Lecture notes Particle Physics II

Quantum Chromo Dynamics

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QCD in Eight Lectures

1. Review of SU(2) and SU(3) symmetry.
2. Lagrangian formalism and U(1) local gauge invariance.
3. The SU(2) (Yang-Mills) and SU(3) (QCD) invariant Lagrangian.
4. Colour factors.
5. The running coupling constant and asymptotic freedom in QCD.
6. Infrared and collinear singularities.
7. The structure of the proton.
8. The QCD improved parton model and DGLAP evolution.

The lecture notes can be found on

`http://www.nikhef.nl/user/h24/qcdcourse`

Background material

The lectures are based on the following books:

Griffiths D. Griffiths, *Introduction to Elementary Particles*, Second Revised Edition, WILEY-VCH, (2008);

H&M F. Halzen and A.D. Martin, *Quarks and Leptons*, John Wiley & Sons, (1984).

You will need these books to supplement the lecture notes, and to successfully complete the exercises.

Some reference is also made to

PP-I W. Hulsbergen and M. Merk, *Lecture notes Particle Physics I.*

The following references were used in the preparation of these notes:

A&H I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics*, IOP Publishing Ltd, Volumes I and II, (2003);

DKS G. Dissertori, I. Knowles and M. Schmelling, *Quantum Chromodynamics*, Oxford University Press, (2003);

ESW R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and Collider Physics*, Cambridge Univeristy Press, (1996);

CTEQ CTEQ Collab. G. Sterman et al., *Handbook of Perturbative QCD*, Version 1.0 (2000), obtainable from `http://www.phys.psu.edu/~cteq`;

Soper D.E. Soper, *Basics of QCD Perturbation Theory*, hep-ph/9702203, (1997);

deWit B. de Wit and J. Smith, *Field Theory in Particle Physics*, Volume 1, North Holland, (1986);

Zee A. Zee, *Quantum Field Theory in a Nutshell*, Second Edition, Princeton University Press, (2010);

Ramond P. Ramond, *Group Theory*, Cambridge University Press, (2010);

Veltman M. Veltman, B. de Wit and G. 't Hooft, *Lie Groups in Physics*, Lecture notes, `http://www.staff.science.uu.nl/~hooft101/lectures/lieg07.pdf`;

Schiff L.I. Schiff, *Quantum Mechanics*, Third Edition, McGraw-Hill, (1968);

Jackson J.D. Jackson, *Classical Electrodynamics*, Second Edition, John Wiley, (1975).

Grading

You may hand-in exercises which will then be graded as follows:

Good	$\rightarrow$	score = 1.0
Reasonable	$\rightarrow$	0.6
Bad	$\rightarrow$	0.3
Not made	$\rightarrow$	0.0

You can **only** hand-in those (sub-)exercises that have a weight factor given in square brackets. The exercises marked with a [$\times$] will help you to better understand the material but you can **not** hand them in and gain bonus points with them. At the end of the course your final score is calculated as the weighted average of the individual scores. Bonus points are then added to the grade of your exam in proportion to your exercise score. As an example, we list below the bonus for an exercise score of 0.8:

Exam grade	0 – 7	8	9	10
Bonus	1.0	0.8	0.4	0

Note that the bonus is less for high exam grades, to avoid that the total grade will exceed the maximum of 10 points.[1]

Because we cannot handle a pile-up of exercises at the end of the course, we make the rule that you cannot hand-in more than 5 exercises at a time. At the day of the exam you can bring your last five exercises.

The exam is 'open book' so that you may consult the lecture notes and the books of Griffiths, Halzen & Martin and Aitchison & Hey, but not the worked-out exercises or any other material.

[1] The bonus is calculated from $B = E \times \min(5 - 0.5\,T, 1.25)$, where $0 \leq E \leq 1$ is your exercise score, and $0 \leq T \leq 10$ is your exam grade.

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Preliminary

This section is not part of the lectures, but a small collection of material that should be familiar from special relativity, electrodynamics, quantum mechanics and the lecture series Particle Physics I.

Also included is a summary of group theory, but still very incomplete.

Units and conversion factors

In particle physics, energy is measured in units of GeV $= 10^6$ eV, where 1 eV $= 1.6 \times 10^{-19}$ J is the change in kinetic energy of an electron when it traverses a potential difference of one volt. From the relation $E^2 = p^2c^2 + m^2c^4$ it follows that the units of momentum and mass are GeV/c and GeV/c^2, respectively. The dimension of $\hbar$ is energy×time so that the unit of time is $\hbar$/GeV; $\hbar c$ has dimension energy×length so that length has unit $\hbar c$/GeV.

One often works in a system of units where $\hbar$ and c have a numerical value of one, so that these constants can be omitted in expressions, as in $E^2 = p^2 + m^2$. A disadvantage is that the dimensions carried by $\hbar$ (energy×time) and c (length/time) also disappear but these can always be restored, if necessary, by a dimensional analysis afterward.

Here are some useful conversions.

	Conversion	$\hbar = c = 1$ units	Natural units
Mass	1 kg $= 5.61 \times 10^{26}$	GeV	GeV/c^2
Length	1 m $= 5.07 \times 10^{15}$	GeV^{-1}	$\hbar c$/GeV
Time	1 s $= 1.52 \times 10^{24}$	GeV^{-1}	$\hbar$/GeV
Charge	$e = \sqrt{4\pi\alpha}$	dimensionless	$\sqrt{\hbar c}$

1 TeV $= 10^3$ GeV $= 10^6$ MeV $= 10^9$ KeV $= 10^{12}$ eV

1 fm $= 10^{-15}$ m $= 10^{-13}$ cm $= 5.07$ GeV^{-1}

1 barn $= 10^{-28}$ m^2 $= 10^{-24}$ cm^2

1 fm^2 $= 10$ mb $= 10^4$ μb $= 10^7$ nb $= 10^{10}$ pb

1 GeV^{-2} $= 0.389$ mb

$\hbar c = 197$ MeV fm

$(\hbar c)^2 = 0.389$ GeV^2 mb

$\alpha = e^2/(4\pi\hbar c) \approx 1/137$

Covariant notation ($c = 1$)

- Contravariant space-time coordinate: $x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x})$
- Covariant space-time coordinate: $x_\mu = (x_0, x_1, x_2, x_3) = (t, -\boldsymbol{x})$
- Contravariant derivative: $\partial_\mu \equiv \partial/\partial x^\mu = (\partial_t, +\nabla)$
- Covariant derivative: $\partial^\mu \equiv \partial/\partial x_\mu = (\partial_t, -\nabla)$
- Metric tensor: $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- Index raising/lowering: $a_\mu = g_{\mu\nu}\, a^\nu, \quad a^\mu = g^{\mu\nu}\, a_\nu$
- Lorentz boost along x-axis:[2] $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

  We have also: $x'_\mu = \Lambda_\mu{}^\nu\, x_\nu$ with $\Lambda_\mu{}^\nu(\beta) = \Lambda^\mu{}_\nu(-\beta) \equiv (\Lambda^\mu{}_\nu)^{-1}$
- Inproduct (Lorentz scalar): $a \cdot b = a^\mu\, b_\mu = a^0\, b_0 - \boldsymbol{a} \cdot \boldsymbol{b} = a_\mu\, b^\mu$
- $a^2 > 0$ time-like 4-vector $\rightarrow$ possible causal connection
  $a^2 = 0$ light-like 4-vector
  $a^2 < 0$ space-like 4-vector $\rightarrow$ no causal connection
- 4-momentum: $p^\mu = (E, \boldsymbol{p}), \quad p_\mu = (E, -\boldsymbol{p})$
- Invariant mass: $p^2 = p^\mu\, p_\mu = p_\mu\, p^\mu = E^2 - \boldsymbol{p}^2 = m^2$
- Particle velocity: $\gamma = E/m, \quad \beta = |\boldsymbol{p}|\,/E$

[2] This is the relation between the coordinates x^μ of an event observed in a system S and the coordinates x'^μ of that *same* event observed in a system S' that moves with a velocity $+\beta$ along the x-axis of S.

Vector calculus

$$\nabla \times (\nabla \psi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

$$\int_V \nabla \cdot \boldsymbol{A} \, \mathrm{d}V = \int_S \boldsymbol{A} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S \qquad \text{(Divergence theorem)}$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, \mathrm{d}V = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{\boldsymbol{n}} \, dS \qquad \text{(Green's theorem)}$$

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \qquad \text{(Stokes' theorem)}$$

- In the above, S is a closed surface bounding V, with $\hat{\boldsymbol{n}}$ the outward normal unit vector at the surface element $\mathrm{d}S$.
- In Stokes' theorem, the direction of $\hat{\boldsymbol{n}}$ is related by the right-hand rule to the sense of the contour integral around C.

Maxwell's equations in vacuum

- Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \rho \qquad\qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t = 0 \qquad \nabla \times \boldsymbol{B} - \partial \boldsymbol{E}/\partial t = \boldsymbol{j}$$

- Continuity equation

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$$

- The potentials V and $\boldsymbol{A}$ are defined such that the second and third of Maxwell's equations are automatically satisfied

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \rightarrow \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \nabla V \qquad \rightarrow \qquad \nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$$

- Gauge transformations leave the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields invariant

$$V' = V + \frac{\partial \lambda}{\partial t} \qquad \text{and} \qquad \boldsymbol{A}' = \boldsymbol{A} - \nabla \lambda$$

- Maxwell's equations in 4-vector notation

4-vector potential	$A^\mu = (V, \boldsymbol{A})$
4-vector current	$j^\mu = (\rho, \boldsymbol{j})$
Electromagnetic tensor	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
Maxwell's equations	$\partial_\mu F^{\mu\nu} = j^\nu$
Continuity equation	$\partial_\mu j^\mu = 0$
Gauge transformation	$A^\mu \rightarrow A^\mu + \partial^\mu \lambda$

- Lorentz gauge and Coulomb condition

Lorentz gauge	$\partial_\mu A^\mu = 0$	$\rightarrow$	$\partial_\mu \partial^\mu A^\nu = j^\nu$
Coulomb condition	$A^0 = 0$	or equivalently	$\nabla \boldsymbol{A} = 0$

The Lagrangian in classical mechanics

In classical mechanics, the Lagrangian is the difference between the kinetic and potential energy: $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \equiv T - V$. The coordinates $\boldsymbol{q}(t) = \{q_1(t), \ldots, q_N(t)\}$ fully describe the system at any given instant t. The number N of coordinates is called the **number of degrees of freedom** of the system.

Let the system move from $A(t_1)$ to $B(t_2)$ along some given path. The **action** S[path] is defined by the integral of the Lagrangian along the path:

$$S[\text{path}] = \int_{t_1}^{t_2} \mathrm{d}t \; L(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

The action S assigns a number to each path and is thus a function of the path. In mathematics, S is called a **functional**.

The **principle of least action** states that the system will evolve along the path that minimises the action.

Let $q(t)$ be a path and $q(t) + \delta q(t)$ be some deviating path between the *same* points $A(t_1)$ and $B(t_2)$. That is, $\delta q(t_1) = \delta q(t_2) = 0$. The variation in the action is then given by

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \; \delta L(q, \dot{q}) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \underset{\text{Iwant}}{=} 0.$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q},$$

we find, by partial integration,

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \int_{t_1}^{t_2} \mathrm{d} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \underset{\text{Iwant}}{=} 0.$$

The second integral vanishes because $\delta q(t_1) = \delta q(t_2) = 0$.

The first integral vanishes for all δq if and only if the term in brackets vanishes, leading to the **Euler-Lagrange equations**, for N degrees of freedom:

$$\boxed{\frac{\delta S}{\delta q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, N}$$

Solving the EL equations for a given Lagrangian lead to the **equations of motion** of the system.

The Hamiltonian in classical mechanics

If L does not explicitly depend on time we have for the time derivative

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial \dot{q}}\,\ddot{q} + \frac{\partial L}{\partial q}\,\dot{q}$$

Substituting $\partial L/\partial q$ from the Euler-Lagrange equations gives

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial \dot{q}}\,\ddot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right) \;\rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q} - L\right) = 0$$

The term in brackets is the **Legendre transform** of L and is called the **Hamiltonian**:

$$H \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}}\,\dot{q} - L = p\dot{q} - L \qquad \text{with} \qquad p \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}},$$

where we have also introduced the **canonical momentum** p. The Hamiltonian is identified with the total energy $E = T + V$ which is thus conserved in the time evolution of the system. This is an example of a conservation law.

In the Lagrangian, the dependence on $\dot{q}$ resides in the kinetic energy term T while the dependence on q is contained in the potential energy V. Thus if $V = 0$ (or a constant) we have in the EL equations

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\mathrm{d}p}{\mathrm{d}t} = 0$$

Thus the momentum p is conserved in a system that is not under the influence of an external potential. This is another example of a conservation law.

The Hamiltonian equations of motion are

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q}}$$

This can be derived as follows. Consider the total differential

$$\mathrm{d}L = \frac{\partial L}{\partial q}\mathrm{d}q + \frac{\partial L}{\partial \dot{q}}\mathrm{d}\dot{q}$$

Now

$$\frac{\partial L}{\partial q} = \dot{p} \text{ (from EL)}, \quad \frac{\partial L}{\partial \dot{q}} = p \text{ (by definition)},$$

and thus, using $p\mathrm{d}\dot{q} = d(p\dot{q}) - \dot{q}\mathrm{d}p$, we obtain

$$\mathrm{d}L = \dot{p}\mathrm{d}q + \mathrm{d}(p\dot{q}) - \dot{q}\mathrm{d}p \quad \rightarrow \quad \mathrm{d}(p\dot{q} - L) = \mathrm{d}H = \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q,$$

from which the Hamiltonian equations immediately follow.

Dirac δ-function

- The Dirac δ-function can be defined by[3]

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(x) \mathrm{d}x = 1$$

- Generalisation to more dimensions is trivial, like $\delta(\boldsymbol{r}) \equiv \delta(x)\delta(y)\delta(z)$.

- For $x \to 0$ we may write $f(x)\delta(x) = f(0)\delta(x)$ so that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- For a linear transformation $y = k(x-a)$ we have

$$\delta(y) = \frac{1}{|k|}\,\delta(x-a)$$

  This is straight-forward to prove by showing that $\delta(y)$ satisfies the definition of the δ-function given above.

- Likewise, if $\{x_i\}$ is the set of points for which $f(x_i) = 0$, then it is easy to show by Taylor expansion around the x_i that

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$$

- There exist many representations of the δ-function, for instance,

$$\delta(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k} \qquad \text{or} \qquad \delta(x) = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x},$$

$$\text{with} \quad \theta(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \geq 0 \end{cases} \quad \text{(Heaviside step function).}$$

[3] A more rigorous mathematical definition is usually in terms of a limiting sequence of functions.

Green functions

- Let Ω be some linear differential operator. A **Green function** of the operator Ω is a solution of the differential equation

$$\Omega\, G(\boldsymbol{r}) = \delta(\boldsymbol{r})$$

  These Green functions can be viewed as some potential caused by a point source at $\boldsymbol{r}$.

- Once we have the Green function we can immediately solve the differential equation for any source density $s(\boldsymbol{r})$

$$\Omega\, \psi(\boldsymbol{r}) = s(\boldsymbol{r})$$

  By substitution it is easy to see that (ψ_0 is the solution of $\Omega\, \psi_0 = 0$)

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \int G(\boldsymbol{r} - \boldsymbol{r}')s(\boldsymbol{r}')\ \mathrm{d}\boldsymbol{r}'$$

  Here it is clearly seen that $G(\boldsymbol{r} - \boldsymbol{r}')$ 'propagates' the contribution from the source element $s(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$ to the potential $\psi(\boldsymbol{r})$.

- A few well-known Green functions are ...

$$\nabla^2\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G(\boldsymbol{r}) = -1/(4\pi r)$$

$$(\nabla^2 + \boldsymbol{k}^2)\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G^{\pm}(\boldsymbol{r}) = -\exp(\pm ikr)/(4\pi r)$$

$$(\nabla^2 - m^2)\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G(\boldsymbol{r}) = -\exp(-mr)/(4\pi r)$$

- ... and here are their Fourier transforms

$$G(\boldsymbol{r}) = -1/(4\pi r) \qquad \tilde{G}(\boldsymbol{q}) = -1/q^2$$

$$G^{+}(\boldsymbol{r}) = -\exp(ikr)/(4\pi r) \qquad \tilde{G}^{+}(\boldsymbol{q}) = 1/(k^2 - q^2 + i\varepsilon)$$

$$G(\boldsymbol{r}) = -\exp(-mr)/(4\pi r) \qquad \tilde{G}(\boldsymbol{q}) = -1/(q^2 + m^2)$$

Non-relativistic scattering theory I

- Classical relation $E = p^2/2m + V$ with substitution $E \to i\partial/\partial t$ and $\boldsymbol{p} = -i\nabla$ gives the Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\left[\frac{\nabla^2}{2m} - V(\boldsymbol{r},t)\right]\psi(\boldsymbol{r},t)$$

- Separate $\psi(\boldsymbol{r},t) = \phi(t)\psi(\boldsymbol{r})$. Dividing through by $\phi\psi$ gives

$$\frac{i\partial_t\phi(t)}{\phi(t)} = -\frac{[\nabla^2 - 2mV(\boldsymbol{r},t)]\,\psi(\boldsymbol{r})}{2m\psi(\boldsymbol{r})}$$

  Assume now that V does not depend on t. The left and right-hand side must then be equal to a constant, say E, and we have

$$\frac{\partial\phi(t)}{\partial t} = -iE\,\phi(t) \quad \to \quad \phi(t) = e^{-iEt}$$

$$[\nabla^2 + k^2]\,\psi(\boldsymbol{r}) = 2mV(\boldsymbol{r})\psi(\boldsymbol{r})$$

  where we have set $k^2 = 2mE$. Using Green functions we get

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

- For large $r \gg r'$ we have $|\boldsymbol{r}-\boldsymbol{r}'| \approx r - \hat{\boldsymbol{r}}\boldsymbol{r}'$ so that

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\pi}\frac{e^{ikr}}{r}\int e^{-ik\hat{\boldsymbol{r}}\boldsymbol{r}'}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

- We set $\boldsymbol{k}' \equiv k\hat{\boldsymbol{r}}$ and write, formally,

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + f(\boldsymbol{k}')\frac{e^{ikr}}{r} \quad \text{with} \quad f(\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{-i\boldsymbol{k}'\boldsymbol{r}'}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

  The function $f(\boldsymbol{k}')$ is called the **scattering amplitude**.

Non-relativistic scattering theory II

- An incoming plane wave $\psi_{\rm in} = Be^{ikz}$ describes beam particles moving along the z axis with momentum k. The wave function is normalised such that $\rho = \psi^*\psi = |B|^2$ is the particle density (number of particles per unit volume). The current density is

$$\boldsymbol{j}_{\rm in} = \frac{1}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = |B|^2\frac{\boldsymbol{k}}{m} = \rho\,\frac{\boldsymbol{k}}{m} = \rho\boldsymbol{v}$$

  with $\boldsymbol{v}$ the velocity of the particle. The number of beam particles passing per second through an area A is $R_{\rm in} = \rho v A = |\boldsymbol{j}_{\rm in}|A$. Likewise, the number of scattered particles that pass per second through an area $r^2\mathrm{d}\Omega$ is $R_{\rm sc} = |\boldsymbol{j}_{\rm sc}|r^2\mathrm{d}\Omega$.

- We now imagine a hypothetical area $\mathrm{d}\sigma$ such that the number of beam particles that pass through that area is equal to the number of particles that scatter in the solid angle $\mathrm{d}\Omega$. We then have, by definition, $|\boldsymbol{j}_{\rm in}|\,\mathrm{d}\sigma = |\boldsymbol{j}_{\rm sc}|\,r^2\mathrm{d}\Omega$, or

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r^2\,|\boldsymbol{j}_{\rm sc}|}{|\boldsymbol{j}_{\rm in}|}}$$

  The quantity $\mathrm{d}\sigma/\mathrm{d}\Omega$ is called a **differential cross section**.

- For our scattered wave $\psi_{\rm sc} = f(\boldsymbol{k}')\,e^{ikr}/r$ we find

$$j_{\rm sc} = \frac{1}{2mi}\left(\psi^*\frac{\partial\psi}{\partial r} - \psi\frac{\partial\psi^*}{\partial r}\right) = |f(\boldsymbol{k}')|^2\frac{k}{mr^2}$$

  and thus

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\boldsymbol{k}')|^2}$$

  where we have assumed $\rho = 1$ and $k_{\rm sc} = k_{\rm in}$ (elastic scattering).

Non-relativistic scattering theory III

- Recall that for scattering on a potential, the outgoing wave is

$$\psi_{\text{out}}(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + f(\boldsymbol{k}')\frac{e^{ikr}}{r}$$

with

$$f(\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{-i\boldsymbol{k}'\boldsymbol{r}'}V(\boldsymbol{r}')\psi_{\text{out}}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

Here $\boldsymbol{k}'$ is the momentum vector of the scattered particle.

- The problem now is that ψ_{out} occurs on both sides of the equation above. A first order approximation is achieved by setting in the scattering amplitude $\psi_{\text{out}} \approx \psi_{\text{in}} = e^{ikz} = e^{i\boldsymbol{k}\boldsymbol{r}}$. This gives

$$f(\boldsymbol{k},\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}'}V(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' = -\frac{m}{2\pi}\int e^{-i\boldsymbol{q}\boldsymbol{r}'}V(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

where we have set the momentum transfer $\boldsymbol{q} \equiv \boldsymbol{k}' - \boldsymbol{k}$. In this so-called **Born approximation**, the scattering amplitude $f(\boldsymbol{k},\boldsymbol{k}') \equiv f(\boldsymbol{q})$ is thus the Fourier transform of the potential.

- Example: Yukawa potential $V(r) = Q_1Q_2\,e^{-ar}/r$

$$f(\boldsymbol{q}) = -\frac{mQ_1Q_2}{2\pi}\int \frac{e^{-ar'}}{r'}\,e^{-i\boldsymbol{q}\boldsymbol{r}'}\mathrm{d}\boldsymbol{r}' = \cdots = \frac{2mQ_1Q_2}{q^2+a^2}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\boldsymbol{q})|^2 = \left[\frac{2mQ_1Q_2}{q^2+a^2}\right]^2$$

- Example: Coulomb potential $V(r) = Q_1Q_2/r$ set $a = 0$ above:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left[\frac{2mQ_1Q_2}{q^2}\right]^2 = \left[\frac{Q_1Q_2}{2mv^2\sin^2(\theta/2)}\right]^2$$

This is the famous formula for **Rutherford scattering**.

Dirac's bra-ket notation I

- A state vector ψ_α can be represented by a column vector of complex numbers in a Hilbert space and is denoted by the ket $|\alpha\rangle$. To each ket is associated a bra vector $\langle\alpha|$ in a dual Hilbert space. This bra is represented by the conjugate transpose $\psi_\alpha^\dagger$, that is, by the row vector of complex conjugates. The operation of **Hermitian conjugation** turns a bra into a ket and *vice versa*

  $$|\alpha\rangle^\dagger = \langle\alpha| \quad \text{and} \quad (c|\alpha\rangle)^\dagger = \langle\alpha|c^* \qquad (c \text{ any complex number})$$

  Note that the Hermitian conjugate of a c-number is the complex conjugate. The inproduct $\psi_\alpha^\dagger \cdot \psi_\beta$ is denoted by $\langle\alpha|\beta\rangle$ and is a c-number so that

  $$\langle\beta|\alpha\rangle \equiv \langle\alpha|\beta\rangle^\dagger = c^\dagger = c^* = \langle\alpha|\beta\rangle^*$$

- An operator O transforms a ket $|\alpha\rangle$ into another ket, say $|\gamma\rangle$. The operator and its Hermitian conjugate are then defined by

  $$O|\alpha\rangle = |\gamma\rangle \qquad \text{and} \qquad \langle\alpha|O^\dagger = \langle\gamma|$$

  Multiplying from the left with $\langle\beta|$ and from the right with $|\beta\rangle$ we find the relation between the **matrix elements** of O and $O^\dagger$

  $$O_{\beta\alpha} \equiv \langle\beta|O|\alpha\rangle = \langle\beta|\gamma\rangle$$
  $$O^\dagger_{\alpha\beta} \equiv \langle\alpha|O^\dagger|\beta\rangle = \langle\gamma|\beta\rangle = \langle\beta|\gamma\rangle^* = \langle\beta|O|\alpha\rangle^* = O^*_{\beta\alpha}$$

- An operator for which $O = O^\dagger$ is called **self-adjoint** or **Hermitian**. Observable quantities are always represented by Hermitian operators. Indeed, the **expectation value** $\langle\alpha|O|\alpha\rangle$ is then real, as it should be, since

  $$\langle\alpha|O|\alpha\rangle \equiv \langle\alpha|O^\dagger|\alpha\rangle = \langle\alpha|O|\alpha\rangle^*$$

Dirac's bra-ket notation II

- An orthonormal basis is written as $|e_i\rangle$ with $\langle e_i|e_j\rangle = \delta_{ij}$. On this basis, a state $|\alpha\rangle$ is given by the linear combination

$$|\alpha\rangle = \sum_i |e_i\rangle\langle e_i|\alpha\rangle$$

  The operator $|e_i\rangle\langle e_i|$ is called a **projection operator**, for obvious reasons. The **closure relation** reads $\sum_i |e_i\rangle\langle e_i| = 1$

- We denote the wave function $\psi_\alpha(\boldsymbol{r})$ by $\langle \boldsymbol{r}|\alpha\rangle$ and its Hermitian conjugate $\psi^\dagger_\alpha(\boldsymbol{r})$ by $\langle\alpha|\boldsymbol{r}\rangle$. In particular, the wave function of a momentum eigenstate is $\langle \boldsymbol{r}|\boldsymbol{k}\rangle \propto e^{i\boldsymbol{k}\boldsymbol{r}}$.

- For the complete set of states $|\boldsymbol{r}\rangle$ the closure relation reads

$$\int |\boldsymbol{r}\rangle\langle\boldsymbol{r}|\,\mathrm{d}\boldsymbol{r} = 1$$

  From this, we nicely recover the expression for the inproduct of two wave functions

$$\langle\alpha|\beta\rangle = \int \langle\alpha|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\beta\rangle\,\mathrm{d}\boldsymbol{r} = \int \psi^*_\alpha(\boldsymbol{r})\psi_\beta(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$

  that of the delta function

$$\delta(\boldsymbol{k}-\boldsymbol{k}') = \langle\boldsymbol{k}'|\boldsymbol{k}\rangle = \int \langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}\rangle \mathrm{d}\boldsymbol{r} \propto \frac{1}{(2\pi)^3}\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

  and also that of Fourier transforms

$$\psi(\boldsymbol{k}) = \langle\boldsymbol{k}|\psi\rangle = \int \langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\psi\rangle \mathrm{d}\boldsymbol{r} \propto \int e^{-i\boldsymbol{k}\boldsymbol{r}}\psi(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$$

Dirac equation

- Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

$$\underbrace{(\not{p} - m)u = 0}_{\text{particle in}}, \quad \underbrace{\bar{u}(\not{p} - m) = 0}_{\text{particle out}}, \quad \underbrace{(\not{p} + m)v = 0}_{\text{antiparticle out}}, \quad \underbrace{\bar{v}(\not{p} + m) = 0}_{\text{antiparticle in}}$$

$$\overline{\psi} = \psi^\dagger \gamma^0, \qquad \not{a} = \gamma^\mu a_\mu$$

- Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad \sigma_i^\dagger = \sigma_i = \sigma_i^{-1}, \quad [\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k$$

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b})$$

$$\exp(i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}) = \cos|\boldsymbol{\theta}| + i(\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\sigma}) \sin|\boldsymbol{\theta}|$$

- Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i, \quad \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad \left(\gamma^5\right)^2 = 1$$

Elements of Group Theory

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This section gives a brief overview of some group theoretical concepts and terminology that is often used in the particle physics literature.

[This section is still incomplete]

Definition of a group

A **group** G is a collection of **elements** $\{a, b, c, \ldots\}$ with a composition rule ab, often called the **multiplication** of a and b, that satisfies:

1. For every element a and b of G, the product ab is also an element of G;

2. The multiplication is associative $(ab)c = a(bc)$;

3. There is a unique **unit element** e, with $ea = ae = a$, for all elements a;

4. Each element a has a unique **inverse** a^{-1} in G, with $aa^{-1} = a^{-1}a = e$.

This is of course quite an abstract definition[4] since it is not specified what these group elements are, and what the group multiplication stands for. In physics, we can think of a group as a set of transformations of some kind, such as translations or rotations in Euclidian space, Lorentz transformations in space-time, or—more abstract—transformations in quark flavour or color space.

A group can be *discrete*, with the group elements labeled by a set of indices, or *continuous*, with the elements labeled by a set of continuous parameters.

An example of a discrete group is the set of integers, with addition as the group multiplication. The number zero is then the unit element and the negative integers are the inverse of the positive integers (and *vice versa*). This group obviously has an infinite number of elements. An example of a continuous group is that of rotations in two dimensions, with each element labeled by a rotation angle. Here the group operation is the addition of rotation angles. The unit element is a rotation over zero angle, and the inverse element is a rotation with the angle reversed.

Another distinction is that of **Abelian groups** where the group operation commutes ($ab = ba$ for all elements a and b) and **non-Abelian** groups where the group operation does not always commute. For instance, the group of rotations in two dimensions is Abelian, but that of rotations in three dimensions is not.

We will now use the finite discrete *cyclic group* to illustrate some basic ideas.

[4] The definition, as stated here, is somewhat redundant because e and a^{-1} *must* be unique by virtue of their definitions and the requirements (1) and (2). We leave it as an exercise to prove this.

The cyclic group

As an example of a finite group, take the set

$$\mathsf{G} = \{1, i, -1, -i\}, \tag{0.1}$$

with ordinary complex multiplication as the group operation. The number of elements of a discrete group is called the **order** of the group, sometimes denoted by $[\mathsf{G}]$. Thus, the group above is of order four.

A finite group is completely specified by its **multiplication table** which for our group $\mathsf{G} = \{e, a, b, c\}$ is given by

G	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

A multiplication table usually is not very instructive but some characteristic features can easily be spotted: (i) Each element of the group occurs only once in each row or column. This is because ab and ac cannot map onto the same element. Indeed, if $ab = ac$ we find, multiplying from the left with a^{-1}, that $b = c$; (ii) The table above is symmetric around the diagonal which shows that the group G is Abelian; (iii) Elements with e on the diagonal are its own inverse.

We can also write (0.1) as

$$\mathsf{G} = \{1, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\},$$

which shows that G can be realised by rotations over $\{0, 90, 180, 270\}$ degrees. In this realisation, the group operation is the *addition* of rotation angles. A rotation of a 2-dimensional coordinate system over an angle θ, measured counterclockwise from the x-axis, is described by the rotation matrix[5]

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{0.2}$$

Setting $\theta = \{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}$, we can represent the group G by the matrices

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}, \tag{0.3}$$

[5]Note that this is a **passive** rotation of the coordinate system where the *same* vector is described in the primed and unprimed systems. An **active** transformation rotates the vector and is related to the passive transformation by inverting the sign of θ.

with matrix multiplication as the group operation. This is called a 2-dimensional *representation* of G.

► An n-dimensional **representation** of a group G is a mapping of each element g_i onto a non-singular $n \times n$ matrix M_i that preserves the group multiplication

$$g_i g_j = g_k \quad \rightarrow \quad M_i M_j = M_k.$$

Don't confuse the *dimension* of a representation of G with the *order* of G. ◄

It is clear that rotations by multiples of 90° leave a square invariant. If we label the corners of the square $\{1, 2, 3, 4\}$ then we see that G can also be realised by the following four *permutations*:

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\}. \tag{0.4}$$

► Every element of a finite group of order n corresponds to a permutation of n objects. ◄

When we arrange the objects in an n-dimensional vector, the permutations can be expressed as $n \times n$ matrices, thus yielding a **regular representation** of the group (*i.e.* a representation with a dimension equal to the order of the group):

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}. \tag{0.5}$$

If we introduce *complex* matrices, we can say that (0.1) is a 1×1 complex representation of G. In this somewhat un-systematic fashion we have thus found a 1-, 2- and 4-dimensional representation of G. It is an important (and nontrivial) task of group theory to find all representations of a group or, to be more precise, all so-called **irreducible representations** since these serve as basic building blocks to construct all others.

Taking powers of a we see that G can be written as

$$\mathsf{G} = \{e, a, a^2, a^3\} \quad \text{with} \quad a^4 = e. \tag{0.6}$$

Thus a generates all elements of the group and is called the **generator** of G. For obvious reasons, G is called the *cyclic group* of order four, denoted by Z_4.

► The **cyclic group** Z_n of order n is generated by a 2-dimensional rotation over the angle $2\pi/n$. The group leaves an n-sided regular polygon invariant. ◄

Some basic concepts

▶ A **subgroup** $\mathsf{H} \subset \mathsf{G}$ is a set of elements of G that satisfy the group conditions. The unit element e is obviously shared by G and all its subgroups. ◀

▶ The **left coset** $g\mathsf{H}$ is obtained by multiplying all elements of H from the left by an element g which is *not* in H. Likewise we define the **right coset** $\mathsf{H}g$. Note that the left and right cosets are of the same order as H but are not subgroups of G since they do not contain the unit element. ◀

▶ A subgroup H and its left (or right) coset have no element in common. ◀

This can easily been seen as follows: Let $gh_1 = h_2 \in \mathsf{H}$. Then $g = h_2 h_1^{-1} \in \mathsf{H}$ which leads to a contradiction since g is, by definition, *not* in H. Let us now take another element g' which is *not* in H and also not in $g\mathsf{H}$. It is easy to show (homework) that $g'\mathsf{H}$ has no element in common with $g\mathsf{H}$ (and H). Now we can pick another element g'' not in H or in any of the two cosets to build another completely disjunct coset $g''\mathsf{H}$. In this way we can continue till we have divided the entire group G into H and cosets $g\mathsf{H}$ which all have the same number of elements, and no elements in common. We just have proven

▶ **Lagrange's theorem**: The order m of a subgroup $\mathsf{H} \subset \mathsf{G}$ is an integer division of the order n of G. The ratio $k = m/n$ is called the **index** of H in G. It directly follows that groups of prime order cannot have any subgroups. ◀

Another very important operation is that of **conjugation**.

▶ The **conjugate** of any element a with respect to any other element g is defined by a so-called **similarity transformation**

$$\tilde{a} = gag^{-1}. \tag{0.7}$$

When $ab = c$ then $\tilde{a}\tilde{b} = \tilde{c}$, that is, conjugation preserves the group multiplication. Clearly, the elements a and $\tilde{a}$ are *each other's* conjugate since $a = g^{-1}\tilde{a}g$. Note that the elements of an Abelian group are their own conjugate $\tilde{a} = a$. ◀

Conjugation is an example of a *one-to-one* mapping of group elements onto another set of elements that have the same multiplication table. Such a mapping is called an **isomorphism**: $\mathsf{G} \cong \mathsf{F}$. A **homomorphism** ($\mathsf{G} \sim \mathsf{F}$) is a mapping of G to F that is not one-to-one, but still preserves the multiplication table.

Conjugation splits a group G into disjunct *classes*:

▶ A **class** C_a is the set of conjugates $\tilde{a}$ with respect to every element g of G:

$$C_a = \{gag^{-1} \ \forall g \in \mathsf{G}\}.$$

It is easy to show (homework) that if b is *not* an element of C_a then C_a and C_b have no element in common. Note that a class is *not* a subgroup, except when $a = e$. The classes of an Abelian group contain exactly one element $C_a = a$. ◀

A **normal** or **invariant subgroup** $\mathsf{H} \subset \mathsf{G}$ maps onto itself by conjugation with respect to any element g:[6]

$$ghg^{-1} \in \mathsf{H}, \quad \forall h \in \mathsf{H},\ \forall g \in \mathsf{G}.$$

Because $gh_1g^{-1} = h_2$ it follows that for each element h_1 of a normal subgroup another element h_2 can be found such that $gh_1 = h_2g$. From this it is clear that

▶ The left and right cosets of a normal subgroup are identical: $g\mathsf{H} = \mathsf{H}g$. ◀

When G contains a normal subgroup H, we can set-up a correspondence $\mathsf{G} \mapsto \mathsf{G}'$ by mapping all elements of H onto e' and all elements of a coset $g\mathsf{H} = \mathsf{H}g$ onto the element g'. We now multiply elements of H and its cosets with each other and see what happens to the images in G'.

$$\begin{array}{lcl} h_1h_2 = h_3 & \mapsto & e'e' = e', \\ h_1(ah_2) = h_1(h_3a) = (h_1h_3)a = h_4a & \mapsto & e'a' = a', \\ (h_1a)(h_2b) = h_1(ah_2)b = h_1(h_3a)b = (h_1h_3)(ab) = h_4c & \mapsto & a'b' = c', \end{array}$$

where we have set $ah_2 = h_3a$, $h_1h_3 = h_4$ and $ab = c$. Thus G and G' have the same multiplication table so that G' is a homomorphic image of G, called the **factor group** G/H. The normal subgroup H maps onto the unit element of G/H and is called the **kernel** of the mapping. From the above it is easy to see that the following statement is true.

