Elements of Group Theory

Michiel Botje
Nikhef, Science Park, Amsterdam

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This section gives a brief overview of some group theoretical concepts and terminology that is often used in the particle physics literature.

[This section is still incomplete]

Definition of a group

A **group** G is a collection of **elements** $\{a, b, c, \ldots\}$ with a composition rule ab, often called the **multiplication** of a and b, that satisfies:

1. For every element a and b of G, the product ab is also an element of G;

2. The multiplication is associative $(ab)c = a(bc)$;

3. There is a unique **unit element** e, with $ea = ae = a$, for all elements a;

4. Each element a has a unique **inverse** a^{-1} in G, with $aa^{-1} = a^{-1}a = e$.

This is of course quite an abstract definition[4] since it is not specified what these group elements are, and what the group multiplication stands for. In physics, we can think of a group as a set of transformations of some kind, such as translations or rotations in Euclidian space, Lorentz transformations in space-time, or—more abstract—transformations in quark flavour or color space.

A group can be *discrete*, with the group elements labeled by a set of indices, or *continuous*, with the elements labeled by a set of continuous parameters.

An example of a discrete group is the set of integers, with addition as the group multiplication. The number zero is then the unit element and the negative integers are the inverse of the positive integers (and *vice versa*). This group obviously has an infinite number of elements. An example of a continuous group is that of rotations in two dimensions, with each element labeled by a rotation angle. Here the group operation is the addition of rotation angles. The unit element is a rotation over zero angle, and the inverse element is a rotation with the angle reversed.

Another distinction is that of **Abelian groups** where the group operation commutes ($ab = ba$ for all elements a and b) and **non-Abelian** groups where the group operation does not always commute. For instance, the group of rotations in two dimensions is Abelian, but that of rotations in three dimensions is not.

We will now use the finite discrete *cyclic group* to illustrate some basic ideas.

[4]The definition, as stated here, is somewhat redundant because e and a^{-1} *must* be unique by virtue of their definitions and the requirements (1) and (2). We leave it as an exercise to prove this.

The cyclic group

As an example of a finite group, take the set

$$\mathsf{G} = \{1, i, -1, -i\}, \tag{0.1}$$

with ordinary complex multiplication as the group operation. The number of elements of a discrete group is called the **order** of the group, sometimes denoted by $[\mathsf{G}]$. Thus, the group above is of order four.

A finite group is completely specified by its **multiplication table** which for our group $\mathsf{G} = \{e, a, b, c\}$ is given by

G	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

A multiplication table usually is not very instructive but some characteristic features can easily be spotted: (i) Each element of the group occurs only once in each row or column. This is because ab and ac cannot map onto the same element. Indeed, if $ab = ac$ we find, multiplying from the left with a^{-1}, that $b = c$; (ii) The table above is symmetric around the diagonal which shows that the group G is Abelian; (iii) Elements with e on the diagonal are its own inverse.

We can also write (0.1) as

$$\mathsf{G} = \{1, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\},$$

which shows that G can be realised by rotations over $\{0, 90, 180, 270\}$ degrees. In this realisation, the group operation is the *addition* of rotation angles. A rotation of a 2-dimensional coordinate system over an angle θ, measured counterclockwise from the x-axis, is described by the rotation matrix[5]

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{0.2}$$

Setting $\theta = \{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}$, we can represent the group G by the matrices

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}, \tag{0.3}$$

[5]Note that this is a **passive** rotation of the coordinate system where the *same* vector is described in the primed and unprimed systems. An **active** transformation rotates the vector and is related to the passive transformation by inverting the sign of θ.

with matrix multiplication as the group operation. This is called a 2-dimensional *representation* of G.

