0}^{\pi} R^2 (\phi)|_0^{2\pi} sin(\theta) d\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} R^2 2\pi(-cos(\theta))|_0^{\pi} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} R^2 4\pi \\
&= \frac{Q}{\epsilon_0}
\end{aligned} \tag{34}
$$

Probably this intermezzo did not increase your confidence in Gauss' Law. It showed however the basic steps which many textbooks tend to skip. In the following we often skip such steps; they contain too much mathematical detail and so distract from our physics case. What was the physics case? Well, the symmetry in the behavior of the electrical field, namely the $1/r^2$ dependence, and that of the size of the spherical surface growing with r^2 leads to a cancellation.

3.3 Validity of Gauss' Law

We checked Gauss' Law for a spherical surface with a point charge located at its center. Figure 15a shows a weird shaped closed surface with a point charge somewhere in its volume. Also for this shape Gauss' Law is valid; it is the 'dot' product which does the job. Put the point charge in the origin. Its electrical field has only radial components. The dot-product with the normal vector of the surface kills all non-radial contributions. In addition, the distance of the surface with respect to the charge is irrelevant: the field drops with $1/r^2$ and the radial projection of the surface grows with r^2. Also, we could have put many point charge at different location within the closed

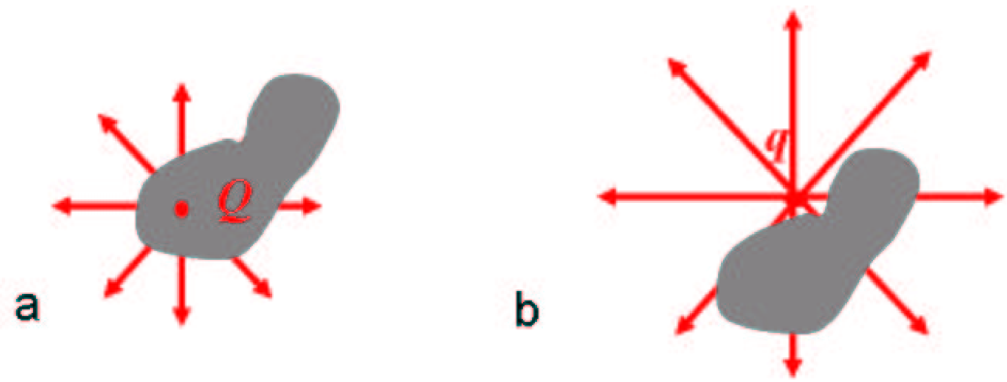

Figure 15: *(a) Point charge Q in a closed surface. (b) Point charge Q outside a closed surface.*

surface. Obviously for each point charge individually the contribution to the flux is its charge (divided by ϵ_0), and the total flux is given by the sum of all charges ($/\epsilon_0$).

Figure 15b shows a closed surface with a charge located outside its volume. Use the above arguments to deduce (qualitatively) that $\Phi = 0$ in this case.

3.4 Applications of Gauss's law

'Gauss' Law is always valid, but not always useful'. What does this mean? Well, Gauss' Law can be applied in some cases to evaluate the electrical field in an elegant way. Below, we describe a few of these configurations.

3.4.1 The line charge

Figure 16 shows an infinitely long and infinitely thin line charge with uniform charge density λ. The optimal Gaussian surface is a cylinder. To obtain this insight, first deduce the shape of the electrical field. It needs some practice to acquire a feeling for this, but we can give some hints.

- Imagine the line is build from small point charges $d\lambda$. Every point charge produces a electrical field point in the radial (spherical-wise) direction. Consider a position somewhere near the line. At this position you 'feel' an equal amount of field lines from above as from below. Hence, the field in the z direction cancels. Obviously there are no components in the ϕ direction. There can only be a component pointing away from the line charge.

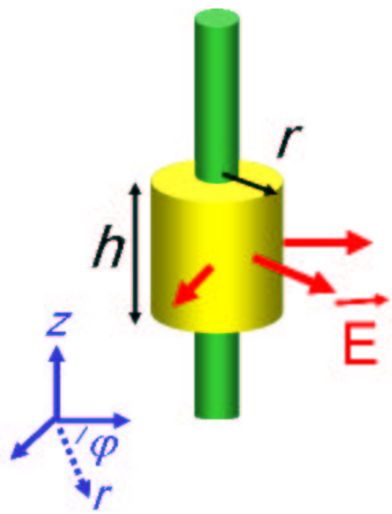

Figure 16: *A line charge (dark gray). A Gaussian cylinder (light gray) with height h and radius r is also drawn.*

- Look at the symmetry of the line charge. Suppose there is a field component in the z direction. Remember that the line is infinitely long and imagine that you mirror the configuration in the $r\phi$ plane. This does not change the physical configuration, but the 'would be' z component of the field has changed sign. There is only one possibility: there is no z component.

We conclude that the electrical field has only a radial (cylinder-wise) component. Therefore we try a cylindrical Gaussian surface with arbitrary height h and radius r. The flux through the cylinder is given by the sum of the contributions of the curved body and the ends:

$$\Phi = \int_{cylinder} \vec{E} \cdot d\vec{o} = \int_{ends} \vec{E} \cdot d\vec{o} + \int_{curved-body} \vec{E} \cdot d\vec{o} \qquad (35)$$

The normal vector on the ends of the cylinder point (only) in the z direction. Hence, the dot-product filters out the z component of the electrical field, which is zero. So, we are left with the contribution of the flux through the curved body of the cylinder. The normal vector is radial (cylinder-wise) and thus filters out the one and only radial component of the electrical field, $(E_r(r))$.We obtain:

$$\Phi = \int_{curved-body} \vec{E} \cdot d\vec{o} = \int_{curved-body} E_r(r) do = \int_{\phi=0}^{2\pi} \int_{z=0}^{z=h} E_r(r) dz r d\phi \qquad (36)$$

The electrical field is independent on the integration variables and we may write:

$$\Phi = E_r(r) \int_{\phi=0}^{2\pi} \int_{z=0}^{z=h} dz r d\phi = E_r(r) 2\pi r h \tag{37}$$

Now we apply Gauss' Law:

$$\Phi = E_r(r) 2\pi r h = \frac{Q_{enclosed}}{\epsilon_0} = \frac{\lambda h}{\epsilon_0} \tag{38}$$

For the electrical field of a line charge we find that:

$$\begin{aligned} E_z &= 0 \\ E_\phi &= 0 \\ E_r(r) &= \frac{\lambda}{2\pi\epsilon_0 r} \end{aligned} \tag{39}$$

If the line has a finite length L this method can not be applied. However, it then still provides a good estimate of the field close to the wire, i.c. $r << L$. If you are not convinced, use the result of the direct integration in Section 2.2.2 to show this. Also when the charge density is not constant in the z direction we can shake it. When the line has a finite thickness of radius ρ this method can still be applied as we will see later.

3.4.2 A flat surface charge

Figure 17 shows an, infinitely large and infinitely thin, surface charge with uniform charge density σ. What are the components of the electrical field?

- Imagine the plate constituted of small point charges $d\sigma = \sigma do$. Every point charge produces an electrical field point in the radial (spherical-wise) direction. Consider a position somewhere near the plate. At this position you 'feel' an equal amount of field lines from above as from below and from left as from right. Hence, the field in the x and z direction is zero. There can only be a component perpendicular to the surface charge.

- We can also used arguments based on symmetry. Suppose there is a field component in the x and/or z direction. Turn the configuration around its y axis. The plate is infinitely large and thus remains physically the same. The would be components would have changed direction, while the physics is invariant.

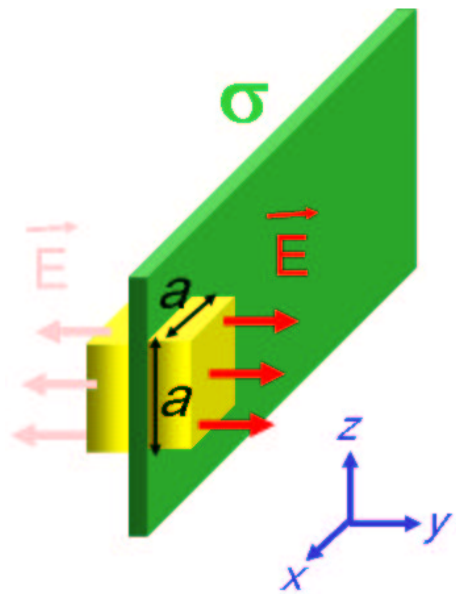

Figure 17: *A flat surface charge (dark-gray). A Gaussian cubical surface is also indicated.*

- In addition, a shift of the plate in the $x - z$ plane does not change the electrical configuration. This implies that the electrical field does not depend on x or z and thus can only be dependent on y.

We conclude that the electrical field has only a component in the y direction, opposite for the region $\pm y$. We try a cubical Gaussian surface (a pill-box) with ribs sized a. The flux through the sides with normal vector in the x and z direction is zero. We only need to calculate the flux through the top-covers with normal vector in the y direction. The normal vector in the $-y$ region is opposite to that in the $+y$ region, but the electrical field direction also swaps. Hence,

$$\Phi = \int_{top-covers} \vec{E} \cdot d\vec{o} = 2 \int_{top-cover} E_y(y) do = 2a^2 E_y(y) \tag{40}$$

Make sure you understand the steps above. From Gauss' Law follows

$$\Phi = 2E_y(y)a^2 = \frac{Q_{enclosed}}{\epsilon_0} = \frac{a^2\sigma}{\epsilon_0} \tag{41}$$

And for the size of the electrical field we obtain $E_y = \frac{\sigma}{2\epsilon_0}$ and thus $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{y}$. Note that the field is constant, but opposite for positive and negative y values respectively.