▶ The kernel H of a homomorphic mapping $\mathsf{G} \mapsto \mathsf{G}'$ is a normal subgroup of G. The factor group G/H is then isomorphic to G'. Note that the factor group is not a subgroup of G but an *image* of G. ◀

Can a factor group also have a normal subgroup so that it can be factorised further? Yes, this is certainly possible but it can be shown that (homework):

▶ If $\mathsf{H} \subset \mathsf{G}$ is the *largest* normal subgroup of G then the factor group G/H has *no* normal subgroup (except e). Because H has the largest possible order it follows that G/H has the smallest possible order. ◀

A group that has no normal subgroup other than e is called **simple** and the above gives a prescription to map any non-simple group onto a simple group

[6] The additive group of integers, for example, contains the normal subgroup of even integers. What about the set of odd integers? (homework).

(of lower order). This is the reason why mathematicians only consider simple groups to be of fundamental interest.

Above we have encountered several ways to dissect a group so let us now introduce the *direct product* (also called Kronecker product) to enlarge a group.

► The **direct product** $\mathsf{F} \times \mathsf{G}$ is the set of pairs

$$(a, b), \quad a \in \mathsf{F},\ b \in \mathsf{G} \quad \text{with} \quad (a, b)(c, d) \equiv (ac, bd). \tag{0.8}$$

With the multiplication thus defined, it is easy to see that $\mathsf{F} \times \mathsf{G}$ is a group. ◄

Finally, let us repeatedly multiply an element by itself. Suppose we make a list

$$a, a^2, a^3, \ldots$$

of powers of some element $a \neq e$ of a finite group G. Clearly the length of such a list has no bound but since the number of elements of G is finite we must have it occur twice at some point in the list, that is, for some $n > m$ we have

$$a^n = a^m \quad \rightarrow \quad a^{n-m} = a^k = e.$$

The power k is called the **order** of a and the set $\{a^n\}$ is called the **orbit** of a. The above implies that:

► Every element $a \neq e$ of a finite group G of order n generates a cyclic subgroup $Z_k \subseteq \mathsf{G}$ with $2 \leq k \leq n$. An element that is its own inverse generates Z_2. ◄

Now because Lagrange's theorem tells us that groups of prime order cannot have any subgroup it follows that we must have $k = n$ when n is prime:

► The only possible finite group of prime order n is the cyclic group Z_n. ◄

The SO(3) group of rotations in three dimensions

Rotations in three dimensions form a continuous group, represented by the **special orthogonal group** SO(3) of 3×3 unimodular (unit determinant) orthogonal matrices R. The study of this group is of interest because rotation is a very common transformation, and also because several important concepts related to continuous groups can be nicely introduced.

We take the convention to rotate the coordinate system so that a vector $\boldsymbol{r}$ with coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ in a reference system O, has coordinates $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$ in the rotated system O'. Here and in the following we will use the

summation convention of summing over repeated indices so that we may write for $\boldsymbol{x}' = R\,\boldsymbol{x}$

$$x_i' = R_{ij}\,x_j.$$

The orthogonality condition reads $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$, in components,

$$R_{ji}R_{jk} = \delta_{ik} \qquad R_{ij}R_{kj} = \delta_{ik}.$$

It follows that a rotation preserves the inproduct $\boldsymbol{x}\cdot\boldsymbol{y}$ of two 3-vectors,

$$x_i'y_i' = R_{ij}x_j\,R_{ik}y_k = \delta_{jk}\,x_jy_k = x_jy_j.$$

The orthogonality condition implies $R^{-1} = R^{\mathrm{T}}$ so that each rotation indeed has an inverse. The unit element is a rotation over zero angle. Furthermore, the product $R_3 = R_2R_1$ of two rotations is again a rotation because

$$R_3^{\mathrm{T}} = R_1^{\mathrm{T}}R_2^{\mathrm{T}} = R_1^{-1}R_2^{-1} = R_3^{-1} \quad \text{and} \quad \det(R_3) = \det(R_2)\det(R_1) = 1.$$

We conclude that 3-dimensional rotations form a group.

Three-dimensional rotations are determined by a rotation axis $\hat{\boldsymbol{u}}$ (unit vector) and a rotation angle α about this axis. We write $\boldsymbol{\alpha} \equiv \alpha\hat{\boldsymbol{u}}$, specified by three parameters $(\alpha_1, \alpha_2, \alpha_3)$. If we rotate the system O counterclockwise by an angle α about the z axis to the system O' we have for the relation between $\boldsymbol{x}$ and $\boldsymbol{x}'$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{0.9}$$

For small angles α/n the rotation matrix can be written as

$$R(\alpha/n) = I + \frac{\alpha}{n}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{O}\left(\frac{\alpha^2}{n^2}\right) \equiv I + \frac{\alpha}{n}\,T + \mathrm{O}\left(\frac{\alpha^2}{n^2}\right).$$

The matrix T is called the **generator** of the rotations about the z axis. Ignoring terms $\mathrm{O}(\alpha^2)$ this gives for a finite rotation

$$R(\alpha) = \lim_{n\to\infty}\left(I + \frac{\alpha}{n}\,T\right)^n = \exp(\alpha T). \tag{0.10}$$

Here the exponent e^A of a matrix should be understood as the series expansion

$$e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{A^n}{n!}.$$

Note that the familiar expression $e^A\, e^B = e^{(A+B)}$ is *only* true when A and B commute. Because 3-dimensional rotations about different axes do *not* commute, it is not obvious that we can write the generator of a rotation about an arbitrary axis as the sum of generators of rotations about the x, y and z axis:

$$R(\boldsymbol{\alpha}) = e^{\alpha_1 T_1}\, e^{\alpha_2 T_3}\, e^{\alpha_3 T_3} \stackrel{?}{=} e^{\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3}.$$

However, for an infinitesimal rotation of a vector $\boldsymbol{r}$ about $\boldsymbol{\alpha}$ we can write

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{\alpha} \times \boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r} \times \boldsymbol{\alpha}.$$

Our convention is that we do not rotate the vector but the coordinate system (over an angle $-\alpha$) so that the coordinate transformation is

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{x} \times \boldsymbol{\alpha}.$$

Introducing the antisymmetric tensor ϵ_{ijk},[7] this reads in components

$$x_i' = x_i + \epsilon_{ijk} x_j \alpha_k = [\, \delta_{ij} + \alpha_k \epsilon_{ijk} \,]\, x_j = [\, \delta_{ij} + \alpha_k (T_k)_{ij} \,]\, x_j.$$

From this we identify the three generators $(T_k)_{ij} = \epsilon_{ijk}$:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{0.11}$$

and write $R(\boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha} \cdot \boldsymbol{T})$. Note that the generators are traceless and anti-orthogonal: $T^{\mathrm{T}} = -T$. Dividing by i makes the generators **Hermitian**[8] ($L^\dagger = L$) and the defining equation for the generators becomes

$$R(\boldsymbol{\alpha}) = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{L}), \tag{0.12}$$

with, for SO(3),

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{0.13}$$

Note that

$$(L_i)_{jk} = -i\epsilon_{ijk} \tag{0.14}$$

[7] The tensor ϵ_{ijk} is $+1$ for even permutations of (123), -1 for even permutations of (231) and zero otherwise.
[8] The Hermitian conjugate of a matrix is defined by $H^\dagger = (H^*)^{\mathrm{T}}$. A matrix is called Hermitian when $H^\dagger = H$.

Let us at this point make a few remarks.

► A continuous group whose elements are continuously connected to the identity is called a **Lie group**. The elements of a Lie group are related to the generators of the group by the limiting equation (0.10). ◄

The rotation group SO(3) is obviously a Lie group, but the group O(3), that includes orthogonal matrices with determinant -1 (reflections) is *not* a Lie group since reflections are not connected to the identity (there is no such thing as an infinitesimal reflection).

► The number of generators of a Lie group is equal to the number of parameters of that group. ◄

The group SO(3) has three parameters and therefore three generators. The number of generators has nothing to do with the dimension of the defining SO(3) matrices, which happens to be three also.

Is is seen from (0.13) that the generators L_i are Hermitian and traceless. They are Hermitian because R is orthogonal (homework) and traceless because $\det(R) = 1$. The latter follows from a theorem of linear algebra:

► For matrices $U = \exp(A)$ that can be brought into diagonal form, the determinant is given by $\det(U) = \exp(\mathrm{Tr}A)$. ◄

For a rotation $s\boldsymbol{\alpha}$, with s a real number, we find

$$R(s\boldsymbol{\alpha}) = \exp(is\boldsymbol{\alpha}\cdot\boldsymbol{L}) = R(\boldsymbol{\alpha})^s \quad \text{so that} \quad R(s\boldsymbol{\alpha})R(t\boldsymbol{\alpha}) = R[(s+t)\boldsymbol{\alpha}].$$

► Rotations about a fixed axis define a commuting subgroup of SO(3). ◄

Because the product of two rotations is again a rotation it follows that

$$R(\boldsymbol{\alpha})R(\boldsymbol{\beta}) = R(\boldsymbol{\gamma}) \tag{0.15}$$

where $\boldsymbol{\gamma}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a (non-trivial) function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. From the fact that such a function must exist it can be shown that the commutator of any two generators must be a linear combination of the generators

$$[L_i, L_j] = c_{ij}^k L_k. \tag{0.16}$$

For SO(3) the commutation relations are, from (0.13),

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{0.17}$$

The c_{ij}^k are called the **structure constants** of the group. Note from (0.14) that the SO(3) structure constants are also matrix elements of the representation of the generators and this is no coincidence, as we will see below.

Eq. (0.15) can be written as

$$\exp(i\boldsymbol{\gamma}\cdot\boldsymbol{L}) = \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{L})\exp(i\boldsymbol{\beta}\cdot\boldsymbol{L}) = \exp[i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{L} + f(\boldsymbol{L})],$$

where $f(\boldsymbol{L})$ is a function of repeated commutators like $[L_i, L_j]$, $[[L_i, L_j], L_k]$, *etc.* From this it can be shown that $f(\boldsymbol{L})$ depends only on the structure constants.

► Structure constants determine the multiplication structure of a Lie group. The commutation relations (0.16) thereby define a so-called **Lie algebra**. ◄

For any triplet of $n \times n$ matrices A, B and C, the **Jacobi identity** states that

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

which is easy to prove by writing out the commutators, and enjoying the cancellations. In terms of the structure constants, the Jacobi identity reads

$$c_{ij}^m\, c_{mk}^n + c_{jk}^m\, c_{mi}^n + c_{ki}^m\, c_{mj}^n = 0.$$

Now define the matrices C_i with elements

$$(C_i)_j^k = -c_{ij}^k. \tag{0.18}$$

From (0.16) it is seen that $c_{ij}^k = -c_{ji}^k$, and the Jacobi identity becomes

$$\begin{aligned} c_{ij}^m\, c_{mk}^n - c_{jk}^m\, c_{im}^n + c_{ik}^m\, c_{jm}^n &= -c_{ij}^m (C_m)_k^n - (C_j)_k^m (C_i)_m^n + (C_i)_k^m (C_j)_m^n \\ &= -c_{ij}^m (C_m)_k^n - (C_j\, C_i)_k^n + (C_i\, C_j)_k^n = 0 \end{aligned}$$

or

$$[C_i, C_j] = c_{ij}^k C_k$$

which is the same commutation relation as (0.16). Thus the matrices C_i are a representation, called the **adjoint representation**, that has a dimension equal to the number of generators. This is in contrast to the so-called **fundamental representation** (0.13), that has the dimension of the defining linear space which is the 3-dimensional Euclidian space in case of SO(3). From (0.14) it is clear that for SO(3) the fundamental and the adjoint representations coincide, but this is certainly not true in general.

Let us give, at this point, two useful relations for the ϵ tensors (the first is the Jacobi identity, the second can trivially be shown to be true by giving the values (1,2,3) to two of the indices).

$$\epsilon_{ijm}\,\epsilon_{mkn} + \epsilon_{jkm}\,\epsilon_{min} + \epsilon_{kim}\,\epsilon_{mjn} = 0 \tag{0.19}$$

$$\epsilon_{ijm}\,\epsilon_{mkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk} \tag{0.20}$$

SO(3) transformations in higher dimensions

In this section we take two 3-vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and use these to build objects of dimensions larger than three. Their transformation under rotations will then yield higher-dimensional representations $D(R)$ of SO(3), other than the fundamental (R) and adjoint representations that we have found up to now.

The simplest composite object we can build has 6 components and is defined by $\boldsymbol{v} = \boldsymbol{x} \oplus \boldsymbol{y} \stackrel{\text{def}}{=} (x_1, x_2, x_3, y_1, y_2, y_3) = (v_1, v_2, v_3, v_4, v_5, v_6)$. It transforms under rotations as

$$\boldsymbol{v}' = \begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = D(R)\,\boldsymbol{v}. \tag{0.21}$$

Clearly $D(R_1)D(R_2) = D(R_1R_2)$, so that D is indeed is a *representation* of SO(3). It is also clear that the components v_1, v_2 and v_3 will never mix with the components v_4, v_5 and v_6 and we say that $D(R)$ is **reducible** into a **direct sum** of two 3-dimensional transformations: $\mathbf{6} = \mathbf{3} \oplus \mathbf{3}$. A block-diagonal representation like (0.21) is the hallmark of reducibility but if we would have defined $\boldsymbol{v} = (x_1, y_1, x_2, y_2, x_3, y_3)$, for instance, then $D(R)$ would not be block-diagonal but of course still be reducible into $\mathbf{3} \oplus \mathbf{3}$ since v_1, v_3 and v_5 will not mix with v_2, v_4 and v_6.

► A representation that cannot be brought into block-diagonal form by a similarity transformation (change of basis) is called **irreducible**. ◄

We can build another object by taking the **outer product** of $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$T_{ij} = (\boldsymbol{x} \otimes \boldsymbol{y})_{ij} \stackrel{\text{def}}{=} x_i y_j \qquad \text{with} \qquad T'_{ij} = x'_i y'_j = R_{ik}x_k R_{jl}x_l = R_{ik}R_{jl}T_{kl}.$$

This tensor $\boldsymbol{T}$ has $3 \times 3 = 9$ components and the transformation $R_{ik}R_{jl}$ can be arranged into a 9×9 matrix $D(R)$ with, again, $D(R_1)D(R_2) = D(R_1R_2)$. The representation D is reducible because some linear combinations of the tensor elements have specific behaviour under rotations, as we will now show. For instance the trace of $\boldsymbol{T}$ is just the inproduct of $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\mathrm{Tr}(\boldsymbol{T}) = \delta_{ij}T_{ij} = T_{ii} = x_i y_i = \boldsymbol{x} \cdot \boldsymbol{y},$$

and is therefore invariant under rotations. The antisymmetric sum

$$a_i = \epsilon_{ijk}T_{jk} = \epsilon_{ijk}x_j y_k = (\boldsymbol{x} \times \boldsymbol{y})_i$$

is a component of the cross product of $\boldsymbol{x}$ and $\boldsymbol{y}$ and thus transforms as the component of a vector: $a'_i = R_{ij}a_j$. Having identified one scalar component

and three vector components there remain $9-1-3=5$ components of $\boldsymbol{T}$ that transform as a proper tensor. This suggests that we may write the decomposition of the tensor product as $\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}$ where, by construction, the vector component $\mathbf{3}$ is antisymmetric in the tensor indices.

Note, in this respect, that any tensor T_{ij} can be decomposed into a symmetric part $S_{ij}=S_{ji}$ and an antisymmetric part $A_{ij}=-A_{ji}$ as follows

$$S_{ij}=\tfrac{1}{2}(T_{ij}+T_{ji}),\quad A_{ij}=\tfrac{1}{2}(T_{ij}-T_{ji}).$$

Now the (anti)symmetric components transform into (anti)symmetric components as is easy to show: If we denote by $\tilde{\boldsymbol{T}}$ the transpose of $\boldsymbol{T}$ then we have, by definition, $\boldsymbol{S}-\tilde{\boldsymbol{S}}=0$ and $\boldsymbol{A}+\tilde{\boldsymbol{A}}=0$. Because the transformation $D(R)$ is linear we can write

$$\begin{aligned}\boldsymbol{S}'-\tilde{\boldsymbol{S}}' &= D(\boldsymbol{S})-D(\tilde{\boldsymbol{S}})=D(\boldsymbol{S}-\tilde{\boldsymbol{S}})=D(0)=0\\ \boldsymbol{A}'+\tilde{\boldsymbol{A}}' &= D(\boldsymbol{A})+D(\tilde{\boldsymbol{A}})=D(\boldsymbol{A}+\tilde{\boldsymbol{A}})=D(0)=0\end{aligned}$$

so that, indeed, $\boldsymbol{S}'=\tilde{\boldsymbol{S}}'$ and $\boldsymbol{A}'=-\tilde{\boldsymbol{A}}'$. The representation $D(R)$ thus decomposes into $\mathbf{3}\otimes\mathbf{3}=\{\mathbf{6}\}\oplus[\mathbf{3}]$, where we have introduced the notation $\{\boldsymbol{n}\}$ and $[\boldsymbol{m}]$ to indicate a representation that transforms as a symmeteric or as an antisymmetric tensor.[9]

We have seen above that the trace is invariant so that the symmetric component is still reducible into $\{\mathbf{6}\}=\{\mathbf{5}\}\oplus\mathbf{1}$. It thus makes sense to formally isolate the trace and write the expansion of a tensor as

$$T_{ij}=\tfrac{1}{2}\underbrace{\epsilon_{ijk}(\epsilon_{klm}T_{lm})}_{T_{ij}-T_{ji}}+\tfrac{1}{2}(T_{ij}+T_{ji}-\tfrac{2}{3}\delta_{ij}T_{kk})+\tfrac{1}{3}\delta_{ij}T_{kk},\qquad(0.22)$$

where use of (0.20) has been made to express the antisymmetric component in terms of ϵ-tensors. This component is traceless by definition, and by using the identity $\delta_{ii}=3$ it is immediately clear that the second term is traceless, too. The decomposition (0.22) shows that the 9-dimensional tensor representation of SO(3) splits into three irreducible representations

$$\mathbf{3}\otimes\mathbf{3}=[\mathbf{3}]\oplus\{\mathbf{5}\}\oplus\mathbf{1}.$$

To summarize, we can write the decomposition of our tensor $\boldsymbol{T}=\boldsymbol{x}\otimes\boldsymbol{y}$ as

$$\boldsymbol{T}\to\begin{cases}T &= \tfrac{1}{3}(\boldsymbol{x}\cdot\boldsymbol{y}) & \text{(scalar, one component)}\\ T_i &= \tfrac{1}{2}(\boldsymbol{x}\times\boldsymbol{y})_i & \text{(vector, three components)}\\ T_{ij} &= \tfrac{1}{2}(x_iy_j+x_jy_i)-\tfrac{1}{3}\delta_{ij}(\boldsymbol{x}\cdot\boldsymbol{y}) & \text{(tensor, five componenents)}\end{cases}$$

[9] This is the same notation as that of anticommutation $\{A,B\}\equiv AB+BA$ (symmetric) and commutation $[A,B]\equiv AB-BA$ (antisymmetric). Of course $\mathbf{1}=\{\mathbf{1}\}$, so there we do not put brackets.

where the first term transforms as a scalar (invariant under rotations), the second term as a vector

$$T_i' = R_{ij} T_j,$$

and the third term as a tensor, according to

$$T_{ij}' = \tfrac{1}{2}\left(R_{ik}R_{jl} + R_{il}R_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right) T_{kl}.$$

The number of indices is called the **rank** or **order** of a tensor; apart from rank-2 tensors we thus have also encountered tensors of rank zero (scalars) and one (vectors). Note that tensors are defined by their SO(3) transformation properties so that a rank-2 tensor not necessarily *is* an outer product, but *behaves* as an outer product of two vectors.

A scalar is, by definition, invariant under SO(3) transformations but there exist also higher order tensors that are invariant. For instance,

$$\delta_{ij}' = R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} = (RR^{\mathrm{T}})_{ij} = \delta_{ij}$$

and[10]

$$\epsilon_{ijk}' = R_{il}R_{jm}R_{kn}\epsilon_{lmn} = \epsilon_{ijk}\det(R) = \epsilon_{ijk}.$$

Quite some more to come ...

[10] We denote the first row of a 3×3 matrix A by the vector $\boldsymbol{a}_1 \equiv (A_{11}, A_{12}, A_{13})$, and similar for the second ($\boldsymbol{a}_2$) and third row and ($\boldsymbol{a}_3$). The determinant is then given by the volume $\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \epsilon_{lmn} A_{1l} A_{2m} A_{3n}$. The determinant changes sign under the interchange of two row-indices while no two row-indices can be equal. This can be encoded by setting the indices $\{1,2,3\}$ to $\{i,j,k\}$ and writing $\epsilon_{lmn}A_{il}A_{jm}A_{kn} = \epsilon_{ijk}\det(A)$.

Lecture notes Particle Physics II

Quantum Chromo Dynamics

1. Introduction

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November 12, 2013

The quest for elementary particles and forces

- Particle physics is the search for the fundamental constituents of matter and their interactions.
- The idea that matter is built from indivisible constituents dates back to the Greek philosopher Demokritos (400 BC) but it took a long time to prove him right: the proof that chemical elements are indeed made of atoms came only at the beginning of the last century, together with the development of statistical mechanics and quantum theory.
- The discovery of radioactivity indicated, however, that atoms could not be the fundamental constituents of matter and, indeed, after the experiments of Rutherford (1909) it was realised that atoms are complex objects with electrons orbiting a small heavy nucleus.
- The discovery of the neutron by Chadwick (1932) showed that atomic nuclei are made up of protons and neutrons. It was also clear that, in addition to gravitation and the electromagnetic force, there should exist two short-range forces in nature: a strong force which binds the nucleons together and a weak force which is responsible for radioactive β-decay. These forces had to be short-range because they were not felt at atomic scales.
- So nowadays four types of interaction are known: the **strong interaction** (~ 1), the **electromagnetic interaction** ($\sim 10^{-2}$), the **weak interaction** ($\sim 10^{-6}$) and **gravity** ($\sim 10^{-38}$), where we have indicated the relative strength in brackets.

Interactions

- Today we know that nucleons are made up of quarks and that the strong force between the nucleons in an atomic nucleus is a van-der-Waals type residual force of a more fundamental strong interaction between quarks. The field theory of this interaction between quarks is called **Quantum Chromodynamics** (QCD).
- QCD is a so-called **gauge theory**, like quantum electrodynamics (QED) and the theory of the weak interactions. In such a theory, the constituent fields are described by representations of a symmetry group while the interaction between the fields is described by the exchange of so-called **gauge bosons**. These interactions follow from the requirement that the Lagrangian is invariant under arbitrary local symmetry transformations of the constituent fields.
- The underlying U(1) symmetry of QED gives rise to the photon γ as the gauge boson. The weak interaction is governed by an SU(2) symmetry and is mediated by the three vector bosons $W^{\pm}$ and Z^0. The SU(3) symmetry of QCD generates generates eight types g_a of gluon as the quanta of the gauge field. Here are the properties of these so-called **intermediate vector bosons** (spin 1).

Interaction	Boson	Spin	Q	A	L	Mass
Electromagnetic	γ	1	0	0	0	0
Weak (CC)	$W^{\pm}$	1	± 1	0	0	80.4 GeV
Weak (NC)	Z^0	1	0	0	0	91.2 GeV
Strong	$g_1, \ldots, g_8$	1	0	0	0	0

- Note that all particles participate in the weak interaction, all charged particles in the electromagnetic interaction and that only the quarks participate in the strong interaction. Gravity is so weak that it can be neglected at subatomic scales.

Elementary particles

- The elementary particles can be classified into **leptons** (without strong interaction) and **quarks** (with strong interaction):

Lepton	Spin	Q	L_e	L_μ	L_τ	Mass	Lifetime
e	$\frac{1}{2}$	–1	1	0	0	0.5 MeV	
ν_e	$\frac{1}{2}$	0	1	0	0	~ 0	
μ	$\frac{1}{2}$	–1	0	1	0	106 MeV	2×10^{-6}
ν_μ	$\frac{1}{2}$	0	0	1	0	~ 0	
τ	$\frac{1}{2}$	–1	0	0	1	1.8 GeV	3×10^{-13}
ν_τ	$\frac{1}{2}$	0	0	0	1	~ 0	

Quark	Spin	Q	A	I_3	S	C	B	T	Mass
d	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	0	~ 7 MeV
u	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	0	0	~ 3 MeV
s	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	–1	0	0	0	~ 120 MeV
c	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	1	0	0	~ 1.2 GeV
b	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	–1	0	~ 4.3 GeV
t	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0	1	~ 172 GeV

- The additive quantum numbers Q, L, A, I_3, S, C, B, T all change sign under charge conjugation (particle $\rightarrow$ antiparticle).[11]
- The charge (Q), lepton number ($L_{\mathrm{e},\nu,\tau}$) and baryon number (A) are aways conserved in every type of interaction (electromagnetic, weak, strong).

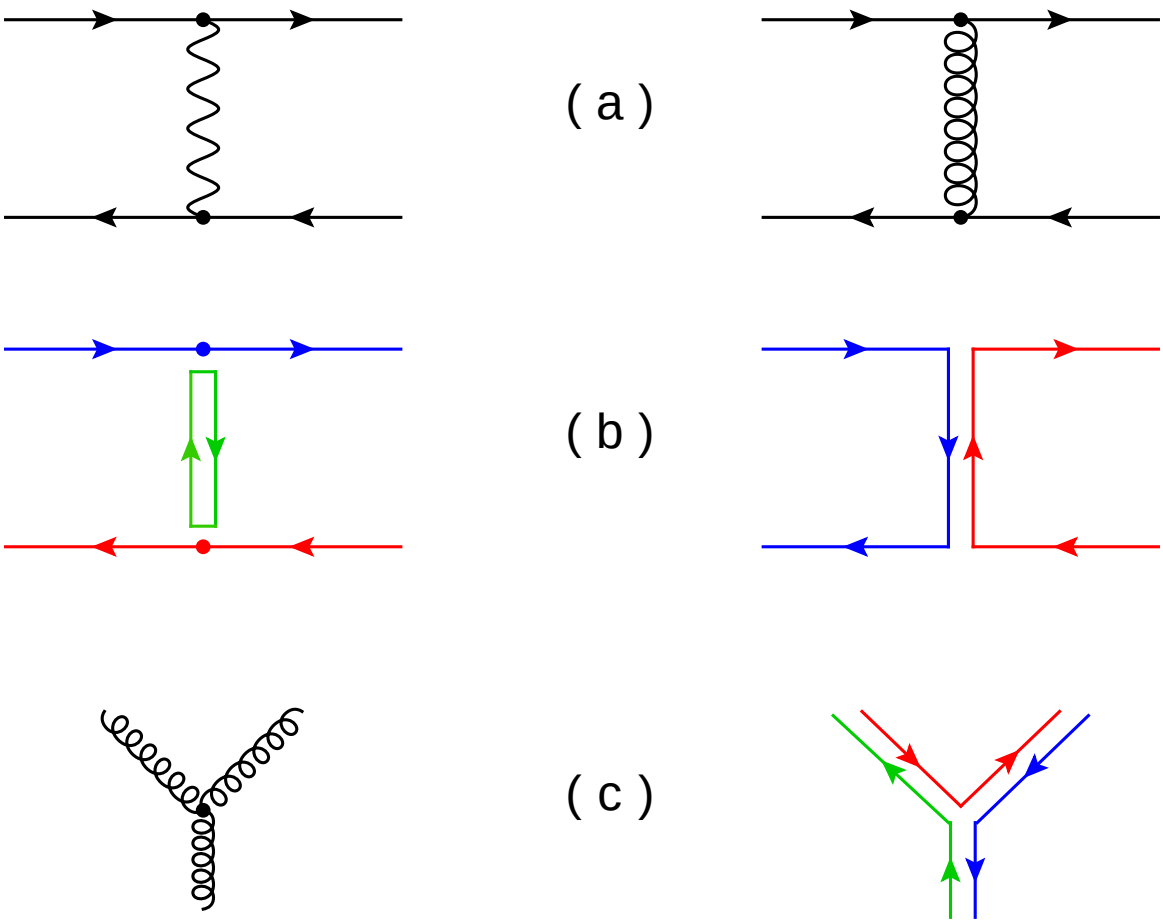
[11] Note that the flavour quantum numbers I_3, S, C, B, T have, by convention, the same sign as the charge.

Colour charge

- Free quarks have never been observed because their coupling is so strong that with increasing separation it becomes easier to produce a quark-antiquark pair than to isolate the quark.
- Quarks therefore bind permanently into **hadrons** which can be classified as **mesons** ($q\bar{q}$) and **baryons** (qqq).
- A problem with this is that there exist baryons such as the spin $\frac{3}{2}$ resonance $\Delta^{++} = u \uparrow u \uparrow u \uparrow$ with a ground state wave function that is fully symmetric under the exchange of two quarks. But for fermions (half-integer spin) the wave function should be antisymmetric.
- A way-out is provided by the **colour hypothesis** which states that each quark comes in one of three colours red (r) , green (g) or blue (b). Antiquarks are anticoloured: $\bar{\text{r}}$, $\bar{\text{g}}$ and $\bar{\text{b}}$. The hypothesis furthermore states that hadrons are **colour singlets** ('white'), that is, they are invariant under rotations in colour space. The colour hypothesis thus naturally explains the existence of $q\bar{q}$ and qqq hadronic states[12] and also that of particles like the Δ^{++} since its colour wave function can always be made fully antisymmetric.
- In QCD, colour plays the role of charge, and gluons are the quanta of the colour gauge field that binds the quarks into hadrons. Unlike the photons in QED, the gluons themselves carry a colour charge, so that self-coupling 3- and 4-gluon vertices do exist ($\rightarrow$ **fig**).
- As we will see later, this self-coupling of gluons has dramatic implications for the effect of *charge screening* in QCD, which turns out to be completely different from that in QED.

[12] It also allows for $q\bar{q}[q\bar{q}]^n$ and $qqq[q\bar{q}]^n$ exotic states. It is controversial if such states actually exist.

Basic diagrams of QED and QCD



(a) Electromagnetic interaction of a quark and an antiquark through photon exchange (left). Strong interaction of a quark and an antiquark through gluon exchange (right).

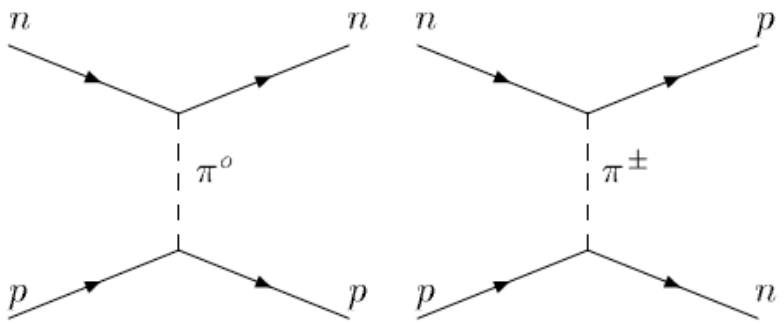
(b) Two possible colour flows in the $q\bar{q}$ strong interaction.

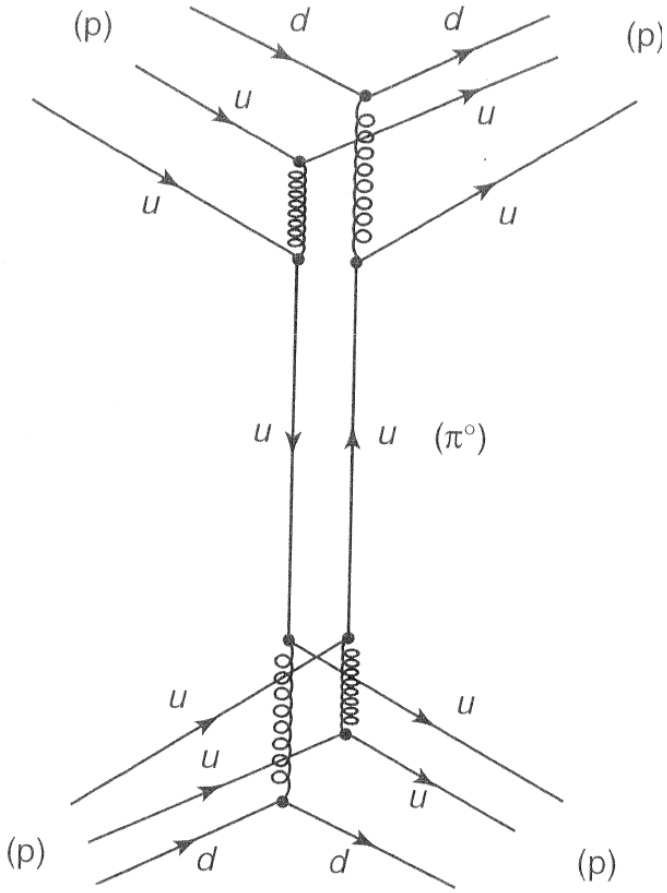
(c) Gluon interaction (left) and a possible colour flow through the 3-gluon vertex (right). Note that gluons always carry one unit of colour and one unit of anticolour.

Two paradoxes

- To explain the short range of the nuclear force, Yukawa (1934) proposed that this force is mediated by the exchange of *massive* field quanta which he called mesons ($\rightarrow$ **fig**). In his theory, the range of the force is inversely proportional to the mass of the intermediate vector boson. He estimated a mass of about 140 MeV and indeed a candidate (the π meson) was later found in cosmic rays (1937).
- But, as we will see later, massive gauge field quanta break the gauge symmetry so that the exchanged boson must necessarily be massless. For instance, the U(1) symmetry of the QED Lagrangian forces the photon to be massless, which indeed it is. As a consequence the electromagnetic interaction has an infinite range.
- It follows that the SU(3) gauge symmetry of the QCD Lagrangian forces the gluons to be also massless, like the photon. But if these gluons are massless, how can the strong force then be short-range?
- Another puzzle came with a series of high-energy electron-proton scattering experiments at SLAC ($\sim$ 1970) which proved the existence of quarks but also showed that they seemed to behave like free particles, in spite of the fact that they are strongly bound inside the proton.
- The solution to both these paradoxes was found by Gross, Politzer and Wilczek by their discovery of **asymptotic freedom**. They could explain why, as Wilczek put it in his Nobel lecture, 'Quarks are Born Free, but Everywhere They are in Chains'.
- As we will see in these lectures, the phenomena of asymptotic freedom and confinement are caused by the self-interaction of gluons which, in turn, is a consequence of the non-abelian nature of SU(3).