▶ An n-dimensional **representation** of a group G is a mapping of each element g_i onto a non-singular $n \times n$ matrix M_i that preserves the group multiplication

$$g_i g_j = g_k \quad \to \quad M_i M_j = M_k.$$

Don't confuse the *dimension* of a representation of G with the *order* of G. ◀

It is clear that rotations by multiples of 90° leave a square invariant. If we label the corners of the square $\{1, 2, 3, 4\}$ then we see that G can also be realised by the following four *permutations*:

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\}. \tag{0.4}$$

▶ Every element of a finite group of order n corresponds to a permutation of n objects. ◀

When we arrange the objects in an n-dimensional vector, the permutations can be expressed as $n \times n$ matrices, thus yielding a **regular representation** of the group (*i.e.* a representation with a dimension equal to the order of the group):

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}. \tag{0.5}$$

If we introduce *complex* matrices, we can say that (0.1) is a 1×1 complex representation of G. In this somewhat un-systematic fashion we have thus found a 1-, 2- and 4-dimensional representation of G. It is an important (and nontrivial) task of group theory to find all representations of a group or, to be more precise, all so-called **irreducible representations** since these serve as basic building blocks to construct all others.

Taking powers of a we see that G can be written as

$$\mathsf{G} = \{e, a, a^2, a^3\} \quad \text{with} \quad a^4 = e. \tag{0.6}$$

Thus a generates all elements of the group and is called the **generator** of G. For obvious reasons, G is called the *cyclic group* of order four, denoted by Z_4.

▶ The **cyclic group** Z_n of order n is generated by a 2-dimensional rotation over the angle $2\pi/n$. The group leaves an n-sided regular polygon invariant. ◀

Some basic concepts

▶ A **subgroup** $\mathsf{H} \subset \mathsf{G}$ is a set of elements of G that satisfy the group conditions. The unit element e is obviously shared by G and all its subgroups. ◀

▶ The **left coset** $g\mathsf{H}$ is obtained by multiplying all elements of H from the left by an element g which is *not* in H. Likewise we define the **right coset** $\mathsf{H}g$. Note that the left and right cosets are of the same order as H but are not subgroups of G since they do not contain the unit element. ◀

▶ A subgroup H and its left (or right) coset have no element in common. ◀

This can easily been seen as follows: Let $gh_1 = h_2 \in \mathsf{H}$. Then $g = h_2 h_1^{-1} \in \mathsf{H}$ which leads to a contradiction since g is, by definition, *not* in H. Let us now take another element g' which is *not* in H and also not in $g\mathsf{H}$. It is easy to show (homework) that $g'\mathsf{H}$ has no element in common with $g\mathsf{H}$ (and H). Now we can pick another element g'' not in H or in any of the two cosets to build another completely disjunct coset $g''\mathsf{H}$. In this way we can continue till we have divided the entire group G into H and cosets $g\mathsf{H}$ which all have the same number of elements, and no elements in common. We just have proven

▶ **Lagrange's theorem**: The order m of a subgroup $\mathsf{H} \subset \mathsf{G}$ is an integer division of the order n of G. The ratio $k = m/n$ is called the **index** of H in G. It directly follows that groups of prime order cannot have any subgroups. ◀

Another very important operation is that of **conjugation**.

▶ The **conjugate** of any element a with respect to any other element g is defined by a so-called **similarity transformation**

$$\tilde{a} = gag^{-1}. \tag{0.7}$$

When $ab = c$ then $\tilde{a}\tilde{b} = \tilde{c}$, that is, conjugation preserves the group multiplication. Clearly, the elements a and $\tilde{a}$ are *each other's* conjugate since $a = g^{-1}\tilde{a}g$. Note that the elements of an Abelian group are their own conjugate $\tilde{a} = a$. ◀

Conjugation is an example of a *one-to-one* mapping of group elements onto another set of elements that have the same multiplication table. Such a mapping is called an **isomorphism**: $\mathsf{G} \cong \mathsf{F}$. A **homomorphism** ($\mathsf{G} \sim \mathsf{F}$) is a mapping of G to F that is not one-to-one, but still preserves the multiplication table.