3.4.3 A spherical surface charge

Figure 18: *Illustration of a spherical surface charge density σ with radius R.*

We consider a spherical charged surface (or shell) with radius R and surface charge density σ as illustrated in Fig 18.

What are the components of the electrical field? Suppose there are non-radial (spherical-wise) components. Rotate the configuration around its center such that the non-radial components change direction. Realize that the physical configuration is invariant which allows no non-radial components. The same argument can be used to deduce that the radial component of the field only depend on r. Hence, $\vec{E} = E_r(r)\hat{r}$. We did not specify whether we discussed the field inside or outside the shell. Well, it doesn't matter. Both inside and outside the shell we can use the above arguments.

For the electrical flux inside the shell follows:

$$\Phi = \int_{spherical-surface} \vec{E} \cdot d\vec{o} = E_r(r) \int_{spherical-surface} do = E_r(r) 4\pi r^2 \quad (42)$$

There is no enclosed charge, so we have:

$$\Phi = E_r(r) 4\pi r^2 = 0 \quad (43)$$

There is only one possibility: $E(r) = 0$ inside the surface.

Outside the shell, $r > R$, we find the same expression for the flux. The enclosed charge is $Q = \int_{surface} \sigma do = \sigma 4\pi R^2$. We obtain:

$$\Phi = E_r(r) 4\pi r^2 == \frac{1}{\epsilon_0} \sigma 4\pi R^2 \quad (44)$$

For the electrical field follows

$$\vec{E} = E_r(r)\hat{r} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r} \quad r > R \quad (45)$$

In fact, the electrical field outside the shell is identical to that of a point charge that carries the same charge as present on the shell. This can be easily shown. We substitute $\sigma = \frac{Q}{4\pi R^2}$ in equation 45 and find:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad (46)$$

which is the field of a point charge, as expected.

3.4.4 A massive spherical charge

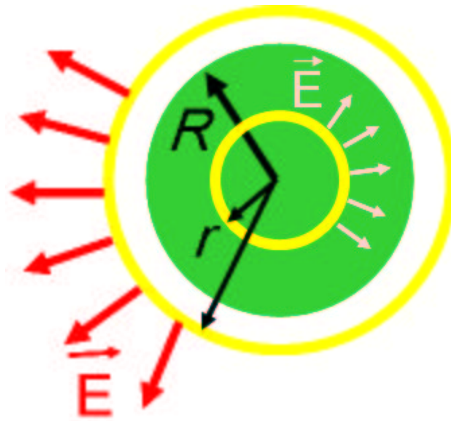

Figure 19: *A spherical volume with radius R carrying a uniform charge density ρ (dark-gray). Two Gaussian spherical surfaces are indicated. One inside and one outside the charge density.*

Figure 19 shows a spherical volume charge density ρ. The volume has radius R. Like in the previous example, we have spherical symmetry. The electrical field has only a radial component and depends only on the radial distance: $\vec{E} = E_r(r)\hat{r}$.

The electrical flux outside the sphere, $r > R$, is given by:

$$\Phi = \int_{spherical-surface} \vec{E} \cdot d\vec{o} = E_r(r) \int_{spherical-surface} do = E_r(r) 4\pi r^2 \quad (47)$$

Outside the spherical charge the enclosed charge is $\int_{volume} \rho dv = \rho \frac{4}{3}\pi R^3$. For the electrical field we find:

$$\vec{E} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r} \quad r > R \quad (48)$$

Inside the spherical charge, $r < R$, we have in principal two contributions to the electrical field. One contribution from the inner sphere (surrounded by the Gaussian surface) with radius r and a contribution of the shell between r and R. In the previous section we calculated that the electrical field contribution inside a charged shell is zero. This implies that we only have to account for the contribution from the inner sphere. The Gaussian spherical surface encloses a charge $\rho \frac{4}{3}\pi r^3$. For the electrical flux we find:

$$E_r(r) 4\pi r^2 = \rho \frac{4}{3}\pi r^3 \quad (49)$$

This leads to an electrical field of

$$\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r} \quad r < R \tag{50}$$

Figure 20 shows a graph of size of the electrical field as function of r. Starting

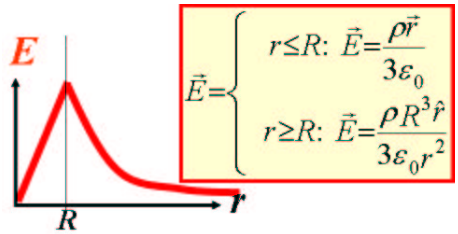

Figure 20: *The size of the electrical field of a spherical uniform charge density as function of* r.

form the center, the field grows linearly with r till the surface of the spherical charge is reached. Then it drops with $1/r^2$, similar to the field of a point charge in the origin.

3.5 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- Gauss' Law:

$$\int_{closed-surface} \vec{E} \cdot d\vec{o} = \sum_{charge-enclosed} \frac{Q_i}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{volume} \rho dv \tag{51}$$

  which is always valid.

- In electrical configuration with a symmetry between the field and the charge distribution, Gauss' Law can be used to determine the electrical field.

- You can apply Gauss' Law for a line, flat surface, and spherical charge. You know how to use Cartesian, cylinder and spherical coordinates to perform surface and volume integrals.

In addition, make the corresponding exercises of this section, which you can find in Appendix A.

4 The divergence of the electrical field

The divergence of the electrical field is defined as

$$\vec{\nabla} \cdot \vec{E} = (\partial_x, \partial_y, \partial_z) \cdot (E_x, E_y, E_z) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{52}$$

In this Section we will introduce this quantity in a natural way and describe its link with Gauss' Law.

4.1 Flux and divergence

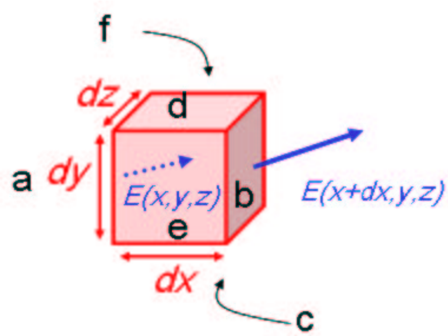

Figure 21: *An infinitesimal box with volume= $dxdydz$ in an electrical field $\vec{E}$. The covers a to f are indicated. Also, the field vectors on cover a and b are shown.*

Figure 21 shows a infinitesimally small box, which is placed in an electrical field $\vec{E}$. We work in Cartesian coordinates and calculate the electrical flux through the box. The flux $\int_{cover} \vec{E} \cdot d\vec{o}$ through each of the covers a to f of the box is given by:

$$\begin{aligned}
\text{a}: &\quad -E_x(x,y,z)dydz \\
\text{b}: &\quad E_x(x+dx,y,z)dydz \\
\text{c}: &\quad -E_y(x,y,z)dxdz \\
\text{d}: &\quad E_y(x,y+dy,z)dxdz \\
\text{e}: &\quad -E_z(x,y,z)dxdy \\
\text{f}: &\quad E_z(x,y,z+dz)dxdy
\end{aligned} \tag{53}$$

Note the relative minus sign for opposite covers which comes from the opposite direction of the normal vectors on these covers. Now we add all contributions to obtain the flux through the box:

$$\int_{box} \vec{E} \cdot d\vec{o} = \begin{aligned} &[E_x(x+dx,y,z) - E_x(x,y,z)]dydz \\ &[E_y(x,y+dy,z) - E_y(x,y,z)]dxdz \\ &[E_z(x,y,z+dz) - E_z(x,y,z)]dxdy \end{aligned} \tag{54}$$

Remember the rule of elementary calculus that $df = f(x+dx) - f(x)$. The above equation can be rewritten as:

$$\int_{box} \vec{E} \cdot d\vec{o} = dE_x dydz + dE_y dxdz + dE_z dxdy \tag{55}$$

Now multiply the part with E_x, E_y and E_z with $\frac{dx}{dx}$, $\frac{dy}{dy}$ and $\frac{dz}{dz}$ respectively, which is mathematically equivalent to multiplying with unity. We find

$$\begin{aligned} \int_{box} \vec{E} \cdot d\vec{o} &= \frac{dE_x}{dx}dxdydz + \frac{dE_y}{dy}dxdydz + \frac{dE_z}{dz}dxdydz \\ &= \left[\frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz}\right] dxdydz \\ &= \vec{\nabla} \cdot \vec{E} \, \mathrm{volume}_{\mathrm{box}} \end{aligned} \tag{56}$$

Hence, we derived a relation between the flux through infinitesimal box and the divergence of the electrical field. The relation is however valid for any volume. To see this, glue boxes together to make you any volume as illustrated in Fig. 22. We obtain:

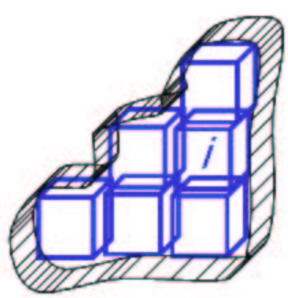

Figure 22: *Any volume consists of a collection of infinitesimal boxes.*

$$\int_{surface} \vec{E} \cdot d\vec{o} = \int_{volume} [\vec{\nabla} \cdot \vec{E}] dV \tag{57}$$

Although we derived this expression for the electric field, it is valid for any vector field and was first derived by Gauss. We will refer to this expression as Gauss Theorem.