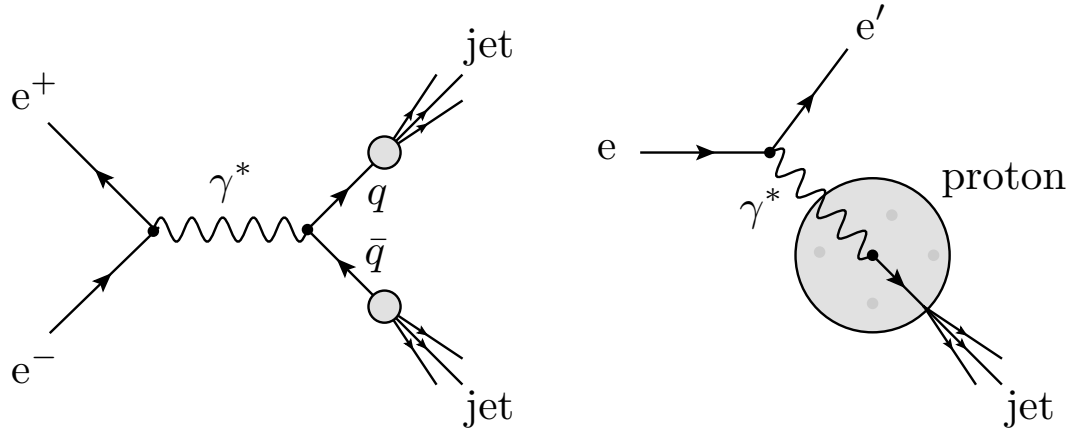
Old and modern views of pion exchange





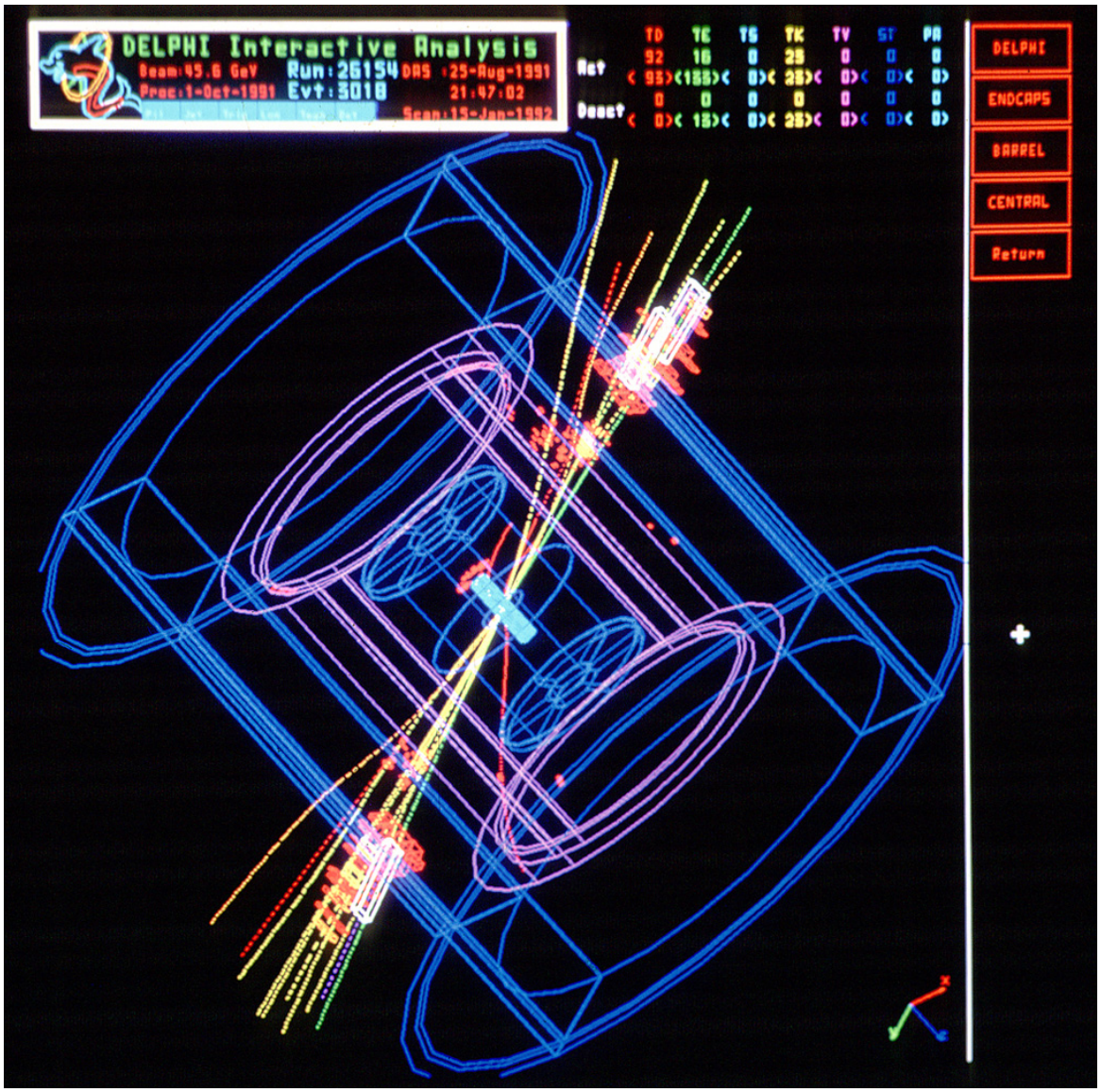
In the lower diagram π^0 exchange in a proton-proton interaction is described in terms of constituent quarks by the exchange of an u$\bar{\text{u}}$ pair.

Are quarks real?



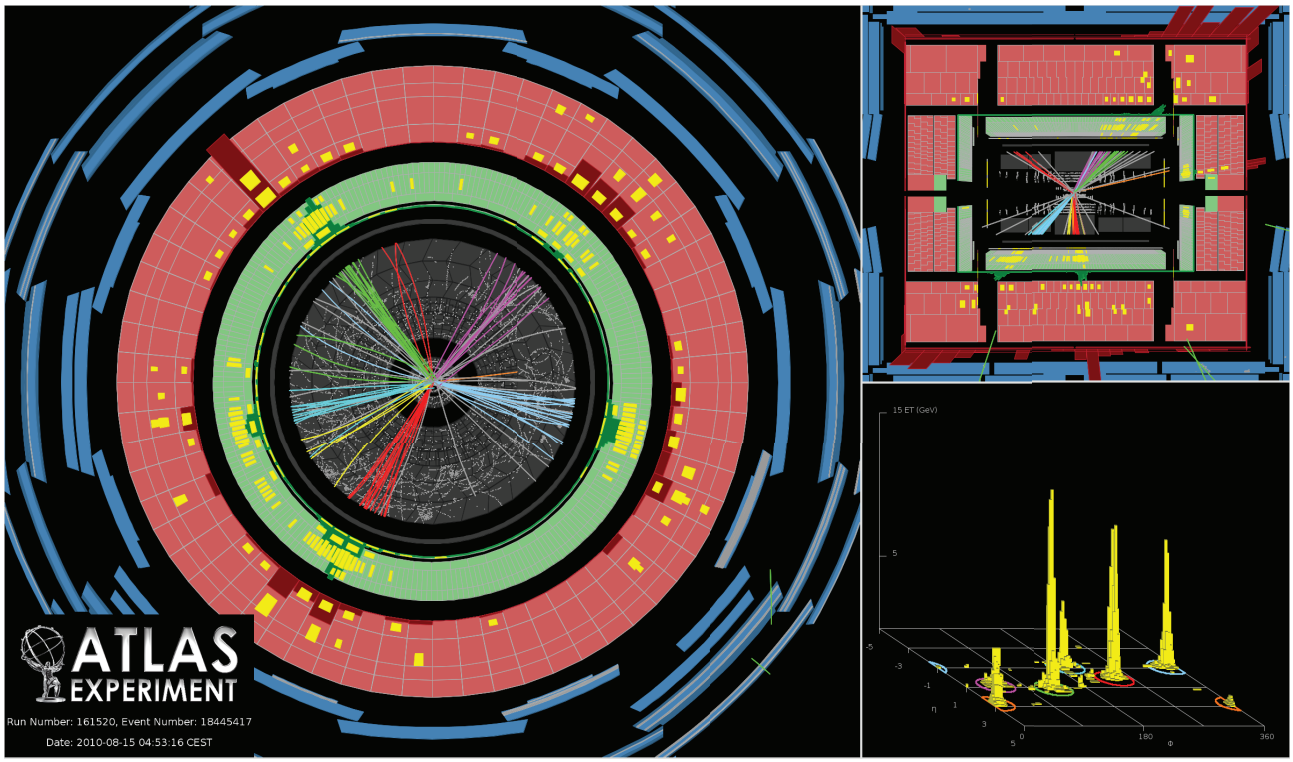
- If quarks cannot be observed in isolation, how do we know that they actually exist and are not mere theoretical constructs?
- One way is to resolve quarks by illuminating protons with photons of large momentum Q and therefore small Compton wavelength $1/Q$. These very short wavelength photons are radiated off highly energetic electrons when they scatter on protons (right-hand diagram above). This process is called **deep inelastic scattering** which indeed acts as a microscope to reveal the internal quark structure of the proton. How this actually works, will be the subject of the last two lectures in this course.
- Furthermore it turns out that highly energetic quarks produced in hard e^+e^-, $\bar{p}p$ and pp scattering hadronise into collimated sprays of particles, known as jets (left-hand diagram above). Thus we can more or less directly probe the dynamics of quarks by measuring jets in experiments at high energy colliders ($\rightarrow$ **fig**). Jet production is clearly a very important tool to confront QCD with experiment and can certainly produce spectacular events in particle colliders ($\rightarrow$ **fig**) but, unfortunately, we cannot cover the large field of jet physics in these eight lectures.

Two-jet event in a e^+e^- collision at LEP



Two back-to-back jets observed by the DELPI experiment at LEP in an e^+e^- collision at 90 GeV centre-of-mass energy.

Six-jet event in a proton-proton collision at the LHC



Candidate six-jet event recorded by the Atlas experiment in a 7 TeV proton-proton collision at the LHC.

About this course

- QCD, and strong interaction physics in general, is a huge subject, so no two courses on QCD are the same since they necessarily reflect the choices made by the lecturer.
- My first choice is to devote ample time (three lectures) to build the QCD Lagragian, much based on Chapter 10 of Griffiths. Here a good understanding is important because QCD lectures which you may attend later (*e.g.* at the CERN summer school) often start from the Lagrangian, without much further ado ($\rightarrow$ **fig**). Of course there will be some overlap with PP-I but I prefer to tell the full story instead of relying on what you presumably know already.
- QCD calculations quickly become technically complicated so we have to limit ourselves to some simple colour factor calculations which, however, nicely explain why the colour force is attractive for meson and baryon quark states and repulsive for others.
- Of course, asymptotic freedom is a crucial property of QCD and we will devote a full lecture to the running coupling constant and its implications. Here we will encounter ultraviolet singularities which will be dealt with by a simple cut-off regularisation.
- Infrared singularities are the subject of the another lecture where we will explain how they are related to long-distance physics, as opposed to short-distance perturbative QCD.
- The last two lectures are devoted to the structure of the proton, the quark-parton model, and the so-called QCD improved quark-parton model. This subject is not covered in Griffiths but extensively treated in Halzen and Martin.

First page of a QCD course at CERN

Feynman rules of QCD

- Feynman rules follow from QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m_q)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

  $F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

  Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_s \equiv g^2/4\pi$. Numbers f^{ABC} $(A, B, C = 1, ..., 8)$ are structure constants of the SU(3) colour group. Quark fields q_a $(a = 1, 2, 3)$ are in triplet colour representation, while gluon fields $\mathcal{A}^A_\alpha$ are in adjoint representation.

- D is covariant derivative:

$$\begin{aligned} (D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha\right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}^C_\alpha)_{AB} \end{aligned}$$

Flying start on the first page of the course 'Introduction to QCD' in the CERN postdoc lecture series, given by Bryan Webber in October 2003.

Lecture notes Particle Physics II

Quantum Chromo Dynamics

2. SU(2) and SU(3) Symmetry

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November 12, 2013

Symmetry in (particle) physics

- If the Lagrangian of the world would be fully known we could derive the equations of motion from it, and the symmetries of nature and the conservation laws would automatically follow.
- For instance the Maxwell Lagrangian yields, via the Maxwell equations, all the symmetries and conservation laws of electrodynamics.
- In subatomic physics the Lagrangians are not so obvious, and symmetry considerations provide essential clues to construct them.
- It can be shown that an invariance of the Lagrangian under a symmetry operation leads to a conserved quantity (Noether's theorem). Thus, if a symmetry is found, the hunt is open for the related conservation law, and if a conservation law is found, the hunt is open for the related symmetry. For instance we know that electric charge is conserved in all reactions of elementary particles, but what symmetry is responsible for this charge conservation? (The answer will be given in the next lecture.)
- As will become clear later, it turns out that discrete symmetries lead to *multiplicative* conserved quantum numbers (*e.g.* reflection symmetry $\rightarrow$ parity conservation $\rightarrow$ multiplication of parities) while continuous symmetries lead to *additive* conserved quantum numbers (*e.g.* rotation invariance $\rightarrow$ angular momentum conservation $\rightarrow$ addition of angular momentum quantum numbers).
- We will now use some elementary non-relativistic quantum mechanics to establish the relation between symmetries and constants of motion.

When is an observable conserved?

- The **expectation value** of a quantum mechanical operator F is

$$\langle F\rangle \equiv \langle\psi|F|\psi\rangle \quad \text{with Hermitian conjugate} \quad \langle F\rangle^* \equiv \langle\psi|F^\dagger|\psi\rangle$$

- The expectation value of an **observable** is a real number so that the operator of an observable should be **Hermitian**

$$F = F^\dagger \qquad \text{if } \langle F\rangle \text{ is observable}$$

- Because energy is an observable the Hamiltonian H is Hermitian. We have for the Schrödinger equation and its Hermitian conjugate

$$i\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \qquad \text{and} \qquad -i\frac{\partial\langle\psi|}{\partial t} = \langle\psi|H^\dagger = \langle\psi|H$$

- This immediately leads to

$$\frac{\partial\langle F\rangle}{\partial t} = i\,\langle\psi|HF - FH|\psi\rangle = 0 \quad \Leftrightarrow \quad HF - FH = 0$$

An observable constant of motion F is
Hermitian and commutes with the Hamiltonian

- When H is known, we can find observable constants of motion by searching for Hermitian operators that commute with H.
- However, when H is *not* fully known, it is sufficient to establish (or postulate) the invariance of H, or the Lagrangian, under a **symmetry operation**, as we will now show.

Symmetry operators

- A **transformation operator** U transforms one wave function into another

$$|\psi'\rangle = U|\psi\rangle$$

- Wave functions are always **normalized** so that we must have

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1$$

- It follows that the transformation operator must be **unitary**

$$U^\dagger U = UU^\dagger = I$$

- We call U a **symmetry operator** when $|\psi'\rangle$ obeys the same Schrödinger equation as $|\psi\rangle$. Then, with U time independent,

$$i\frac{\partial U|\psi\rangle}{\partial t} = HU|\psi\rangle \quad \rightarrow \quad i\frac{\partial|\psi\rangle}{\partial t} = U^{-1}HU|\psi\rangle \underset{\text{Iwant}}{=} H|\psi\rangle$$

and thus

$$U^{-1}HU = H \quad \text{or} \quad [H,U] = 0$$

A symmetry operator U is unitary and commutes with the Hamiltonian

- Thus U commutes with the Hamiltonian, as does a constant of motion. However, we cannot identify U with an observable since it is unitary, and not necessarily Hermitian.

Discrete symmetries

- There is a class of unitary transformations with the property

$$U^2 = I$$

  Multiplying from the right with $U^\dagger$ and using $UU^\dagger = I$ we find that $U = U^\dagger$: the operator is both unitary and Hermitian.

- Thus if U is a symmetry of (commutes with) the Hamiltonian we can directly conclude that it is an observable constant of motion.

- Examples of this the are the charge conjugation operator C (exchange of particles and antiparticles) and the parity operator P (reflection of the spatial coordinates).[13]

- Remark: C and P are not the only operators that are both unitary and Hermitian. This is, for instance, also true for the Pauli spin matrices, as is straight-forward to check.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- If $|\psi\rangle$ is an eigenvector of both U_1 and U_2 then

$$U_{1,2}|\psi\rangle = \lambda_{1,2}|\psi\rangle \qquad \text{and} \qquad U_1 U_2|\psi\rangle = U_2 U_1|\psi\rangle = \lambda_1\lambda_2|\psi\rangle$$

  The quantum numbers of a discrete symmetry are *multiplicative*.

- In these lectures we are not so much interested in **discrete** transformations (like C, P, T) but, instead, in **continuous** transformations. These transformations are unitary (by definition), but not necessarily Hermitian. But the **generator** of a unitary continuous transformation is Hermitian, as we will see.

[13] The time reversal operator T also has $T^2 = I$ but it is antiunitary, and not unitary.

Continuous transformations

- There is a large class of **continuous transformations** that depend on one or more continuous parameters, say α

$$|\psi'\rangle = U(\alpha)|\psi\rangle$$

  An example is the transformation induced by a rotation over an angle α of the coordinate system (passive rotation), or of the wave function (active rotation).

- Such transformations have the property that they can be written as a succession of infinitesimal deviations from the identity

$$U(\alpha) = \lim_{n\to\infty} \left(I + \frac{i\alpha}{n} F \right)^n = \exp(i\alpha F)$$

  The factor 'i' is a matter of definition but important (see below). In the above, F is called the **generator** of U.[14]

- Now if U is unitary we have, to first order in α,

$$U^\dagger U = (I - i\alpha F^\dagger)\,(I + i\alpha F) = I + i\alpha(F - F^\dagger) = I$$

  so that $F = F^\dagger$. In other words,

The generator of a unitary operator is Hermitian

- Now we also understand the factor 'i' in the definition of a generator: without it the generator $G \equiv iF$ of a unitary operator would not be Hermitian but **anti-Hermitian**:

$$G = -G^\dagger$$

[14] Exponentiation of an operator F should be interpreted as $\exp(i\alpha F) = I + i\alpha F + \frac{1}{2!}(i\alpha F)^2 + \cdots$ But watch out, the familiar relation $e^A e^B = e^{A+B}$ is *only* true when A and B commute.

Generators as conserved observables

- We have seen that a symmetry operator U commutes with the Hamiltonian so it remains to show that its generator will then also commute with H. The proof is very simple:
- First, if $U(\alpha)$ is a symmetry operator then the infinitesimal transformation $U(\epsilon)$ will also be a symmetry operator. Expanding to the first order in ϵ obtains

$$[H, U] \doteq [H, I + i\epsilon F] = \underbrace{[H, I]}_{0} + i\epsilon\,[H, F] = 0 \quad \rightarrow \quad [H, F] = 0$$

> If U is a unitary operator that commutes with the Hamiltonian then its generator F is a Hermitian operator that also commutes with the Hamiltonian

- We now have the work plan to find the relation between a continuous symmetry of H and the corresponding conserved observable:
  1. Find the generator F of the symmetry transformation U.
  2. The expectation value of F is a constant of motion
- Clearly a multiplication of continuous symmetry operators corresponds to the addition of their generators in the exponent. The conserved quantum numbers, which are related to F and not to U, are therefore *additive*.
- We will now proceed with the introduction of some concepts of **group theory** which is the mathematical framework to systematically describe and classify symmetry operations.

Exercise 2.1:

Show that (consult a quantum mechanics book if necessary)

(a) [0.5] Invariance for translations in space leads to the conservation of momentum.

(b) [0.5] Invariance for translations in time leads to the conservation of energy.

(c) [0.5] Rotational invariance leads to the conservation of angular momentum.

Group theory

- It is clear that a combination of two symmetry operations—that each leaves the system unchanged—is again a symmetry operation. And there is of course the trivial symmetry operation, namely, 'do nothing'. Furthermore, we can assume that each symmetry operation can be undone. We say, in fact, that symmetry operations form a **group**.

- What is a group? It is a set of elements $\{g_i\}$,
  - with a composition law $g_i \cdot g_j = g_k$
  - that is associative $(g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$
  - with a unit element e such that $e \cdot g_i = g_i \cdot e = g_i$
  - and with an inverse g_i^{-1} such that $g_i \cdot g_i^{-1} = g_i^{-1} \cdot g_i = e$

- Examples:

  The set $\{1, i, -1, -i\}$ under multiplication (discrete, 4 elements)

  The set of integers under addition (discrete, infinite # elements)

  Rotations in 3 dimensions (continuous, 3 parameters)

  Lorentz transformations (continuous, 6 parameters: which ones?)

- A group is called **Abelian** when the group operation is commutative $g_i \cdot g_j = g_j \cdot g_i$ (*e.g.* 2-dim rotations). Non-commutative groups are called non-Abelian (*e.g.* 3-dim rotations).

- A systematic study of symmetries is provided by a branch of mathematics called **group theory**. We will not present group theory in these lectures, but only a few basic concepts.[15]

[15] A nice summary of group theory can be found in A&H-II, Appendix M.

Representation of a group

- In these lectures, we will be concerned with groups of *matrices*.
- It may be the case, of course, that the group '*is*' a set of matrices. For instance, the group SO(2) of orthogonal 2×2 matrices with determinant 1, that describe 2-dimensional rotations.
- But a matrix representation may also come from mapping each element g_i of some group to an $n \times n$ matrix M_i (why must M be square?), such that the multiplication structure is preserved

$$g_1 \cdot g_2 = g_3 \qquad \rightarrow \qquad M_1 M_2 = M_3$$

  This is called an n-dimensional **representation** of the group $\{g\}$. Thus, SO(2) is *defined* by 2×2 matrices, (the **fundamental representation**) but it has also representations in higher dimensions.
- Two groups with the same multiplication structure are said to be **isomorphic ($\cong$)** if the elements map one-to-one. If the mapping is not one-to-one, they are called **homomorphic** ($\sim$).
- **Exercise 2.2**: [0.5] Show that

$$\{1, i, -1, -i\} \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

- From an n and an m-dimensional representation we can always construct an $(n+m)$-dimensional representation through

$$M_i^{(n+m)} = \begin{bmatrix} M_i^{(n)} & 0 \\ 0 & M_i^{(m)} \end{bmatrix} \equiv \boldsymbol{n} \oplus \boldsymbol{m}$$

  but this does not classify as a new representation. The relevant representations are the so-called **irreducible** ones which cannot be decomposed in block diagonal form. It is a (non-trivial) task of group theory to find all the irreducible representations of a group.

Lie groups

- On page 2–10 we have encountered discrete groups (elements labelled by an index, or a set of indices) and continuous groups where the elements are labelled by a set of continuous parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. Important groups of transformations $U(\boldsymbol{\alpha})$ are those which can be written as a succession of infinitesimal deviations from the identity transformation (see also page 2–7):

$$U(\boldsymbol{\alpha}) = \lim_{n\to\infty} \left[\, 1 + i(\boldsymbol{\alpha}/n) \cdot \boldsymbol{T} \,\right]^n = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{T})$$

  Such a group is called a **Lie group**,[16] and the matrices $\boldsymbol{T}$ are called the **generators** of the group.[17] The number of generators is equal to the number of parameters that label the group elements.

  Example: Rotations are a Lie group but reflections are not since these are not continuously connected to the identity.

- There is a theorem which states that the commutator of two generators is always a linear combination of the generators

$$[T_i, T_j] = f_{ij}^k \, T_k \qquad \text{(summation over } k \text{ implied)}$$

  These commutation relations are called the **algebra**, and the (complex) numbers f_{ij}^k are called the **structure constants** of the group. It can be shown that these structure constants fully characterise the multiplication structure of a Lie group.

- On page 2–7 we have shown that if U is unitary then $T_i = T_i^\dagger$. In other words, the generators of a unitary operator are Hermitian.

[16] The formal definition of a Lie group states first of all that the number of parameters is finite, and furthermore that $U(\alpha_1) \cdot U(\alpha_2) = U(\alpha_3)$, with α_3 an *analytic* function of α_1 and α_2.

[17] Discrete groups also have generators: *e.g.* repeated rotation over $2\pi/n$ generates the cyclic group $\mathcal{Z}_n$.

The 2-state nucleon system

- After the discovery of the neutron by Chadwick in 1932, the near equality of its mass (939.5 MeV) to that of the proton (938.3 MeV) suggested to Heisenberg that, as far as the strong interactions are concerned, these are two nearly degenerate states of one particle: the **nucleon**.
- This 'isospin symmetry' of the strong force is further supported by, for instance, the observation of very similar energy levels in **mirror nuclei** (the number of protons in one, is equal to number of neutrons in the other, and *vice versa*, like in $^{13}_{7}$N and $^{13}_{6}$C).
- In addition, apart from the p-n doublet, there are other particles that are nearly degenerate in mass, like the pion triplet ($\sim$140 MeV) and the quadruplet of Δ resonances ($\sim$1.23 GeV) $\rightarrow$ **Fig**. This looks like the doublet, triplet and quadruplet structure of spin-$\frac{1}{2}$, spin-1 and spin-$\frac{3}{2}$ systems built from spin-$\frac{1}{2}$ states, and is thus strongly suggestive of hadronic substructure.
- We know today that hadrons are built up from quarks and we can explain isospin symmetry from the fact that the strong interaction is insensitive to the quark flavour. The mass differences within the nucleon, π and Δ multiplets are, after electromagnetic correction, believed to be due to the difference in the u and d quark masses.
- The invariance for p to n transitions obeys the mathematics of ordinary spin, hence the term '**isospin**'. The reason is that transitions in *any* 2-state quantum mechanical system are described by the special unitary group SU(2), as will become clear next.

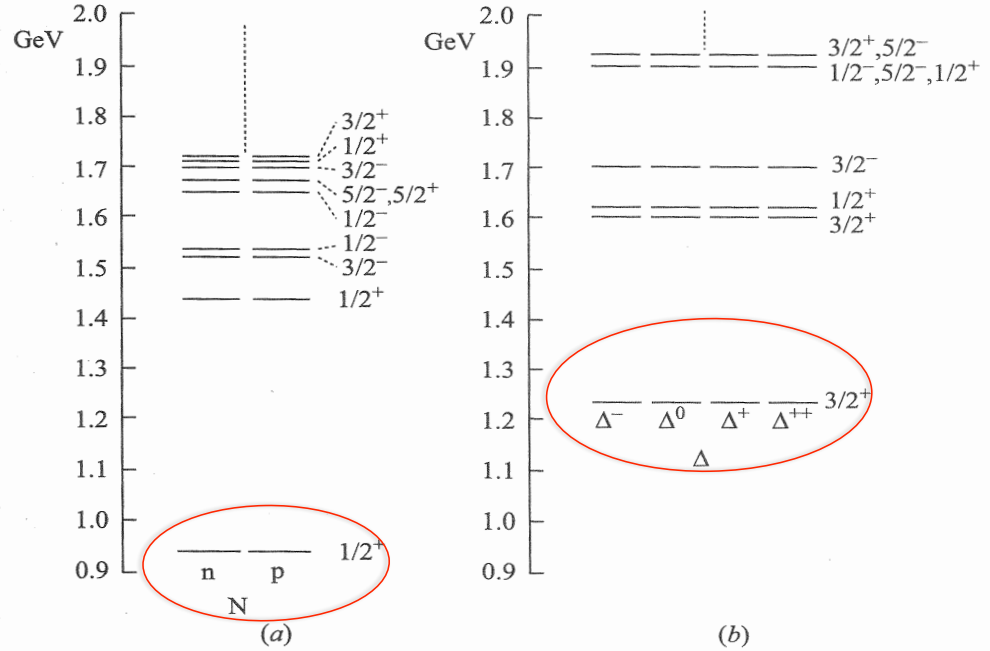


Figure 1.10. Baryon energy levels: (*a*) doublets (N); (*b*) quartets (Δ).

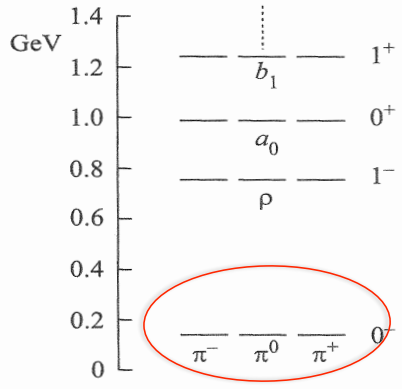


Figure 1.11. Meson triplets.

Isospin symmetry

- We work in a 2-dim Hilbert space spanned by the basis vectors[18]

$$|\mathrm{p}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\mathrm{n}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  The Hermitian conjugates are $\langle \mathrm{p}| = (1,0)$ and $\langle \mathrm{n}| = (0,1)$. An arbitrary state is written as the linear combination

$$|\psi\rangle = \alpha\,|\mathrm{p}\rangle + \beta\,|\mathrm{n}\rangle$$

  Because $|\alpha|^2$ is the probability to find the system in a $|p\rangle$ state and $|\beta|^2$ the same for the $|n\rangle$ state we must have, for any state $|\psi\rangle$,

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

- We have seen already that a transformation $|\psi'\rangle = U|\psi\rangle$ must preserve the norm so that U must be unitary: $U^\dagger U = 1$.

- Taking determinants we find

$$\det(U^\dagger U) = \det(U^\dagger)\,\det(U) = \det(U)^*\,\det(U) = 1$$

  Therefore $\det(U) = e^{i\phi}$ with ϕ some arbitrary phase factor.

- So we may set $U = e^{i\phi}V$ with $\det(V) = 1$. Invariance for phase shifts is called a U(1) invariance and leads to charge conservation, as we will see later. The charge conserved in the p-n case here is not electrical charge, but **baryon number**

$$A = (N_{\mathrm{p}} - N_{\bar{\mathrm{p}}}) + (N_{\mathrm{n}} - N_{\bar{\mathrm{n}}})$$

- Putting U(1) invariance aside, we have to deal with unitary 2×2 matrices V with unit determinant, that is, with the group SU(2).

[18] When we talk about quarks we will use the notation $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ instead.

The group SU(2)

- The mathematics of SU(2) is well known from the treatment of ordinary spin in quantum mechanics. A transformation can be written as $U = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{I})$ with the three generators $\boldsymbol{I} \equiv \boldsymbol{\tau}/2$ given by the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  These generators are clearly Hermitian ($\tau_i^\dagger = \tau_i$), as they should be, since U is unitary. It can be shown (Exercise 2.3) that, quite in general, $\det[\exp(A)] = \exp[\mathrm{Tr}(A)]$ so that the traces of the τ_i vanish because the SU(2) transformations have unit determinant.

> The generators of a unitary matrix group with unit determinant are Hermitian and traceless

- By matrix multiplication you may check the commutation relations

$$[I_i, I_j] = i\,\epsilon_{ijk} I_k$$

  with ϵ_{ijk} the antisymmetric tensor (+1 for cyclic permutations of 123 and −1 for cyclic permutations of 213, zero otherwise).

- SU(2) has one so-called **Casimir operator** that commutes with all the generators, and is always some non-linear function of the generators. For SU(2) this is the total isospin operator:

$$I^2 = I_1^2 + I_2^2 + I_3^2$$

  A state can then be a simultaneous eigenstate[19] of I^2 with eigenvalue $i(i+1)$, $i = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and of I_3 with eigenvalue $m = -i, \ldots, +i$. The eigenvalues label the state, like $|\psi\rangle = |i, m\rangle$.

[19] A Hermitian matrix has the property that it can always be diagonalised by a unitary transformation. Hermitian matrices can be *simultaneously* diagonalised by a single transformation if they commute.

Exercise 2.3:

In this exercise we will review a few easy-to-prove properties of matrices and of **matrix transforms** (also called **similarity transforms**) defined by

$$A' = SAS^{-1},$$

where S is a non-singular transformation matrix. Such transforms can come in very handy in a calculation because they allow you to transform matrices to convenient forms, such as a transformation to diagonal form which is used for the proof in (e) below.

(a) [0.1] Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

(b) [0.2] Show that a matrix transform preserves the algebra of a Lie group. Representations that are related by similarity transformations are therefore called **equivalent**.

(c) [0.2] Show that a matrix transform preserves the product, determinant and trace, that is,

$$(AB)' = A'B', \quad \det(A') = \det(A) \quad \text{and} \quad \mathrm{Tr}(A') = \mathrm{Tr}(A).$$

What about Hermitian conjugation: $(A')^\dagger \stackrel{?}{=} (A^\dagger)'$.

(d) [0.2] Show that a matrix transform preserves the terms in a power series, that is,

$$(A^n)' = (A')^n \quad \rightarrow \quad (\exp A)' = \exp(A').$$

(e) [0.3] Now show that

$$\det[\exp(A)] = \exp[\mathrm{Tr}(A)]$$

for all matrices A that can be brought into diagonal form.

Exercise 2.4:

(a) [0.5] Show that $\tau_i \tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k$. Together with the fact that the τ are Hermitian, we thus have $\tau_i^\dagger = \tau_i = \tau_i^{-1}$.

(b) [0.5] Now show that $(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\tau}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ and, from this, that $(\boldsymbol{\theta}\cdot\boldsymbol{\tau})^2 = |\boldsymbol{\theta}|^2$.

(c) [0.5] Use the above, and the Taylor expansions of exp(), sin() and cos(), to show that $\exp(i\boldsymbol{\theta}\cdot\boldsymbol{\tau}) = \cos|\boldsymbol{\theta}| + i(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\tau})\sin|\boldsymbol{\theta}|$. Here $\hat{\boldsymbol{\theta}}$ is the unit vector along $\boldsymbol{\theta}$.

(d) [0.25] Instead of $|\mathrm{p}\rangle$ and $|\mathrm{n}\rangle$ we will write $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ to reflect isospin symmetry on the quark level. Verify that

$$I_3\,|\mathrm{u}\rangle = \tfrac{1}{2}|\mathrm{u}\rangle, \quad I_3\,|\mathrm{d}\rangle = -\tfrac{1}{2}|\mathrm{d}\rangle$$

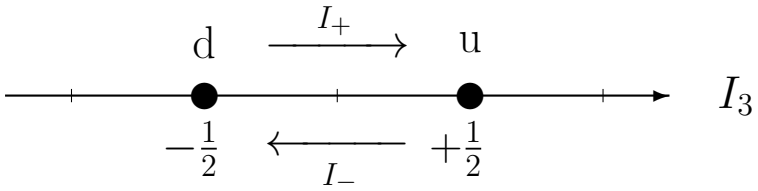
and that the Casimir operator $I^2 = I_1^2 + I_2^2 + I_3^2$ is a multiple of the unit operator, with

$$I^2\,|\mathrm{u}\rangle = \tfrac{3}{4}\,|\mathrm{u}\rangle, \qquad I^2\,|\mathrm{d}\rangle = \tfrac{3}{4}\,|\mathrm{d}\rangle$$

(e) [0.25] Define the step operators $I_\pm = I_1 \pm i\,I_2$ and verify that

$$I_+|\mathrm{u}\rangle = 0, \qquad I_+|\mathrm{d}\rangle = |\mathrm{u}\rangle, \qquad I_-|\mathrm{u}\rangle = |\mathrm{d}\rangle, \qquad I_-|\mathrm{d}\rangle = 0$$

We can now draw a, kind of trivial, **weight diagram** like



Composite states

- The rules for addition of angular momenta from quantum mechanics carry straight over to the addition of isospins. We will not derive here the mathematics but will only indicate how it works.

- Addition of two states $|i_1, m_1\rangle$ and $|i_2, m_2\rangle$, results in $(2i_1 + 1) \times (2i_2 + 1)$ different states which can be classified according to the eigenvalue label i of the Casimir operator I^2 which ranges from $|i_1 - i_2|$ to $i_1 + i_2$, and the eigenvalues m of the I_3 operator that, for each state i, range from $-i$ to $+i$. Here $m = m_1 + m_2$. Formally, the combined state can be written as

$$|i, m\rangle = \sum \langle i_1, i_2, m_1, m_2 | i, m\rangle |i_1, m_1\rangle |i_2, m_2\rangle$$

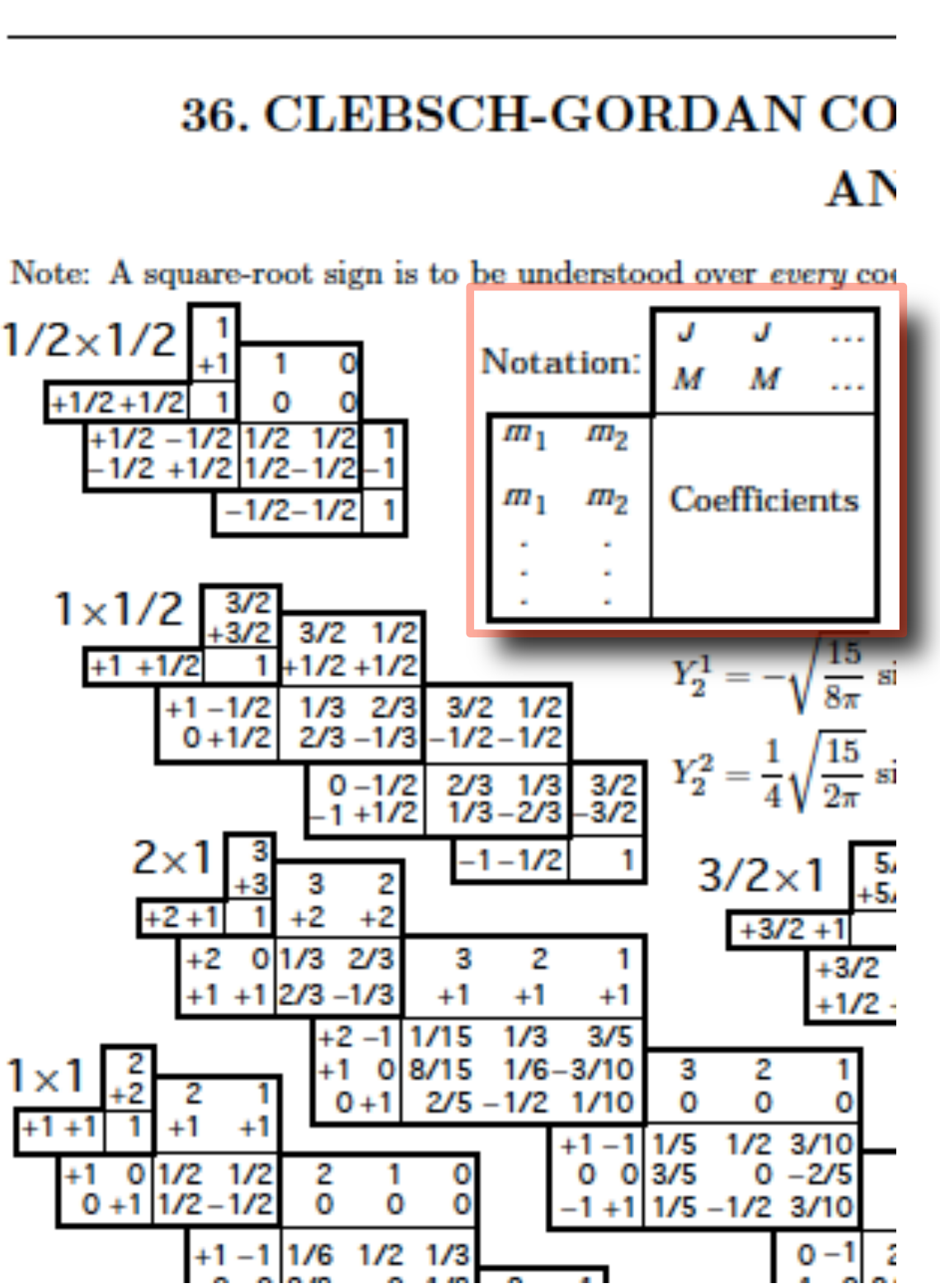
  The Clebsch-Gordan coefficients $\langle \cdot | \cdot \rangle$ can be found in the Particle Data Book $\rightarrow$ **Fig**. For a nucleon-nucleon system we get

$$\begin{aligned} |I, I_3\rangle &= |0, \quad 0\rangle &&= (\mathrm{pn} - \mathrm{np})/\sqrt{2} \\ &= |1, \quad 1\rangle &&= \mathrm{pp} \\ &= |1, \quad 0\rangle &&= (\mathrm{pn} + \mathrm{np})/\sqrt{2} \\ &= |1, -1\rangle &&= \mathrm{nn} \end{aligned}$$

- **<u>Exercise 2.5</u>**: [1.0] Use exchange symmetry arguments or the step operators $I_\pm \equiv I_\pm^{(1)} + I_\pm^{(2)}$ to justify the decomposition above.[20] <u>Hint</u>: See H&M Exercise 2.1.