Conjugation splits a group G into disjunct *classes*:

▶ A **class** C_a is the set of conjugates $\tilde{a}$ with respect to every element g of G:

$$C_a = \{gag^{-1}\ \forall g \in \mathsf{G}\}.$$

It is easy to show (homework) that if b is *not* an element of C_a then C_a and C_b have no element in common. Note that a class is *not* a subgroup, except when $a = e$. The classes of an Abelian group contain exactly one element $C_a = a$. ◀

A **normal** or **invariant subgroup** $\mathsf{H} \subset \mathsf{G}$ maps onto itself by conjugation with respect to any element g:[6]

$$ghg^{-1} \in \mathsf{H}, \quad \forall h \in \mathsf{H},\ \forall g \in \mathsf{G}.$$

Because $gh_1g^{-1} = h_2$ it follows that for each element h_1 of a normal subgroup another element h_2 can be found such that $gh_1 = h_2g$. From this it is clear that

▶ The left and right cosets of a normal subgroup are identical: $g\mathsf{H} = \mathsf{H}g$. ◀

When G contains a normal subgroup H, we can set-up a correspondence $\mathsf{G} \mapsto \mathsf{G}'$ by mapping all elements of H onto e' and all elements of a coset $g\mathsf{H} = \mathsf{H}g$ onto the element g'. We now multiply elements of H and its cosets with each other and see what happens to the images in G'.

$$\begin{array}{lcl} h_1h_2 = h_3 & \mapsto & e'e' = e', \\ h_1(ah_2) = h_1(h_3a) = (h_1h_3)a = h_4a & \mapsto & e'a' = a', \\ (h_1a)(h_2b) = h_1(ah_2)b = h_1(h_3a)b = (h_1h_3)(ab) = h_4c & \mapsto & a'b' = c', \end{array}$$

where we have set $ah_2 = h_3a$, $h_1h_3 = h_4$ and $ab = c$. Thus G and G' have the same multiplication table so that G' is a homomorphic image of G, called the **factor group** G/H. The normal subgroup H maps onto the unit element of G/H and is called the **kernel** of the mapping. From the above it is easy to see that the following statement is true.

▶ The kernel H of a homomorphic mapping $\mathsf{G} \mapsto \mathsf{G}'$ is a normal subgroup of G. The factor group G/H is then isomorphic to G'. Note that the factor group is not a subgroup of G but an *image* of G. ◀

Can a factor group also have a normal subgroup so that it can be factorised further? Yes, this is certainly possible but it can be shown that (homework):

▶ If $\mathsf{H} \subset \mathsf{G}$ is the *largest* normal subgroup of G then the factor group G/H has *no* normal subgroup (except e). Because H has the largest possible order it follows that G/H has the smallest possible order. ◀

A group that has no normal subgroup other than e is called **simple** and the above gives a prescription to map any non-simple group onto a simple group

[6]The additive group of integers, for example, contains the normal subgroup of even integers. What about the set of odd integers? (homework).

(of lower order). This is the reason why mathematicians only consider simple groups to be of fundamental interest.

Above we have encountered several ways to dissect a group so let us now introduce the *direct product* (also called Kronecker product) to enlarge a group.

▶ The **direct product** $\mathsf{F} \times \mathsf{G}$ is the set of pairs

$$(a,b), \quad a \in \mathsf{F},\ b \in \mathsf{G} \quad \text{with} \quad (a,b)(c,d) \equiv (ac, bd). \tag{0.8}$$

With the multiplication thus defined, it is easy to see that $\mathsf{F} \times \mathsf{G}$ is a group. ◀

Finally, let us repeatedly multiply an element by itself. Suppose we make a list

$$a, a^2, a^3, \ldots$$

of powers of some element $a \neq e$ of a finite group G. Clearly the length of such a list has no bound but since the number of elements of G is finite we must have it occur twice at some point in the list, that is, for some $n > m$ we have

$$a^n = a^m \quad \to \quad a^{n-m} = a^k = e.$$

The power k is called the **order** of a and the set $\{a^n\}$ is called the **orbit** of a. The above implies that:

▶ Every element $a \neq e$ of a finite group G of order n generates a cyclic subgroup $Z_k \subseteq \mathsf{G}$ with $2 \leq k \leq n$. An element that is its own inverse generates Z_2. ◀

Now because Lagrange's theorem tells us that groups of prime order cannot have any subgroup it follows that we must have $k = n$ when n is prime:

▶ The only possible finite group of prime order n is the cyclic group Z_n. ◀

The SO(3) group of rotations in three dimensions

Rotations in three dimensions form a continuous group, represented by the **special orthogonal group** SO(3) of 3×3 unimodular (unit determinant) orthogonal matrices R. The study of this group is of interest because rotation is a very common transformation, and also because several important concepts related to continuous groups can be nicely introduced.

We take the convention to rotate the coordinate system so that a vector $\boldsymbol{r}$ with coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ in a reference system O, has coordinates $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$ in the rotated system O'. Here and in the following we will use the

summation convention of summing over repeated indices so that we may write for $\boldsymbol{x}' = R\,\boldsymbol{x}$

$$x'_i = R_{ij}\,x_j.$$

The orthogonality condition reads $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$, in components,

$$R_{ji}R_{jk} = \delta_{ik} \qquad R_{ij}R_{kj} = \delta_{ik}.$$

It follows that a rotation preserves the inproduct $\boldsymbol{x}\cdot\boldsymbol{y}$ of two 3-vectors,

$$x'_i y'_i = R_{ij}x_j\,R_{ik}y_k = \delta_{jk}\,x_j y_k = x_j y_j.$$

The orthogonality condition implies $R^{-1} = R^{\mathrm{T}}$ so that each rotation indeed has an inverse. The unit element is a rotation over zero angle. Furthermore, the product $R_3 = R_2R_1$ of two rotations is again a rotation because

$$R_3^{\mathrm{T}} = R_1^{\mathrm{T}}R_2^{\mathrm{T}} = R_1^{-1}R_2^{-1} = R_3^{-1} \quad \text{and} \quad \det(R_3) = \det(R_2)\det(R_1) = 1.$$

We conclude that 3-dimensional rotations form a group.

Three-dimensional rotations are determined by a rotation axis $\hat{\boldsymbol{u}}$ (unit vector) and a rotation angle α about this axis. We write $\boldsymbol{\alpha} \equiv \alpha\hat{\boldsymbol{u}}$, specified by three parameters $(\alpha_1, \alpha_2, \alpha_3)$. If we rotate the system O counterclockwise by an angle α about the z axis to the system O' we have for the relation between $\boldsymbol{x}$ and $\boldsymbol{x}'$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{0.9}$$

For small angles α/n the rotation matrix can be written as

$$R(\alpha/n) = I + \frac{\alpha}{n}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{O}\left(\frac{\alpha^2}{n^2}\right) \equiv I + \frac{\alpha}{n}T + \mathrm{O}\left(\frac{\alpha^2}{n^2}\right).$$

The matrix T is called the **generator** of the rotations about the z axis. Ignoring terms $\mathrm{O}(\alpha^2)$ this gives for a finite rotation

$$R(\alpha) = \lim_{n\to\infty}\left(I + \frac{\alpha}{n}T\right)^n = \exp(\alpha T). \tag{0.10}$$

Here the exponent e^A of a matrix should be understood as the series expansion

$$e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{A^n}{n!}.$$

Note that the familiar expression $e^A\, e^B = e^{(A+B)}$ is *only* true when A and B commute. Because 3-dimensional rotations about different axes do *not* commute, it is not obvious that we can write the generator of a rotation about an arbitrary axis as the sum of generators of rotations about the x, y and z axis:

$$R(\boldsymbol{\alpha}) = e^{\alpha_1 T_1}\, e^{\alpha_2 T_3}\, e^{\alpha_3 T_3} \stackrel{?}{=} e^{\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3}.$$