4.2 Gauss' Law and Gauss' Theorem

Gauss' Law for the electrical flux:

$$\int_{surface} \vec{E} \cdot d\vec{o} = \frac{1}{\epsilon_0} Q_{enclosed} = \frac{1}{\epsilon_0} \int_{volume} \rho dv \tag{58}$$

can be combined with Gauss theorem:

$$\int_{surface} \vec{E} \cdot \vec{o} = \int_{volume} \vec{\nabla} \cdot \vec{E} dv \tag{59}$$

This leads to the following expression:

$$\int_{surface} \vec{E} \cdot \vec{o} = \int_{volume} \vec{\nabla} \cdot \vec{E} dv = \frac{1}{\epsilon_0} \int_{volume} \rho dv \tag{60}$$

which implies

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{61}$$

This relation is Gauss' Law in differential form. It locally related the charge density and the electrica field.

4.3 Gauss' Law for a charged sphere

Given the electrical field, we can 'simply' determine the charge density using Gauss' Law (in differential form). We start with the known field of a uniformly charged sphere with density ρ. The electrical field inside the sphere is:

$$\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r} = \frac{\rho}{3\epsilon_0} \vec{r} \quad r < R \tag{62}$$

Now take the divergence.

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{3\epsilon_0} \vec{\nabla} \cdot \vec{r} = \frac{\rho}{3\epsilon_0} (\partial_x, \partial_y, \partial_z) \cdot (x, y, z) & (63) \\ &= \frac{\rho}{3\epsilon_0} (\partial_x x + \partial_y y + \partial_z z) = \frac{\rho}{3\epsilon_0} 3 = \frac{\rho}{\epsilon_0} & (64) \end{aligned}$$

Yes!

Outside the charge sphere $\rho = 0$, the electrical field is $\vec{E} = \frac{\rho R^3}{3\epsilon_0 r^3} \vec{r}$. We calculate the divergence of $\frac{\vec{r}}{r^3}$:

$$\begin{aligned} \vec{\nabla} \cdot \frac{\vec{r}}{r^3} &= \vec{\nabla} \cdot [\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} (x, y, z)] \\ &= \vec{\nabla} \cdot (\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
&= \left[\frac{\partial x}{\partial x}\right]\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + x\left[\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \partial_y .... + \partial_z ..... \\
&= 1\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + x\left[-\frac{3}{2}2x\frac{1}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right] + \partial_y .... + \partial_z ..... \\
&= \frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} - 3(x^2+y^2+z^2)\frac{1}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
&= 0
\end{aligned} \tag{65}
$$

Not convinced? Try it yourself!

4.4 The loop integral of the electrical field

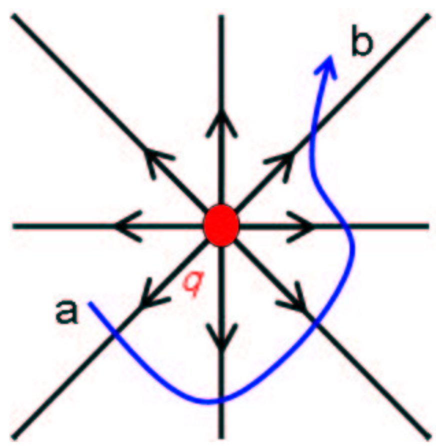

Figure 23: *An illustration of a path integral in the electrical field of a point charge q.*

Another important characteristic of the electric field emerges when we consider the path integral of the field a point charge q:

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\hat{r} \cdot d\vec{l} \tag{66}$$

The electrical field is pointing purely radially. No matter what the exact path is followed from a to b, the dot-product filters out the radial component ($\hat{r} \cdot d\vec{l} = dr$). Thus we can replace the path integral by:

$$\int_{r_a}^{r_b} E_r \cdot dr = \frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}dr = \frac{q}{4\pi\epsilon_0}\frac{-1}{r}\Big|_{r_a}^{r_b} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{67}$$

For a closed path $r_a = r_b$ we obtain:

$$\int \vec{E} \cdot d\vec{l} = 0 \tag{68}$$

Thus, independent of the path we followed, the integral of a closed path of the electric field is zero. Using the superposition principle we can argue that this relation derived for a point charge is valid for any charge density.

The expression $\int \vec{E} \cdot d\vec{l} = 0$ is the second electrical field equation in integral form and has no historical name. The differential form $\vec{\nabla} \times \vec{E} = \vec{0}$ we will derive later.

4.5 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- You understand Gauss theorem and the relation between the electrical field an the charge density:

$$\int_{surface} \vec{E} \cdot \vec{o} = \int_{volume} \vec{\nabla} \cdot \vec{E} dv = \frac{1}{\epsilon_0} \int_{volume} \rho dv \tag{69}$$

- You can apply the differential field equation:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{70}$$

  which is Gauss' Law in differential form.

- You can derive and use the loop integral of the electrical field:

$$\int \vec{E} \cdot d\vec{l} = 0, \tag{71}$$

  independent of the path we followed.

In addition, make the corresponding exercises of this section, which you can find in Appendix A.

5 The electric potential

In this Section we will introduce the electric potential. It turns out that the electric potential is a powerful quantity to calculate the electric field of complex charge configurations. However, within the scope of this report we have to limit to more straightforward but elegant examples.

5.1 Work in a gravitational field

To refresh your memory we first consider work and potential energy in a gravitational field. The work, W_{person} when you lift an object with mass m from the ground to height h is

$$W = \int_0^h \vec{F} \cdot d\vec{l} \tag{72}$$

with the force, $|\vec{F}| = mg$, the mass times the gravitational constant. To lift the object we need to apply a force $\vec{F} = -\vec{F}_G = mg$ and tus the the total amount of work needed is:

$$W_{person} = mgh \tag{73}$$

The potential energy of the object equals $U = W_{person} = mgh$. In the following Section we will apply this principle to the electrical field.

5.2 Potential energy in an electric field

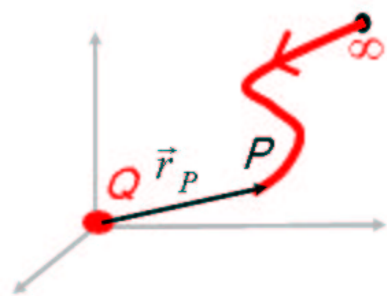

Figure 24: *A test charge q in the field of a source charge Q is brought in from infinity to P.*

Consider a (test) charge q in the electrical field of a point charge Q at position P, see also Fig. 24. The electrical force on the test charge is $\vec{F}_{elec} = q\vec{E}$. The (minimum) force you must exert on q to move it opposite to the electrical field is -$q\vec{E}$. The potential energy of the configuration is defined as the minimal work needed (for you) to bring the test charge q from infinity to P.

$$U_P = W_{person} = -\int_{\infty}^{P} q\vec{E} \cdot d\vec{l} \tag{74}$$

We substitute the electrical field of a point charge and obtain:

$$U_P = -q\int_{\infty}^{r_P} \vec{E}\cdot d\vec{r} = \frac{-qQ}{4\pi\epsilon_0}\int_{\infty}^{r_P} \frac{1}{r^3}\vec{r}\cdot d\vec{r} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{r}\Big|_{\infty}^{r_P} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{r_P} \tag{75}$$

A definition of the 'potential' V is the potential energy of a charge unit in the field of the source charge Q:

$$V_P = \frac{U_P}{q} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r_P} \tag{76}$$

where the reference point or 'gauge-point' is implicitly taken in infinity.

The most general definition of the potential is

$$V_P = -\int_{gauge}^{P} \vec{E}\cdot d\vec{l} \tag{77}$$

where the gauge-point can be chosen where-ever you want. You can verify yourself that for a point charge and gauge-point at infinity you find back equation 76.

We have defined the potential starting the potential energy of a simple charge configuration. Fine, the potential an the potential energy are related; that is useful to know. But, what about the general definition of the potential? It depends in general on the free choice of a gauge-point. How can such freedom be useful for describing physics? Indeed, the potential itself has no physical interpretation. However, potential difference and the gradient of the potential are relevant physics quantities.

5.3 The potential and the electrical field

From the previous section we know how to calculate the potential from the electrical field. Is is also possible to determine the electrical field given the potential. To find this relation, consider the difference in potential between points A and B:

$$V_{AB} = V_B - V_A = \int_B^{\infty} \vec{E}\cdot d\vec{l} - \int_A^{\infty} \vec{E}\cdot d\vec{l} = -\int_B^{A} \vec{E}\cdot d\vec{l} \tag{78}$$

where we swapped the upper and lower boundary of the integral, so keep track of the 'plus and minus' signs! Note that the potential difference V_{AB} is uniquely defined, independent of the gauge-point.

The potential is just a scalar function. Hence,

$$\begin{aligned} V_B - V_A = \int_A^B dV &= \int_A^B \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= \int_A^B \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)\cdot(dx, dy, dz) \end{aligned} \tag{79}$$

With the standard definition of the gradient:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{80}$$

we find:

$$\begin{aligned} V_b - V_A &= \int_A^B \vec{\nabla} V \cdot (dx, dy, dz) \\ &= \int_A^B \vec{\nabla} V \cdot d\vec{l} \end{aligned} \tag{81}$$

Combining Equations 78 and 81 we obtain:

$$\vec{E} = -\vec{\nabla} V \tag{82}$$

Thus the gradient of V has a physical interpretation; it is the electrical field (with a minus sign). This closes the circle. We can now determine the potential from the electric field and vice versa.