- This splitting of the combination of two 2-component states into a singlet and a triplet state is often written as $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. The significance of such a decomposition is that under a SU(2) transformation the substates of the **1** and **3** representation will transform among themselves.

[20] In full, the step operator is defined by $I_\pm |i, m\rangle = \sqrt{i(i+1) - m(m \pm 1)}\, |i, m \pm 1\rangle$.



Clebsch-Gordan coefficients from the Particle Data Book. Given in the tables is the *square* of the coefficients, so you should take the square root.

$\mathbf{SU(2)_f}$ for antiquarks

- If $|\psi\rangle$ is a particle state then the complex conjugate is identified with the corresponding antiparticle state:[21] $|\bar{\psi}\rangle \equiv |\psi\rangle^*$. An antiquark state therefore transforms in the complex conjugate representation of SU(2), denoted by $\mathbf{2}^*$ or $\bar{\mathbf{2}}$.

  $$|\bar{\psi}'\rangle = U^*|\bar{\psi}\rangle = \exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}^*/2)\,|\bar{\psi}\rangle \equiv \exp(i\boldsymbol{\alpha}\cdot\bar{\boldsymbol{\tau}}/2)\,|\bar{\psi}\rangle$$

  The two representations are thus related by $\bar{\boldsymbol{\tau}} = -\boldsymbol{\tau}^*$.

- To combine a quark with an antiquark we could calculate from scratch the Clebsch-Gordan coefficients of $\mathbf{2}\otimes\bar{\mathbf{2}}$ but we can save us the effort by using a trick that, by the way, only works for SU(2).

- Just replace $\bar{\mathrm{u}}$ by $-\bar{\mathrm{d}}$ and $\bar{\mathrm{d}}$ by $\bar{\mathrm{u}}$ in $|\bar{\psi}\rangle$, that is, define

  $$|\tilde{\psi}\rangle \equiv C|\bar{\psi}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix} = \begin{pmatrix} -\bar{\mathrm{d}} \\ \bar{\mathrm{u}} \end{pmatrix}$$

  It is now straight-forward to show (Exercise 2.7) that $|\tilde{\psi}\rangle$ transforms as a quark state $|\tilde{\psi}'\rangle = U|\tilde{\psi}\rangle$ so that we just can use the Clebsch-Gordans of the $\mathbf{2}$ representation.[22]

- **Exercise 2.6**: [×] Take the $|qq\rangle$ states given on page 2–19 (substitute u for p and d for n), to arrive at $|q\bar{q}\rangle$ meson states that properly transform under SU(2):

  $$\begin{aligned}
  \omega &= |0,\ 0\rangle &&= (\mathrm{u}\bar{\mathrm{u}} + \mathrm{d}\bar{\mathrm{d}})/\sqrt{2} \\
  \pi^+ &= |1,\ 1\rangle &&= -\mathrm{u}\bar{\mathrm{d}} \\
  \pi^0 &= |1,\ 0\rangle &&= (\mathrm{u}\bar{\mathrm{u}} - \mathrm{d}\bar{\mathrm{d}})/\sqrt{2} \\
  \pi^- &= |1,-1\rangle &&= \mathrm{d}\bar{\mathrm{u}}
  \end{aligned}$$

[21] We use here $|\bar{\psi}\rangle$ to indicate an antiparticle; please do not confuse it with a conjugate Dirac spinor $\overline{\psi}$.

[22] In fact, for SU(2) the generators $\bar{\tau}_i$ and τ_i are related by the similarity transformation $\bar{\tau}_i = C^{-1}\tau_i C$ so that they are equivalent, that is, they are not regarded as different representations, see also Exercise 2.3.

Exercise 2.7:

(a) [1.0] Use isospin invariance to show that the ratio

$$\frac{\sigma(\mathrm{pp} \to \pi^+ \mathrm{d})}{\sigma(\mathrm{pn} \to \pi^0 \mathrm{d})} = 2$$

Here the deuteron has isospin $I = 0$ and the pion isospin $I = 1$. You may assume that the cross section is

$$\sigma \sim |\text{amplitude}|^2 = \sum_I |\langle I', I_3'|A|I, I_3\rangle|^2 = A^2 \sum_I |\langle I', I_3'|I, I_3\rangle|^2 .$$

Hint: See H&M Exercise 2.3.

(b) [0.2] Show that the generators $\bar{\boldsymbol{\tau}}$ are a representation of SU(2).

(c) [$\times$] Verify that $I_3(\bar{\mathrm{u}}) = -\frac{1}{2}$ and $I_3(\bar{\mathrm{d}}) = +\frac{1}{2}$.

(d) [0.3] Show that

$$|\tilde{\psi}\rangle = C|\bar{\psi}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix}$$

transforms as a particle state.

The group $\mathbf{SU(3)_f}$ I

- To accommodate strange quarks, our space has to be extended

$$\text{from} \quad \begin{pmatrix} \text{u} \\ \text{d} \end{pmatrix} \qquad \text{to} \quad \begin{pmatrix} \text{u} \\ \text{d} \\ \text{s} \end{pmatrix}$$

- Like in the (iso)spin case we can write a unitary transformation as

$$|\psi'\rangle = U|\psi\rangle = \exp(i\boldsymbol{a}\cdot\boldsymbol{\lambda}/2)\,|\psi\rangle \equiv \exp(i\boldsymbol{a}\cdot\boldsymbol{T})\,|\psi\rangle$$

  but the generators $\boldsymbol{\lambda}$ are now Hermitian 3×3 matrices. A complex 3×3 matrix is characterised by 18 numbers but only 8 are independent because the matrices are Hermitian, and traceless since $\det U = 1$. Thus there are 8 independent generators.

- The 8 **Gell-Mann matrices** (with Pauli matrices inside!) are

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_1} \quad \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_2} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_3} \quad \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{\lambda_4}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}}_{\lambda_5} \quad \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\lambda_6} \quad \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{\lambda_7} \quad \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\lambda_8}$$

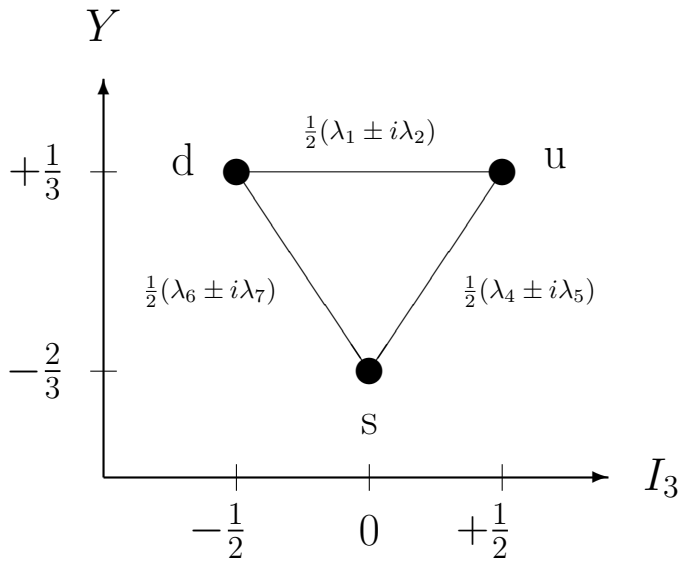
- The algebra of the SU(3) group is given by the commutation relation of the matrices $T_a = \lambda_a/2$: $\quad [T_a, T_b] = i f_{ab}^c T_c$

The group SU(3)$_{\rm f}$ II

- The structure constants f^c_{ab} are antisymmetric in the exchange of two indices (see Exercise 2.8); the non-zero ones are

$$\begin{aligned} f^3_{12} &= 1 \\ f^7_{14} = f^5_{16} = f^6_{24} = f^7_{25} = f^5_{34} = f^6_{37} &= \tfrac{1}{2} \\ f^8_{45} = f^8_{67} &= \tfrac{1}{2}\sqrt{3} \end{aligned}$$

- It is seen that λ_3 and λ_8 are simultaneously diagonal so that we can label quark states by the simultaneous eigenvalues of the **isospin** operator $T_3 = \lambda_3/2$ and the **hypercharge** operator $Y = 2T_8/\sqrt{3} = \lambda_8/\sqrt{3}$. This gives rise to following weight diagram for the quark states (see Exercise 2.8 for antiquarks):



- As mentioned on page 2–16 there is one Casimir operator for SU(2), but there are two Casimirs for SU(3). By definition, these commute with all the λ_i. One of them is is the total 'isospin' operator $\sum \lambda_i^2$ while the other is a rather complicated trilinear function of the λ_i which can be found in A&H-II, Appendix M.5.

Exercise 2.8:

(a) [0.5] The λ matrices are normalised such that $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Check this for a few matrices λ_a and λ_b.

(b) [0.5] Show that $\mathrm{Tr}(\lambda_c[\lambda_a, \lambda_b]) = 4if^c_{ab}$. By changing the order of the λ, and using $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, show that the structure constants f^c_{ab} are antisymmetric in the exchange of two indices.

(c) [0.5] Plot the eigenvalues of the isospin and hypercharge operator for the u, d and s quarks in an I_3-Y diagram. Check the Gell-Mann Nishijima formula $Q = I_3 + \frac{1}{2}Y$ and also that $Y = S + B$. Repeat the exercise for antiquarks in the $\bar{\mathbf{3}}$ representation.

(d) [0.5] Write down the matrices for the step operators $\frac{1}{2}(\lambda_1 \pm i\lambda_2)$, $\frac{1}{2}(\lambda_4 \pm i\lambda_5)$ and $\frac{1}{2}(\lambda_6 \pm i\lambda_7)$ and justify their position in the weight diagram on page 2–24.

Exercise 2.9: The adjoint representation of SU(3)

- We have encountered the algebra of the groups SU(2) and SU(3) in terms of the two-dimensional Pauli matrices and the three-dimensional Gell-Mann matrices, respectively. These matrices are, together with the 2- or 3-dim vectors on which they act, called the **fundamental representation** of SU(2) or SU(3).
- However, the structure constants of a Lie group automatically generate a representation with a dimension that is equal to the number of generators, *e.g.* 8×8 for SU(3). This is called the **adjoint representation**. Below we let you find out how this works.

(a) [×] Verify the **Jacobi identity** for matrices A, B and C:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

(b) [×] Now show that in terms of the SU(3) structure constants the Jacobi identity reads

$$f_{ij}^m \, f_{mk}^n + f_{jk}^m \, f_{mi}^n + f_{ki}^m \, f_{mj}^n = 0$$

(c) [×] Verify that $f_{ij}^k = -f_{ji}^k$

(d) [1.0] Define the 8×8 matrices C_i with elements

$$(C_i)_j^k = -f_{ij}^k$$

and show that the C_i obey the SU(3) algebra

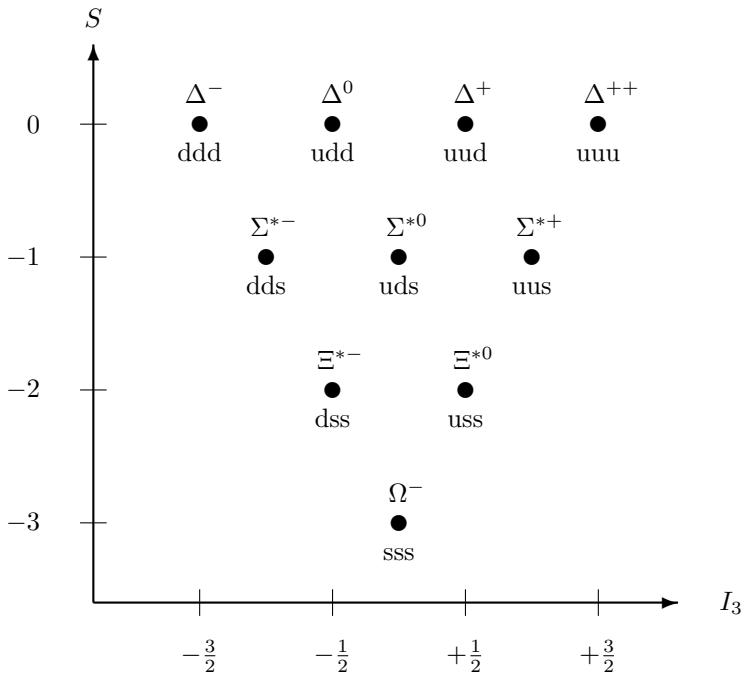
$$[C_i, C_j] = f_{ij}^k C_k$$

In this way, we have constructed the adjoint representation of SU(3) from its structure constants. We will see later that coloured quarks are described by the fundamental representation of SU(3), of dimension 3, and gluons by the adjoint representation, of dimension 8.

The Eightfold Way

- Because our interest in SU(3) lies in the fact that it is an exact (colour) symmetry of QCD, we will not present here how $\mathrm{SU}(3)_\mathrm{f}$ is used to classify the hadrons (the Eightfold Way). This is treated in great detail in H&M Chapter 2, and also in Griffiths Chapter 5.
- We just mention that the mesons $|q\bar{q}\rangle$ can be grouped into octets and singlets ($\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$) and baryons $|qqq\rangle$ can be grouped into decuplets, octets and singlets ($\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$).
- Nevertheless, let us have a look at the spin 3/2 baryon decuplet, because it provides us with an argument to introduce the colour quantum number.

The need for a colour quantum number



- In this spin 3/2 baryon decuplet, the flavour wave functions at the corners are obviously symmetric under the exchange of two quarks. Although this is not apparent from the labels, all wave functions of the decuplet are symmetric, as you will discover in Exercise 2.10.
- But now we have a problem: the *total* wave function

$$\psi = \psi_{\text{space}}(L=0) \;\times\; \psi_{\text{spin}}(\uparrow\uparrow\uparrow) \;\times\; \psi_{\text{flavour}}(q_1 q_2 q_3)$$

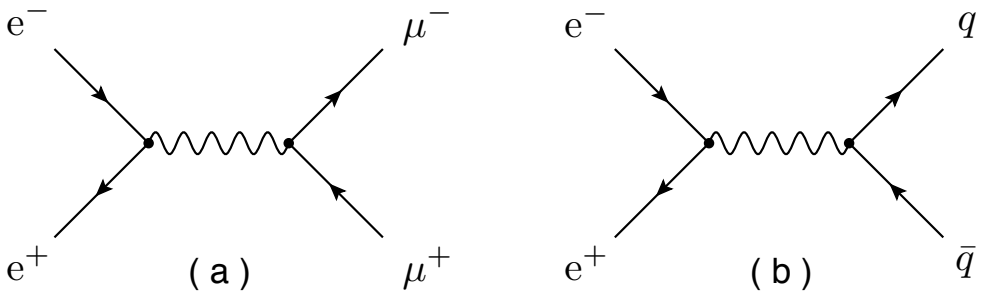
  is symmetric under the exchange of two quarks, while it should be anti-symmetric, since baryons are fermions (half-integer spin).
- The solution is to assign a 'colour' quantum number (r, g, b) to each quark so that the quarks can be distinguished by their colour, provided, of course, that we do not allow two quarks in a baryon to have the same colour. Thus the *three* colours are always present and we say that baryons are 'white', or **colour singlets** (= invariant under $\mathrm{SU(3)_c}$ transformations). By anti-symmetrising the wave function in colour space, over-all anti-symmetry is established.

Exercise 2.10:

(a) [0.5] Use the step operators defined in the weight diagram on page 2–24 (and also in Exercise 2.8d) to generate all quark states of the baryon decuplet, starting from one of the corner states (ddd), (uuu) or (sss). You will not obtain the correct normalisation in this way, but that is not so important here (you can always normalise the wave functions afterwards, if you wish). The point of this exercise is to note that all wave functions that you obtain by stepping through the diagram are *symmetric* in the exchange of two quarks.

(b) [0.5] Construct a wave function $\psi_{\text{colour}}(c_1, c_2, c_3)$ that is fully antisymmetric in the exchange of two colours.

Experimental evidence for colour I



- The cross section for the left diagram is given in PP-I section 8.3:

$$\sigma(\mathrm{e^+e^-} \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

  Here particle masses are neglected and if we do the same for the right diagram, we obtain the cross section for $q\bar{q}$ production simply by putting the correct charge at the $\gamma q\bar{q}$ vertex

$$\sigma(\mathrm{e^+e^-} \to q_i\ \bar{q}_i) = \frac{4\pi\alpha^2 e_i^2}{3s}$$

- Because quarks fragment with 100% probability into hadrons, we can sum over all available quark species to get the observable

$$\sigma(\mathrm{e^+e^-} \to \mathrm{hadrons}) = N_\mathrm{c} \sum_i \frac{4\pi\alpha^2 e_i^2}{3s}$$

- Here the sum runs over all quark flavours that can be produced at a given energy $\sqrt{s}$, and N_c counts the number of coloured duplicates of each quark. Thus $N_\mathrm{c} = 3$ for the quark colours q_r, q_g and q_b.

Experimental evidence for colour II

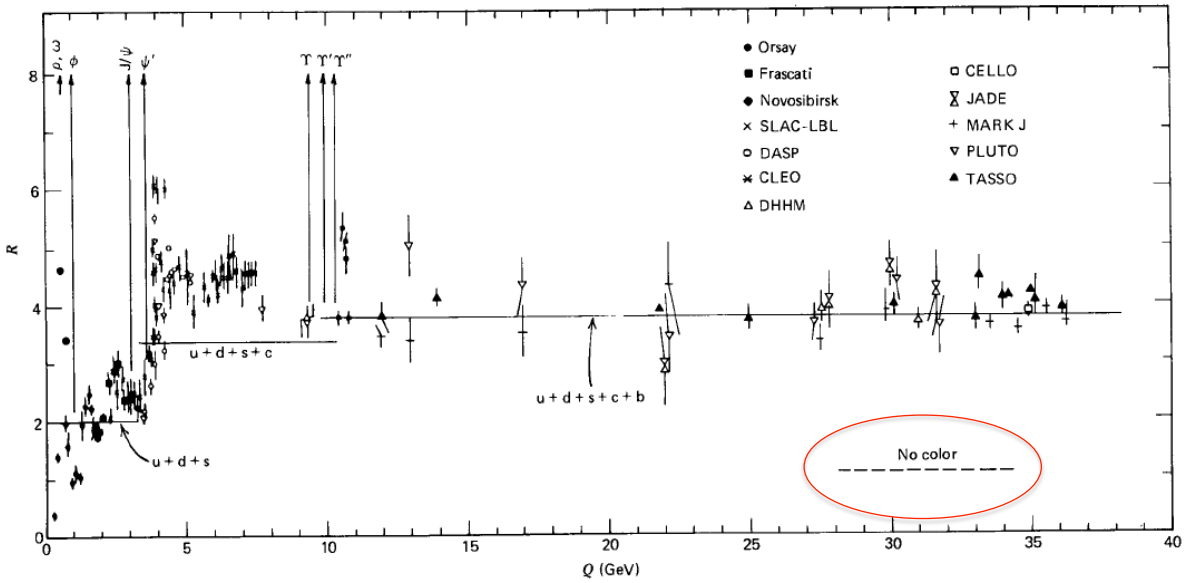


Fig. 11.3 Ratio R of (11.6) as a function of the total $e^- e^+$ center-of-mass energy. (The sharp peaks correspond to the production of narrow 1^- resonances just below or near the flavor thresholds.)

- This plot shows, as a function of $\sqrt{s}$, measurements of the ratio

$$R = \frac{\sigma(\mathrm{e^+e^-} \to \text{hadrons})}{\sigma(\mathrm{e^+e^-} \to \mu^+\mu^-)} = N_\mathrm{c} \sum_i e_i^2 = 3 \sum_i e_i^2$$

- The data are consistent with $N_\mathrm{c} = 3$ and certainly exclude $N_\mathrm{c} = 1$.
- Remark: There is quite some structure in this plot, in particular around the thresholds of heavy quark production where $q\bar{q}$ pairs are produced with little relative momentum so that they can form bound states, like the J/ψ family ($\mathrm{c\bar{c}}$) at about 3 GeV, and the Υ family ($\mathrm{b\bar{b}}$) at about 10 GeV.

Lecture notes Particle Physics II

Quantum Chromo Dynamics

3. U(1) Local Gauge Invariance

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November 16, 2013

Electric charge conservation

- In subatomic physics it is customary to express electric charge in units of the **elementary charge** $e = 1.6 \times 10^{-19}$ Coulomb. The electron then has charge -1, the positron $+1$, the up quark $+\frac{2}{3}$, the down quark $-\frac{1}{3}$, *etc.*, see the table on Page 1–5.
- As far as we know, total electric charge is the same in the initial and final state of any elementary reaction, and this **charge conservation** is experimentally verified to great accuracy.
- For instance electron decay

  $$\mathrm{e} \to \gamma\, \nu_{\mathrm{e}}$$

  is allowed by all known conservation laws but is forbidden by charge conservation and it indeed has never been observed. In fact, the life time of the electron is measured to be larger than 5×10^{26} years.

- We have seen that conserved quantities are related to symmetries in the Hamiltonian, or the Lagrangian, so the question is now which symmetry causes this charge conservation. Charge is obviously an *additive* conserved quantity so that the symmetry transformation must be *continuous*.
- The answer, as we will see, is that a so-called **gauge symmetry** is responsible for the charge conservation. Gauge transformations enter when interactions are described in terms of potentials, instead of forces. A well known example is from classical electrodynamics where we can transform the scalar and vector potentials in such a way that the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are unaffected.

Gauge transformation in electrodynamics

- In electrodynamics the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are related to the scalar and vector potentials V and $\boldsymbol{A}$ by

$$\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \nabla V \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- A gauge transformation leaves the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields invariant

$$V' = V - \partial \Lambda/\partial t \qquad \boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda$$

  Here $\Lambda(\boldsymbol{x}, t)$ is an arbitrary function of $\boldsymbol{x}$ and t.

- To this gauge transformation corresponds a unitary operator that transforms the wave function of a particle in an electromagnetic field. We can write this transformation as (see page 2–7)

$$|\psi\rangle' = \exp(i\epsilon G)|\psi\rangle$$

  where the generator G is to be identified later. Since Λ is an arbitrary function of $\boldsymbol{x}$ and t we require that ϵ is also an arbitrary function of $\boldsymbol{x}$ and t. Because ϵ can vary in space-time, we speak of a **local gauge transformation**.

- Now consider the Schrödinger equation of a particle in a static electric field before and after our gauge transformation

$$\begin{aligned} i\frac{\partial|\psi\rangle}{\partial t} &= \left(-\frac{\nabla^2}{2m} + q\,V\right)|\psi\rangle \\ i\frac{\partial|\psi\rangle'}{\partial t} &= \left(-\frac{\nabla^2}{2m} + q\,V'\right)|\psi\rangle' \end{aligned}$$

  Here q is the charge of the particle.

- Because of **gauge invariance**, both equations should apply and this fixes the generator G, as we will now show.

From local gauge invariance to charge conservation

- Let us work out the transformed Schrödinger equation (for clarity we write ψ instead of $|\psi\rangle$). To simplify the mathematics we will take ϵ to be a function of t only, instead of $\boldsymbol{x}$ and t:

$$\begin{aligned}
i\frac{\partial}{\partial t}\left(e^{i\epsilon G}\,\psi\right) &= \left(-\frac{\nabla^2}{2m} + q\,V - q\frac{\partial\epsilon}{\partial t}\right) e^{i\epsilon G}\,\psi \\
ie^{i\epsilon G}\left(iG\,\psi\,\frac{\partial\epsilon}{\partial t} + \frac{\partial\psi}{\partial t}\right) &= e^{i\epsilon G}\left(-\frac{\nabla^2}{2m} + q\,V\right)\psi - e^{i\epsilon G}\,q\,\psi\,\frac{\partial\epsilon}{\partial t} \\
-e^{i\epsilon G}\,G\,\psi\,\frac{\partial\epsilon}{\partial t} + ie^{i\epsilon G}\,\frac{\partial\psi}{\partial t} &= ie^{i\epsilon G}\,\frac{\partial\psi}{\partial t} - e^{i\epsilon G}\,q\,\psi\,\frac{\partial\epsilon}{\partial t} \\
-e^{i\epsilon G}\,G\,\psi\,\frac{\partial\epsilon}{\partial t} &= -e^{i\epsilon G}\,q\,\psi\,\frac{\partial\epsilon}{\partial t} \\
G\,\psi &= q\,\psi
\end{aligned}$$

- We find that G is the charge operator Q! This is due to the cancellations that occur because ϵ is *local* (*i.e.* a function of t in our derivation); all this would not work if ϵ would be a constant.
- Clearly if H and Q commute, then it follows that the expectation value $\langle Q\rangle$ is conserved, in other words, charge is conserved.
- It is straight-forward to extend the derivation above to local transformations that depend on both $\boldsymbol{x}$ and t, instead of on t alone, but we will not do this here since it brings a lot of additional algebra and is not very illuminating.
- The family of phase transformations $U(\alpha) \equiv e^{i\alpha}$, with real α, forms a unitary Abelian group called U(1). Phase invariance is therefore also known as U(1) invariance.

Lagrangian formalism

- Gauge theories, or field theories in general, are usually defined in terms of a **Lagrangian**. This is a well-known concept from classical mechanics; a brief summary can be found on page 0–7.
- In classical mechanics the Lagrangian is the difference between the kinetic and potential energy and is written as the function $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of a set of N coordinates q_i and velocities $\dot{q}_i$ that fully describe the system at any instant t. N is called the number of degrees of freedom of the system.
- The **action** is defined by

$$S[\text{path}] = \int_{t_1}^{t_2} \mathrm{d}t \; L(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

  where the integral is taken along some path from $\boldsymbol{q}(t_1)$ to $\boldsymbol{q}(t_2)$.
- The **principle of least action** states that the system will evolve along the path that minimises the action. The equations of motion then follow from the **Euler-Lagrange** equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, N$$

- Example: Mass m in a central potential $V(r)$

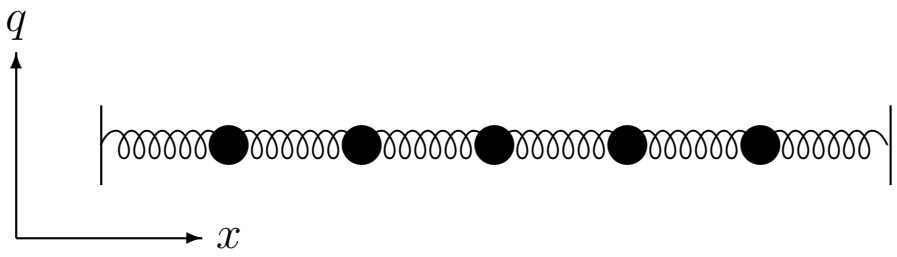
$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \tfrac{1}{2} m \dot{\boldsymbol{r}}^2 - V(r) \quad \rightarrow \quad m\ddot{\boldsymbol{r}} = -\nabla V(r)$$

- Example: Harmonic oscillator

$$L(x, \dot{x}) = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 \quad \rightarrow \quad m\ddot{x} = -kx$$

Infinite number of degrees of freedom

- Consider small *transverse* vibrations of a system of N masses m connected by springs.



  The state of this system is described by the vertical deviations $q_1(t), \ldots, q_N(t)$ from the equilibrium position.

- We can let $N \to \infty$ in such a way that we obtain a vibrating string that can be described by a function $q(x,t)$.
- Such a function is called a **field**, a *displacement field* in this case.
- For our field, the Lagrangian is a function of q, $\dot{q}$, and the gradient $\mathrm{d}q/\mathrm{d}x$, and is written as the integral of a **Lagrangian density**

$$L(q, \dot{q}, \mathrm{d}q/\mathrm{d}x) = \int \mathrm{d}x \; \mathcal{L}(q, \dot{q}, \mathrm{d}q/\mathrm{d}x)$$

  Generalising to 3 dimensions, the action integral reads

$$S[\text{path}] = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \; \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \nabla \boldsymbol{q})$$

- In 4-vector notation this gives for the action integral of a field $\phi(x^\mu)$

$$S[\text{path}] = \int \mathrm{d}^4x \; \mathcal{L}(\phi, \partial_\mu \phi)$$

- In this notation the Euler-Lagrange equation reads

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

Exercise 3.1:

The Lagrangian of a vibrating string is:

$$\mathcal{L} = (\partial\phi/\partial t)^2 - (\partial\phi/\partial x)^2.$$

(a) [0.25] Write this Lagrangian in 4-vector notation.

(b) [0.25] Now use the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

to derive the wave equation of a vibrating string.

Remark: When you have to derive a field equation from a Lagrangian but do not feel confident in manipulating upper and lower Lorentz indices to keep track of the signs, you can always resort to writing it all out into the components (t, x, y, z). This is elaborate, but it works. Here is the conversion of the derivative indices

$$(\partial_0, \partial_1, \partial_2, \partial_3) = (\partial^0, -\partial^1, -\partial^2, -\partial^3) = (\partial_t, \partial_x, \partial_y, \partial_z)$$

And here is that of four-vector fields A, if present

$$(A^0, A^1, A^2, A^3) = (A_0, -A_1, -A_2, -A_3) = (A_t, A_x, A_y, A_z)$$

You may find it useful to also make conversion tables for $F^{\mu\nu}$ and $F_{\mu\nu}$.

A few Lagrangians ...

- Here are a few well-known Lagrangians that yield—via the E-L equations—several field equations of interest.
- Klein-Gordon Lagrangian for a real scalar field (spin 0).

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 \quad \underset{\text{E–L}}{\rightarrow} \quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

- KG for a complex scalar field (take ϕ and ϕ^* as independent).

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi \quad \underset{\text{E–L}}{\rightarrow} \quad \begin{cases} \partial_\mu\partial^\mu\phi + m^2\phi = 0 \\ \partial_\mu\partial^\mu\phi^* + m^2\phi^* = 0 \end{cases}$$

- Dirac Lagrangian for a spin $\frac{1}{2}$ spinor field (ψ and $\overline{\psi}$ independent).

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \quad \underset{\text{E–L}}{\rightarrow} \quad \begin{cases} (i\gamma^\mu\partial_\mu - m)\psi = 0 \\ (i\gamma^\mu\partial_\mu + m)\overline{\psi} = 0 \end{cases}$$

- Proca Lagrangian for a vector field (spin 1).

$$\mathcal{L} = -\tfrac{1}{4}(F^{\mu\nu})(F_{\mu\nu}) + \tfrac{1}{2}m^2A^\nu A_\nu \quad \underset{\text{E–L}}{\rightarrow} \quad \partial_\mu F^{\mu\nu} + m^2A^\nu = 0,$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

For massless fields we recover the Maxwell equations in empty space (no sources or currents)

$$\partial_\mu F^{\mu\nu} = 0.$$

Exercise 3.2:

(a) [1.0] Derive the field equations from the KG, complex and Dirac Lagrangians given on page 3–9.

(b) [1.0] The Proca Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}(F^{\mu\nu})(F_{\mu\nu}) + \tfrac{1}{2}m^2 A^\nu A_\nu$$

The field tensor is defined by $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

- Show that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) = -F^{\mu\nu}$$

  Hint: Work this out for two components, $(\mu = 0, \nu = 1)$ and $(\mu = 1, \nu = 2)$, for instance, and then generalise to the result above.[23] Remember that $\partial^\mu = (\partial_t, -\nabla)$ and $\partial_\mu = (\partial_t, +\nabla)$.

- Show that

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu$$

- Now write down the field equation.

(c) [0.5] The Maxwell Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu$$

- Show that the Euler-Lagrangian equation leads to the Maxwell equations (see page 0–6 for the Maxwell equations in 4-vector notation):

$$\partial_\mu F^{\mu\nu} = j^\nu$$

- Show that the current is conserved: $\partial_\mu j^\mu = 0$.

[23] For a shorter (but more tricky) derivation see H&M, comment on Exercise 14.3 and 14.4, page 374.

Global phase invariance of the Dirac Lagrangian

- The Dirac Lagrangian $i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$ is manifestly invariant under a global phase change $\psi' = e^{i\alpha}\,\psi$ and $\overline{\psi}' = e^{-i\alpha}\,\overline{\psi}$.

- According to **Noether's theorem** this implies the existence of a conserved quantity. To find out what this is, consider the infinitesimal transformation

$$\begin{aligned} \psi' = (1+i\alpha)\psi \quad &\rightarrow \quad \delta\psi = +i\alpha\,\psi \\ \overline{\psi}' = (1-i\alpha)\overline{\psi} \quad &\rightarrow \quad \delta\overline{\psi} = -i\alpha\,\overline{\psi} \end{aligned}$$

- The variation in $\mathcal{L}$ is

$$\begin{aligned} \delta\mathcal{L} &= \tfrac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\delta\partial_\mu\psi + \tfrac{\partial\mathcal{L}}{\partial\overline{\psi}}\,\delta\overline{\psi} + \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\psi}}\,\delta\partial_\mu\overline{\psi} \\ &= i\alpha\left[\tfrac{\partial\mathcal{L}}{\partial\psi}\,\psi + \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\partial_\mu\psi - \tfrac{\partial\mathcal{L}}{\partial\overline{\psi}}\,\overline{\psi} - \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\psi}}\,\partial_\mu\overline{\psi}\right] \\ &= i\alpha\left[\left(\tfrac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\,\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right)\psi + \left(\partial_\mu\,\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right)\psi + \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\partial_\mu\psi - \cdots\right] \end{aligned}$$

  Now the first term in brackets is zero (Euler-Lagrange) and the next two terms combine into

$$\left(\partial_\mu\,\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right)\psi + \tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\partial_\mu\psi = \partial_\mu\left(\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\psi\right)$$

  The same is true for the $\overline{\psi}$ terms so that we obtain

$$\delta\mathcal{L} = i\alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\psi - \overline{\psi}\,\frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\psi}}\right) \propto \partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right) \underset{\text{Iwant}}{=} 0$$

- Phase invariance leads to (electric) current conservation!

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = q\,\overline{\psi}\gamma^\mu\psi \qquad (q \text{ is the electric charge})$$

Local charge conservation

- We have seen that global phase invariance leads to the continuity equation $\partial_\mu j^\mu = 0$ which reads in 3-vector notation

$$\frac{\partial \rho}{\partial t} = -\nabla \boldsymbol{j}$$

- The meaning of this continuity equation becomes clear after integration over a volume V

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = -\int_V \nabla \boldsymbol{j} \,\mathrm{d}V = -\int_S \boldsymbol{j}\cdot\hat{\boldsymbol{n}} \,\mathrm{d}S$$

  which tells us that the change of charge in some volume should be accounted for by the net flow of charge in or out of that volume. However, we can make this volume as small as we please because we know that charge is really *locally* conserved. Indeed, as we have already mentioned on page 3–3, the decay

$$\mathrm{e} \to \gamma\, \nu_\mathrm{e}$$

  has never been observed since it violates charge conservation. The electron is a point charge, so we cannot get more local than this!