However, for an infinitesimal rotation of a vector $\boldsymbol{r}$ about $\boldsymbol{\alpha}$ we can write

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{\alpha} \times \boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r} \times \boldsymbol{\alpha}.$$

Our convention is that we do not rotate the vector but the coordinate system (over an angle $-\alpha$) so that the coordinate transformation is

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{x} \times \boldsymbol{\alpha}.$$

Introducing the antisymmetric tensor ϵ_{ijk},[7] this reads in components

$$x_i' = x_i + \epsilon_{ijk} x_j \alpha_k = [\,\delta_{ij} + \alpha_k \epsilon_{ijk}\,]\, x_j = [\,\delta_{ij} + \alpha_k (T_k)_{ij}\,]\, x_j.$$

From this we identify the three generators $(T_k)_{ij} = \epsilon_{ijk}$:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{0.11}$$

and write $R(\boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha} \cdot \boldsymbol{T})$. Note that the generators are traceless and anti-orthogonal: $T^{\mathrm{T}} = -T$. Dividing by i makes the generators **Hermitian**[8] ($L^\dagger = L$) and the defining equation for the generators becomes

$$R(\boldsymbol{\alpha}) = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{L}), \tag{0.12}$$

with, for SO(3),

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{0.13}$$

Note that

$$(L_i)_{jk} = -i\epsilon_{ijk} \tag{0.14}$$

[7] The tensor ϵ_{ijk} is $+1$ for even permutations of (123), -1 for even permutations of (231) and zero otherwise.

[8] The Hermitian conjugate of a matrix is defined by $H^\dagger = (H^*)^{\mathrm{T}}$. A matrix is called Hermitian when $H^\dagger = H$.

Let us at this point make a few remarks.

▶ A continuous group whose elements are continuously connected to the identity is called a **Lie group**. The elements of a Lie group are related to the generators of the group by the limiting equation (0.10). ◀

The rotation group SO(3) is obviously a Lie group, but the group O(3), that includes orthogonal matrices with determinant -1 (reflections) is *not* a Lie group since reflections are not connected to the identity (there is no such thing as an infinitesimal reflection).

▶ The number of generators of a Lie group is equal to the number of parameters of that group. ◀

The group SO(3) has three parameters and therefore three generators. The number of generators has nothing to do with the dimension of the defining SO(3) matrices, which happens to be three also.

Is is seen from (0.13) that the generators L_i are Hermitian and traceless. They are Hermitian because R is orthogonal (homework) and traceless because $\det(R) = 1$. The latter follows from a theorem of linear algebra:

▶ For matrices $U = \exp(A)$ that can be brought into diagonal form, the determinant is given by $\det(U) = \exp(\mathrm{Tr}A)$. ◀

For a rotation $s\boldsymbol{\alpha}$, with s a real number, we find

$$R(s\boldsymbol{\alpha}) = \exp(is\boldsymbol{\alpha}\cdot\boldsymbol{L}) = R(\boldsymbol{\alpha})^s \quad \text{so that} \quad R(s\boldsymbol{\alpha})R(t\boldsymbol{\alpha}) = R[(s+t)\boldsymbol{\alpha}].$$

▶ Rotations about a fixed axis define a commuting subgroup of SO(3). ◀

Because the product of two rotations is again a rotation it follows that

$$R(\boldsymbol{\alpha})R(\boldsymbol{\beta}) = R(\boldsymbol{\gamma}) \tag{0.15}$$

where $\boldsymbol{\gamma}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a (non-trivial) function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. From the fact that such a function must exist it can be shown that the commutator of any two generators must be a linear combination of the generators

$$[L_i, L_j] = c^k_{ij} L_k. \tag{0.16}$$

For SO(3) the commutation relations are, from (0.13),

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \tag{0.17}$$

The c^k_{ij} are called the **structure constants** of the group. Note from (0.14) that the SO(3) structure constants are also matrix elements of the representation of the generators and this is no coincidence, as we will see below.