5.3.1 The electrical potential and field of a point charge

We determined the potential for a point charge to be:

$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} \tag{83}$$

Now we check whether the expression $\vec{E} = -\vec{\nabla} V$ returns the correct electrical field. We start with:

$$\begin{aligned} \vec{\nabla}\frac{1}{r} &= \vec{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{\partial}{\partial y}.....\hat{y} + \frac{\partial}{\partial z}.....\hat{z} \\ &= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{x} - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{y} - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{z} \\ &= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x, y, z) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2} \end{aligned} \tag{84}$$

and conclude that

$$\vec{\nabla} V = \frac{Q}{4\pi\epsilon_0}\vec{\nabla}\frac{1}{r} = -\frac{Q}{4\pi\epsilon_0} = -\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} = -\vec{E} \tag{85}$$

Most expressions in electrodynamics involve spherical symmetric function. For such a function, $f(r)$, you can quickly determine the gradient using the following relation.

$$\vec{\nabla} f(r) = \frac{df}{dr}\hat{r} \tag{86}$$

5.3.2 The potential of a charged sphere

In Section 3.4.4 we determined the electrical field inside and outside a massive spherical charge with radius R and charge density ρ.

$$\begin{aligned}
\vec{E} &= \frac{\rho r}{3\epsilon_0}\hat{r} \quad r < R \\
\vec{E} &= \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r} \quad r > R
\end{aligned} \tag{87}$$

As an example we calculate the potential inside the spherical charge.

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = -\int_{\infty}^{R} \vec{E}\cdot d\vec{l} - \int_{R}^{r} \vec{E}\cdot d\vec{l} \tag{88}$$

We substitute the expression for the electrical field and find:

$$\begin{aligned}
V(r) &= -\int_{\infty}^{R} \frac{\rho R^3}{3\epsilon_0 r^2} dr - \int_{R}^{r} \frac{\rho r}{3\epsilon_0} dr \\
&= \frac{\rho R^3}{3\epsilon_0 r}\Big|_{\infty}^{R} - \frac{\rho r^2}{6\epsilon_0}\Big|_{R}^{r} \\
&= \frac{\rho R^3}{3\epsilon_0 R} - \frac{\rho r^2}{6\epsilon_0} + \frac{\rho R^2}{6\epsilon_0} \\
&= \frac{\rho R^2}{2\epsilon_0} - \frac{\rho r^2}{6\epsilon_0}
\end{aligned} \tag{89}$$

At this point it may be not yet clear why the potential is relevant anyway. Don't worry about that now and make sure you understand the mathematics.

5.3.3 The eletrical field of a dipole

We introduced the potential and perhaps the idea came to your mind 'where do we need that for'. Well, to calculate the electrical field of complex charge configuration can be a difficult task, even numerically. Starting with a calculation of the potential and then calculate the electrical field is often much easier.

We will illustrate this approach by calculating the field of an electrical dipole. Figure 25 shows a electrical dipole configuration, consisting of two opposite point charges at a distance $2d$. The potential in a point $P(r,\theta)$ has a contribution from the positive and negative point charge. Thus,

$$V_P(r,\theta) = V_+ + V_- = \frac{q}{4\pi\epsilon_0}\frac{1}{r_+} + \frac{-q}{4\pi\epsilon_0}\frac{1}{r_-} \tag{90}$$

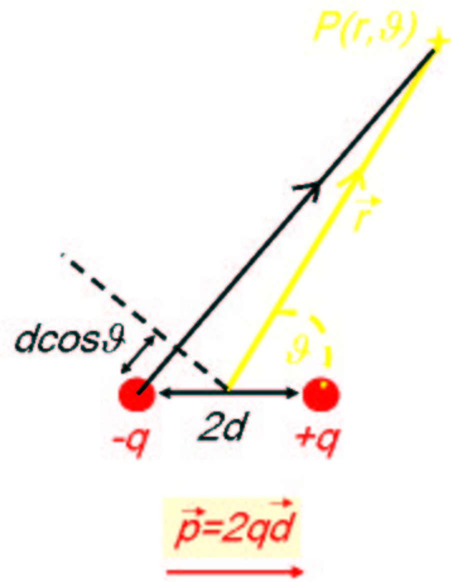

Figure 25: *An electrical dipole consisting of two opposite point charge at distance 2d.*

with $r_{\pm}$ the distance between the corresponding point charge and point P. When $r >> d$ we can make the approximation:

$$r_{\pm} = r \mp dcos(\theta) \tag{91}$$

Verify this yourself. For the potential in P we obtain:

$$V_P(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r - dcos(\theta)} - \frac{1}{r + dcos(\theta)}\right) \tag{92}$$

We can simplify this expression, using a 'cunning' trick. Multiply the nominators and denominators with $r + dcos(\theta)$ and $r - dcos(\theta)$ respectively and realize that $(r + dcos(\theta))(r - dcos(\theta)) = r^2 - d^2cos^2(\theta)$. Remember that we work in approximation $r >> d$ and thus $r^2 - d^2cos^2(\theta) \simeq r^2$. The leads to

$$\begin{aligned} V_P(r,\theta) &= \frac{q}{4\pi\epsilon_0}\frac{2dcos(\theta)}{r^2} \\ &= \frac{2qdcos(\theta)}{4\pi\epsilon_0 r^2} \\ &\equiv \frac{pcos(\theta)}{4\pi\epsilon_0 r^2} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} \end{aligned} \tag{93}$$

where $\vec{p} = 2q\vec{d}$. The quantity $\vec{p}$ is called the dipole moment. Mathematical dipoles have distance $d \to 0$ and $q \to \infty$, such that $p = 2qd$ remains constant.

Now we determine the electrical field by taking the gradient of the potential:

$$\begin{aligned}
\vec{E}_P &= -\vec{\nabla} V_P = -\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}\right) \\
&= -\frac{1}{4\pi\epsilon_0}\left(\frac{\partial}{\partial x}\frac{xp_x + yp_y + zp_z}{r^3}, \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\right) \\
&= -\frac{1}{4\pi\epsilon_0}\left(\frac{p_x}{r^3} - \frac{3x(xp_x + yp_y + zp_z)}{r^5}, \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{-\vec{p} + 3\hat{r}(\hat{r}\cdot\vec{p})}{r^3} \qquad (94)
\end{aligned}$$

The drop of the magnitude of the field with r^3 is characteristic for a dipole field.

5.4 The energy of a charge configuration

We have already discussed that the work needed to bring one point charge, q from infinity to a point P in the field of another point charge equals:

$$U = W = -\int_{\infty}^{P} q\vec{E}\cdot d\vec{l} = \int_{\infty}^{P} q\vec{\nabla} V \cdot d\vec{l} = qV_P \qquad (95)$$

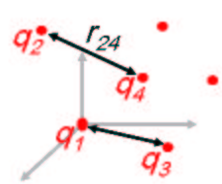

Figure 26: *Illustration of a collection of point charges.*

Figure 26 shows a collection of point charges. To determine the energy of a charge collection (of point charges) we have to calculate how much work is required to assemble such collection. The first charge takes no force (there is no field yet) and thus physically no work is done. To place the second charge, we require $W_2 = q_2 V(r_{12}) = \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}}$, where r_{12} represents the distance between charge q_1 and q_2. When we bring in the third charge we feel the field of the first an second charge, thus:

$$W_3 = q_3 V(r_{13}) + q_3 V(r_{23}) = q_3 \left(\frac{q_1}{4\pi\epsilon_0 r_{13}} + \frac{q_2}{4\pi\epsilon_0 r_{23}} \right) \tag{96}$$

The total energy of our collection so-far is

$$W_{123} = W_1 + W_2 + W_3 = 0 + \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} + \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}} \tag{97}$$

For a collection of N point charges we find:

$$W_N = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i V(r_i) \tag{98}$$

Note that all combinations of q_i and q_j appear twice, which is accounted for by the factor $\frac{1}{2}$. Another remark we have to make is that the self-energy to make the point charges is completely ignored in the above expressions.

5.5 The energy of continuous charge distribution

For a volume charge density we can generalize Equation 98 and obtain:

$$W = \frac{1}{2} \int_{volume} \rho V dv \tag{99}$$

Consider the following question of a smart student. There appears a contribution from the charge ρdv, sensing its own potential in infinitesimal volume dv. Such contribution from the self energy is not present in equation 98. Wouldn't this contribution lead to unphysical results? To check the size of this contribution, we imagine that our continuous charge distribution consists of infinitesimal charged spheres. We calculate the self energy of a uniformly charged sphere with (infinitesimal) radius R, due to its own potential (see equation 89):

$$\begin{aligned} W_{self} &= \frac{1}{2} \int_{sphere} \rho V_{inside} dv = \frac{1}{2} \int_0^R \rho \left(\frac{\rho R^2}{2\epsilon_0} - \frac{\rho r^2}{6\epsilon_0}\right) 4\pi r^2 dr \\ &= \frac{1}{2} \int_0^R \left(\frac{4\pi\rho^2 R^2 r^2}{2\epsilon_0} - \frac{4\pi\rho^2 r^4}{6\epsilon_0}\right) dr \\ &= \frac{1}{2} \left(\frac{4\pi\rho^2 R^5}{6\epsilon_0} - \frac{4\pi\rho^2 R^5}{30\epsilon_0}\right) \\ &= \frac{4\pi\rho^2 R^5}{15\epsilon_0} \end{aligned} \tag{100}$$

Remember that our sphere is infinitesimal ($R \to 0$) and thus the contribution $W_{self} = 0$. Hence, equation 99 correctly represents the energy of a continuous charge distribution.