- Local charge conservation suggests that the Lagrangian should not only be invariant under *global* phase transformations but also under *local* ones:

$$\psi' = e^{i\alpha(x)}\,\psi$$

- On Page 3–4 we have already investigated local phase invariance of the Schrödinger equation of a particle in a static electric field, but let us now investigate what happens when this local invariance is imposed on the Dirac Lagrangian.

Local phase invariance

- Take the Dirac Lagrangian

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

and consider a *local* transformation

$$\psi'(x) = e^{-ig_e\alpha(x)}\,\psi(x)$$

where we have introduced a strength parameter g_e (the electromagnetic coupling constant).

- The second term in $\mathcal{L}$ is clearly invariant but not the first term. This is because $\partial_\mu\psi$ depends on the infinitesimal neighbourhood of x where, by construction, ψ transforms differently than at x itself.

- This effect is seen in

$$\partial_\mu\psi' = \partial_\mu e^{-ig_e\alpha}\psi = e^{-ig_e\alpha}\,[\partial_\mu - ig_e(\partial_\mu\alpha)]\,\psi \neq e^{-ig_e\alpha}\,\partial_\mu\psi$$

- To restore local gauge invariance we can construct a **covariant derivative** which has the desired transformation property

$$D_\mu\psi \to D'_\mu\psi' \underset{\text{Iwant}}{=} e^{-ig_e\alpha}\,D_\mu\psi$$

- We can get this by introducing a **gauge field** A^μ such that

$$D_\mu\psi = (\partial_\mu + ig_eA_\mu)\,\psi.$$

- Indeed, provided that A_μ transforms as

$$A'_\mu = A_\mu + \partial_\mu\alpha$$

we find that, as you can easily check,

$D'_\mu\psi' = (\partial_\mu + ig_eA'_\mu)\,e^{-ig_e\alpha}\,\psi = e^{-ig_e\alpha}\,(\partial_\mu + ig_eA_\mu)\,\psi = e^{-ig_e\alpha}\,D_\mu\psi$

Exercise 3.3: [×] Well, please check it.

Locally invariant Dirac Lagrangian

- So we can now propose, as a first step, the Lagrangian

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi = \underbrace{i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi}_{\text{free term}} - \underbrace{g_{\rm e}(\overline{\psi}\gamma^\mu\psi)A_\mu}_{\text{interaction term}}$$

  which is invariant under local phase transformations and has acquired an interaction term $j^\mu A_\mu$ in addition to the free Lagrangian.

- We have a free term for the Dirac field, which suggests that we should add a free term (Proca Lagrangian) for the gauge field A_μ

$$\mathcal{L} = -\tfrac{1}{4}(F^{\mu\nu})(F_{\mu\nu}) + \tfrac{1}{2}m^2A^\nu A_\nu$$

- **Exercise 3.4**: [0.5] Check that the first term is invariant under the gauge transformation $A'_\mu = A_\mu + \partial_\mu\alpha$ but not the second term.

- To maintain gauge invariance we are thus forced to set $m = 0$ and consider only a massless gauge field which, of course, turns out to be the electromagnetic (photon) field.

- We have, in fact, found here a restriction that also applies to the SU(2) and SU(3) gauge invariant Lagrangians that we will consider later on:

> To maintain gauge invariance, the gauge field must be massless

The Lagrangian of QED

- We now can write-down the QED Lagrangian describing the interaction of Dirac particles with the electromagnetic field

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= \overline{\psi}(i\not{D} - m)\psi - \tfrac{1}{4}(F^{\mu\nu})(F_{\mu\nu}) \\ &= \overline{\psi}(i\not{\partial} - m)\psi - g_{\text{e}}(\overline{\psi}\gamma^{\mu}\psi)A_{\mu} - \tfrac{1}{4}(F^{\mu\nu})(F_{\mu\nu}) \end{aligned}$$

  In the expression above, we have introduced the usual shorthands $\not{\partial} \equiv \gamma^{\mu}\partial_{\mu} = \gamma_{\mu}\partial^{\mu}$ and $\not{D} \equiv \gamma^{\mu}D_{\mu} = \gamma_{\mu}D^{\mu}$.

- Note that the last two terms in the QED Lagrangian correspond to Maxwell Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^{\mu}A_{\mu}$$

- This Lagrangian leads to the Maxwell equations (see Exercise 3.2)

$$\partial_{\mu}F^{\mu\nu} = j^{\nu}$$

  with j^{μ} the Dirac current $g_{\text{e}}(\overline{\psi}\gamma^{\mu}\psi)$.

From Lagrangian to Feynman rules

- The Lagrangians we have thus far considered may describe *classical* as well as *quantum* fields. Field quantisation is the realm of **quantum field theory** which is outside the scope of these lectures. In QFT, particles emerge as quanta of the associated fields; photons are then the quanta of the electromagnetic field A^μ, leptons and quarks are the quanta of the Dirac field ψ, and gluons are the quanta of an $\mathrm{SU(3)_c}$ gauge field, as we will see. Field quantisation does not require a modification of the Lagrangian or the field equations, which stay formally the same.

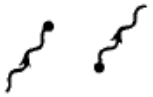
- To each Lagrangian corresponds a particular set of **Feynman rules**. The derivation of these rules is part of QFT and beyond the scope of these lectures. We just mention at this point that the QED Lagrangian contains two types of terms, as we have seen: free terms for the participating fields, and interaction terms that were generated through local gauge invariance. In general, we have the following correspondence:

$$\begin{aligned} \text{Free Lagrangian} &\rightarrow \text{propagator} \\ \text{Interaction term} &\rightarrow \text{vertex factor} \end{aligned}$$

- For the Feynman rules of QED, you can have a look at PP-I section 8, Griffiths section 7.5 and appendix D, or H&M section 6.17 (reproduced on the next page).

Feynman rules for QED

TABLE 6.2
Feynman Rules for $-i\mathcal{M}$

		Multiplicative Factor
• **External Lines**		
Spin 0 boson (or antiboson)		1
Spin $\frac{1}{2}$ fermion (in, out)		u, $\bar{u}$
antifermion (in, out)		$\bar{v}$, v
Spin 1 photon (in, out)		ε_μ, ε_μ^*
• **Internal Lines—Propagators (need $+i\varepsilon$ prescription)**		
Spin 0 boson		$\dfrac{i}{p^2 - m^2}$
Spin $\frac{1}{2}$ fermion		$\dfrac{i(\not{p} + m)}{p^2 - m^2}$
Massive spin 1 boson		$\dfrac{-i\left(g_{\mu\nu} - p_\mu p_\nu / M^2\right)}{p^2 - M^2}$
Massless spin 1 photon (Feynman gauge)		$\dfrac{-ig_{\mu\nu}}{p^2}$
• **Vertex Factors**		
Photon—spin 0 (charge $-e$)		$ie(p + p')^\mu$
Photon—spin $\frac{1}{2}$ (charge $-e$)		$ie\gamma^\mu$

Loops: $\int d^4k/(2\pi)^4$ over loop momentum; include -1 if fermion loop and take the trace of associated γ-matrices
Identical Fermions: -1 between diagrams which differ only in $e^- \leftrightarrow e^-$ or initial $e^- \leftrightarrow$ final e^+

Lecture notes Particle Physics II

Quantum Chromo Dynamics

4. SU(2) and SU(3) Gauge Invariance

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A hypothetical 2-component Dirac field

- Consider two fields ψ_1 and ψ_2 that obey the Dirac equations

$$(i\partial\!\!\!/ - m_1)\,\psi_1 = 0 \quad \text{and} \quad (i\partial\!\!\!/ - m_2)\,\psi_2 = 0$$

- The total Lagrangian is then simply the sum

$$\mathcal{L} = \underbrace{\overline{\psi}_1(i\partial\!\!\!/ - m_1)\psi_1}_{\mathcal{L}_1} + \underbrace{\overline{\psi}_2(i\partial\!\!\!/ - m_2)\psi_2}_{\mathcal{L}_2}$$

- We introduce the compact notation

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2), \quad M = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$$

  and set $m_1 = m_2$ so that $M = mI$, and write

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$

  but we have to remember that ψ and $\overline{\psi}$ are now 2-component objects, each component being itself a 4-component spinor.[24]

- We immediately see that $\mathcal{L}$ is invariant under a global unitary transformation $\psi' = U\psi$ in our 2-dimensional space because

$$\overline{\psi}'\partial\!\!\!/\psi' = \overline{\psi}U^\dagger\partial\!\!\!/ U\psi = \overline{\psi}\partial\!\!\!/\psi, \quad \text{and} \quad \overline{\psi}'\psi' = \overline{\psi}U^\dagger U\psi = \overline{\psi}\psi$$

  Note that for $m_1 \neq m_2$ the term $\overline{\psi}M\psi$ would *not* be invariant because then $U^\dagger MU \neq MU^\dagger U$.

- We will now follow the original idea of Yang and Mills (1954), and investigate what happens if this *global* invariance for unitary transformations is made *local*.

[24] The notation $\partial\!\!\!/\psi$ should here be understood as $\begin{pmatrix}\partial\!\!\!/\psi_1\\ \partial\!\!\!/\psi_2\end{pmatrix}$.

Yang-Mills theory

- The Yang-Mills theory describes pairs of spin-$\frac{1}{2}$ particles of equal mass, and Yang and Mills originally had the proton and neutron in mind as such a pair. A problem, however, is that the quanta of the Yang-Mills field must be *massless* in order to maintain gauge invariance (we have seen this already for the photon field on page 3–14). The massless quanta should have long-range effects, like the photon, and for this reason the theory was abandoned as a candidate theory of the strong interaction, which is short-range.
- However, the Yang-Mills theory is still important because it serves as a prototype of **non-Abelian gauge theories**, that is, theories for which the generators of the underlying symmetry group do not commute.
- Indeed, like SU(3) is a generalisation of SU(2), we will see that QCD is a generalisation of Yang-Mills.
- So we will first present the nuts and bolts of Yang-Mills as an important step towards building the QCD Lagrangian. We may summarise this in the road map below where we will follow, of course, the branch leading to QCD.

$$e^{i\phi}\, e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}} = \mathrm{U}(1) \times \mathrm{SU}(2)$$

$$\mathrm{U}(1)\times\mathrm{SU}(2) \begin{cases} \mathrm{U}(1) \;\rightarrow\; \text{QED} \\ \mathrm{SU}(2) \;\rightarrow\; \text{Yang-Mills} \begin{cases} \rightarrow\; \text{Electroweak theory} \\ \rightarrow\; \mathrm{SU}(3) \;\rightarrow\; \mathbf{QCD} \end{cases} \end{cases}$$

Recap of SU(2)

- We will use several SU(2) formula which were presented earlier in these lectures, or derived in exercises. Here is a summary:
- Unitary SU(2) matrix $\boldsymbol{U} = \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)$ with $\boldsymbol{U}^\dagger\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\dagger = \mathbf{1}$
- Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \tau_i^\dagger &= \tau_i = \tau^{-1} \\ \mathrm{Tr}(\tau_i) &= 0 \\ \tau_i\tau_j &= \delta_{ij} + \epsilon_{ijk}\tau_k \\ (\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau}) &= \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\tau}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \\ \exp(i\boldsymbol{\theta}\cdot\boldsymbol{\tau}) &= \cos|\boldsymbol{\theta}| + i(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\tau})\sin|\boldsymbol{\theta}| \end{aligned}$$

- Commutation relations for $\boldsymbol{T} = \boldsymbol{\tau}/2$: $\quad [T_i, T_j] = i\epsilon_{ijk}T_k$.

Local SU(2) invariance

- We want to make the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi \quad \text{with} \quad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \text{and} \quad \overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$$

  invariant under local SU(2) transformations

$$U(x) = \exp[-ig_{\rm w}\,\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)]$$

  Here $g_{\rm w}$ is some strength parameter (coupling constant).

- As in the U(1) case, we replace ∂_μ by a covariant derivative D_μ and require that the Lagrangian is invariant:

$$\mathcal{L}' = \overline{\psi}\,U^\dagger\,(iD\!\!\!/' - m)\,U\,\psi \underset{\text{Iwant}}{=} \overline{\psi}(iD\!\!\!/ - m)\psi = \mathcal{L},$$

  which is the case if $U^\dagger D'_\mu U = D_\mu$, or $D'_\mu U = U D_\mu$.

- In analogy with the U(1) case we set

$$D_\mu = \partial_\mu + ig_{\rm w}\,\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu$$

  which introduces *three* gauge fields $\boldsymbol{A}_\mu = [(A_1)_\mu, (A_2)_\mu, (A_3)_\mu]$.

- The transformation property of $\boldsymbol{A}$ we find from the requirement $D'_\mu U = U D_\mu$:

$$(\partial_\mu + ig_{\rm w}\boldsymbol{\tau}\cdot\boldsymbol{A}'_\mu)\,U\psi \underset{\text{Iwant}}{=} U(\partial_\mu + ig_{\rm w}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu)\,\psi$$

  Exercise 4.1: [1.0] Show that this gives the transformation rule:

$$\boldsymbol{\tau}\cdot\boldsymbol{A}'_\mu = U(\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu)U^{-1} + \frac{i}{g_{\rm w}}\,(\partial_\mu U)U^{-1}$$

Exercise 4.2:

(a) [0.4] The transformation rule for the gauge fields is

$$\boldsymbol{\tau} \cdot \boldsymbol{A}'_\mu = U(\boldsymbol{\tau} \cdot \boldsymbol{A}_\mu)U^{-1} + \frac{i}{g_\mathrm{w}} (\partial_\mu U)U^{-1}.$$

Expand to first order $U \approx \mathbf{1} - i g_\mathrm{w} \boldsymbol{\tau} \cdot \boldsymbol{\alpha}$ and show that the transformation rule can be approximated by

$$\boldsymbol{\tau} \cdot \boldsymbol{A}'_\mu \approx \boldsymbol{\tau} \cdot \boldsymbol{A}_\mu + i g_\mathrm{w} \left[\boldsymbol{\tau} \cdot \boldsymbol{A}_\mu \, , \boldsymbol{\tau} \cdot \boldsymbol{\alpha}\right] + \boldsymbol{\tau} \cdot \partial_\mu \boldsymbol{\alpha}$$

(b) [0.4] Use the expression for $(\boldsymbol{a} \cdot \boldsymbol{\tau})(\boldsymbol{b} \cdot \boldsymbol{\tau})$ on page 4–5 to evaluate the commutator

$$\left[\boldsymbol{\tau} \cdot \boldsymbol{A}_\mu \, , \boldsymbol{\tau} \cdot \boldsymbol{\alpha}\right] = -2i \boldsymbol{\tau} \left(\boldsymbol{\alpha} \times \boldsymbol{A}_\mu\right)$$

(c) [0.2] Now substitute the commutator and multiply with $\boldsymbol{\tau}^{-1}$ to get

$$\boldsymbol{A}'_\mu \approx \boldsymbol{A}_\mu + \partial_\mu \boldsymbol{\alpha} + 2 g_\mathrm{w} \left(\boldsymbol{\alpha} \times \boldsymbol{A}_\mu\right)$$

The SU(2) invariant Lagrangian

- Substituting $D_\mu = \partial_\mu + i g_\mathrm{w} \boldsymbol{\tau} \cdot \boldsymbol{A}_\mu$, we get for our SU(2) invariant Lagrangian

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi = \underbrace{\overline{\psi}(i\not{\partial} - m)\psi}_{\text{free term}} - \underbrace{(g_\mathrm{w}\, \overline{\psi}\gamma^\mu \boldsymbol{\tau}\psi) \cdot \boldsymbol{A}_\mu}_{\text{interaction term}}$$

- But we still have to add a free term for the gauge fields $\boldsymbol{A}_\mu$

$$\begin{aligned} \mathcal{L}_\text{free} &= -\tfrac{1}{4}\left[(F_1)^{\mu\nu}(F_1)_{\mu\nu} + (F_2)^{\mu\nu}(F_2)_{\mu\nu} + (F_3)^{\mu\nu}(F_3)_{\mu\nu}\right] \\ &= -\tfrac{1}{4}\boldsymbol{F}^{\mu\nu} \cdot \boldsymbol{F}_{\mu\nu} \end{aligned}$$

- Now we have to look for a definition of $\boldsymbol{F}^{\mu\nu}$ that makes $\mathcal{L}_\text{free}$ invariant under the (infinitesimal) gauge transformation

$$\boldsymbol{A}'_\mu \approx \boldsymbol{A}_\mu + \partial_\mu \boldsymbol{\alpha} + 2 g_\mathrm{w}\, (\boldsymbol{\alpha} \times \boldsymbol{A}_\mu)$$

  It can be shown (elaborate, but straight forward algebra) that the sought-after gauge field tensor is

$$\boldsymbol{F}^{\mu\nu} \equiv \partial^\mu \boldsymbol{A}^\nu - \partial^\nu \boldsymbol{A}^\mu - 2 g_\mathrm{w}\, (\boldsymbol{A}^\mu \times \boldsymbol{A}^\nu)$$

- The SU(2) invariant Lagrangian is now[25]

$$\boxed{\mathcal{L}_\mathrm{YM} = \overline{\psi}(i\not{\partial} - m)\psi - \tfrac{1}{4}\boldsymbol{F}^{\mu\nu} \cdot \boldsymbol{F}_{\mu\nu} - (g_\mathrm{w}\, \overline{\psi}\gamma^\mu \boldsymbol{\tau}\psi) \cdot \boldsymbol{A}_\mu}$$

- Note that the last two terms are like a Maxwell Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}\boldsymbol{F}^{\mu\nu} \cdot \boldsymbol{F}_{\mu\nu} - \boldsymbol{j}^\mu \cdot \boldsymbol{A}_\mu$$

  but now with *three* Dirac currents $\boldsymbol{j}^\mu = g_\mathrm{w}\, \overline{\psi}\gamma^\mu \boldsymbol{\tau}\psi$ and with *three* gauge fields $\boldsymbol{A}_\mu$.

[25] Here we have talked only about *infinitesimal* gauge transformations but it can be shown that $\mathcal{L}$ is also invariant for *finite* transformations.

Global SU(3)$_c$ invariance

- We have seen on page 2–28 the need to introduce the quark colour quantum number and, on page 2–31, how this shows up experimentally in the ratio $R = \sigma(e^+e^- \to \text{hadrons}) \,/\, \sigma(e^+e^- \to \mu^+\mu^-)$.
- So we have now *three* spin-$\frac{1}{2}$ Dirac fields

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix} \qquad \text{and} \qquad \overline{\psi} = (\overline{\psi}_r, \overline{\psi}_g, \overline{\psi}_b)$$

- The free Lagrangian is, again,

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$

  but we have to remember that ψ and $\overline{\psi}$ represent 3-component objects, with colour index (r, g, b), and that each component is by itself a 4-component Dirac spinor.[26]

- This Lagrangian is manifestly invariant under U(3) = U(1)×SU(3) global transformations. The U(1) phase invariance was already explored so we only investigate here SU(3) invariance

$$\psi' = U\psi \qquad \text{and} \qquad \overline{\psi}' = \overline{\psi}\, U^\dagger$$

- To make $\mathcal{L}$ invariant under *local* SU(3) transformations is now a relatively easy task since we just can replace the 2×2 SU(2) matrices in the Yang-Mills theory by 3×3 SU(3) matrices.

[26] We assume here that quarks of all flavours are identical by having the same mass. This is not true, of course, and we should introduce a flavour index $f = (d, u, s, c, b, t)$, and different masses m_f. We will not do that here to keep the notation simple.

Local SU(3)$_c$ invariance

- We want to make the Lagrangian invariant under local SU(3) transformations (g_s is the strong coupling constant)

$$U(x) = \exp[i g_s \boldsymbol{\lambda} \cdot \boldsymbol{\alpha}(x)]$$

  Here we have *eight* angles $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_8)$ and the eight Gell-Mann matrices $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_8)$ that are given on page 2–23.

- We can now simply repeat the steps made in the Yang-Mills theory and define the covariant derivative

$$D_\mu = \partial_\mu + i g_s \boldsymbol{\lambda} \cdot \boldsymbol{A}_\mu$$

  where we have now 8 gauge fields $\boldsymbol{A} = (A_1, \ldots, A_8)$.

- For infinitesimal transformations, the gauge fields transform as

$$\boldsymbol{A}'_\mu \approx \boldsymbol{A}_\mu + \partial_\mu \boldsymbol{\alpha} + 2 g_s (\boldsymbol{\alpha} \times \boldsymbol{A}_\mu)$$

  but here we have to use the general expression for the cross product

$$(\boldsymbol{a} \times \boldsymbol{b})_i = f_{ijk}\, a_j\, b_k \qquad \text{(summation over } j \text{ and } k \text{ implied)}$$

  with f_{ijk} the structure constants of SU(3), see page 2–23.[27]

- The gauge field tensor is given by

$$\boldsymbol{F}^{\mu\nu} \equiv \partial^\mu \boldsymbol{A}^\nu - \partial^\nu \boldsymbol{A}^\mu - 2 g_s (\boldsymbol{A}^\mu \times \boldsymbol{A}^\nu)$$

  where we have to take the SU(3) cross product for the last term.

- Now we can write down the Lagrangian of QCD.

[27] You may check that for 3-vectors $(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk}\, a_j\, b_k$.

The QCD Lagrangian

- The QCD Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\partial\!\!\!/ - m)\psi - \tfrac{1}{4}\boldsymbol{F}^{\mu\nu}\cdot\boldsymbol{F}_{\mu\nu} - (g_{\mathrm{s}}\,\overline{\psi}\gamma^{\mu}\boldsymbol{\lambda}\psi)\cdot\boldsymbol{A}_{\mu} + \cdots$$

- We have now eight colour fields $\boldsymbol{A}_{\mu}$ (gluon fields) and eight colour currents $\boldsymbol{j}^{\mu} = g_{\mathrm{s}}\,\overline{\psi}\gamma^{\mu}\boldsymbol{\lambda}\psi$ that act as sources for the colour fields, like the electric current is the source for the electromagnetic field.
- In the first term we recognise the free Dirac Lagrangian, just like in QED. It will give rise to quark propagators.
- The last term also looks familiar: it is an interaction term that gives rise to the quark-gluon vertex.
- The second term is the free Lagrangian of the gluon fields, which also looks familiar from QED, but has a much richer structure. As we will see, it gives rise to the gluon propagator, like the photon propagator in QED, but also to 3- and 4-gluon vertices, which is something that does not exist in QED. We will see in Section 6 that these gluon self-interactions are responsible for a characteristic feature of QCD interactions: asymptotic freedom.
- Note that the QCD Lagrangian given above is not complete and so-called **gauge-fixing terms** and **ghost fields** must be introduced to make the theory consistent. This is highly technical and way beyond the scope of these lectures.
- Up to now we have expressed Lagrangians in some kind of vector notation which is compact, but not commonly used. So let us now first present the QCD Lagrangian in (colour) index notation.

Colour space indices

- We have seen that quarks come in three colours $i = (\mathrm{r}, \mathrm{g}, \mathrm{b})$ so that the wave function can be written as

$$\psi_i = \begin{cases} c_i \, u_f^{(s)}(p^\mu) & \text{incoming quark} \\ c_i^\dagger \, \bar{u}_f^{(s)}(p^\mu) & \text{outgoing quark} \\ c_i^\dagger \, \bar{v}_f^{(s)}(p^\mu) & \text{incoming antiquark} \\ c_i \, v_f^{(s)}(p^\mu) & \text{outgoing antiquark} \end{cases}$$

  Expressions for the 4-component spinors u and v can be found in Griffiths p.233–4. We have here explicitly indicated the Lorentz index $\mu = (0, 1, 2, 3)$, the spin index $s = (1, 2) = (\text{up}, \text{down})$ and the flavour index $f = (\mathrm{d}, \mathrm{u}, \mathrm{s}, \mathrm{c}, \mathrm{b}, \mathrm{t})$. To not overburden the notation we will suppress these indices in the following.

- The colour index $i = (1, 2, 3) = (\mathrm{r}, \mathrm{g}, \mathrm{b})$ is taken care of by defining the following basis vectors in **colour space**

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

  for red, green and blue, respectively. The Hermitian conjugates $c_i^\dagger$ are just the corresponding row vectors.

- A colour transition like $\psi_\mathrm{r} \to \psi_\mathrm{g}$ can now be described as an SU(3) matrix operation in colour space. Recalling the SU(3) step operators (page 2–24 and Exercise 2.8d) we may write

$$\psi_\mathrm{g} = (\lambda_1 - i\lambda_2)\,\psi_\mathrm{r} \quad \text{or, in colour space,} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The QCD Lagrangian in colour index notation

- Here is QCD Lagrangian with all colour indices shown.[28]

$$\mathcal{L}_{\rm QCD} \;=\; \overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - \tfrac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} - g_{\rm s}\,\overline{\psi}_i\lambda^a_{ij}\psi_j\,\gamma^\mu A^a_\mu$$

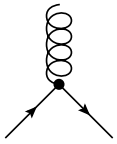
$$F_a^{\mu\nu} \;=\; \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - 2g_{\rm s}\,f_{abc}A^\mu_b A^\nu_c$$

  The colour index $i = (1,2,3)$ labels the quark fields, while the colour index $a = (1,\ldots,8)$ labels the gluon fields and the corresponding generators.

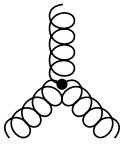
- Here are all the propagators and vertices of a QCD Feynman diagram; the ones for the gluon become visible when you multiply-out the field tensor contraction $F_a^{\mu\nu}F^a_{\mu\nu}$ in the Lagrangian:

$\bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i$ quark propagator

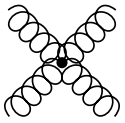
$(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$ gluon propagator



$g_{\rm s}\,\bar{\psi}_i\lambda^a_{ij}\psi_j\gamma^\mu A^a_\mu$ quark-gluon vertex



$g_{\rm s}\,(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)f_{abc}A^b_\mu A^c_\nu$ 3-gluon vertex



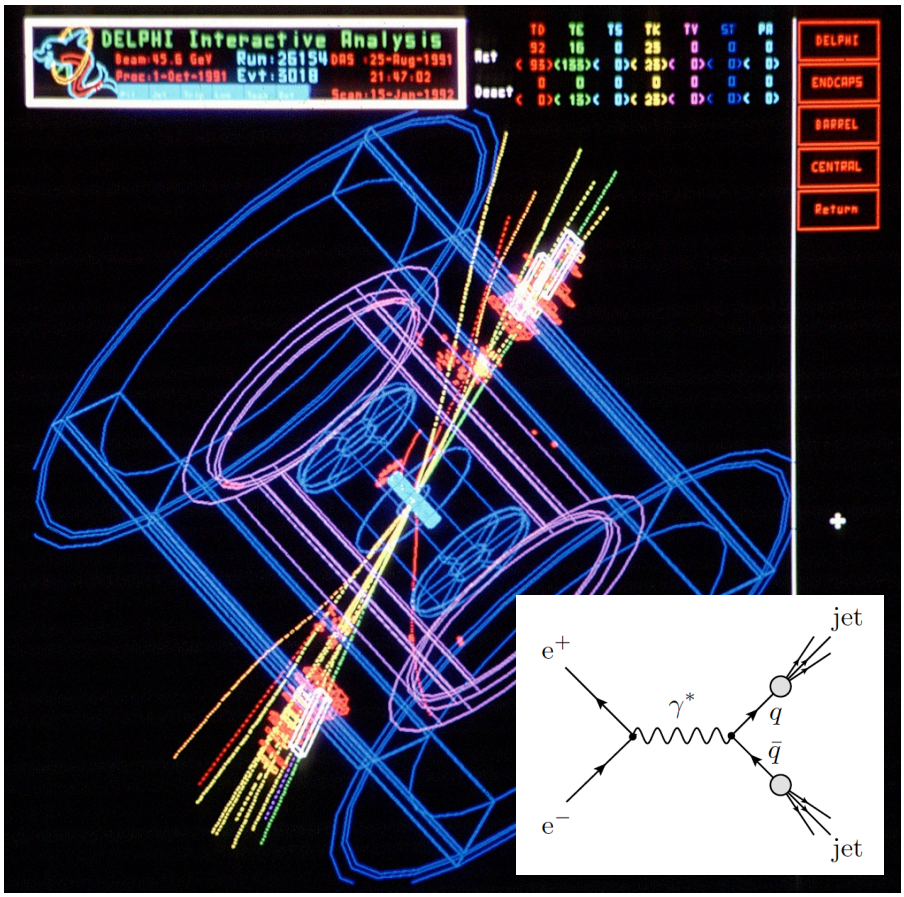
$g_{\rm s}^2\,f_{abc}A^\mu_b A^\nu_c\,f_{ade}A^d_\mu A^e_\nu$ 4-gluon vertex

[28] Summation over repeated indices is implied, irrespective of their position (upper or lower); the colour indices are just placed wherever the Lorentz indices leaves room for them.

Probing the strong quark interactions

- Of course all hard scattering phenomena involving hadrons probe, in one way or another, the constituent quarks. In the introduction we have mentioned that energetic quarks tend to fragment into jets of particles. The study of those jets is indeed an important part of the experimental programmes at high energy $\bar{\mathrm{p}}\mathrm{p}$ and pp colliders.
- The largest centre of mass energies are reached at the Tevatron 2 TeV $\bar{\mathrm{p}}\mathrm{p}$ collider (Fermilab, Chicago, 1985–2011) and the pp collider LHC at CERN, Geneva, which came into operation in November 2009 and is presently upgraded from 8 TeV to 14 TeV centre-of-mass energy.
- The advantage of these machines is the high collision energies that can be reached but a disadvantage is that the (anti)proton beams at these energies can be regarded as complicated streams of quarks and gluons, so that the initial state of the collision is non-trivial.
- A clean initial state is provided by $\mathrm{e}^+\mathrm{e}^-$ collisions, like at the CERN LEP collider (1989–2000) with centre of mass energies of up to 200 GeV. The figure on page 4–15 shows an $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}$ event, where the quarks fragment into two back-to-back jets. Here we can really 'see' the quark and antiquark flying apart.

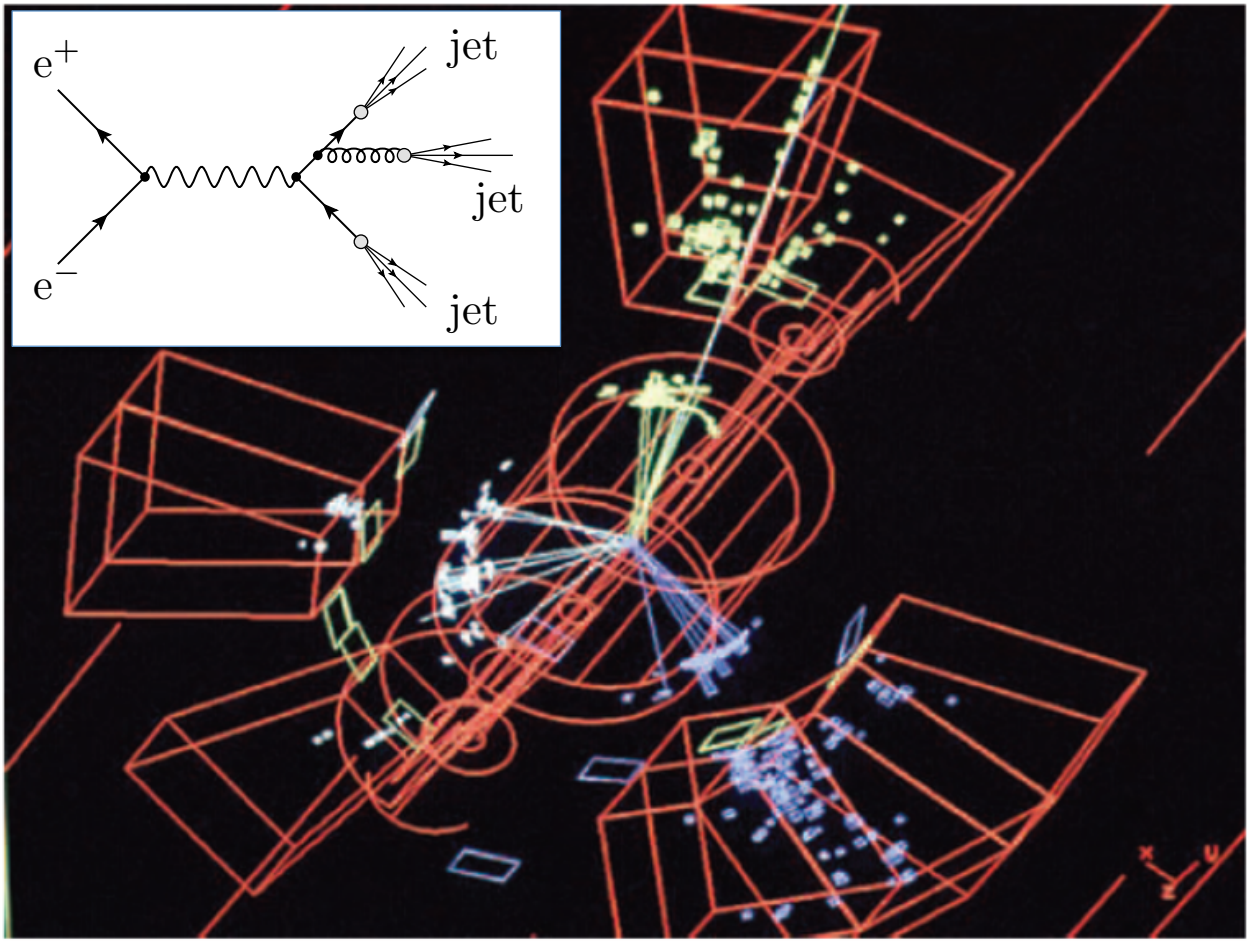
Two-jet event in e^+e^-



Here we see quarks, but can we also see gluons?

Yes!

Here they are (the gluons)



Three-jet event in an e^+e^- collision, recorded by L3 at LEP.

Lecture notes Particle Physics II

Quantum Chromo Dynamics

5. Colour factors

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Nikhef, Science Park, Amsterdam

December 2, 2013

From the Lagrangian to Feynman graphs

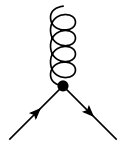
- We show here once more the QCD Lagrangian with the colour indices $i = (1, 2, 3) = (\mathrm{r}, \mathrm{g}, \mathrm{b})$ that label the quarks fields ψ_i and $\overline{\psi}_i$, and the index $a = (1, \ldots, 8)$ that label the eight gluon fields A^a_μ. In the expression below, g_s is the strong coupling constant, and the f_{abc} are the structure constants of SU(3).

$$\begin{aligned} \mathcal{L}_\mathrm{QCD} &= \overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - \tfrac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} - g_\mathrm{s}\,\overline{\psi}_i\lambda^a_{ij}\psi_j\,\gamma^\mu A^a_\mu \\ F^{\mu\nu}_a &= \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - 2g_\mathrm{s}\,f_{abc}A^\mu_b A^\nu_c \end{aligned}$$

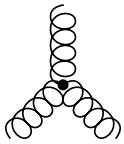
- And here are again the propagators and vertices in the Lagrangian.