Eq. (0.15) can be written as

$$\exp(i\boldsymbol{\gamma}\cdot\boldsymbol{L}) = \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{L})\exp(i\boldsymbol{\beta}\cdot\boldsymbol{L}) = \exp[i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{L} + f(\boldsymbol{L})],$$

where $f(\boldsymbol{L})$ is a function of repeated commutators like $[L_i, L_j]$, $[[L_i, L_j], L_k]$, *etc.* From this it can be shown that $f(\boldsymbol{L})$ depends only on the structure constants.

▶ Structure constants determine the multiplication structure of a Lie group. The commutation relations (0.16) thereby define a so-called **Lie algebra**. ◀

For any triplet of $n \times n$ matrices A, B and C, the **Jacobi identity** states that

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

which is easy to prove by writing out the commutators, and enjoying the cancellations. In terms of the structure constants, the Jacobi identity reads

$$c_{ij}^m\, c_{mk}^n + c_{jk}^m\, c_{mi}^n + c_{ki}^m\, c_{mj}^n = 0.$$

Now define the matrices C_i with elements

$$(C_i)_j^k = -c_{ij}^k. \tag{0.18}$$

From (0.16) it is seen that $c_{ij}^k = -c_{ji}^k$, and the Jacobi identity becomes

$$\begin{aligned} c_{ij}^m\, c_{mk}^n - c_{jk}^m\, c_{im}^n + c_{ik}^m\, c_{jm}^n &= -c_{ij}^m (C_m)_k^n - (C_j)_k^m (C_i)_m^n + (C_i)_k^m (C_j)_m^n \\ &= -c_{ij}^m (C_m)_k^n - (C_j\, C_i)_k^n + (C_i\, C_j)_k^n = 0 \end{aligned}$$

or

$$[C_i, C_j] = c_{ij}^k C_k$$

which is the same commutation relation as (0.16). Thus the matrices C_i are a representation, called the **adjoint representation**, that has a dimension equal to the number of generators. This is in contrast to the so-called **fundamental representation** (0.13), that has the dimension of the defining linear space which is the 3-dimensional Euclidian space in case of SO(3). From (0.14) it is clear that for SO(3) the fundamental and the adjoint representations coincide, but this is certainly not true in general.

Let us give, at this point, two useful relations for the ϵ tensors (the first is the Jacobi identity, the second can trivially be shown to be true by giving the values (1,2,3) to two of the indices).

$$\epsilon_{ijm}\,\epsilon_{mkn} + \epsilon_{jkm}\,\epsilon_{min} + \epsilon_{kim}\,\epsilon_{mjn} = 0 \tag{0.19}$$

$$\epsilon_{ijm}\,\epsilon_{mkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk} \tag{0.20}$$

SO(3) transformations in higher dimensions

In this section we take two 3-vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and use these to build objects of dimensions larger than three. Their transformation under rotations will then yield higher-dimensional representations $D(R)$ of SO(3), other than the fundamental (R) and adjoint representations that we have found up to now.

The simplest composite object we can build has 6 components and is defined by $\boldsymbol{v} = \boldsymbol{x} \oplus \boldsymbol{y} \stackrel{\text{def}}{=} (x_1, x_2, x_3, y_1, y_2, y_3) = (v_1, v_2, v_3, v_4, v_5, v_6)$. It transforms under rotations as

$$\boldsymbol{v}' = \begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = D(R)\,\boldsymbol{v}. \tag{0.21}$$

Clearly $D(R_1)D(R_2) = D(R_1R_2)$, so that D is indeed is a *representation* of SO(3). It is also clear that the components v_1, v_2 and v_3 will never mix with the components v_4, v_5 and v_6 and we say that $D(R)$ is **reducible** into a **direct sum** of two 3-dimensional transformations: $\mathbf{6} = \mathbf{3} \oplus \mathbf{3}$. A block-diagonal representation like (0.21) is the hallmark of reducibility but if we would have defined $\boldsymbol{v} = (x_1, y_1, x_2, y_2, x_3, y_3)$, for instance, then $D(R)$ would not be block-diagonal but of course still be reducible into $\mathbf{3} \oplus \mathbf{3}$ since v_1, v_3 and v_5 will not mix with v_2, v_4 and v_6.