5.6 The energy in the electrical field

Starting point is the energy of the continuous charge distribution.

$$W = \frac{1}{2}\int_{volume} \rho V dv \tag{101}$$

Substitute Gauss's law $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$:

$$W = \frac{\epsilon_0}{2}\int_{volume} (\vec{\nabla}\cdot\vec{E})V dv \tag{102}$$

We can further simplify this expression. Therefore we first consider the expression:

$$\begin{aligned} \vec{\nabla}\cdot(\vec{E}V) &= \frac{\partial(E_x V)}{\partial x} + ... \\ &= (\frac{\partial E_x}{\partial x})V + (\frac{\partial V}{\partial x})E_x + ... \\ &= (\vec{\nabla}\cdot\vec{E})V + \vec{E}\cdot(\vec{\nabla}V) \end{aligned} \tag{103}$$

We return to the energy (equation 102) and use equation 103 to write:

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int_{volume}\left(\vec{\nabla}\cdot(\vec{E}V) - \vec{E}\cdot(\vec{\nabla}V)\right) dv \\ &= \frac{\epsilon_0}{2}\int_{volume}\left(\vec{\nabla}\cdot(\vec{E}V) + \vec{E}\cdot\vec{E}\right) dv \\ &= \frac{\epsilon_0}{2}\int_{volume}\left(\vec{\nabla}\cdot(\vec{E}V)\right) dv + \frac{\epsilon_0}{2}\int_{volume}\left(\vec{E}^2\right) dv \end{aligned} \tag{104}$$

Using Gauss' Theorem 57, we obtain:

$$W = \frac{\epsilon_0}{2}\int_{surface}\left(\vec{E}V\right) d\vec{o} + \frac{\epsilon_0}{2}\int_{volume} \vec{E}^2 dv \tag{105}$$

The integral of the surface equals zero. Why? Suppose we consider the field of a point charge. The electrical field drops with $1/r^2$ and the potential with $1/r$. We can write:

$$\int_{surface} d\vec{o}\cdot\vec{E}V \approx \int_{surface} d\vec{o}\cdot 1/r^3 \approx \int_{surface} d\phi d\theta sin(\theta) 1/r \tag{106}$$

We should consider a surface enclosing all space, thus $r \to \infty$ and thus

$$\int_{surface} d\phi d\theta sin(\theta) 1/r = 0 \tag{107}$$

Finally, we obtain for the energy of the electrical field:

$$W = \frac{\epsilon_0}{2}\int_{volume} \vec{E}^2 dv \tag{108}$$

5.6.1 $\vec{E}$, V and Energy of a spherical surface charge

In Section 3.4.3 we calculated the electrical field of a charged spherical surface (or shell) with radius R and charge density σ. We found that outside the shell:

$$\vec{E} = E_r \hat{r} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r} \quad r > R \tag{109}$$

Inside the the sphere no charged is enclosed and thus $E_r = 0$.

What is the potential as function of r? We use $V(r) = -\int_\infty^r \vec{E} \cdot d\vec{l}$ and write:

$$V(r) = -\int_\infty^r dr \frac{\sigma R^2}{\epsilon_0 r^2} = \frac{\sigma R^2}{\epsilon_0 r} \quad r > R \tag{110}$$

Now, calculate the potential inside the sphere (where the electrical field is zero):

$$V(r) = -\int_\infty^R \vec{E} \cdot d\vec{l} - \int_R^r \vec{E} \cdot d\vec{l} = V(R) + 0 = \frac{\sigma R}{\epsilon_0} \quad r < R \tag{111}$$

You could always check the results for the potential by calculating the electrical field using $\vec{E} = -\vec{\nabla} V$. The results for the electrical field are also shown in Fig 27, where you can see that inside the surface the electrical field becomes zero, while the potential is constant, $V(R)$.

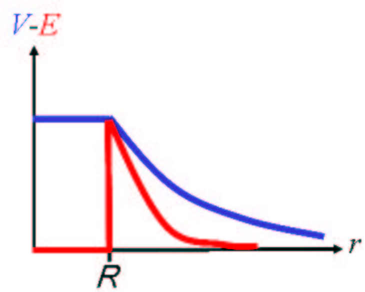

Figure 27: *The electrical field and the potential of a spherical surface charge*

Now we can calculate the energy of this configuration in two ways. We start with the expression for the energy based on the potential (equation 99). The volume charge density ρ is zero everywhere, except on the spherical surface where it is σ. Hence,

$$W = \frac{1}{2}\int_{volume} \rho V dv \rightarrow \frac{1}{2}\int_{surface} \sigma V(R) do \tag{112}$$

$$= \frac{1}{2}\int_{surface} \sigma \frac{\sigma R}{\epsilon_0} do = \frac{1}{2} 4\pi R^2 \sigma \frac{\sigma R}{\epsilon_0} = 2\pi \frac{\sigma^2 R^3}{\epsilon_0} \tag{113}$$

The other expression for the energy with the electrical field in quadrature (equation 108) should yield the same result. Let's check that.

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int_{volume}\vec{E}^2 dv = \frac{\epsilon_0}{2}\int_{r>R}\vec{E}^2 dv \\ &= \frac{\epsilon_0}{2}\int_{r>R}(\frac{R^2\sigma}{r^2\epsilon_0})^2 dv = \frac{4\pi\epsilon_0}{2}\int_{r>R} dr(\frac{R^2\sigma}{\epsilon_0})^2\frac{1}{r^2} \\ &= \frac{4\pi\epsilon_0}{2}(\frac{R^2\sigma}{\epsilon_0})^2\frac{-1}{r}|_R^\infty = 2\pi\frac{\sigma^2 R^3}{\epsilon_0} \end{aligned} \tag{114}$$

as expected.

5.7 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- You understand the definition (and the derivation) of the potential:

$$V_P = -\int_{gauge}^{P}\vec{E}\cdot d\vec{l} \tag{115}$$

  For a point charge, the potential is given by:

$$V_P = \frac{U_P}{q} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r_P} \tag{116}$$

- you can also calculate the electrical field given the potential, using $\vec{E} = -\vec{\nabla}V$.

- you understand how we derived the energy of a charge collection:

$$W_N = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i) \tag{117}$$

- For a continuous charge density the energy is given by:

$$W = \frac{1}{2}\int_{volume}\rho V dv \tag{118}$$

  or, alternatively:

$$W = \frac{\epsilon_0}{2}\int_{volume}\vec{E}^2 dv \tag{119}$$

  which you can apply for a spherical charge density.

In addition, make the corresponding exercises of this section, which you can find in Appendix A.

6 Electrical fields in matter

So-far, we considered electrical fields in vacuum. Usually we started with some (symmetrical) charge configuration and then we calculated the electrical field, its potential or energy. What if we place (electrically neutral) objects in the electrical field? What happens inside those objects and what is the effect on the electrical field in and outside the object? In this Section we will discuss these questions and more.

6.1 The Conductor

What is a conductor? For our purposes a conductor is an object that conducts electrical currents because the negative charge carriers (electrons) can move freely inside the material. The number of the free charge carriers is unlimited. Materials that approach these ideal properties are metals like iron, copper and gold. If the conductor is electrically neutral, it contains an equal amount of negative and positive charge. The positive charge is always bound and thus cannot move freely through the material.

Given the above we can deduce what happens when a conductor is placed inside an electrical field as illustrated in Fig. 28.

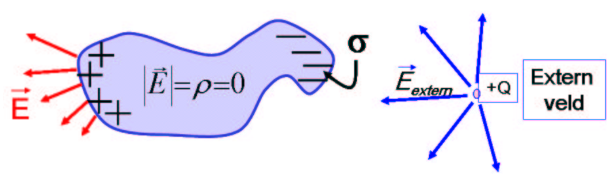

Figure 28: *A conductor is placed in an extern electrical field.*

- What is the electrical field inside a conductor? Suppose there is an electrical field in the conductor. Then, the free electrons would be subjected to the electrical force and start moving. Well, they may do for a short time when the field is just turned on, but we discuss only electrostatic situations. There is only one stable solution to our question and that is that there is no electrical field inside a conductor!

- How can the field be zero inside a conductor? Well, suppose the electrical field is just turned on. Then some free electrons will be attracted by the electrical force and flow to the surface of the conductor (they can not escape). This process goes on till the electrical field inside the conductor has vanished, or better, the original field gets canceled by the field of the free charge sitting on the surface and the nett positive charge that keeps its original position [3]. In general, the nett positive charge will appear on the opposite surface with respect to the side of the free negative charge.

- What is the charge density inside the conductor? We know now that the field inside the conductor is zero. Hence, $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 = 0$ The charge density ρ inside the conductor must be zero. Thus, any nett negative or positive charge must sit on the surface(s) of the conductor.

- What about the electrical field on the surface? Suppose there is an electrical field along the surface of the conductor. The free charge carriers will immediately 'respond' and flow into a configuration with no electrical charge along the surface. suppose there is electrical field perpendicular to the surface. The charge sitting just on the surface will be attracted or repelled by the field but it cannot move inward or outward the conductor. So, the electrical field just outside the conductor is perpendicular to its surface.

- Is there a potential in or over a conductor? No, there can't be, because $V(a) - V(b) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$. At any place inside or at the surface of a conductor the potential is constant.

We seem to know now what happens if we place a conductor inside an electrical field. Well, not really. The (original) field outside the conductor also changes and we have not discussed that. This is a though problem to solve. When there is an obvious symmetry in the configuration we can give the solution.