$\bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i$ quark propagator

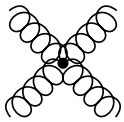
$(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$ gluon propagator



$g_\mathrm{s}\,\bar{\psi}_i\lambda^a_{ij}\psi_j\gamma^\mu A^a_\mu$ quark-gluon vertex



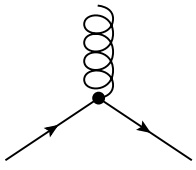
$g_\mathrm{s}\,(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)f_{abc}A^b_\mu A^c_\nu$ 3-gluon vertex



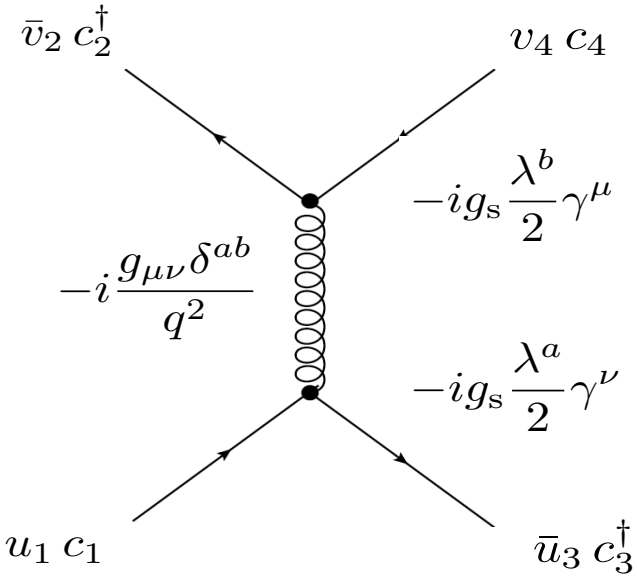
$g^2_\mathrm{s}\,f_{abc}A^\mu_b A^\nu_c\,f_{ade}A^d_\mu A^e_\nu$ 4-gluon vertex

Feynman rules of QCD

- To each element of a Feynman diagram corresponds a mathematical expression which, together with the prescription how to assemble a scattering amplitude, make up the set of **Feynman rules**. Deriving these rules is the subject of Quantum Field Theory and is beyond the scope of these lectures. It was already said before that our QCD Lagrangian is not complete and that so-called **ghost fields** must be introduced to make the theory consistent, but this is also beyond the scope of these lectures. The full set of QCD Feynman rules (without ghosts) can be found in Griffiths p.287–8.
- In the following we will calculate so-called **colour factors** for the leading order $q\bar{q} \to q\bar{q}$ and $qq \to qq$ diagrams. Here is the dictionary that we will need for that calculation:

	$u\,c$	incoming quark
	$\bar{u}\,c^\dagger$	outgoing quark
	$\bar{v}\,c^\dagger$	incoming antiquark
	$v\,c$	outgoing antiquark
	$-i\frac{g_{\mu\nu}\delta^{ab}}{q^2}$	gluon propagator
	$-ig_{\mathrm{s}}\frac{\lambda^a}{2}\gamma^\mu$	quark-gluon vertex

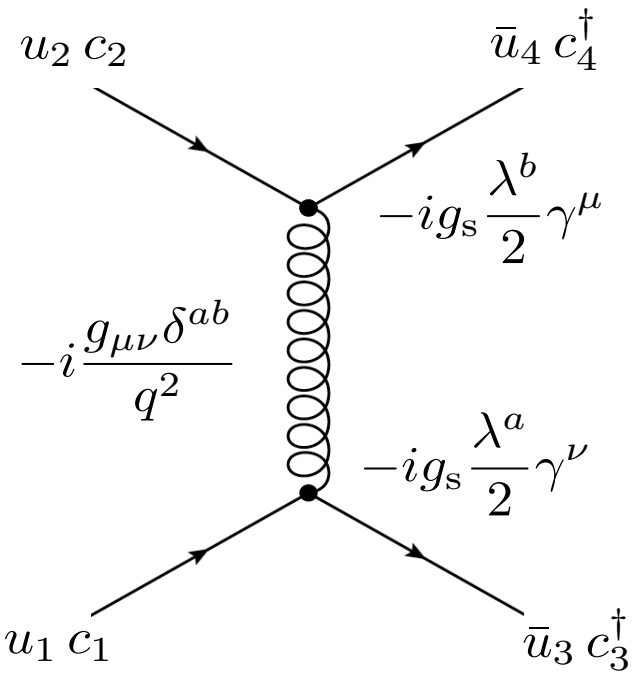
Colour factor for $q\bar{q} \to q\bar{q}$



$$
\begin{aligned}
-i\mathcal{M} &= \bar{u}_3 c_3^\dagger \left(-ig_{\rm s}\frac{\lambda^a}{2}\gamma^\nu\right) u_1 c_1 \\
&\times \left(-i\frac{g_{\mu\nu}\delta^{ab}}{q^2}\right) \\
&\times \bar{v}_2 c_2^\dagger \left(-ig_{\rm s}\frac{\lambda^b}{2}\gamma^\mu\right) v_4 c_4
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M} &= -\frac{g_{\rm s}^2}{q^2}\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{v}_2\gamma_\mu v_4\right) \\
&\times \underbrace{\frac{1}{4}\left(c_3^\dagger\,\lambda^a c_1\right)\left(c_2^\dagger\,\lambda^a c_4\right)}_{\text{colour factor}}
\end{aligned}
$$

Colour factor for $qq \to qq$



$$
\begin{aligned}
-i\mathcal{M} &= \bar{u}_3 c_3^\dagger \left(-ig_{\rm s}\frac{\lambda^a}{2}\gamma^\nu\right) u_1 c_1 \\
&\times \left(-i\frac{g_{\mu\nu}\delta^{ab}}{q^2}\right) \\
&\times \bar{u}_4 c_4^\dagger \left(-ig_{\rm s}\frac{\lambda^b}{2}\gamma^\mu\right) u_2 c_2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M} &= -\frac{g_{\rm s}^2}{q^2}\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_4\gamma_\mu u_2\right) \\
&\times \underbrace{\frac{1}{4}\left(c_3^\dagger\,\lambda^a c_1\right)\left(c_4^\dagger\,\lambda^a c_2\right)}_{\text{colour factor}}
\end{aligned}
$$

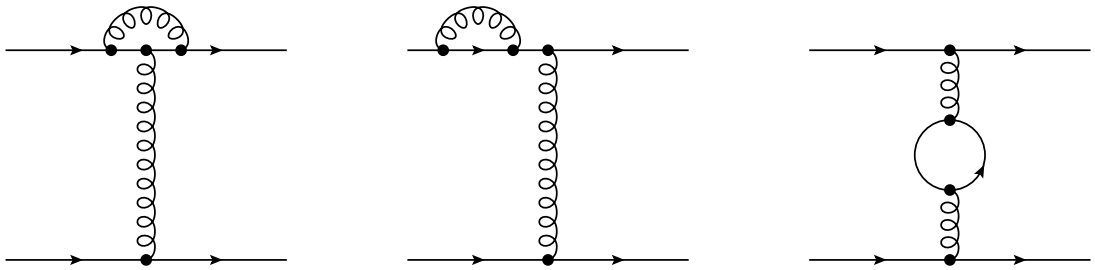
Colour factors

- We see that the leading order QCD amplitudes for $q\bar{q} \to q\bar{q}$ and $qq \to qq$ scattering can be written as ($\alpha_s \equiv g_s^2/4\pi$)

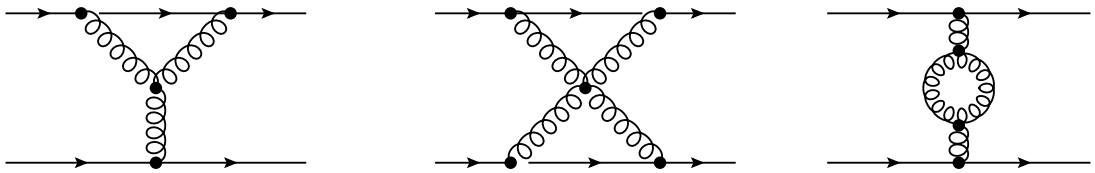
$$\mathcal{M}_{\rm QCD} = \alpha_s \times \text{colourfactor} \times \mathcal{M}_{\rm QED}$$

  This means that we can take the QED result (from PP-I for instance), provided that we replace the fine structure constant α by the strong coupling constant α_s, and multiply by the colour factor.

- This correspondence does not only hold for **tree diagrams** (diagrams without loops) but for any diagram where photons can be replaced by gluons.



- However, it is not true that QCD $\equiv$ colourfactor $\times$ QED because there are QCD diagrams which do not exist in QED.



Colour transition operators

- Instead of working with the matrices λ^a it is sometimes handy to express them in terms of colour transition operators as follows.
- Consider an N-dimensional Hilbert space spanned by the kets $|i\rangle$. In this space live the transition operators $O_{i\to j} = |j\rangle\langle i|$ that transform the state $|i\rangle$ to the state $|j\rangle$. Indeed,

$$O_{i\to j}|i\rangle = |j\rangle\langle i|i\rangle = |j\rangle$$

  A constraint is imposed by the closure relation

$$\sum_i |i\rangle\langle i| = 1$$

  so that there are $N^2 - 1$ independent transition operators.
- In our 3-dimensional colour space we then have transition operators like $O_{\mathrm{r}\to\mathrm{g}} = |G\rangle\langle R|$ and it is straight-forward to express the Gell-Mann matrices given on Page 2–23 in terms of these operators

$$\begin{aligned}
\lambda_1 &= |G\rangle\langle R| + |R\rangle\langle G| \\
\lambda_2 &= i\,(\,|G\rangle\langle R| - |R\rangle\langle G|\,) \\
\lambda_3 &= |R\rangle\langle R| - |G\rangle\langle G| \\
\lambda_4 &= |B\rangle\langle R| + |R\rangle\langle B| \\
\lambda_5 &= i\,(\,|B\rangle\langle R| - |R\rangle\langle B|\,) \\
\lambda_6 &= |B\rangle\langle G| + |G\rangle\langle B| \\
\lambda_7 &= i\,(\,|B\rangle\langle G| - |G\rangle\langle B|\,) \\
\lambda_8 &= \frac{1}{\sqrt{3}}\,(\,|R\rangle\langle R| + |G\rangle\langle G| - 2|B\rangle\langle B|\,)
\end{aligned}$$

Bookkeeping I

- We will write $c_i^\dagger \lambda^a c_j$ as $\langle i|\lambda^a|j\rangle$ which is, of course, just the matrix element λ^a_{ij}.
- The colour factors on Page 5–5 and 5–6 can be calculated from

$$\tfrac{1}{4}\sum_a \langle i|\lambda^a|j\rangle\langle k|\lambda^a|l\rangle = \tfrac{1}{4}\sum_a \lambda^a_{ij}\lambda^a_{kl} = \tfrac{1}{2}\,\delta_{il}\delta_{jk} - \tfrac{1}{6}\,\delta_{ij}\delta_{kl} \equiv f(ijkl)$$

- The colour factor $f(ijkl)$ thus depends on whether pairs of colours are the same or not and we get, from the identity above,

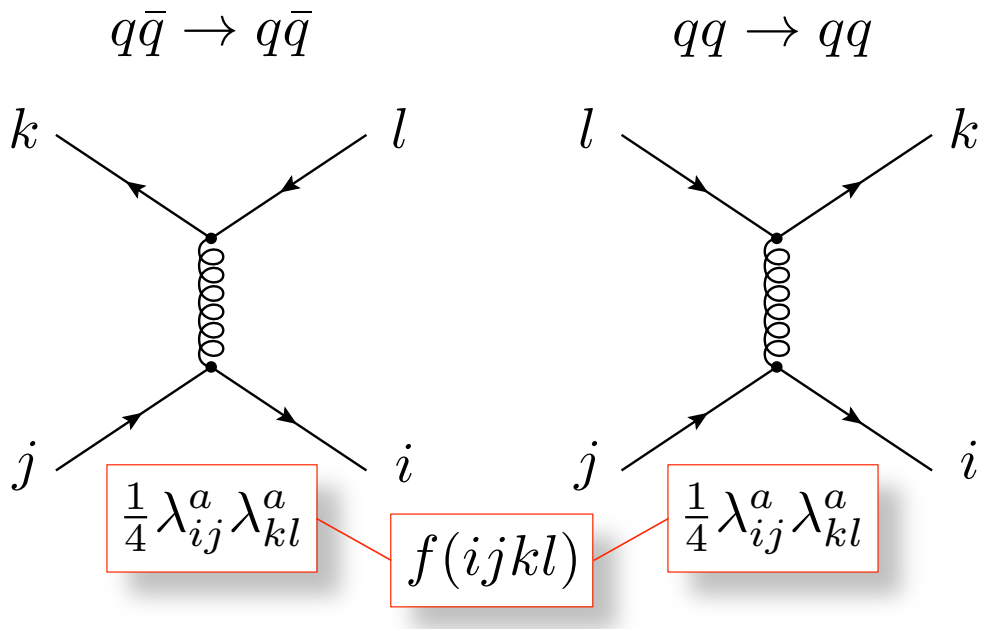
$$f(ijkl) = \begin{cases} -\frac{1}{6} & \text{if } i=j \text{ and } k=l \\ \frac{1}{2} & \text{if } i=l \text{ and } j=k \\ \frac{1}{3} & \text{if } i=j=k=l \\ 0 & \text{otherwise} \end{cases}$$

  For the three non-zero colour factors we will also use the generic notation $f(xxyy)$, $f(xyyx)$ and $f(xxxx)$, respectively.

- **Exercise 5.1**: [1.0] Use the λ matrices on page 2–23 or the operator representation on page 5–8 to check the colour factor $f(ijkl)$.

  Hint: Restrict yourself to two colours (red and green, say) so that i, j, k, l can only take the values 1 and 2. After this it is easy to generalise to the other combinations red-blue and green-blue.

Bookkeeping II



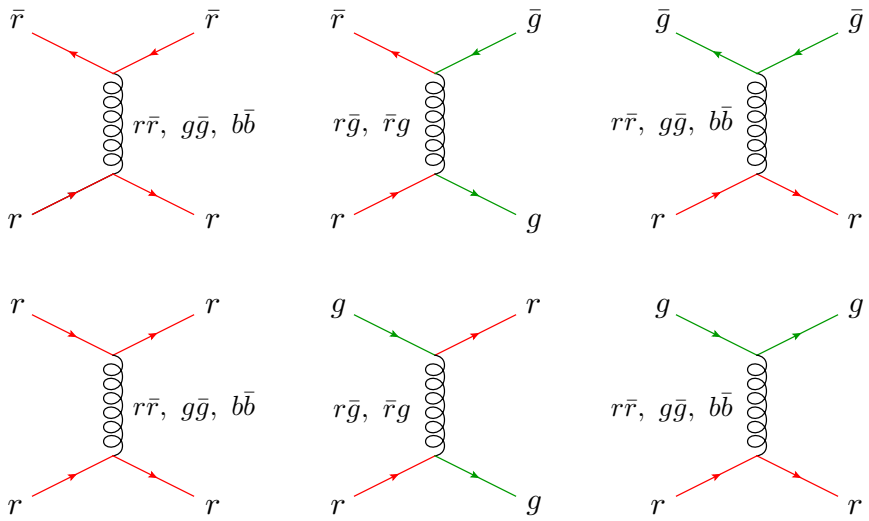
- The colour factors $\frac{1}{4}\lambda^a_{ij}\lambda^a_{kl} = f(ijkl)$ are the same in both diagrams above, but note that the indices l and k in the upper legs are swapped. Thus for a given $f(ijkl)$ the indices i, j, k, l are assigned differently to the incoming and outgoing particles in $q\bar{q} \to q\bar{q}$ and $qq \to qq$ scattering.
- Now we can list all allowed combinations of colour 'x' and 'y':

$q\ \bar{q} \to q\ \bar{q}$ $j\ \ k \to i\ \ l$	$q\ \ q \to q\ \ q$ $j\ \ l \to i\ \ k$	$f(\ ijkl\ )$
$x\ \ x \to x\ \ x$	$x\ \ x \to x\ \ x$	$f(xxxx) = \frac{1}{3}$
$x\ \ x \to y\ \ y$	$x\ \ y \to y\ \ x$	$f(yxxy) = \frac{1}{2}$
$x\ \ y \to x\ \ y$	$x\ \ y \to x\ \ y$	$f(xxyy) = -\frac{1}{6}$
$x\ \ y \to y\ \ x$	$x\ \ x \to y\ \ y$	$f(yxyx) = 0$

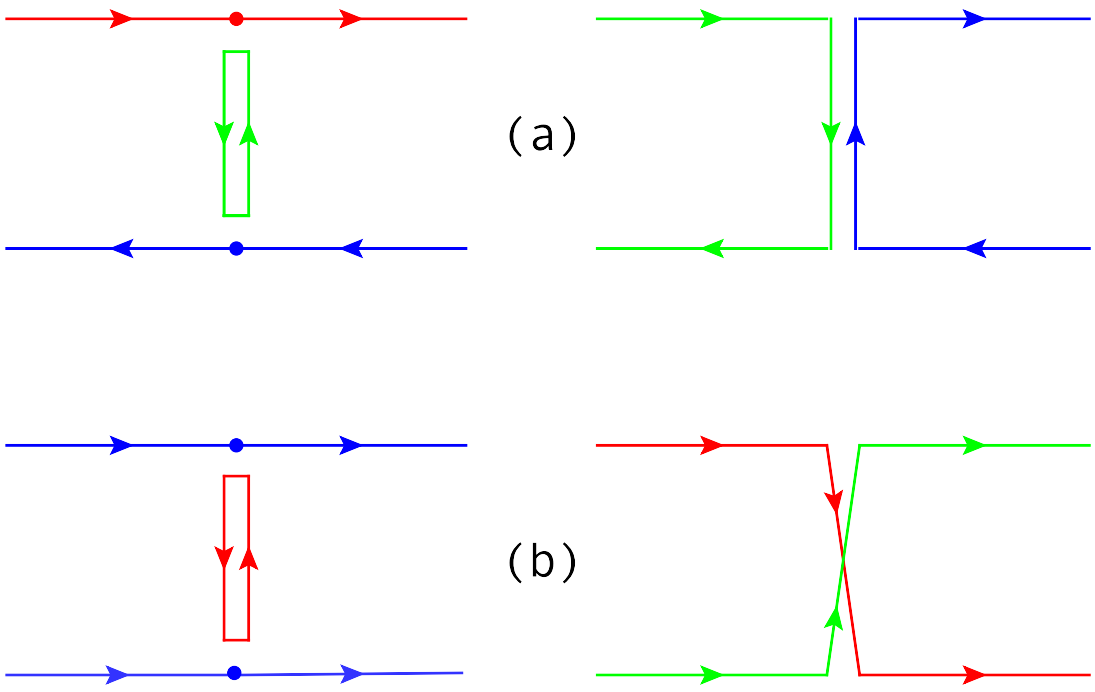
- In fact, the table above reflects **colour flow** through the diagram, as is shown on the next page for the combination red and green.

Colour flow

- $q\bar{q} \to q\bar{q}$ and $qq \to qq$ diagrams showing colour exchange and the possible $c\bar{c}$ combinations carried by the exchanged gluon.



- Here are some colour flow diagrams for (a) $q\bar{q} \to q\bar{q}$ and (b) $qq \to qq$ interactions.



Gluon colour states

- The gluon carries one unit of colour and one unit of anticolour and because $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ (see page 2–27) we can take the singlet as our principal axis in colour-anticolour space and then construct a basis by orthogonalisation. The invariant singlet state is obviously

$$|0\rangle \equiv \sqrt{\tfrac{1}{3}}\,(\mathrm{r\bar{r}} + \mathrm{g\bar{g}} + \mathrm{b\bar{b}}).$$

  By simple orthogonalisation we find a fully orthonormal basis as[29]

	$\mathrm{r\bar{g}}$	$\mathrm{r\bar{b}}$	$\mathrm{g\bar{r}}$	$\mathrm{g\bar{b}}$	$\mathrm{b\bar{r}}$	$\mathrm{b\bar{g}}$	$\mathrm{r\bar{r}}$	$\mathrm{g\bar{g}}$	$\mathrm{b\bar{b}}$	
$\lvert 0\rangle = \sqrt{\frac{1}{3}}$	| 0	0	0	0	0	0	1	1	1	>
$\lvert 1\rangle =$	| 1	0	0	0	0	0	0	0	0	>
$\lvert 2\rangle =$	| 0	1	0	0	0	0	0	0	0	>
$\lvert 3\rangle =$	| 0	0	1	0	0	0	0	0	0	>
$\lvert 4\rangle =$	| 0	0	0	1	0	0	0	0	0	>
$\lvert 5\rangle =$	| 0	0	0	0	1	0	0	0	0	>
$\lvert 6\rangle =$	| 0	0	0	0	0	1	0	0	0	>
$\lvert 7\rangle = \sqrt{\frac{1}{2}}$	| 0	0	0	0	0	0	1	−1	0	>
$\lvert 8\rangle = \sqrt{\frac{1}{6}}$	| 0	0	0	0	0	0	1	1	−2	>

- According to the colour hypothesis (page 1–6) a singlet gluon could exist as a free (unconfined) particle. Such a gluon would then behave as a strongly interacting photon and be able to transmit the strong force over (infinitely) large distances. A singlet gluon is excluded from the list since it has never been observed in isolation, and also because we know that the strong force is short-ranged.[30]

- The quarks transform according to the 3×3 fundamental representation of SU(3) (the λ matrices), while the gluons transform according to the 8×8 **adjoint representation**.

[29] H&M use this basis in Eq(2.93); Griffiths uses another orthonormal basis for the octet state, Eq(8.29).

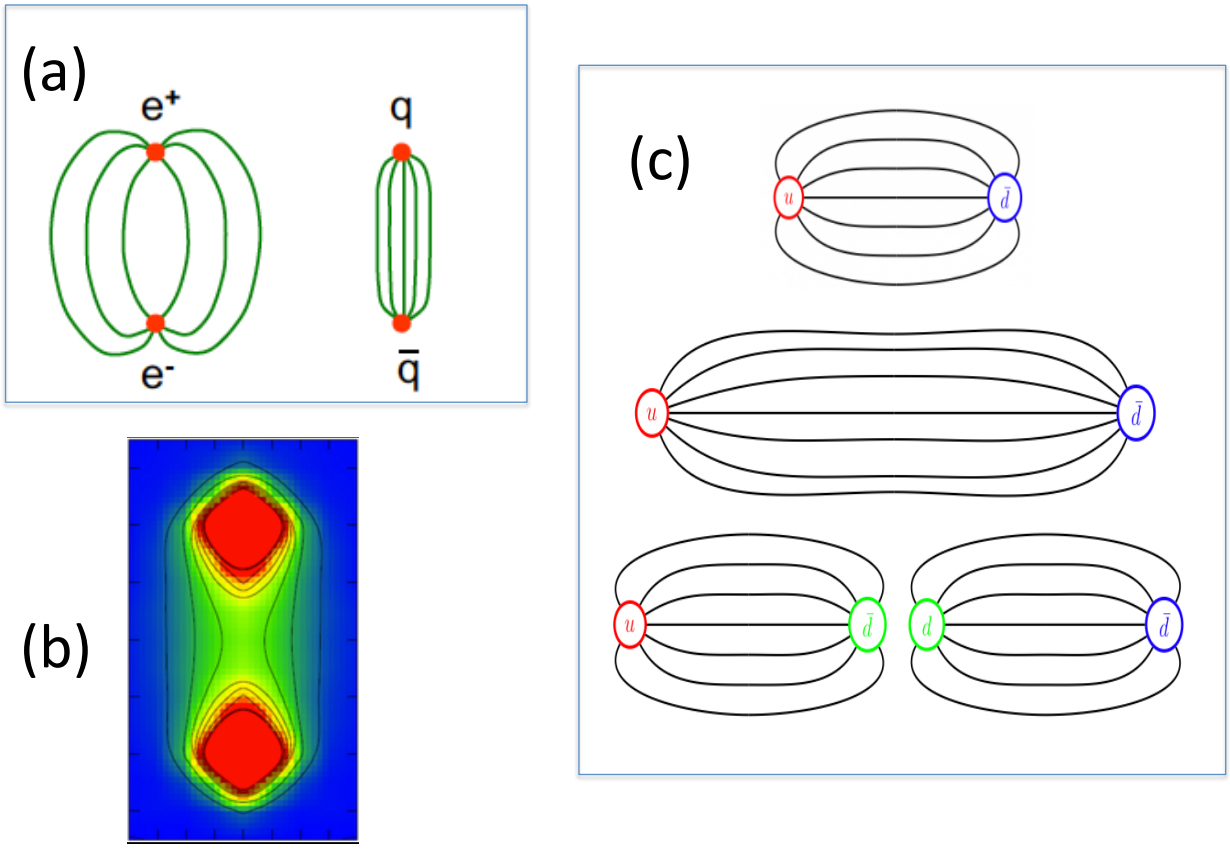
[30] Remember also that QCD is based on the SU(3) colour symmetry which has eight generators and therefore eight (not nine) gauge field quanta: there is no room for a singlet gluon in an SU(3) gauge theory.

Colour interaction

- QCD is formally very reminiscent of QED but there are important differences because, unlike photons, the gluons interact among themselves.
- In QED, the electric field of two oppositely charged particles permeates all space and diminishes quickly when the charges are separated ($\rightarrow$ **Fig** a). On the other hand, when one tries to separate a colour charge the gluon self-interaction causes the colour field between these charges to organise itself into a so-called flux tube or **colour string** ($\rightarrow$ **Fig** a,b). When stretched, the behaviour of such a string is very much like that of a rubber band.
- This behaviour of the gluon field leads to a force which is constant between the colour charges, regardless of their distance. The strength of this force is huge, about 16×10^4 N, or 16 tons.
- It follows that the colour charges cannot be fully separated since that would cost an infinite amount of energy. Instead, it will be energetically more favourable to pull a quark-antiquark pair out of the vacuum and this causes the string to break ($\rightarrow$ **Fig** c).
- EM-like behaviour at small distance, and string behaviour at large distance leads to a QCD potential that behaves roughly as:

$$V(r) \sim \frac{\alpha}{r} + kr$$

Colour strings



(a) Lines of force do spread over the entire space in the electrostatic attraction of two opposite charges (left) but in QCD the gluon self-interactions squeeze the lines of force into a flux tube (right).

(b) Result of a Lattice QCD simulation[31] showing a quark and an antiquark (red colour) bound together into a meson by a string-like gluon field configuration (green colour).[32]

(c) Schematic view of the colour-string breaking when an $u\bar{d}$ quark pair is pulled apart.

[31] In Lattice QCD the field equations are solved numerically on a discrete space-time grid. This technique allows to explore the non-perturbative regime of QCD.

[32] M. Cardoso et al., Phys. Rev. D 81, 034504 (2010).

Is colour exchange attractive or repulsive?

- We have seen that the $2 \to 2$ colour interaction at lowest order is the same as the electromagnetic interaction, provided we replace α by $\alpha_{\rm s}$, and multiply by the QCD colour factor f.
- Now for unlike-sign charges the EM force is attractive and the Coulomb potential is given by $V \sim -\alpha/r$ so that we set for $q\bar{q}$ colour interactions, at short distance,

  $$V_{q\bar{q}}(r) \sim -f\,\frac{\alpha_{\rm s}}{r}$$

  Likewise, the Coulomb potential is repulsive for like-sign charges so that we set

  $$V_{qq}(r) \sim +f\,\frac{\alpha_{\rm s}}{r}$$

- Thus we have

  $$q\bar{q} \to q\bar{q} \;=\; \begin{cases} \text{attractive when } f > 0 \\ \text{repulsive when } f < 0 \end{cases}$$

  $$qq \to qq \;=\; \begin{cases} \text{attractive when } f < 0 \\ \text{repulsive when } f > 0 \end{cases}$$

- Note that this correspondence is based on lowest order perturbation theory which is only valid at short distance where the coupling constant is small (asymptotic freedom, see next lecture). At large distance we do not know how the colour interaction exactly behaves because it is not perturbatively calculable.[33]
- Nevertheless, let us calculate the colour factors for different $q\bar{q}$ and qq colour configurations, and see if we can get some understanding whether the colour force is attractive or repulsive at short distance.

[33] Lattice QCD tries to access the non-perturbative regime by numerically solving the field equations on a discrete space-time lattice. This is notoriously difficult and requires large-scale computing, sometimes with dedicated computers engineered for this purpose.

$q\bar{q}$ in an octet state

- Here the $q\bar{q}$ pair is in one of the octet states (page 5–12), say $\mathrm{r}\bar{\mathrm{g}}$.
- From the flow diagrams on page 5–11 it is seen that, if they are different, an incoming quark colour can only be carried away by an outgoing quark while an incoming anti-quark colour can only be carried away by an outgoing anti-quark. This can also be seen from the table on page 5–10.
- So we have to consider only the transition $rg \to rg$ or, more in general, $xy \to xy$ with, as can be seen from the table, a colour factor of $f(xxyy) = -\frac{1}{6}$.
- To finalise the calculation we have to sum over all possible output states and average over the input states. Since there is only one output state for each input state, and the colour factor is the same for each of the eight members of the octet (see the exercise below), we find that $f = -\frac{1}{6} < 0$.
- We conclude that the colour force between a quark and an anti-quark in an octet state is repulsive.
- **Exercise 5.2**:

  (a) [0.5] The $q\bar{q}$ pair is in the octet state $|8\rangle \equiv |\mathrm{r}\bar{\mathrm{r}} + \mathrm{g}\bar{\mathrm{g}} - 2\mathrm{b}\bar{\mathrm{b}}\rangle/\sqrt{6}$, Calculate the color factor of the transition $|8\rangle \to |8\rangle$.

  (b) [0.5] Argue why all transitions within an $\mathrm{SU(3)_c}$ multiplet must have the same colour factor.

$q\bar{q}$ in a singlet state

- The $q\bar{q}$ singlet state is

$$\frac{1}{\sqrt{3}}(\mathrm{r\bar{r}} + \mathrm{g\bar{g}} + \mathrm{b\bar{b}})$$

- From the diagrams on page 5–11 or from the table on page 5–10 it is seen that an initial state where the quark and anti-quark colours are equal always will give rise to a final state where the quark and anti-quark colours are also equal, for example,

$$\mathrm{r\bar{r}} \to \mathrm{r\bar{r}} \quad \text{or} \quad \mathrm{r\bar{r}} \to \mathrm{g\bar{g}} \quad \text{or} \quad \mathrm{r\bar{r}} \to \mathrm{b\bar{b}}$$

- Thus we always have $c_1 = c_2$ and $c_3 = c_4$ with, as can be seen from the table,

$$\begin{aligned} f = \tfrac{1}{3} \quad &\text{if} \;\; x\;x \;\to\; x\;x \quad (3 \text{ combinations}) \\ f = \tfrac{1}{2} \quad &\text{if} \;\; x\;x \;\to\; y\;y \quad (6 \text{ combinations}) \end{aligned}$$

- Summing over all possible output states, and taking into account the normalisation factor $\sqrt{3}$ we find, since there is only one input state,

$$f = \tfrac{1}{3}\,(3 \times \tfrac{1}{3} + 6 \times \tfrac{1}{2}) = \tfrac{4}{3} > 0$$

- Because $f > 0$ we conclude that the colour force is attractive for a $q\bar{q}$ pair in a singlet state.

- To summarise, we have found that, at least at short distance, the colour force is repulsive for $q\bar{q}$ pairs in the octet state and attractive in the singlet state. So now we understand why bound $q\bar{q}$ states (mesons) would prefer to be colour singlets.

qqq in a singlet state

- We have seen (page 2–28) that baryons are colour singlet qqq states with an antisymmetric wave function in colour space

$$\begin{aligned}\psi(c_1c_2c_3) &= \mathrm{rgb} + \mathrm{brg} + \mathrm{gbr} - \mathrm{grb} - \mathrm{bgr} - \mathrm{rbg} \\ &= (\mathrm{rg} - \mathrm{gr})\mathrm{b} + (\mathrm{br} - \mathrm{rb})\mathrm{g} + (\mathrm{gb} - \mathrm{bg})\mathrm{r}\end{aligned}$$

  Each *pair* of quarks in the singlet 3-quark system is thus in a triplet state which are just the antisymmetric combinations

$$\text{Colour triplet } \bar{\mathbf{3}} = \begin{cases} (\mathrm{rg} - \mathrm{gr})/\sqrt{2} \\ (\mathrm{br} - \mathrm{rb})/\sqrt{2} \\ (\mathrm{gb} - \mathrm{bg})/\sqrt{2} \end{cases}$$

- For each triplet state we have $c_1 \neq c_2$ with, as can be seen from the table on page 5–10, $xy \to xy$ and $xy \to yx$ as the only possible transitions: each triplet state transforms into (minus) itself.

- To calculate the colour factor for a triplet state let us first introduce some shorthand and write the matrix element for $q_aq_b \to q_cq_d$ as

$$\langle c|\lambda^a|a\rangle\langle d|\lambda^a|b\rangle \equiv \langle cd||ab\rangle$$

  Taking only into account the transition $xy \to xy$, we have

$$\langle xy - yx||xy - yx\rangle = \langle xy||xy\rangle - \langle xy||yx\rangle - \langle yx||xy\rangle + \langle yx||yx\rangle$$

- Looking up in the table the colour factor of each term, and taking into account the normalisation factor $\sqrt{2}$ we get

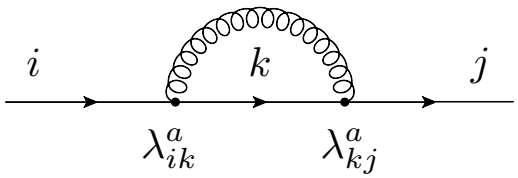
$$f = \tfrac{1}{2} \times (-\tfrac{1}{6} - \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{6}) = -\tfrac{2}{3} < 0$$

  We find that the 2-quark colour interaction in a hadron is attractive.

- **Exercise 5.3**: [1.0] What is the color factor for the transition $xy \to yx$, and why did we not consider this transition?

- We refer to Griffiths Section 8.4.2 for other $(qq)q$ colour states.

SU(N) group invariants



- Results of colour algebra are often expressed in terms of group invariants. Below we will not use the Gell-Mann matrices λ^a but, instead, the SU(N) generators $t^a \equiv \lambda^a/2$. The first invariant is called T_F and is used to fix the normalisation

$$\mathrm{Tr}(t^a t^b) = T_F \delta_{ab} \quad \text{with, by convention,} \quad T_F = \tfrac{1}{2}$$

- The colour factor of the quark self-energy diagram above introduces another invariant, C_F,

$$\sum_a t^a_{ik} t^a_{kj} = \sum_a (t^a t^a)_{ij} \equiv C_F \, \delta_{ij}$$

  where the right-hand side simply expresses the fact that the colours i and j must be the same, because of colour conservation. Taking the trace of the left-hand side gives

$$\mathrm{Tr}(t^a t^a) = \delta_{ab}\mathrm{Tr}(t^a t^b) = T_F \delta_{ab}\delta_{ab} = T_F \delta_{aa} = T_F(N^2-1)$$

  since δ_{aa} (summation over a) just gives the number of generators, which is N^2-1 for SU(N). The δ_{ij} on the right-hand side is simply the $N \times N$ unit matrix which has trace N. Thus we find

$$C_F N = T_F(N^2-1) \quad \text{or} \quad C_F = T_F \, \frac{N^2-1}{N}$$

  For SU(3), $C_F = \frac{4}{3}$.

- A third invariant shows up in the relation

$$\sum_{a,b} f_{abc} f_{abd} = C_A \, \delta_{cd} \quad \text{with} \quad C_A = N$$

Lecture notes Particle Physics II

Quantum Chromo Dynamics

6. Asymptotic Freedom

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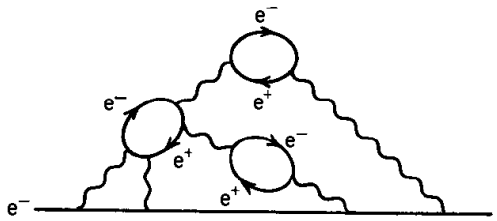
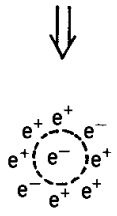
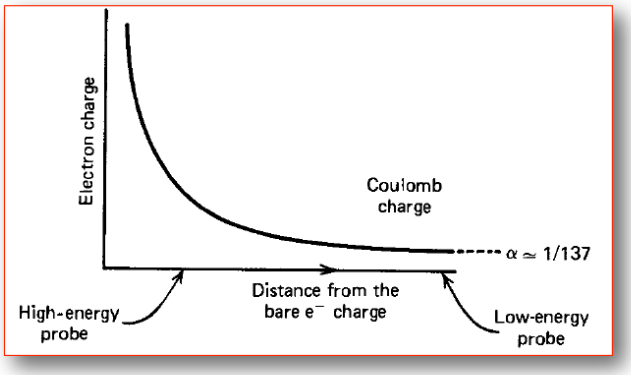
December 2, 2013

Charge screening in QED

- In QED, a charged particle like the electron is surrounded by a cloud of virtual photons and e^+e^- pairs continuously popping in and out of existence. Because of the attraction of opposite charges, the virtual positrons tend to be closer to the electron and screen the electron charge, as is indicated in the figure. This is analogous to the polarisation of a dielectric medium in the presence of a charge and is called **vacuum polarisation**.







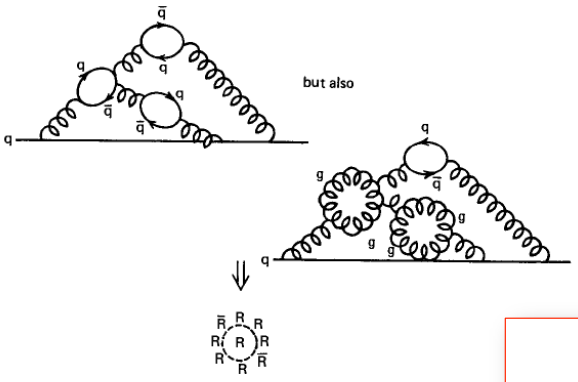
- This gives rise to the notion of an effective charge $e(r)$ that becomes smaller with larger distance.
- One says that the **beta function**

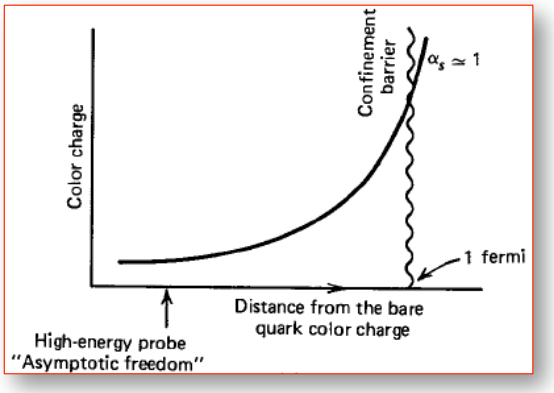
$$\beta(r) \equiv -\frac{\mathrm{d}\, e(r)}{\mathrm{d} \ln r}$$

is positive in QED.