► A representation that cannot be brought into block-diagonal form by a similarity transformation (change of basis) is called **irreducible**. ◄

We can build another object by taking the **outer product** of $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$T_{ij} = (\boldsymbol{x} \otimes \boldsymbol{y})_{ij} \stackrel{\text{def}}{=} x_i y_j \qquad \text{with} \qquad T'_{ij} = x'_i y'_j = R_{ik}x_k R_{jl}x_l = R_{ik}R_{jl}T_{kl}.$$

This tensor $\boldsymbol{T}$ has $3 \times 3 = 9$ components and the transformation $R_{ik}R_{jl}$ can be arranged into a 9×9 matrix $D(R)$ with, again, $D(R_1)D(R_2) = D(R_1R_2)$. The representation D is reducible because some linear combinations of the tensor elements have specific behaviour under rotations, as we will now show. For instance the trace of $\boldsymbol{T}$ is just the inproduct of $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\text{Tr}(\boldsymbol{T}) = \delta_{ij}T_{ij} = T_{ii} = x_i y_i = \boldsymbol{x} \cdot \boldsymbol{y},$$

and is therefore invariant under rotations. The antisymmetric sum

$$a_i = \epsilon_{ijk}T_{jk} = \epsilon_{ijk}x_j y_k = (\boldsymbol{x} \times \boldsymbol{y})_i$$

is a component of the cross product of $\boldsymbol{x}$ and $\boldsymbol{y}$ and thus transforms as the component of a vector: $a'_i = R_{ij}a_j$. Having identified one scalar component and three vector components there remain $9 - 1 - 3 = 5$ components of $\boldsymbol{T}$ that transform as a proper tensor. This suggests that we may write the decomposition of the tensor product as $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$ where, by construction, the vector component $\mathbf{3}$ is antisymmetric in the tensor indices.

Note, in this respect, that any tensor T_{ij} can be decomposed into a symmetric part $S_{ij} = S_{ji}$ and an antisymmetric part $A_{ij} = -A_{ji}$ as follows

$$S_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}), \quad A_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji}).$$

Now the (anti)symmetric components transform into (anti)symmetric components as is easy to show: If we denote by $\tilde{\boldsymbol{T}}$ the transpose of $\boldsymbol{T}$ then we have, by definition, $\boldsymbol{S} - \tilde{\boldsymbol{S}} = 0$ and $\boldsymbol{A} + \tilde{\boldsymbol{A}} = 0$. Because the transformation $D(R)$ is linear we can write

$$\begin{aligned} \boldsymbol{S}' - \tilde{\boldsymbol{S}}' &= D(\boldsymbol{S}) - D(\tilde{\boldsymbol{S}}) = D(\boldsymbol{S} - \tilde{\boldsymbol{S}}) = D(0) = 0 \\ \boldsymbol{A}' + \tilde{\boldsymbol{A}}' &= D(\boldsymbol{A}) + D(\tilde{\boldsymbol{A}}) = D(\boldsymbol{A} + \tilde{\boldsymbol{A}}) = D(0) = 0 \end{aligned}$$

so that, indeed, $\boldsymbol{S}' = \tilde{\boldsymbol{S}}'$ and $\boldsymbol{A}' = -\tilde{\boldsymbol{A}}'$. The representation $D(R)$ thus decomposes into $\mathbf{3} \otimes \mathbf{3} = \{\mathbf{6}\} \oplus [\mathbf{3}]$, where we have introduced the notation $\{\boldsymbol{n}\}$ and $[\boldsymbol{m}]$ to indicate a representation that transforms as a symmeteric or as an antisymmetric tensor.[9]