6.1.1 A conducting plate in a uniform electrical field.

We start with a uniform electrical field in the z direction and then place an infinitely large conducting plate as shown in Fig. 29. After a few nanoseconds an electrostatic configuration exists. Inside the plate the field is zero and we know that the nett charge will sit on the surface. We can replace the plate by the electrical configuration as shown in Fig. 30. Thus we have only one unknown charge density $\sigma = \sigma_+ = -\sigma_-$. We know all the techniques to solve this problem. In the area between the surface charges the electrical field should disappear:

$$\vec{E}_{total} = \vec{0} = \vec{E}_{external}\hat{z} + \vec{E}_{(-)}\hat{z} - \vec{E}_{(+)}\hat{z} \tag{120}$$

[3] This is always possible in one, and only one, way based on the 'uniqueness theorem'. The derivation of this theorem is beyond the scope of this course.

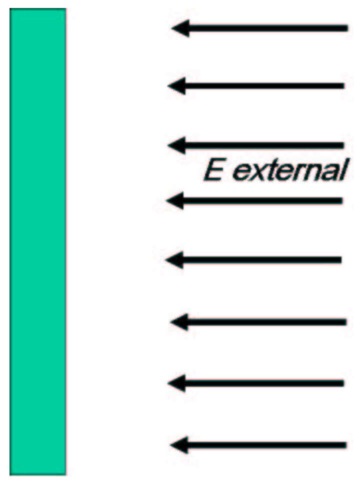

Figure 29: *An infinitely large conducting plate is placed in an external electrical field in the z direction.*

Note that the contribution of the positive charge acquires a minus sign, because it points in the negative z direction! We have already calculated the electrical field of a flat surface charge: $E_{\pm} = \frac{\sigma_{\pm}}{2\epsilon_0}$, which we substitute in the above expression:

$$\vec{0} = \vec{E}_{external}\hat{z} + \frac{-\sigma}{2\epsilon_0}\hat{z} - \frac{\sigma}{2\epsilon}\hat{z} \tag{121}$$

which leads to a charge density:

$$\sigma = \epsilon_0 |\vec{E}_{external}| \tag{122}$$

Return to the original configuration with the plate and check yourself that all conditions for a conductor in an electrical field are fulfilled. Outside the plate the contribution to the electrical field of the positive and negative net charge cancels such that the original field outside the plate has not changed.

6.1.2 A grounded conducting plate

We can also consider a grounded plate. This means that the plate is connected to the earth. For this purpose the earth should be seen as an unlimited source of charges. Thus the earth can pump electrons in or out the plate at no cost without acquiring any net charge itself. Figure 31 shows a grounded plate, together with the charge distribution. Note that no positive net charge is induced. All the necessary electrons are provided by the earth. Check yourself that all boundary condition in and on the conducting plate are satisfied. Note that the field outside the plate has changed.

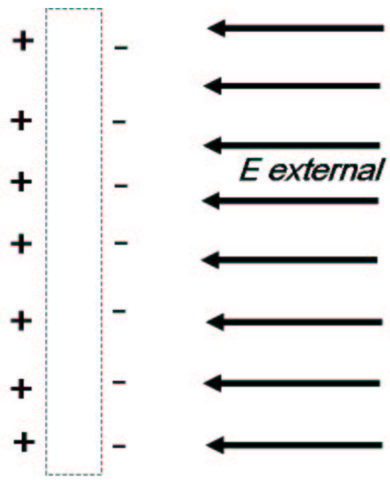

Figure 30: *The electrical charge configuration induced on the conducting plate when placed in an extern electrical field as indicated. The '+' and '−' sign represent flat surface densities σ_+ and σ_- respectively.*

6.1.3 A spherical conducting shell with a point charge inside

We consider a conducting shell with an inner and outer radius of R_i and R_o respectively. We have put a positive point charge q in its center as shown in Fig. 32. What is the charge density in the shell and what is the electrical field everywhere in space? To cancel the electrical field at R_i of the point charge we need negative charge on the inner surface,Q_i. If net negative charge is induced on the inner surface there must be a similar amount of positive charge,$Q_o = -Q_i$ on the outer surface at R_o. Due to the symmetry of this configuration the induced charge will be distributed uniformly over the inner and outer surface. The question is now how to calculate the Q_i (or Q_o). Well, we have calculated the electrical field of a spherical charge density before (see equation45):

$$\vec{E} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r} \quad r > R \tag{123}$$

$$= \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad r > R \tag{124}$$

and $\vec{E} = 0$ inside the shell. Inside the conductor the field should cancel. Hence, when we add the contribution of the point charge, the negative charge density on the inner shell and the positive one on the outer shell, we find:

$$\vec{0} = \vec{E}_q + \vec{E}_{Q_i} + \vec{E}_{Q_o}$$

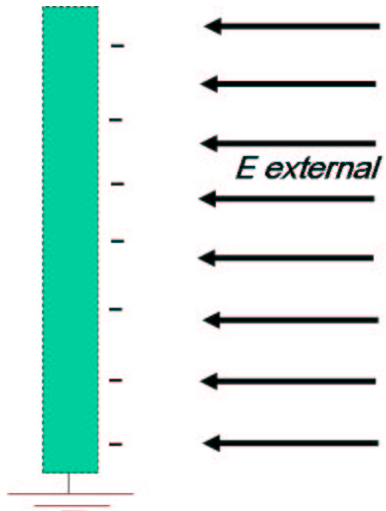

Figure 31: *A grounded conducting plate is placed in an extern electrical field. Note the international symbol for 'grounding' indicated in the figure.*

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2} + \frac{Q_i}{r^2} + 0\right)\hat{r} \tag{125}$$

We conclude that $Q_i = -q$ (and $Q_o = -Q_i = q$). Note that the charge densities on the inner and outer surface, besides the sign, are also in magnitude not the same: $\sigma_i = -q/(4\pi R_i^2)$ and $\sigma_o = q/(4\pi R_o^2)$.

What are the consequences for the electrical field everywhere in space outside the conductor? Well, the field is just the field of the point charge. We could say that the presence of the conductor has not disturbed the field at all!

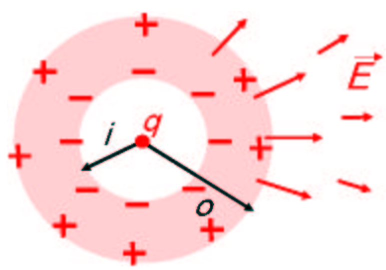

Figure 32: *A positive point charge is positioned in a conducting shell with inner radius R_i and out outer radius R_o.*

6.1.4 Method of images

A powerful technique to calculate the electrical field in many situations is the 'method of images' using a 'mirror charge'. For example we consider a grounded plate and place a point charge q at a distance d as illustrated in Fig. 33. The point charge induces charge on the plate surface with an a priori unknown distribution. For this example the plate is infinitely large and infinitely thin

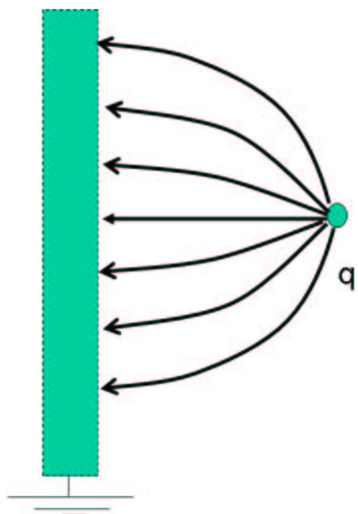

Figure 33: *A grounded conducting plate is placed in the field of a point charge q.*

The technique of the mirror charge is based on the fact that we may change the charge configuration, as long as we respect the boundary conditions. The electrical field of the changed configuration is the same as the original configuration, between the boundaries. Outside the boundaries the fields are incompatible. Suppose we want to calculate the electrical field between the plate and the point charge q. What are the boundary conditions? Well we have a point charge and we know that in the (infinitely thin) plate the electrical field should be zero. Additionally, the electrical field is perpendicular on the surface of the plate. What is an alternative charge distribution that respects these boundary conditions? Figure 34 shows the alternative configuration, existing of a negative charge 'mirrored' in the plate. Check that the boundary conditions are fulfilled. The resulting configuration is the an electrical dipole (on the z axis). The charge density on the plate can be determined using Gauss' Law: $\frac{\sigma}{\epsilon_0} = \vec{E}(ontheplate) \cdot \hat{n}$. We need to know the

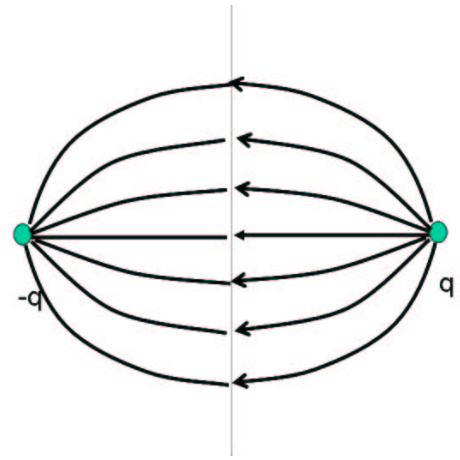

Figure 34: *An alternative charge configuration, existing of a electrical dipole.*

electrical field. Therefore we start with the potential:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{Q}{\sqrt{x^2+y^2+(z+d)^2}}\right) \quad (126)$$

To electrical field is given by:

$$\vec{E}(x,y,0) = -\vec{\nabla}V(x,y,0) = \frac{Q}{4\pi\epsilon_0}\left(\frac{2d}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\hat{z} \quad (127)$$

Now we know the electrical field and thus the charge density. In principle we can go back tot the original configuration and calculate the electrical field everywhere in space.

