Homework Theory 1
Exercises sent to Brightspace on Tuesday 11 February 2020, 17:00
Answers to be submitted individually by Tuesday 18 February 2020 11:00am
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1 Lorentz transformation

a) The Galilean transformation of the space coordinate, from coordinate system $S$ to system $S'$, with relative velocity $v$, is given by $x' = x - vt$. What is the Galilean transformation of the time coordinate, between two inertial observers?

b) The Galilean transformation of the space coordinate $x$, from system $S'$ to $S$, is given by $x = x' + vt$. Let’s find the corresponding transformation if we assume that the speed of light is equal in systems $S$ and $S'$, i.e. $x' = ct'$ and $x = ct$. We modify the Galilean transformation rules, by $x' = \gamma(x - vt)$ and find the expression for $\gamma$:

$$x' = \gamma(x - vt) \quad \Rightarrow \quad \gamma(ct - vt)$$

$$x = \gamma(x' + vt') \quad \Rightarrow \quad \gamma(ct + vt')$$

This leads to:

$$\frac{x'}{\gamma} = \frac{ct'}{\gamma} = \frac{\gamma(ct - vt)}{\gamma} = (ct - vt)$$

Eliminate $t$ in the above expression, and give the expression for $\gamma$.

c) Rewrite the Lorentz transformation,

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{v}{c}x),$$

expressing the velocity as a fraction of the speed of light, $\beta = v/c$, and the time-coordinate as $x^0 \equiv ct$.

d) The time-coordinate, and three space coordinates can be expressed as 4-vectors $x^\mu = (ct, x, y, z)$. Show that the quantity $I = \Sigma_{\mu=0,3} \Sigma_{\nu=0,3} g_{\mu\nu} x^\mu x^\nu = x^\mu x^\mu$ is invariant, i.e. that $I = I'$. (Apply a boost in the direction of $x^1$.)

e) Suppose you want to build a muon collider, and you want to keep your muons about 30 minutes in your accelerator before they decay. What boost (i.e. value for $\gamma$) is then needed for the muons? (The lifetime of muons is 2.2 µs.) To what beam energy does this correspond? (The mass of the muon is 106 MeV/c².)
2 Relativistic momentum

Given 4-vector calculus, we know that \( p_\mu p^\mu = E^2/c^2 - \vec{p}^2 = m_0^2 c^2 \).

a) Show that you get in trouble when you use \( E = mc^2 \) and \( \vec{p} = m\vec{v} \).

b) Show that \( E = \gamma m_0 c^2 \) and \( \vec{p} = \gamma m_0 \vec{v} \) obey \( E^2/c^2 - \vec{p}^2 = m_0^2 c^2 \).

Notice that energy and momentum are “treated” in the same way; both get an extra factor \( \gamma \). Such an “identical treatment” is known as covariance, and implies that the Lorentz transformations and 4-vector description yield a consistent picture.

It is tempting to write the substitution \( m = \gamma m_0 \), to yield the original formulas \( E = mc^2 \) and \( \vec{p} = m\vec{v} \). This is sometimes referred to as “relativistic mass”. However, Albert Einstein himself wrote on 19 June 1948 in a letter to Lincoln Barne (quote from L.B. Okun (1989), p. 42): “It is not good to introduce the concept of the mass \( m = \gamma m_0 \) of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the rest mass \( m_0 \). Instead of introducing \( m \) it is better to mention the expression for the momentum and energy of a body in motion.”

So, from now on every \( m \) we use, refers to the rest mass \( m_0 \). And we will use natural units, \( c = 1 \). Hence, \( E = \gamma m \) !
3 Center-of-mass energy

a) Not only the space and time can be expressed as a 4-vector, but also energy and momentum can be expressed as 4-vectors, \( p^\mu = (E/c, p_x, p_y, p_z) \). Because \( p_\mu p^\mu \) is invariant, this means that the rest-mass \( m_0 \) of a particle does not change under Lorentz transformations. Show that \( p_\mu p^\mu = m_0^2 c^2 \).

b) Let’s consider two colliding particles \( a \) and \( b \), with 4-momenta \( p_a^\mu \) and \( p_b^\mu \). We will use natural units, with \( c = 1 \) and \( \hbar = 1 \), so \( p_a^\mu = (E_a, \vec{p}_a) \). We take the masses of the two colliding particles equal, \( m_a = m_b = m \), and we sit in the center-of-mass frame of the system, \( \vec{p}_a = -\vec{p}_b \). What are the four components of the sum of the two 4-vectors, \( p_{\text{tot}}^\mu = (p_a^\mu + p_b^\mu) \)?

c) The ‘invariant mass’ of the combined system, is often called the ‘center-of-mass energy’ of the collision. If the energy of both particles \( a \) and \( b \) is 4 TeV, what is then the center-of-mass energy, \( \sqrt{s} \equiv \sqrt{p_{\text{tot}}^\mu p_{\mu,\text{tot}}} \)?

d) Let’s consider a fixed-target collision of two protons. One proton has an energy of 4 TeV, and 4-vector \( p_a^\mu \), whereas the other proton is at rest, with 4-vector \( p_b^\mu \). What are the four components of the sum of the two 4-vectors, \( p_{\text{tot}}^\mu = (p_a^\mu + p_b^\mu) \)? Give the expression for the center-of-mass energy of this system.

e) People were afraid that the earth would be destroyed at the start of the LHC, planning for collisions with beams of 7 TeV each. The earth has been bombarded for billions of years with cosmic rays. What is the center-of-mass energy of the highest energetic cosmic rays (\( 10^{21} \) eV) hitting the atmosphere? Was the fear justified?

f) What is the energy of a cosmic ray hitting the atmosphere, that corresponds to the center-of-mass energy of collisions of two lead-ions \( ^{208}\text{Pb} \) with energies of 2.24 TeV per nucleon?

g) Consider relatively low-energy proton-proton collisions, with opposite and equal momenta (ie. the center-of-mass system is at rest). In the process \( p+p \rightarrow p+p+p+\bar{p} \) an extra proton-antiproton pair is created. What is the minimum energy of the protons to create two extra (anti)protons?