We have seen above that the trace is invariant so that the symmetric component is still reducible into $\{\mathbf{6}\} = \{\mathbf{5}\} \oplus \mathbf{1}$. It thus makes sense to formally isolate the trace and write the expansion of a tensor as

$$T_{ij} = \tfrac{1}{2} \underbrace{\epsilon_{ijk}(\epsilon_{klm} T_{lm})}_{T_{ij} - T_{ji}} + \tfrac{1}{2}(T_{ij} + T_{ji} - \tfrac{2}{3}\delta_{ij} T_{kk}) + \tfrac{1}{3}\delta_{ij} T_{kk}, \tag{0.22}$$

where use of (0.20) has been made to express the antisymmetric component in terms of ϵ-tensors. This component is traceless by definition, and by using the identity $\delta_{ii} = 3$ it is immediately clear that the second term is traceless, too. The decomposition (0.22) shows that the 9-dimensional tensor representation of SO(3) splits into three irreducible representations

$$\mathbf{3} \otimes \mathbf{3} = [\mathbf{3}] \oplus \{\mathbf{5}\} \oplus \mathbf{1}.$$

To summarize, we can write the decomposition of our tensor $\boldsymbol{T} = \boldsymbol{x} \otimes \boldsymbol{y}$ as

$$\boldsymbol{T} \to \begin{cases} T = \tfrac{1}{3}(\boldsymbol{x} \cdot \boldsymbol{y}) & \text{(scalar, one component)} \\ T_i = \tfrac{1}{2}(\boldsymbol{x} \times \boldsymbol{y})_i & \text{(vector, three components)} \\ T_{ij} = \tfrac{1}{2}(x_i y_j + x_j y_i) - \tfrac{1}{3}\delta_{ij}(\boldsymbol{x} \cdot \boldsymbol{y}) & \text{(tensor, five componenents)} \end{cases}$$

[9]This is the same notation as that of anticommutation $\{A, B\} \equiv AB + BA$ (symmetric) and commutation $[A, B] \equiv AB - BA$ (antisymmetric). Of course $\mathbf{1} = \{\mathbf{1}\}$, so there we do not put brackets.

where the first term transforms as a scalar (invariant under rotations), the second term as a vector

$$T'_i = R_{ij} T_j,$$

and the third term as a tensor, according to

$$T'_{ij} = \tfrac{1}{2}\left(R_{ik}R_{jl} + R_{il}R_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right) T_{kl}.$$

The number of indices is called the **rank** or **order** of a tensor; apart from rank-2 tensors we thus have also encountered tensors of rank zero (scalars) and one (vectors). Note that tensors are defined by their SO(3) transformation properties so that a rank-2 tensor not necessarily *is* an outer product, but *behaves* as an outer product of two vectors.

A scalar is, by definition, invariant under SO(3) transformations but there exist also higher order tensors that are invariant. For instance,

$$\delta'_{ij} = R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} = (RR^{\mathrm{T}})_{ij} = \delta_{ij}$$

and[10]

$$\epsilon'_{ijk} = R_{il}R_{jm}R_{kn}\epsilon_{lmn} = \epsilon_{ijk}\det(R) = \epsilon_{ijk}.$$

Quite some more to come ...

[10] We denote the first row of a 3×3 matrix A by the vector $\boldsymbol{a}_1 \equiv (A_{11}, A_{12}, A_{13})$, and similar for the second ($\boldsymbol{a}_2$) and third row and ($\boldsymbol{a}_3$). The determinant is then given by the volume $\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \epsilon_{lmn} A_{1l} A_{2m} A_{3n}$. The determinant changes sign under the interchange of two row-indices while no two row-indices can be equal. This can be encoded by setting the indices $\{1, 2, 3\}$ to $\{i, j, k\}$ and writing $\epsilon_{lmn} A_{il} A_{jm} A_{kn} = \epsilon_{ijk} \det(A)$.