6.2 Capacitors

We take two parallel plates of conducting material and charge the plates with opposite charge as indicated in Fig. 35. The plates are separated by a distance d. We ignore the thickness of the plates and assume that the plate have a large surface A with respect to d. We have calculated the electrical field for an infinitely large flat surface charge already in Sec. 3.4.2, which provides a good approximation for the present configuration with $A >> d$. We can write:

$$\vec{E} = \frac{+Q}{2A\epsilon_0}\hat{z} - \frac{-Q}{2A\epsilon_0}\hat{z} = \frac{Q}{A\epsilon_0}\hat{z} \quad (128)$$

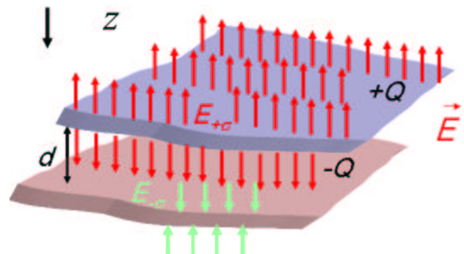

Figure 35: *Two parallel plates of conducting material. The plate have opposite charge Q and are separated by a distance d. The contribution to the electrical field of each plate individual is also indicated.*

The electrical field is constant and points away from the positive charged plate. The contribution of the negative charge obtain an additional minus sign, because if it would be a positive charge the field would point in the negative z direction. Make sure you understand this argument.

The potential difference between the plates is given by:

$$V = V_+ - V_- = -\int_-^+ \vec{E} \cdot dz = -\frac{1}{A\epsilon_0} Q \int_d^0 dz = \frac{d}{A\epsilon_0} Q \qquad (129)$$

What we have shown now, i.c. the potential difference between the conductors is proportional to Q, is generally valid for capacitors.

Is the behavior $V \sim Q$ independent on the shape and size of the conductors? Yes. Suppose you take two conductors of arbitrary shape and size as illustrated in Fig. 36. The electrical field is proportional to the charge

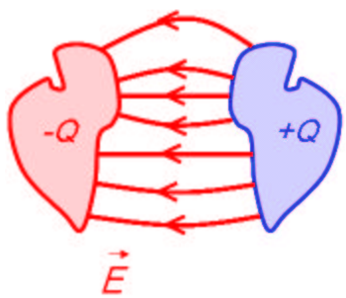

Figure 36: *A capacitor, consisting of two conductors with bizarre shapes.*

density in, or better *on*, the conductors. The charge density is proportional to Q. V is proportional to E and thus also to Q.

6.2.1 Capacitance

We have seen that $V \sim Q$. Now we introduce a a constant of proportionality called capacitance, C, such that

$$C = \frac{Q}{V} \tag{130}$$

The capacitance depends completely on the geometry of the electrical configuration. For the configuration with the two plates, we have shown that $C = \frac{A\epsilon_0}{d}$. The larger A and the smaller d the more charge can be stored in the configuration for the same potential V. The unity of capacitance is called Farad (=Coulomb/Volt), denoted by F. In practice C, measured in Farad is numerically small. For our plate configuration with $d = 1$ mm and $A = 1$ m, $C = 9 \times 10^{-11}$ F. In the newspapers a few years ago, there was an item about a small capacitor with $C = 1$ F, but I haven't figured out yet how to make that. Let me know if you find the 'trick' on the web.

6.2.2 The energy of a capacitor

We start with an uncharged capacitor and move electrons from one conductor to the other to charge it up. The electrons 'sense' the electrical force in this process. Hence, moving them requires energy.

The energy needed to bring some charge dq to the other conductor is $dU = V(q)dq$. With $V(q)$ the potential difference between the two conductors as function of the already moved charge q. Since the capacitance C is a purely geometrical quantity we can write $V(q) = q/C$ and thus $dU = \frac{qdq}{C}$. The total energy of a charge capacitor is then given by:

$$U = \int dU = \int_0^Q \frac{qdq}{C} = \frac{1}{2}\frac{Q^2}{C}|_0^Q = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \tag{131}$$

with Q and V the final charge and potential of the capacitor respectively.

6.2.3 Examples of Capacitors

6.2.4 Cylindrical configuration

Figure 37 shows a cylindrical configuration, consisting of two coaxial conductors with length L. The inner conductor has a radius a. The outer radius is b and is much smaller than the length. The conductors have opposite total charge Q. What is the capacitance of this configuration?

- First, determine the electrical field in the space between the two conductors. We may assume that $L >> b$ which implies that away from

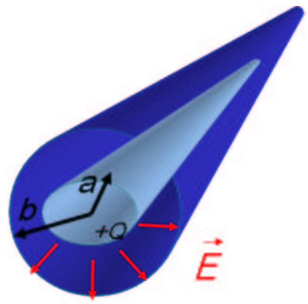

Figure 37: *Two coaxial conductors. The inner and outer conductors have radii a and b respectively. The conductors have opposite total charge Q.*

the edges the field is radial. The charge density on the inner conductor is $\sigma_i = \frac{2\pi a L}{Q}$. We calculate the flux through an imaginary small cylindrical Gaussian surface and obtain:

$$E_r 2\pi r l = \frac{\sigma_i 2\pi a l}{\epsilon_0} \tag{132}$$

Note that in between the conductors there is no contribution to the flux (and field) from the outer conductor. The electrical field is $\vec{E} = \frac{\sigma_i a}{\epsilon_0 r}\hat{r}$

- The potential difference between the two conductors is given by:

$$V = -\int_b^a \vec{E} \cdot d\hat{r} = -\int_b^a \frac{\sigma_i a}{\epsilon_0 r} dr = -\frac{\sigma_i a}{\epsilon_0} ln(r)|_b^a = \frac{\sigma_i a}{\epsilon_0} ln(b/a) \tag{133}$$

- Now we know the potential and it becomes a 'piece of cake' to obtain the capacitance:

$$C = \frac{Q}{V} = \frac{2\pi a L \sigma_i \epsilon_0}{\sigma_i a ln(b/a)} = \frac{2\pi L \epsilon_0}{ln(b/a)} \tag{134}$$

Like the plate-capacitor, the capacitance of this configuration increases when the distance between the two conductors becomes smaller.

6.2.5 Spherical Capacitor

We now consider a capacitor consisting of two spherical surface of a conducting material with radius b. In its center we placed a conducting sphere with radius a. The conductors carry opposite charge Q. What is the capacitance of this configuration? To calculate the electrical field we use a spherical

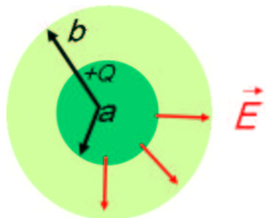

Figure 38: *A spherical capacitors. The inner conductor has radius a and the outer one has radius b. The conductors have opposite charge.*

Gaussian surface with radius r such that $a < r < b$. For the electrical field we obtain:

$$\vec{E} = \frac{\sigma a^2}{\epsilon_0 r^2}\hat{r} \tag{135}$$

Now we calculate the potential difference between the conductors:

$$V = -\int_b^a \vec{E}\cdot d\hat{r} = -\int_b^a \frac{\sigma a^2}{\epsilon_0 r^2}dr = \frac{\sigma a^2}{\epsilon_0 r}|_b^a = \frac{\sigma a^2}{\epsilon_0}(\frac{1}{a} - \frac{1}{b}) \tag{136}$$

for the capacitance follows:

$$C = \frac{Q}{V} = \frac{4\pi a^2 \sigma \epsilon_0}{\sigma a^2(1/a - 1/b)} = 4\pi\epsilon_0 \frac{ab}{a-b} \tag{137}$$

Note that the capacitance is independent on V and/or Q and thus is indeed a purely geometrical quantity.

6.3 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- You understand the properties of a conductor.
- When a conductor is placed in an electrical field you can identify the surface charge densities and calculate the resulting electrical field.
- You understand the basics of the mirror charge technique
- You can explain what capacitors are and deduce the relation:

$$C = \frac{Q}{V} \tag{138}$$

- You can calculate the the capacitance of a parallel plate capacitor.
- The energy of a capacitor is given by

$$U = \frac{1}{2}CV^2 \tag{139}$$

In addition, make the corresponding exercises of this section, which you can find in Appendix A.

7 Insulators

All matter that cannot be classified as a conductor we call an insulator. In insulators there are no free electrons to cancel the electric field inside the 'bulk'. However, by polarization of the molecules and atoms inside an insulator there is some cancellation of the field. When an atom polarizes we can discriminate between a positive and negative side. For this reason, insulators are historically referred to as 'dielectrics'. The properties and behavior of dielectrics in electrical fields is discussed in this Section.

7.1 Polarization of atoms and molecules

What happens microscopically when we electrically polarize an atom? The classical picture of an atom is a big positive nucleus surrounded by tiny electrons orbiting it. For our purpose, that is electrostatics, we adapt this picture by taking the 'average' of the atom over time. Yes, you need some imagination, but this leads to an atom consisting of a small positive nucleus in the center of a uniform cloud of negative charge. Consider the illustration in Fig. 39. When we apply an electrical field as is depicted in the right

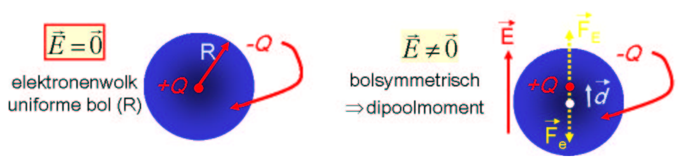

Figure 39: *A view of an electrostatic atom. It consists of a positive nucleus surrounded by a uniform cloud of negative charge. The cloud has radius (R). Left: Unpolarized, the nucleus is centered; Right: Polarized, the nucleus is shifted away from the center of the cloud. This results effectively in a dipole.*

figure, the negative charge is attracted, while the positive nucleus is repelled.