Charge screening in QCD

- Likewise, the QCD vacuum consists of virtual $q\bar{q}$ pairs, and if this would be all, the charge screening mechanism would be the same as in QED, with a *positive* beta function.
- However, due to the gluon self coupling, the vacuum will also be filled with virtual gluon pairs as is indicated in the figure. Because the gluon cloud carries colour charge, it turns out that the effective charge becomes larger with larger distance; the beta function is *negative*. This effect is called **antiscreening**.[34]





- It turns out that the negative contribution wins over the positive contribution, so that the QCD beta function is *negative*, and the effective strong coupling becomes small a short distances.

[34] Antiscreening follows from the calculation of vacuum polarisation in QCD, which is non-trivial and beyond the scope of these lectures; unfortunately it is not so easy to *intuitively* understand the antiscreening effect.

The running coupling constant in QED and QCD

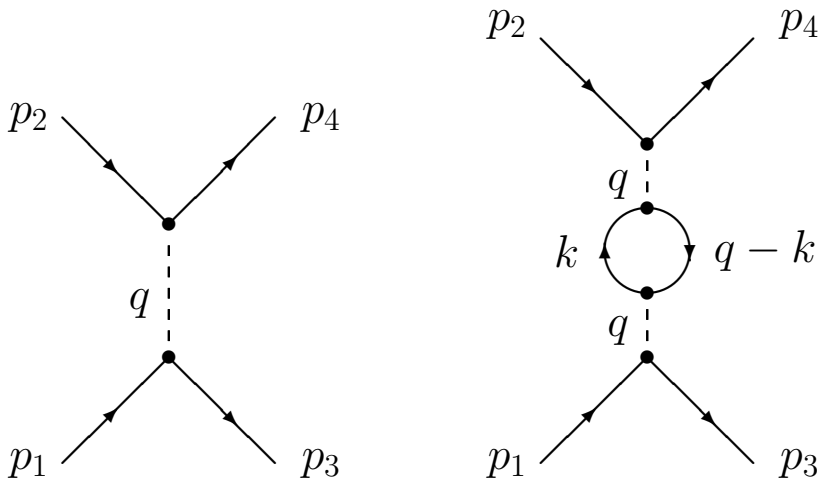
- Charge screening in QED (screening) and QCD (antiscreening) leads to the concept of a **running coupling**. In QED the coupling becomes large at (very) short distance but its effect is small.
- In QCD, the antiscreening effect causes the strong coupling to become small at short distance (large momentum transfer). This causes the quarks inside hadrons to behave more or less as free particles, when probed at large enough energies. This property of the strong interaction is called **asymptotic freedom**. Asymptotic freedom allows us to use perturbation theory, and by this arrive at quantitative predictions for hard scattering cross sections in hadronic interactions.
- On the other hand, at increasing distance the coupling becomes so strong that it is impossible to isolate a quark from a hadron (it becomes cheaper to create a quark-antiquark pair). This mechanism is called **confinement**. Confinement is verified in Lattice QCD calculations but, since it is nonperturbative, not mathematically proven from first principles.[35]
- The discovery of asymptotic freedom (1973) was a major breakthrough for QCD as the theory of the strong interaction, and was awarded the Nobel prize in 2004 to Gross, Politzer and Wilczek.[36]
- To get a more quantitative insight into asymptotic freedom, we will now first discuss the running coupling in QED.

[35] A mathematical proof will gain you a $1M millennium prize from the Clay Mathematics Institute.

[36] The Nobel lecture of Frank Wilczek can be downloaded from `http://www.nobelprize.org` and makes highly recommended reading, both as an exposé of the basic ideas, and as a record of the hard struggle.

Propagator loop correction in QED I

- Consider the two electron-muon scattering diagrams below



- The matrix element of the left diagram is ($q = p_1 - p_3 = p_4 - p_2$)

$$\mathcal{M} = -g_{\rm e}^2 \, [\bar{u}_3 \gamma^\mu u_1] \, \frac{g_{\mu\nu}}{q^2} \, [\bar{u}_4 \gamma^\nu u_2]$$

- The right diagram has a virtual $\mathrm{e}^+\mathrm{e}^-$ loop in the propagator:[37]

$$\mathcal{M} = -g_{\rm e}^2 \, [\bar{u}_3 \gamma^\mu u_1] \, \frac{-iI_{\mu\nu}}{q^4} \, [\bar{u}_4 \gamma^\nu u_2]$$

  with[38]

$$I_{\mu\nu} = -g_{\rm e}^2 \int \frac{{\rm d}^4 k}{2\pi^4} \, \frac{{\rm Tr}\left[\, \gamma_\mu \left(\not{k} + m_{\rm e} \right) \gamma_\nu \left(\not{k} - \not{q} + m_{\rm e} \right) \right]}{\left[k^2 - m_{\rm e}^2 \right] \left[(k-q)^2 - m_{\rm e}^2 \right]}.$$

- Addition of this loop diagram is thus effectively a modification of the photon propagator

$$\frac{g_{\mu\nu}}{q^2} \quad \rightarrow \quad \frac{g_{\mu\nu}}{q^2} - \frac{iI_{\mu\nu}}{q^4}.$$

- Unfortunately the integral $I_{\mu\nu}$ is divergent.

[37] The matrix element $\mathcal{M} \propto g_{\rm e}^4$ but this does not show because we assign a factor $g_{\rm e}^2$ to the propagator $I_{\mu\nu}$.
[38] In this lecture we will skip over much mathematical detail, most of which can be found in H&M Chapter 7.

Propagator loop correction in QED II

$$I_{\mu\nu} = -g_{\rm e}^2 \int \frac{{\rm d}^4k}{2\pi^4} \frac{{\rm Tr}\left[\,\gamma_\mu\,(\not{k}+m_{\rm e})\,\gamma_\nu\,(\not{k}-\not{q}+m_{\rm e})\,\right]}{[k^2-m_{\rm e}^2]\,[(k-q)^2-m_{\rm e}^2]}$$

- Because, after integration, the tensor $I_{\mu\nu}$ only depends on q_μ it must be some linear combination of $g_{\mu\nu}$ and $q_\mu q_\nu$, since these are the only tensors at our disposal. We can thus parameterise $I_{\mu\nu}$ as,

$$I_{\mu\nu} = -ig_{\mu\nu}\,q^2 I(q^2) + q_\mu q_\nu J(q^2),$$

  where $I(q^2)$ and $J(q^2)$ are some functions of q^2, see below.[39]

- **Exercise 6.1**: [1.0] Show that the term $q_\mu q_\nu J(q^2)$ does not contribute to the matrix element $\mathcal{M}$. Hint: Use the Dirac equation, page 0–16.

- Thus only the first term needs to be considered, and the function $I(q^2)$ is found to be, after a lengthy calculation,

$$I(q^2) = \frac{g_{\rm e}^2}{12\pi^2}\left\{\int_{m_{\rm e}^2}^{\infty}\frac{{\rm d}z}{z} - 6\int_0^1 {\rm d}z\, z(1-z)\ln\left[1-\frac{q^2}{m_{\rm e}^2}\,z(1-z)\right]\right\}.$$

  Indeed, the first integral is logarithmically divergent, while all non-divergent contributions are collected in the second integral.

- The first step to handle the infinity is to **regularise** the integral.

[39] The factor $-iq^2$ in the first term is just there for convenience.

Propagator loop correction in QED III

- Because $q^2 < 0$ (Exercise 6.2) we define $Q^2 \equiv -q^2$ and write

$$\boxed{\begin{aligned} I(q^2) &= \frac{g_{\rm e}^2}{12\pi^2}\left[\int_{m_{\rm e}^2}^{\infty}\frac{{\rm d}z}{z} - f\left(\frac{Q^2}{m_{\rm e}^2}\right)\right] \\ f\left(\frac{Q^2}{m_{\rm e}^2}\right) &\equiv 6\int_0^1 {\rm d}z\, z(1-z)\ln\left[1+\frac{Q^2}{m_{\rm e}^2}\,z(1-z)\right] \end{aligned}}$$

- We now impose a cutoff M so that first integral becomes finite

$$\int_{m_{\rm e}^2}^{\infty}\frac{{\rm d}z}{z} \quad\to\quad \int_{m_{\rm e}^2}^{M^2}\frac{{\rm d}z}{z} = \ln\left(\frac{M^2}{m_{\rm e}^2}\right)$$

  so that we obtain

$$I(q^2) = \frac{g_{\rm e}^2}{12\pi^2}\left\{\ln\left(\frac{M^2}{m_{\rm e}^2}\right) - f\left(\frac{Q^2}{m_{\rm e}^2}\right)\right\}.$$

- The modification of the propagator on page 6–6 can be written as

$$\frac{g_{\mu\nu}}{q^2} \quad\to\quad \frac{g_{\mu\nu}}{q^2} - \frac{iI_{\mu\nu}}{q^4} = \frac{g_{\mu\nu}}{q^2}\left[1 - I(q^2)\right]$$

- Because the propagator is always accompanied by the factor $g_{\rm e}^2$ (see $\mathcal{M}$ on page 6–6), its modification can be interpreted as a loop correction to the 'bare' electron charge $g_{\rm e}$:

$$g_{\rm e}^2 \to g_{\rm e}^2\left[1 - I(q^2)\right] = g_{\rm e}^2\left\{1 - \frac{g_{\rm e}^2}{12\pi^2}\left[\ln\left(\frac{M^2}{m_{\rm e}^2}\right) - f\left(\frac{Q^2}{m_{\rm e}^2}\right)\right]\right\}$$

Exercise 6.2:

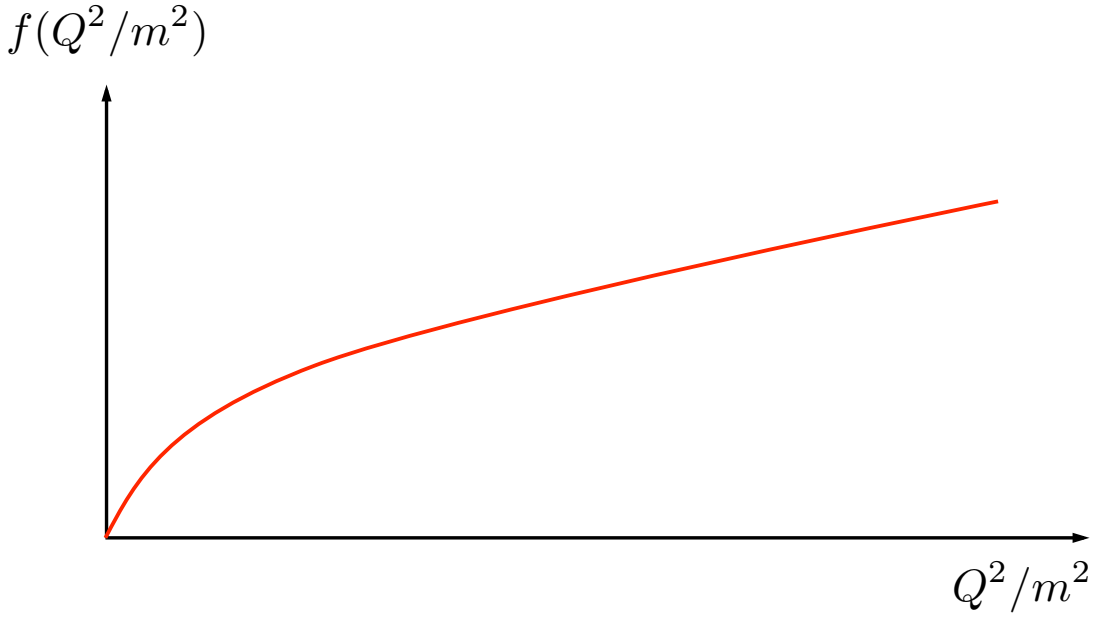
$$f\left(\frac{Q^2}{m_e^2}\right) = 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left[1+\frac{Q^2}{m_e^2}\,z(1-z)\right]$$

(a) [0.5] Show that $q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2 < 0$.

(b) [0.5] Show that,[40] for small and large Q^2,

$$f\left(\frac{Q^2}{m_e^2}\right) = \begin{cases} \frac{1}{5}\frac{Q^2}{m_e^2} & \text{for} \quad Q^2 \ll m_e^2 \\ \ln\left(\frac{Q^2}{m_e^2}\right) & \text{for} \quad Q^2 \gg m_e^2 \end{cases}$$

Note from this that f is an increasing function of Q^2, with $f(0) = 0$. Here is a sketch of f.



[40] For this, I give you the integrals $\int_0^1 \mathrm{d}z\, z(1-z) = \frac{1}{6}$ and $\int_0^1 \mathrm{d}z\, z^2(1-z)^2 = \frac{1}{30}$.

The QED running coupling constant I

- The Q^2 evolution of the bare coupling constant is thus given by

$$g_{\rm e}^2 \to g_{\rm e}^2 \left\{ 1 - \frac{g_{\rm e}^2}{12\pi^2} \left[\ln\left(\frac{M^2}{m_{\rm e}^2}\right) - f\left(\frac{Q^2}{m_{\rm e}^2}\right) \right] \right\}$$

- The first term is called the renormalised coupling constant

$$g_0^2 = g_{\rm e}^2 \left[1 - \frac{g_{\rm e}^2}{12\pi^2} \ln\left(\frac{M^2}{m_{\rm e}^2}\right) \right]$$

  so that we may write

$$g_{\rm e}^2 \to g_0^2 + \frac{g_{\rm e}^4}{12\pi^2} f\left(\frac{Q^2}{m_{\rm e}^2}\right) = g_0^2 \left\{ 1 + \frac{1}{12\pi^2} \frac{g_{\rm e}^4}{g_0^2} f\left(\frac{Q^2}{m_{\rm e}^2}\right) \right\}$$

- Up to terms $\mathrm{O}(g_{\rm e}^4)$ we may set $g_{\rm e}^4 = g_0^4$ inside the braces, so that

$$g_{\rm e}^2 \to g_0^2 \left\{ 1 + \frac{g_0^2}{12\pi^2} f\left(\frac{Q^2}{m_{\rm e}^2}\right) + \mathrm{O}(g_0^4) \right\} \equiv g_{\rm R}^2(Q^2)$$

  Here $g_{\rm R}^2(Q^2)$ is called the **running coupling constant**.

- Because $f(0) = 0$ (Exercise 6.2) we can set $g_0^2 = g_{\rm R}^2(0)$ and thus

$$\boxed{g_{\rm R}^2(Q^2) = g_{\rm R}^2(0) \left\{ 1 + \frac{g_{\rm R}^2(0)}{12\pi^2} f\left(\frac{Q^2}{m_{\rm e}^2}\right) + \mathrm{O}(g_{\rm R}^4) \right\}}$$

- The cutoff M has now disappeared from view since it is absorbed in $g_{\rm R}^2(0)$ which becomes infinitely large when we let $M \to \infty$. The mathematical technique to isolate the singularities in a perturbative calculation is called **regularisation**, cut-off regularisation in our case.

The QED running coupling constant II

- In terms of $\alpha = g_{\rm e}^2/4\pi$, the running coupling becomes

$$\alpha(Q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{Q^2}{m_{\rm e}^2}\right) + \mathrm{O}(\alpha^2)\right\}$$

- The next step is to admit that our theory cannot describe physics at asymptotically small distances so that we must replace the singular part of the calculation by measurement. This is called **renormalisation**.[41] In QED it means that $\alpha(0)$ is replaced by the fine structure constant $\alpha = 1/137$, as measured at 'large' distances of the order of the nuclear scale.

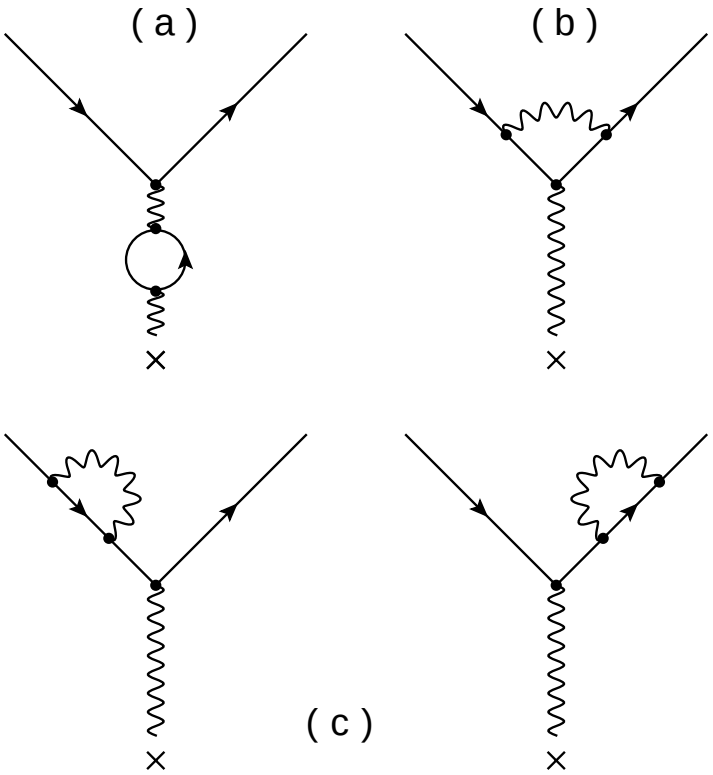
- There remains a finite correction term $f(Q^2)$ which causes the coupling to run with Q^2. This is a consequence of vacuum polarisation, as we have already discussed on page 6–3.

- It turns out that the effect of the running QED coupling constant is really small and can safely be neglected at atomic or nuclear scales. Even at large momentum transfers of $Q^2 \sim 1000$ GeV^2 at the HERA collider, the correction to α is only about 1–2%.

  Exercise 6.3: [0.5] Calculate $\alpha(Q^2)$ for $Q^2 = 1000$ GeV^2.

[41] 't Hooft and Veltman showed that this can be done consistently to all orders, without spoiling gauge invariance: they proved in general that gauge theories are *renormalisable*. They received for this work the Nobel prize in 1999.

More O(α^2) graphs



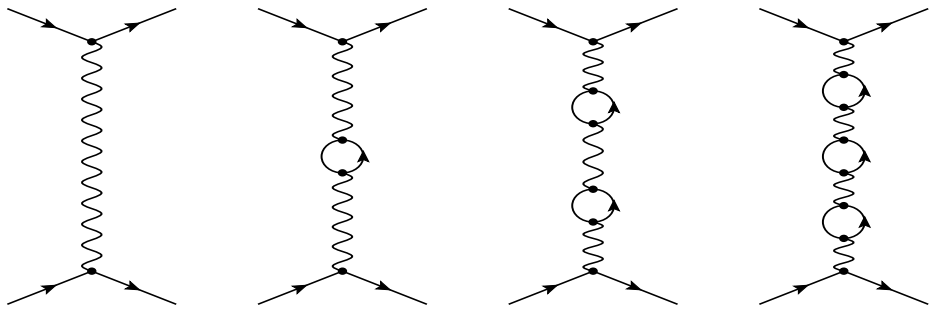
- Apart from the vacuum polarisation graph (a) there are three more divergent graphs to consider. The vertex correction (b) modifies the electron magnetic moment (see H&M Section 7.4) while the graphs (c) renormalise the electron mass.
- The three graphs (b) and (c) also contribute to the renormalisation of the electron charge. However, it turns out that these contributions cancel each other so that our previous calculation, based on diagram (a) alone, remains valid.
- This cancellation is called a **Ward identity** and is quite fortunate: without it, the graphs (c) would cause the coupling constant to be dependent on the lepton mass, and we would have different renormalisation for the electron and the muon electric charge.

The leading log approximation

- For $Q^2 \gg m_e^2$ the one-loop corrected coupling constant is given by

$$\alpha(Q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi}\ln\left(\frac{Q^2}{m_e^2}\right) + \mathrm{O}(\alpha^2)\right\}.$$

- Because of the Ward identities, only propagator loops will contribute at higher orders:



This induces a series

$$1 + X + X^2 + X^3 + \cdots = \frac{1}{1-X}$$

and indeed, from a full calculation one gets[42]

$$\boxed{\alpha(Q^2) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi]\ln(Q^2/m_e^2)} \quad \text{for} \quad m_e^2 \ll Q^2 < Q^2_{\max}}$$

The expression blows up when $\ln(Q^2/m_e^2) = 3\pi/\alpha(0)$, which occurs at an astronomical scale of $Q^2_{\max} = 10^{280}$ MeV^2.

- Although the loops are summed to all orders, there are still more complicated propagator diagrams (like multi-photon exchange between loops), which are ignored. The result given above is thus not exact, and is known as the **leading log approximation**.

[42] This is an example of **resummation** where terms in a perturbative calculation are arranged in a geometric series which is then summed up to all orders.

The renormalisation scale

- We have seen that the running QED coupling constant decreases with decreasing Q^2 (increasing distance) to the asymptotic value $\alpha(0) = 1/137$ at $Q^2 = 0$. However, we could also have specified an input value $\alpha(\mu^2)$ at some arbitrary reference scale μ^2. We will now derive the formula for the coupling constant running from $Q^2 = \mu^2$, instead of from $Q^2 = 0$. This is useful because, as we will see, the reference scale $Q^2 = 0$ cannot be used in QCD.

- From

$$\alpha(Q^2) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi] \ln(Q^2/m_{\rm e}^2)}$$

we have

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi} \ln\left(\frac{Q^2}{m_{\rm e}^2}\right)$$

and

$$\frac{1}{\alpha(\mu^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi} \ln\left(\frac{\mu^2}{m_{\rm e}^2}\right).$$

Subtraction gives

$$\frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(\mu^2)} = -\frac{1}{3\pi}\left[\ln\left(\frac{Q^2}{m_{\rm e}^2}\right) - \ln\left(\frac{\mu^2}{m_{\rm e}^2}\right)\right] = -\frac{1}{3\pi} \ln\left(\frac{Q^2}{\mu^2}\right)$$

and thus

$$\boxed{\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - [\alpha(\mu^2)/3\pi] \ln(Q^2/\mu^2)} \quad \text{for} \quad m_{\rm e}^2 \ll Q^2 < Q^2_{\rm max}}$$

- The reference scale μ^2 where we wish to specify our input value of α is called the **renormalisation scale**. Obviously, the value of $\alpha(Q^2)$ does not depend on what renormalisation scale μ^2 we chose.

The beta function

- The running coupling can be written as

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha(\mu^2)} - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right)$$

  Differentiation to $t = \ln Q^2$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\alpha}\right) = -\frac{1}{\alpha^2}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{1}{3\pi} \quad \text{or} \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} \equiv \beta(\alpha) = \frac{1}{3\pi}\alpha^2$$

- In the above, we have introduced the so-called **beta function**:

$$\frac{\mathrm{d}\alpha(Q^2)}{\mathrm{d}\ln(Q^2)} \equiv \beta(\alpha) = -(\beta_0\alpha^2 + \beta_1\alpha^3 + \beta_2\alpha^4 + \cdots)$$

  Here we have written this function as a series expansion in powers of the coupling constant,[43] where the first term corresponds to the leading log approximation. It is an important task of perturbative QED and QCD to calculate the coefficients in this expansion.[44]

- The QED one-loop beta function is $\beta = \alpha^2/3\pi > 0$. This means that the coupling constant increases with increasing Q^2 (decreasing distance). For QCD it turns out that $\beta < 0$, as we will see.

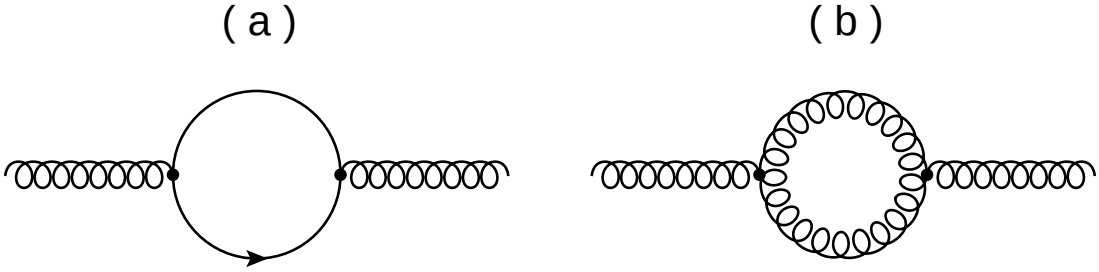
- The one-loop QED coupling constant can now be written as

$$\boxed{\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 + \beta_0\,\alpha(\mu^2)\ln(Q^2/\mu^2)} \quad \text{with} \quad \beta_0 = -\frac{1}{3\pi}}$$

[43] The minus sign in front of the series expansion is a matter of convention.

[44] This is an explosive business: for QCD the 4-loop coefficient β_3 has been calculated (at Nikhef!) by evaluating 50.000 Feynman diagrams, using sophisticated symbolic algebra programs (also developed at Nikhef).

The running strong coupling constant α_s



- To calculate the propagator loop correction in QCD, we do not only have to consider quark loops (a), like electron loops in QED, but also gluon loops (b). The quark loop will give rise to a positive contribution to the beta function (screening) while the gluon loop contribution will be negative (antiscreening), see also the discussion on charge screening on page 6–4.

- The formula for the one-loop running coupling constant in QCD is

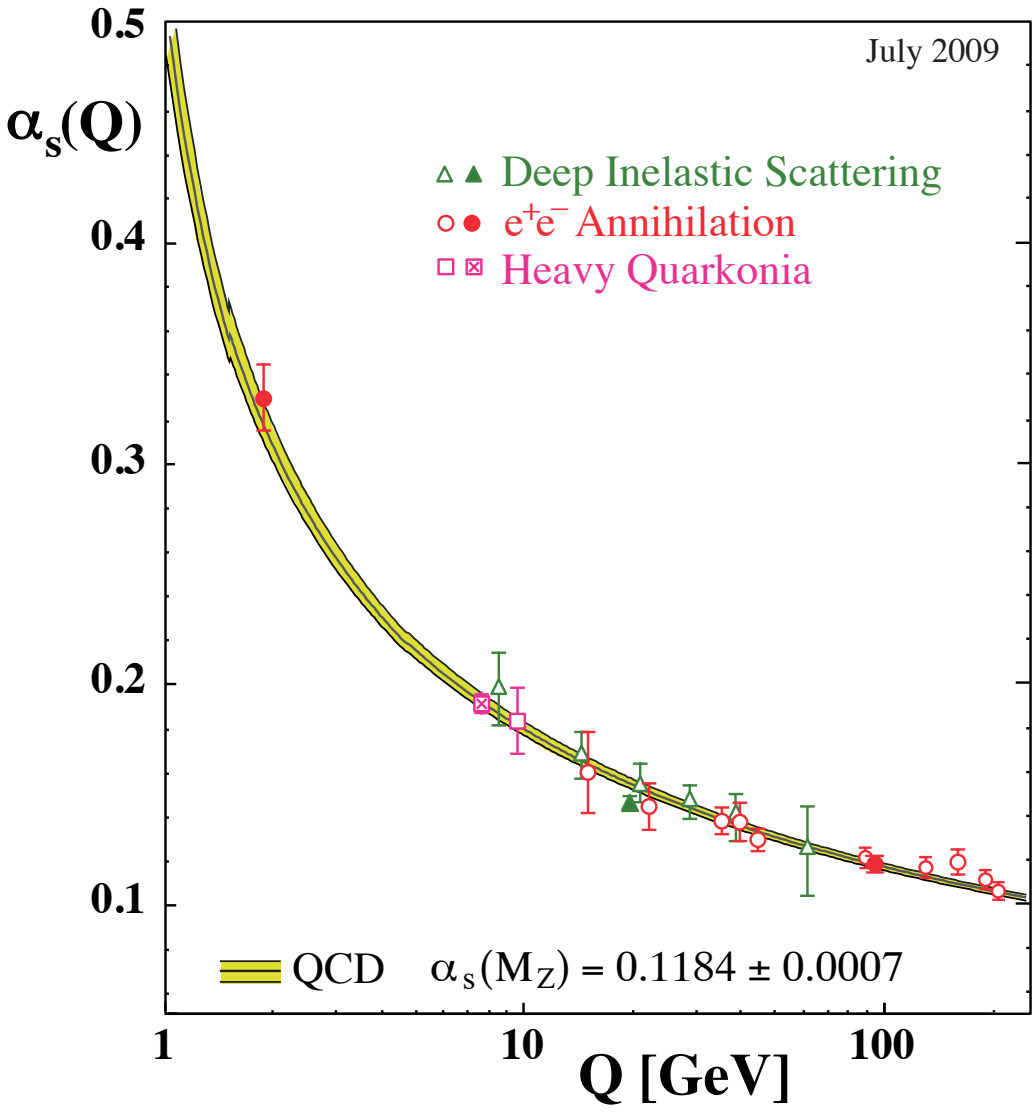
$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0\,\alpha_s(\mu^2)\ln(Q^2/\mu^2)} \quad \text{with} \quad \beta_0 = \frac{11N_c - 2n_f}{12\pi}$$

  Here N_c is the number of colours (3) and n_f is the number of flavours (6 in the standard model).

- The second factor $-2n_f/12\pi$ in β_0 comes from diagram (a). It is the same (modulo a colour factor) as the coefficient $\beta_0 = -1/3\pi$ in QED and causes screening. The first factor $11N_c/12\pi$ comes from diagram (b) and causes anti-screening.

- Clearly with $N_c = 3$ and $n_f = 6$, the antiscreening wins over the screening, with $\beta_0 > 0$ and a slope $\beta(\alpha_s) = -\beta_0\alpha_s^2 < 0$. This means that α_s decreases with Q^2 ($\rightarrow$ **fig**).

The running strong coupling constant



1. Note that α_s is large, compared to the electromagnetic coupling constant $\alpha = 1/137$: strong interactions *are* strong.
2. The running is also strong, compared to a few percent effect at large Q^2 in QED.
3. The running of α_s is beautifully confirmed by experiment.
4. For $Q^2 \sim 1$, $\alpha_s \sim 1$ and perturbative QCD breaks down. Usually $Q^2 \sim$ 5–10 GeV^2 is considered to be reasonable lower bound for perturbation theory to apply.

The QCD scale parameter Λ

$$\alpha_{\rm s}(Q^2) = \frac{\alpha_{\rm s}(\mu^2)}{1+\beta_0\,\alpha_{\rm s}(\mu^2)\ln(Q^2/\mu^2)} \quad \text{with} \quad \beta_0 = \frac{11N_{\rm c}-2n_f}{12\pi}$$

- Because $\beta_0 > 0$ we find that $\alpha_{\rm s} \to 0$ for $Q^2 \to \infty$. This vanishing coupling is called **asymptotic freedom** and is responsible for the fact that quarks behave like free particles at short distances (large momentum transfers) as is observed in deep inelastic scattering experiments.

- The expression for the running coupling constant can be simplified when we define the QCD **scale parameter** Λ as follows

$$\frac{1}{\alpha_{\rm s}(Q^2)} = \frac{1}{\alpha_{\rm s}(\mu^2)} + \beta_0 \ln\left(\frac{Q^2}{\mu^2}\right) \equiv \beta_0 \ln\left(\frac{Q^2}{\Lambda^2}\right)$$

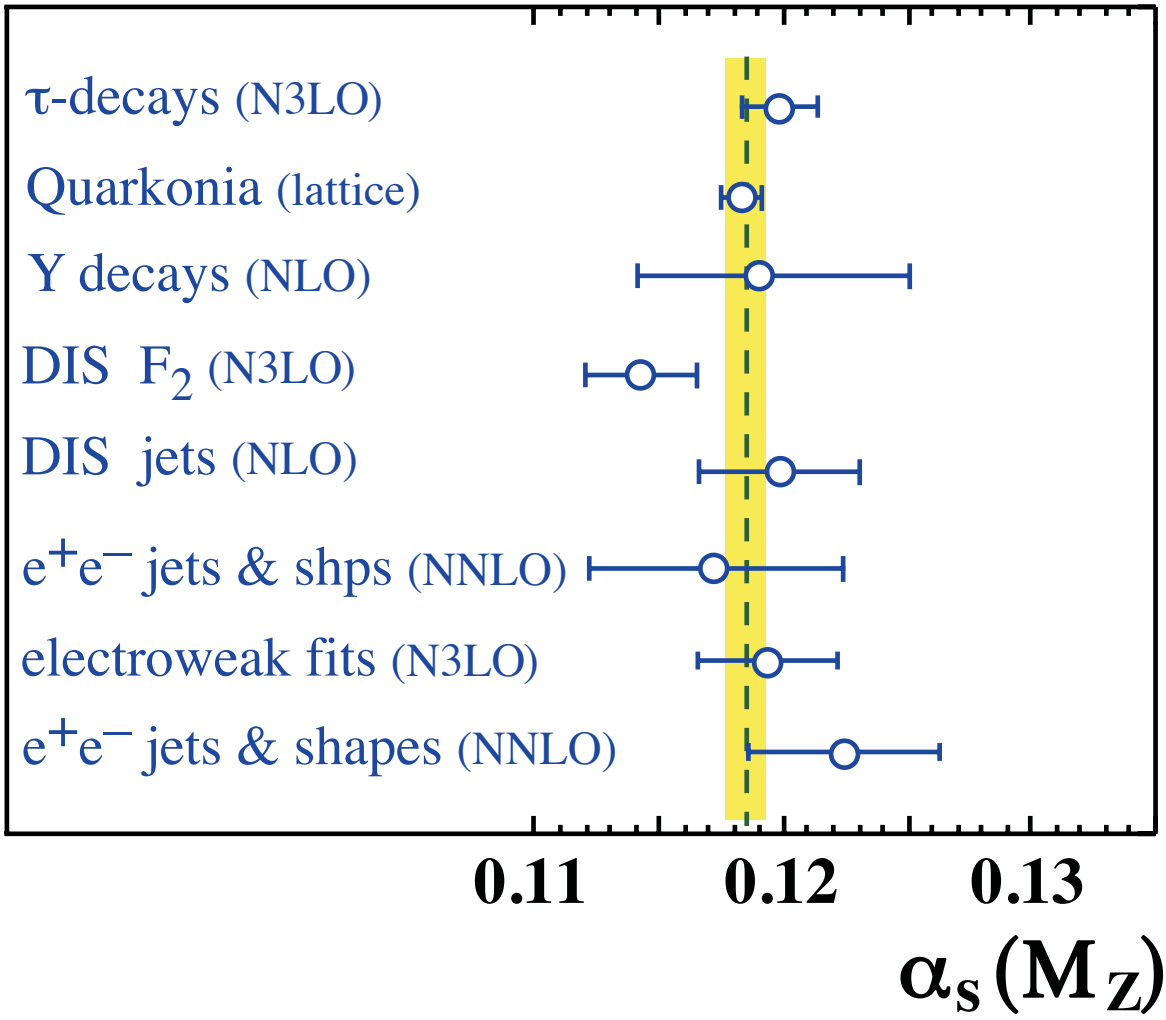
  The parameter Λ is thus equal to the scale where the first term on the right-hand side vanishes, that is, the scale where $\alpha_{\rm s}(\mu^2)$ becomes infinite. Now we may write

$$\boxed{\alpha_{\rm s}(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}}$$

- Experimentally, the value of Λ is found to be about 300 MeV, but the scale parameter is nowadays out of fashion because it cannot be defined unambiguously beyond 1-loop order. Instead, it is now common practise to not quote a value for Λ, but a value for $\alpha_{\rm s}$ at the mass of the Z ($\rightarrow$ **fig**). This is unambiguous at all orders.

- At Q^2 values close to Λ, the coupling constant becomes large and perturbative QCD breaks down.