Consequently, the nucleus is shifted by a distance d with respect from the center of the cloud. We have assumed that the spherical shape of the cloud is conserved. The net effect is that we produced an electrical dipole with $\vec{p} = Q\vec{d}$.

The external field results in a polarization $\vec{p} = Q\vec{d} = \alpha\vec{E}$, with proportionally factor α, called the 'polarizability'. Table 1 lists some experimentally obtained polarizabilities. As expected, the factor α grows with increasing

Atom	Z	α (10^{-30} m^3)
Helium	2	3
Neon	10	5
Argon	18	20
Water vapor	-	500

Table 1: *Several examples of the polarizability for some atoms and water vapor.*

charge Z. For water vapor we observe that α is relatively large. Such a-typical behavior points in the direction of a different physical mechanism. Indeed, there is an additional effect in water vapor and other so called 'polar' molecules. The electrons in water molecules are attracted by the positive oxygen nucleus, more than by the hydrogen nuclei. The result is that water (and other polar molecules) have a built in electrical dipole moment. Figure 40 shows what happens when these molecules are placed in an electrical field. The dipole moments of the water molecules are aligned by the external

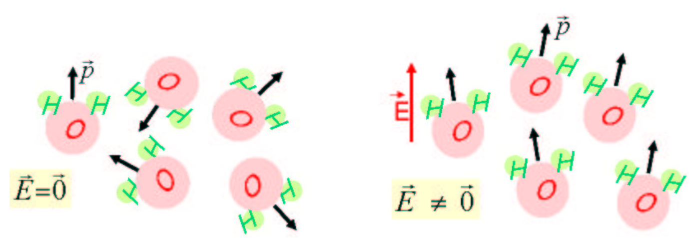

Figure 40: *Left: Water molecules without electrical field. the electrical dipole moment of the molecules is indicated. Right: Water molecules in an external electrical field. The dipole moment of the molecules are aligned by the electrical force.*

electrical field which leads to the relatively large factor α.

7.2 Macroscopic Polarization

What is the electrical field in a dielectric when put in a known external field? Forget for the moment the atoms and molecules in the dielectric and put on your 'abstract glasses'. In electrostatic theory, dielectrics consist of infinitesimally small electrical dipoles. A dielectric can be electrically polarized by putting it in an external field $\vec{E}_0$ as illustrated by Fig. 41. The polarization per unit volume is defined as the polarization $\vec{P}$ which is

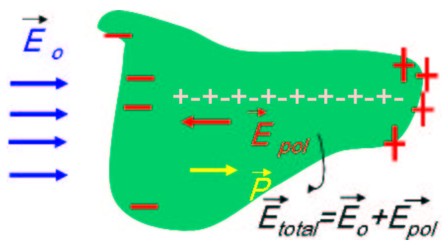

Figure 41: *A dielectric in an external electric field $\vec{E}_0$. The microscopic dipoles polarize (polarization $\vec{P}$). The resulting total field is given by the sum of the original external field and that of the polarized dipoles.*

proportional to the electrical field:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{140}$$

The factor of proportionality χ_e is called the electrical susceptibility for historical reasons. In this expression, the electrical field $\vec{E} = \vec{E}_{total}$, the total field. It is the sum of the external field and the, yet unknown, field from the polarized dipoles ($\vec{E}_{pol}$).

So, what is the field of the polarized dipoles? Well, the polarization leads to net charge separation. When we can quantify this charge, we know how to calculate the field!

Have another look at Fig. 41. Inside the the dielectric there is macroscopically no net charge. The dipoles are aligned in chains of positive and negative charge, but all the '+' are canceled by '−'. This is not the case at 'the start and end of the chains' at the boundary of the dielectric. At the left side, all the chains start with '−', while at the right side all chains end with '+'. There is a net charge separation at the surface of the dielectric. Note that the charges themselves are localized in contrast with the free charge in a conductor. For this reason, the net charge at the surfaces of a dielectric is called bound charge. The amount of bound charge is given by the polarization:

$$\sigma_{bound} = \vec{P} \cdot \hat{n} \tag{141}$$

In principle, we can now deal with any electrical field configuration in a dielectric. In the following Sections we will discuss some examples.

7.2.1 The electrical field in a flat dielectric

We consider a flat dielectric with a given χ_e and we place it in a known uniform electrical field $\vec{E}_0$ as shown in Fig. 42. The dipoles in the dielectric

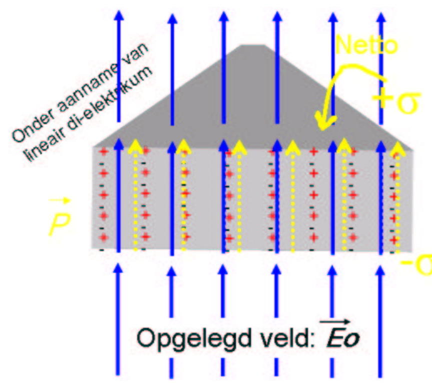

Figure 42: *A piece of dielectric in an external electric field $\vec{E}_0$. The microscopic dipoles polarize (polarization $\vec{P}$). The resulting bound charge at the surface is also indicated.*

polarize, leading to a polarization $\vec{P}$. The bound net charge on the surfaces of the dielectric is $\sigma_b = \vec{P} \cdot \hat{n} = \vec{P}$. Hence, we have a configuration of two oppositely charged flat surfaces, which leads to an induced field: $\vec{E}_{pol} = \frac{-\sigma_{pol}}{\epsilon_0}$. Here we simply used the formula of a plate capacitor, derived in Section 6.2 (but I am sure, you could derive it yourself by now). In the dielectric, the electrical field is now:

$$\begin{aligned} \vec{E} &= \vec{E}_0 + \vec{E}_{pol} \\ &= \vec{E}_0 - \sigma_{pol}/\epsilon_0 \\ &= \vec{E}_0 - \vec{P}/\epsilon_0 \end{aligned} \tag{142}$$

Using equation 140 we can write

$$\vec{E} = \vec{E}_0 - \chi_e \vec{E} = \frac{1}{(\chi_e + 1)} \vec{E}_0 \tag{143}$$

The susceptibility χ_e for glass and plastic like are of order 10. For water the susceptibility is about 80.

7.2.2 Plate capacitor with dielectric

Consider a plate capacitor with a dielectric in between its plates as illustrated in Fig. 43. The plates have surface A and are separated by a distance d. The

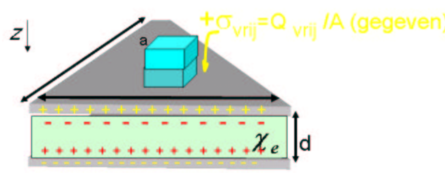

Figure 43: *A plate capacitor with dielectric in between the plates.*

dielectric has an electrical susceptibility χ_e. Given the free charge Q_{free} on the plates, what is the electrical field in the dielectric and what is the capacitance of this capacitor?

Starting point is an *empty* capacitor. Thus the same configuration as already discussed in Section 6.2. We repeat our results:

$$\begin{aligned} \vec{E}_{empty} &= \frac{Q_{free}}{A\epsilon_0}\hat{z} = \frac{\sigma_{free}}{\epsilon_0}\hat{z} \\ V_{empty} &= -\int_-^+ \vec{E}_{empty}\cdot dz = \frac{d}{A\epsilon_0}Q_{free} \\ C_{empty} &= \frac{Q_{free}}{V_{empty}} = \frac{\epsilon_0 A}{d} \end{aligned} \tag{144}$$

Now we place the dielectric. The electrical field between the plates (in the dielectric) changes. According to equation 143 it becomes:

$$\vec{E} = \frac{1}{(\chi_e+1)}\vec{E}_{empty} = \frac{1}{(\chi_e+1)}\frac{\sigma_{free}}{\epsilon_0}\hat{z} \tag{145}$$

With this electrical field we can calculate the potential between the plates:

$$V = -\int_-^+ \vec{E}\cdot dz = \frac{1}{(\chi_e+1)}\frac{d}{A\epsilon_0}Q_{free} = \frac{1}{(\chi_e+1)}V_{empty} \tag{146}$$

for the capacity follows:

$$C = \frac{Q_{free}}{V} = (\chi_e+1)\frac{\epsilon_0 A}{d} = (\chi_e+1)C_{empty} \tag{147}$$

In textbooks the following definition is often used

$$(\chi_e+1)\epsilon_0 = \epsilon \tag{148}$$

With this definition most expression related to dielectrics become similar as the expression in vacuum after substituting ϵ for ϵ_0.

Note that the capacitance between the empty and 'filled' capacitor is just the factor $(\chi_e + 1)$. Thus, for a plastic fillings the capacity grows with a factor of order ten.

7.3 Knowledge and Skills

The knowledge and skills you should have acquired during reading of the previous can be summarized as follows:

- You can explain how a dielectric is polarized in an electrical field, resulting in a net bound charge at its surface.
- The relation between the polarization and the electrical field is

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{149}$$

- With this relation you can calculate the electrical field in a parallel plate capacitor filled with a dielectric:

$$\vec{E} = \frac{1}{(\chi_e + 1)} \vec{E}_{empty} = \frac{1}{(\chi_e + 1)} \frac{\sigma_{free}}{\epsilon_0} \hat{z} \tag{150}$$

  You can also show that the capacitance is:

$$C = (\chi_e + 1) C_{empty} \tag{151}$$

  The capacitance of the filled capacitor is larger than that of the empty capacitor.

In addition, make the corresponding exercises of this section, which you can find in Appendix A.