The strong coupling constant α_s at m_Z^2



Today's average: $\alpha_s(m_Z^2) = 0.1184 \pm 0.0007$ (0.6% accuracy)

Renormalisation scale dependence

$$\alpha_{\rm s}(Q^2) = \frac{\alpha_{\rm s}(\mu^2)}{1 + \beta_0\,\alpha_{\rm s}(\mu^2)\ln(Q^2/\mu^2)}$$

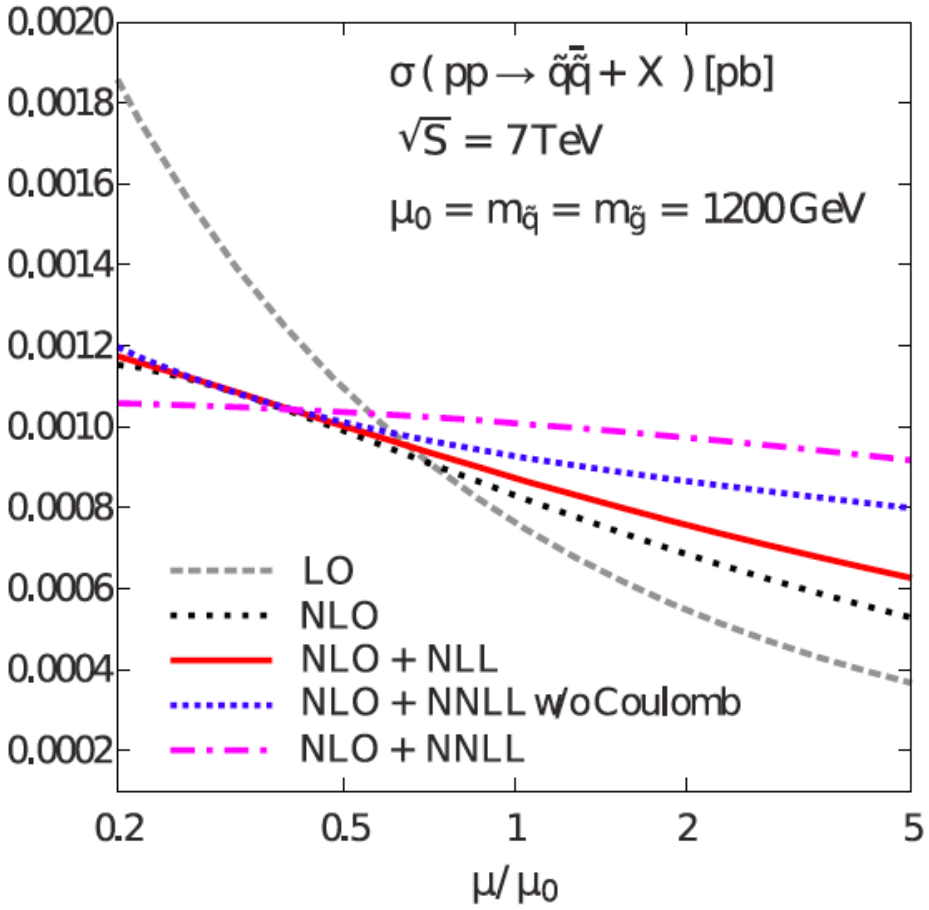
- We have seen that apart from our physical scale Q^2 we have introduced the renormalisation scale μ^2 where we wish to specify our input value of $\alpha_{\rm s}$.
- Clearly it should not matter which scale μ^2 we chose so that any perturbatively calculated observable Σ should satisfy

  $$\frac{\partial\Sigma(Q^2,\mu^2,\ldots)}{\partial\ln\mu^2} = 0,$$

  which is called the **renormalisation group equation**.
- In practice, the renormalisation group equation will not be satisfied since it holds only if we calculate our observable to all orders, which is never the case of course. However, the renormalisation scale dependence can be used to judge the accuracy of the calculation because a vanishing μ^2 dependence is a sign that enough terms are being included in the perturbative expansion ($\rightarrow$ **fig**.)
- Please be aware of the distinction between the Q^2 dependence,[45] which is *physical* (caused by vacuum polarisation, for instance), and the μ^2 dependence, which is an *artefact* of our incomplete perturbative expansion.

[45] The relevant physical scale depends on the process under study and is usually taken to be momentum transfer, centre of mass energy, or transverse momentum.

Squark production at the LHC



This plot shows the renormalisation scale dependence of the squark production cross-section at the LHC. As expected, this dependence becomes weaker as more terms are included in the perturbative expansion (LO and NLO label leading and next-to-leading order in α_s, while NLL and NNLL label two re-summation prescriptions).

Lecture notes Particle Physics II

Quantum Chromo Dynamics

7. Soft and Collinear Singularities

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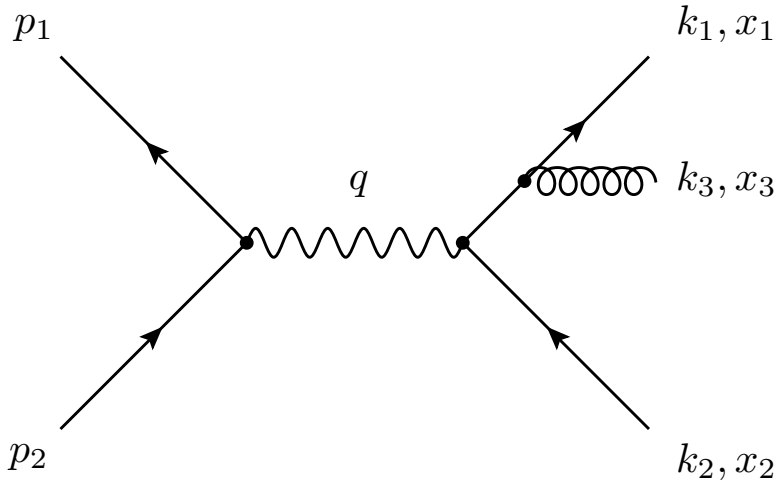
December 2, 2013

Can perturbative QCD predict anything?

- We have seen that asymptotic freedom allows us to use perturbation theory to calculate quark and gluon interactions at short distances. But is this enough to arrive at predictions for experimental observables?
- The answer is 'no', because the detectors in an experiment can only observe hadrons and not the constituent quarks and gluons.
- We will see that we need two more things, if we want to make the connection between theory and experiment:
  1. either infrared safety,
  2. or factorisation.
- These concepts are intimately related to the separation of the short and long distance aspects of the strong interaction.
  - **Infrared Safety**: There is a class of observables that do not depend on long distance physics and are therefore calculable in perturbative QCD.
  - **Factorisation**: There is a wide class of processes that can be factorised in a universal long distance piece (nonperturbative, but process independent) and a short distance piece that is calculable in perturbative QCD.
- To understand these ideas we will, in this section, study the lowest order QCD correction to the process $e^+e^- \to q\bar{q}$.

The process $e^+e^- \to q\bar{q}g$



- Consider the process $e^+e^- \to q\bar{q}g$. We have the following kinematic variables:
  1. The four-momentum $q = p_1 + p_2$ of the virtual photon. The square of the centre-of-mass energy is $s = q^2 = q^\mu q_\mu$.
  2. The outgoing four-momenta k_1, k_2 and k_3. The energies of the outgoing partons[46] in the centre-of-mass frame are $E_i = k_i^0$.
- We define the energy fractions by

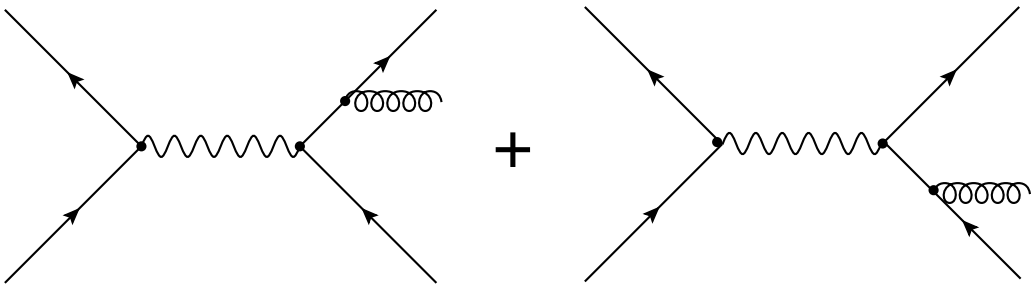
$$x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2q \cdot k_i}{s}$$

Exercise 7.1: [0.5] Show that $q \cdot k_i = E_i \sqrt{s}$ and that $\sum_i x_i = 2$.

From $\sum_i x_i = 2$ it follows that only two of the x_i are independent.

[46] We use the name 'parton' for both quark and gluon.

Singularities in the cross section



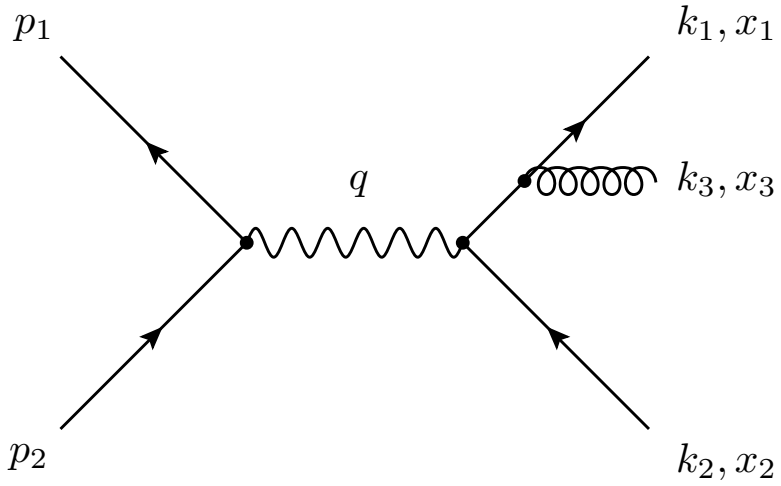
- To calculate the cross section $\sigma(\mathrm{e^+e^-} \to q\bar{q}\mathrm{g})$ two Feynman graphs have to be taken into account, one where the gluon is radiated from the quark and another where it is radiated from the antiquark. The calculation of the cross section is rather lengthy so we will not give it here; you can find it in H&M Section 11.5.
- The result is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = \sigma_0 \frac{2\alpha_\mathrm{s}}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

  Here $\sigma_0 = \sigma(\mathrm{e^+e^-} \to \mathrm{hadrons}) = (4\pi\alpha^2/s)\sum e_i^2$, see page 2–30.
- There are three singularities in this cross section
  1. $(1-x_1) = 0$
  2. $(1-x_2) = 0$
  3. $(1-x_1) = (1-x_2) = 0$

  We will now have a look where these singularities come from.

More kinematics



- In the following we will neglect the quark masses ($k_i^2 = 0$) so that

$$(k_i + k_j)^2 = k_i^2 + k_j^2 + 2k_i \cdot k_j = 2k_i \cdot k_j$$

- Denote by θ_{ij} the angle between the momenta of partons i and j. Then we can relate these angles to the energy fractions as follows

$$2k_1 \cdot k_2 = (k_1 + k_2)^2 = (q - k_3)^2 = s - 2q \cdot k_3 \quad \rightarrow$$

$$2E_1E_2(1 - \cos\theta_{12}) = s(1 - x_3)$$

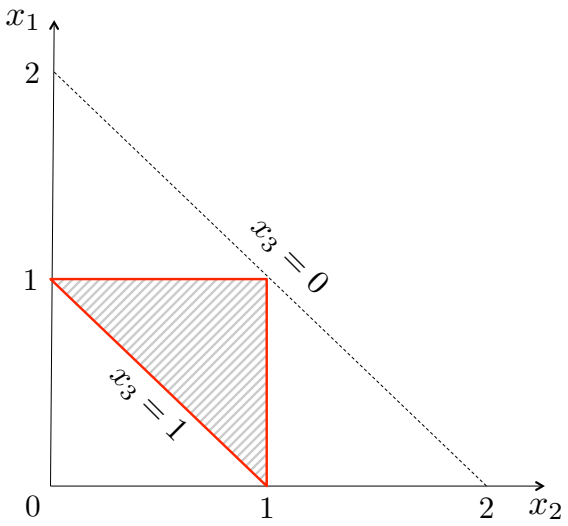
Exercise 7.2: [×] Show that $k_i \cdot k_j = E_iE_j(1 - \cos\theta_{ij})$.

- Dividing by $s/2$ and repeating the above for the angles between other pairs of particles gives

$$x_1x_2(1 - \cos\theta_{12}) = 2(1 - x_3)$$
$$x_2x_3(1 - \cos\theta_{23}) = 2(1 - x_1)$$
$$x_3x_1(1 - \cos\theta_{31}) = 2(1 - x_2)$$

Phase space

- From the above it follows that $0 \leq x \leq 1$. Together with the constraint $x_3 = 2 - x_1 - x_2$ this implies that the allowed region for (x_1, x_2) is the triangle shown below.



- From

$$x_1 x_2 (1 - \cos\theta_{12}) = 2(1 - x_3)$$
$$x_2 x_3 (1 - \cos\theta_{23}) = 2(1 - x_1)$$
$$x_3 x_1 (1 - \cos\theta_{31}) = 2(1 - x_2)$$

we find that the collinear configurations are related to the x_i by

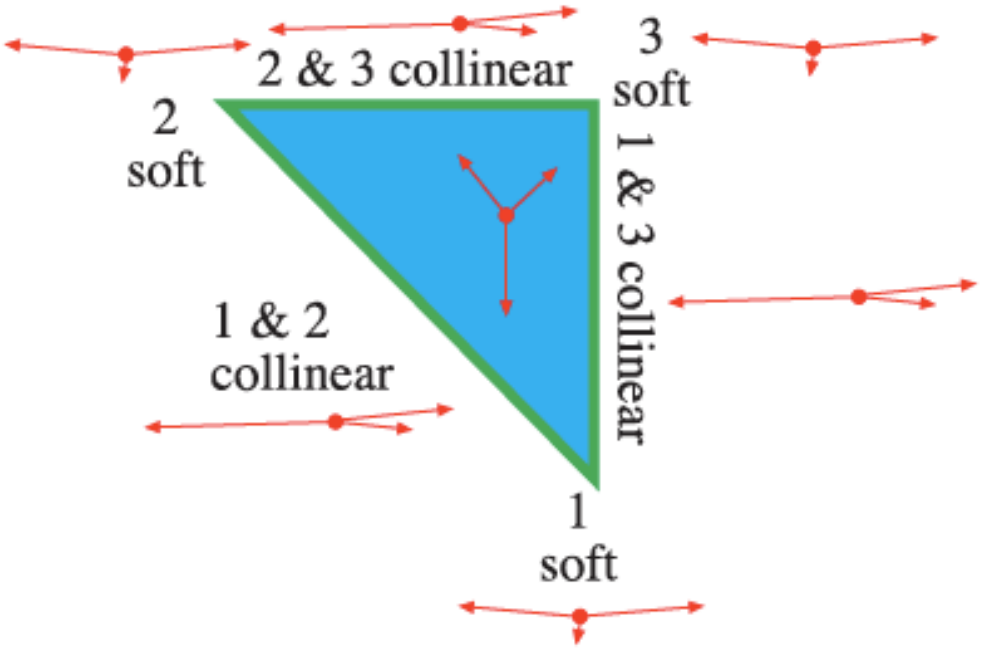
$$\theta_{12} \to 0 \quad \Leftrightarrow \quad x_3 \to 1$$
$$\theta_{23} \to 0 \quad \Leftrightarrow \quad x_1 \to 1$$
$$\theta_{31} \to 0 \quad \Leftrightarrow \quad x_2 \to 1$$

Thus when $x_i \to 1$ then $\theta_{jk} \to 0$, that is, j and k are collinear.

Exercise 7.3: [0.5] Show that when $x_i \to 1$ then i is back-to-back with both j and k. Also show that $x_i \to 0$ implies $E_i \to 0$: particle i becomes soft. What can you say about the relative directions of the particles j and k in this case?

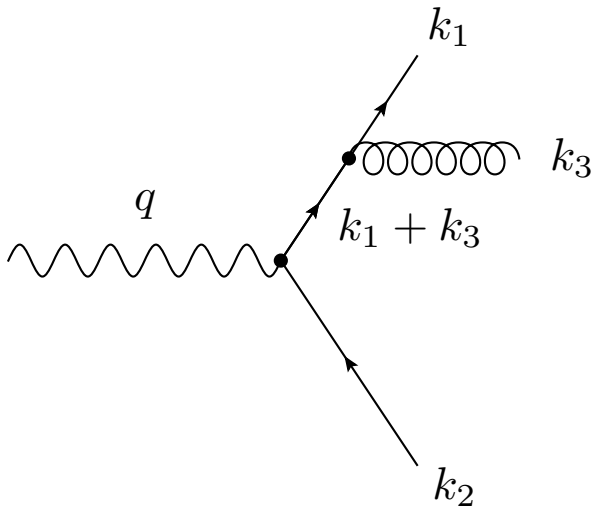
Three-parton configurations

- This plot shows the three-parton configurations at the boundaries of phase space.



- Edges: two partons collinear: $\theta_{ij} \to 0 \;\Leftrightarrow\; x_k \to 1$.
- Corners: one parton soft $x_i \to 0 \;\Leftrightarrow\; E_i \to 0$ (other two partons are back-to-back).
- Note that at the boundaries of phase space $2 \to 3$ kinematics goes over to $2 \to 2$ kinematics.

Origin of the singularities



- Where do the singularities actually come from? This is easy to see by noting that internal quark momentum is $(k_1 + k_3)$, giving a propagator term $\sim 1/(k_1 + k_3)^2$ in the cross section. Now

$$(k_1 + k_3)^2 = 2k_1 \cdot k_3 = 2E_1 E_3 \,(1 - \cos\theta_{31})$$

  so that the propagator term evidently is singular when $\theta_{31} \to 0$ and when $E_3 \to 0$.

- The collinear singularity at $\theta_{31} \to 0$ and $E_3 \to 0$ can be made explicit by rewriting the cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_3\,\mathrm{d}\cos\theta_{31}} = \sigma_0\,\frac{2\alpha_\mathrm{s}}{3\pi}\,\frac{f(E_3,\theta_{31})}{E_3\,(1-\cos\theta_{31})}.$$

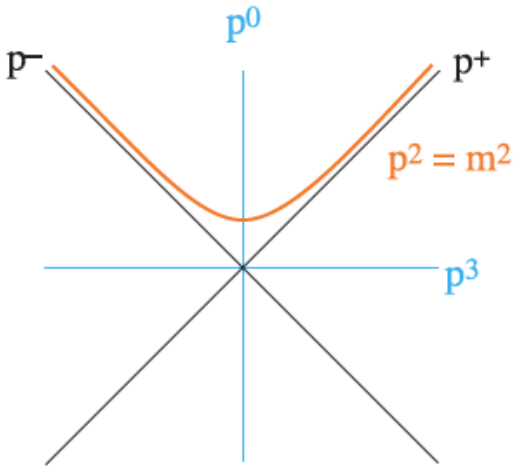
  Here $f(E_3, \theta_{31})$ is a rather complicated function that turns out to be finite when $E_3 \to 0$ or $\theta_{31} \to 0$.

- Clearly we get a logarithmic divergence when we attempt to integrate over θ_{31} with E_3 kept fixed or over E_3 with θ_{31} kept fixed.

Infrared singularities

- Are we seeing here a breakdown of perturbative QCD? The answer is no: the problem is that we are trying to work with cross sections on the parton level that are not **infrared safe**.
- These infrared problems always show up when $2 \to 3$ kinematics becomes $2 \to 2$ kinematics. We have seen that this happens at the edges of phase space when two partons become collinear or one parton becomes soft. Another way of stating this is that the internal propagator goes on shell: $(k_1 + k_3)^2 \to 0$.
- Please note that infrared divergences are omnipresent in QCD (and also in QED) and are by no means limited to $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}\mathrm{g}$.
- It is useful to get a space-time picture with the help of **light cone coordinates**. We will then see that the divergences are caused by long distance interactions.

Intermezzo: light cone coordinates



- The light cone components of a four-vector a are defined by

$$a^{\pm} = (a^0 \pm a^3)/\sqrt{2}$$

The vector is then be written as $a = (a^+, a^-, a^1, a^2) = (a^+, a^-, \boldsymbol{a}_{\rm T})$.[47]

- **Exercise 7.4**: [1.0] Show that

$$a \cdot b = a^+ b^- + a^- b^+ - \boldsymbol{a}_{\rm T} \cdot \boldsymbol{b}_{\rm T}, \quad \text{and} \quad a^2 = 2a^+ a^- - \boldsymbol{a}_{\rm T}^2$$

Show that the vector transforms under boosts along the z axis like

$$a'^+ = a^+ e^{\psi}, \quad a'^- = a^- e^{-\psi}, \quad \boldsymbol{a}'_{\rm T} = \boldsymbol{a}_{\rm T}$$

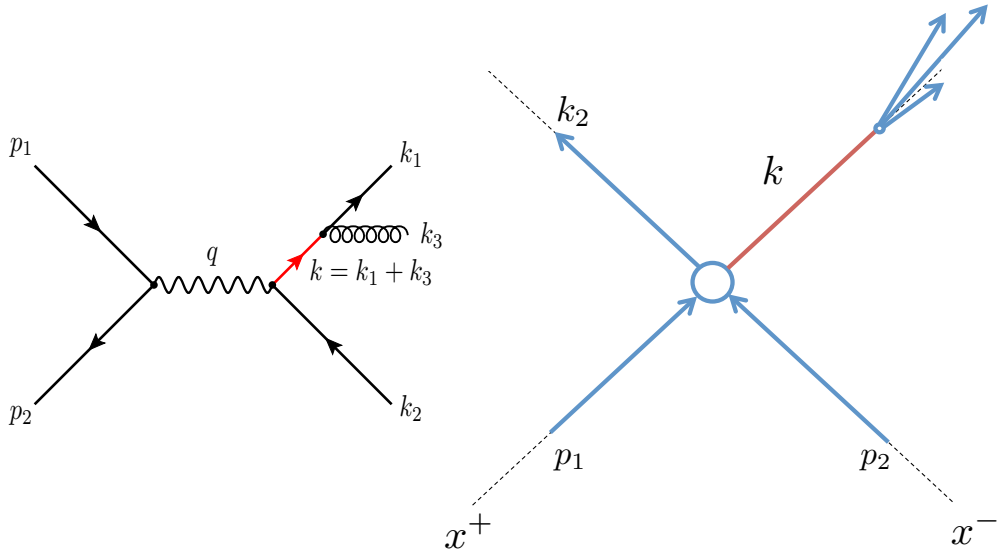
with $\psi = \frac{1}{2}\ln[(1-\beta)/(1+\beta)]$. How does a transform under two successive boosts β_1 and β_2?

- One often chooses the z axis such that, perhaps after a boost, a particle or a group of particles have large momenta along that axis. For these particles p^+ is large and (since they are on the mass shell)

$$p^- = \frac{m^2 + \boldsymbol{p}_{\rm T}^2}{2p^+} \quad \text{is small.}$$

[47] Note that a^+ and a^- are *not* 4-vector components.

Space-time picture of the singularities



- To see what happens when $k^2 = (k_1 + k_3)^2$ becomes small (goes on-shell), we choose the z axis along k with k^+ large and $\boldsymbol{k}_\mathrm{T} = 0$. Thus $k^2 = 2k^+k^- \to 0$ when k^- becomes small.
- In QFT, the Green functions (propagators) in momentum space are related to those in coordinate space by a Fourier transform:

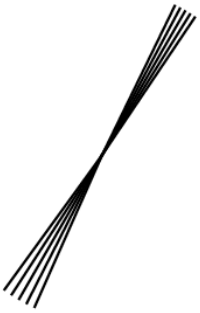
$$\begin{aligned} S_F(x) &= \int \mathrm{d}^4k \, \exp(-ikx) S_F(k) \\ &= \int \mathrm{d}^4x \exp[-i(k^+x^- + k^-x^+ - \boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{x}_\mathrm{T})] \, S_F(k) \end{aligned}$$

  Because k^+ is large and k^- is small, the contributing values of x have small x^- and large x^+. This means that the quark propagates a long distance in the x^+ direction before decaying in a quark-gluon pair, as is indicated in the space-time diagram above.
- It follows that the singularities that can lead to divergent perturbative cross-sections arise from interactions that happen a long time after the creation of a quark-antiquark pair.

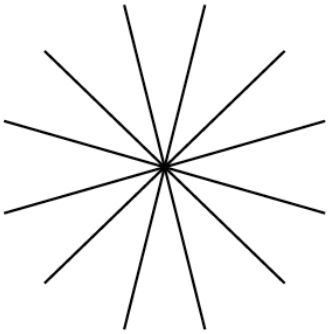
Infrared safe observables

- We have seen that soft/collinear singularities arise from interactions that happen a long time after the creation of the quark-antiquark pair and that perturbation theory cannot handle this long-time physics. But a detector is a long distance away from the interaction so we must somehow take long-time physics into account in our theory.
- Fortunately there are measurements that are *insensitive* to long-time physics. These are called **infrared safe** observables. We have seen that soft/collinear singularities appear when $2 \to 3$ kinematics reduces to $2 \to 2$ kinematics at the boundaries of phase space. Therefore a meaningful infrared safe observable must be insensitive to the indistinguishable $2 \to 2$, $2 \to 3$ origin of the long-distance interactions.
- The most well known example of an infrared safe observable is the total cross section $\sigma(e^+e^- \to \text{hadrons})$, see page 2–30. This cross section is infrared safe because it is a totally *inclusive* quantity (we sum over all particles in the final state and don't care how many there are) and the transition from partons to the hadronic final state in a given event always occurs with unit probability, whatever the details of the long-time hadronisation process.
- As an example of another infrared safe variable used in the analysis of e^+e^- collisions, we mention the thrust event shape variable.

Thrust



pencil-like: $T \lesssim 1$



spherical: $T \gtrsim 1/2$

- Thrust is an event shape variable, used to discriminate between pencil-like and spherical events.
- Thrust is defined by

$$T = \max_{\hat{\boldsymbol{u}}} \frac{\sum_i |\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}|}{\sum_i |\boldsymbol{p}_i|}$$

  Here the sum runs over all particles i in the event, and the unit vector $\hat{\boldsymbol{u}}$ is varied to maximise the sum of the projections of the 3-momenta $\boldsymbol{p}_i$ on $\hat{\boldsymbol{u}}$.
- So why is thrust infrared safe?

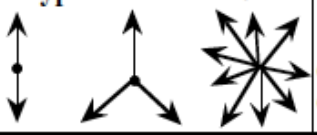
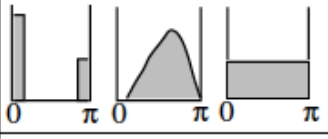
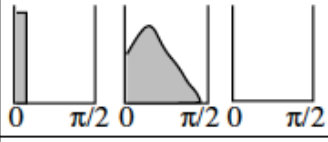
  1. Zero-momentum particles do not contribute to T.
  2. A collinear splitting does not change the trust:

$$|(1-\lambda)\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}| + |\lambda \boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}| = |\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}|$$

$$|(1-\lambda)\boldsymbol{p}_i| + |\lambda \boldsymbol{p}_i| = |\boldsymbol{p}_i|$$

IR safe observables used in e^+e^- physics

Here is a list of more infrared safe observables.

Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\Sigma_i \lvert \vec{p}_i \vec{n} \rvert}{\Sigma_i \lvert \vec{p}_i \rvert} \right)$	1	≥2/3	≥1/2	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	≤1/3	≤1/√2	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	≤1/2	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	≤1/3	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \le ... \le Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\Sigma_i p_i^\alpha p_i^\beta}{\Sigma_i p_i^2}$	0	≤3/4	≤1	none (not infrared safe)
Aplanarity	$A = 1.5\,Q_1$	0	0	≤1/2	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = \left(\Sigma^i E_i^2 - \Sigma_i \vec{p}_i^2\right)_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	 0 0	 ≤1/3 ≤1/3	 ≤1/2 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\sum_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2 \sum_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	≤1/(2√3) ≤1/(2√3)	≤1/(2√2) ≤1/(2√3)	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi + \frac{\Delta\chi}{2}}^{\chi - \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$				(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$				$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

Lecture notes Particle Physics II

Quantum Chromo Dynamics

8. The Structure of the Proton

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December 3, 2013

QCD as a predictive theory

- We have seen that perturbative QCD suffers from collinear and soft singularities but that so-called infrared-safe observables are insensitive to the number of partons in the final state which means that they are insensitive to the collinear merging of two daughter partons or the disappearance of one daughter parton in the soft limit. It can be shown that for infrared-safe observables there is a precise cancellation of the soft and collinear divergences in the contributing Feynman diagrams at all orders.[48]
- However, it is clear that if QCD predictions would be restricted to infrared-safe observables only, it would not be a very useful theory. Fortunately, there is a large class of cross-sections that **factorises** into a perturbatively calculable infrared-safe short-distance (hard) part, and a long-distance (soft) part which is infrared-singular but has the virtue of being *universal*, that is, process-independent.
- An example is the **deep-inelastic scattering** (DIS) cross-section

  $$\sigma(\ell\,\mathrm{p} \to \ell' X)$$

  where a lepton ℓ (electron, muon or neutrino) scatters on a proton which breaks-up into the (uspecified) system X. The DIS cross-section can be factorised in a hard lepton-quark cross-section and a so-called **parton distribution** which is process independent, but non-perturbative and infrared-singular. However, it can be replaced by a measurement, like we did for the running coupling where *ultraviolet divergences* were replaced by a measured value of the coupling constant at some renormalisation scale (Section 6).
- In this section we will study DIS in more detail and see how it leads to the famous **quark-parton model** of the proton.

[48] So-called Bloch-Nordsieck and Kinoshita-Lee-Lauenberg theorems.

Inward bound

- One way to probe the internal structure of matter is to bombard it with high energy particles, and then see what happens. For instance, in the Rutherford experiment (1911), alpha particles (helium nuclei) were deflected on a thin gold foil. Rutherford found that the deflections followed his famous inverse $\sin^4(\theta/2)$ law (see page 0–13), and concluded that the alpha particles were scattered from electrically charged point-like nuclei inside the gold atoms.
- Experiments using probes of higher energy later revealed that the point-like scattering distributions were damped by **form factors** which are essentially the Fourier transform of a charge *distribution*. This clearly showed that nuclei are not point-like and indeed, after the discovery of the neutron by Chadwick (1932), it became clear that nuclei are bound states of protons and neutrons.
- Also the protons and neutrons were found not to be point-like and a real breakthrough came with a series of *deep inelastic scattering* experiments in the 1960's at SLAC, where electron beams were scattered on proton targets at energies of about 20 GeV, large enough to reveal the proton's internal structure.
- The SLAC experiments showed that the electrons were scattering off quasi-free point-like constituents inside the proton which were soon identified with quarks. This was the first time that quarks were shown to be dynamical entities, instead of bookkeeping devices to classify the hadrons (Gell-Mann's eightfold way). The Nobel prize was awarded in 1990 for this spectacular discovery, and the lectures of Friedman, Kendall and Taylor are a fascinating record of the struggle to understand what these data did tell us.

Deep inelastic scattering (DIS)

- The pioneering SLAC experiments were followed by a series of other **fixed-target** experiments[49] with larger energies at CERN (Geneva) and at Fermilab (Chicago), using electrons, muons, neutrino's and anti-neutrinos as probes.

- The largest centre-of-mass energies were reached at the HERA collider in Hamburg (1992–2007) with counter-rotating beams of 27 GeV electrons and 800 GeV protons.

  Exercise 8.1: [1.0] Calculate the centre-of-mass energies at SLAC (20 GeV electrons on stationary protons) and at HERA (27 GeV electrons on 800 GeV protons). You can neglect the electron mass and, at HERA, also the proton mass.

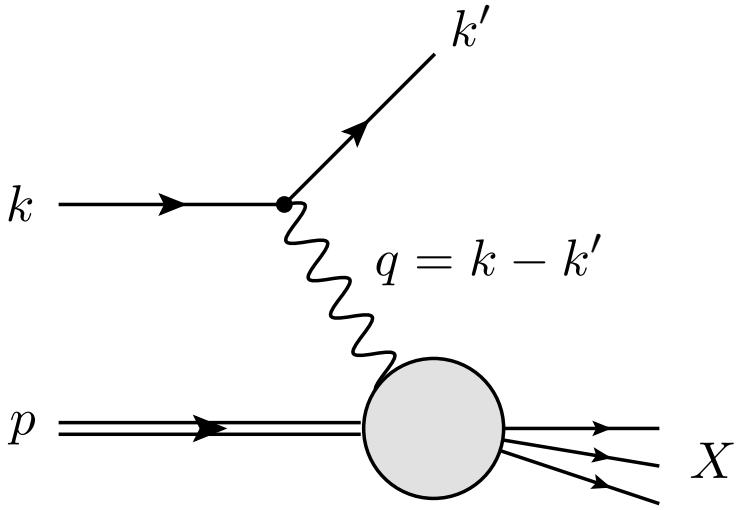
- Deep inelastic scattering (DIS) data are very important since they provide detailed information on the momentum distributions of the **partons** (quarks and gluons) inside the proton.

- Parton distributions are crucial ingredients in theoretical predictions of scattering cross-sections at hadron colliders like the Tevatron (Fermilab, proton-antiproton at 2 TeV) or the LHC (CERN, proton-proton at 8–14 TeV). The reason for this is simple: the colliding (anti)protons have a fixed centre-of-mass energy but not the colliding partons, since their momenta are distributed inside the (anti)proton. Clearly one has to fold-in this momentum spread to compare theoretical predictions with the data.

- Apart from providing parton distributions, DIS is also an important testing ground for perturbative QCD, as we will see.

[49] In a fixed-target experiment, beam particles interact with a stationary target in the laboratory, and the debris is recorded in a downstream detector. In a collider experiment, on the other hand, counter-rotating beams collide in the centre of a detector, which is built around the interaction region.

DIS kinematics I



- The graph above shows the kinematics of deep inelastic scattering:

  k = incoming lepton — k' = outgoing lepton

  p = incoming proton — X = hadronic final state

  $q = k - k'$ momentum transfer

- The interaction between the exchanged photon (or W in case of $\nu\mathrm{p} \to \ell X$ neutrino scattering) and the proton depends on p and q, from which we can build the two Lorentz scalars:

$$\boxed{Q^2 = -q^2 \quad \text{and} \quad x = \frac{Q^2}{2p \cdot q}}$$

- Other scalars that are often used to characterise the event are

$M^2 = p^2$	Proton mass squared
$s = (p + k)^2$	Centre of mass energy squared
$W^2 = (p + q)^2$	Invariant mass of X squared
$y = (p \cdot q)/(p \cdot k)$	Fractional energy transfer in the lab
$\nu = (p \cdot q)/M$	Energy transfer in the lab system

DIS kinematics II

- In case of **elastic scattering**, the proton leaves the collision without breaking-up or being in an excited state. Thus we have, for elastic scattering, ep $\to$ e′p′ with[50]

$$p^2 = p'^{\,2} = (p+q)^2 = W^2 = M^2.$$

- Here are a few useful relations, which we will prove below

1. $q^2 < 0$ (hence the minus sign in the definition of Q^2)
2. $0 \le x \le 1$
3. $0 \le y < 1$
4. $W^2 = M^2 + Q^2(1-x)/x \ge M^2$

- Proof

  Since all kinematic variables are Lorentz invariants, it is often useful to calculate them in frames which are convenient.

  1. In the rest frame of the incoming electron with the outgoing electron along the z axis we have $k = (m,0,0,0)$ and $k' = (E',0,0,k')$ so that

     $$q^2 = (m-E',0,0,-k')^2 = m^2 - 2mE' + E'^2 - k'^2 = m^2 - 2mE' + m^2 = 2m(m-E') < 0.$$

  2. Obviously $x = 0$ when $Q^2 = 0$. For the other limit set $W^2 = M^2$. This gives

     $$W^2 = (p+q)^2 = p^2 + 2p\cdot q + q^2 = M^2 + 2p\cdot q - Q^2 = M^2 \quad \to \quad Q^2 = 2p\cdot q \quad \to \quad x = 1.$$

  3. For the limits on y it is easiest to work in the lab frame where the proton is at rest and the electron comes in at the z direction. We then have $k = (E,0,0,k)$, $p = (M,0,0,0)$ and $q = (E-E', q_x, q_y, q_z)$ so that $p\cdot k = ME$ and $p\cdot q = M(E-E')$. Therefore

     $$y = \frac{p\cdot q}{p\cdot k} = \frac{E-E'}{E} \quad \text{with} \quad m_{\rm e} \le E' \le E.$$

     From this it immediately follows that $0 \le y < 1$.

  4. With $x = Q^2/(2p\cdot q)$ we find

     $$W^2 = (p+q)^2 = p^2 + 2p\cdot q + q^2 = M^2 + Q^2/x - Q^2 = M^2 + Q^2(1-x)/x.$$

[50] From relation (4) below it follows that the elastic scattering limit is also given by $x = 1$.

Exercise 8.2:

(a) [0.5] Show that $Q^2 \approx xys$ for large $s \gg M^2$ (so that we can neglect the proton mass). What is, in this approximation, the largest Q^2 that can be reached at the SLAC experiments ($\sqrt{s} = 6.4$ GeV) and at HERA ($\sqrt{s} = 294$ GeV).[51]

(b) [1.5] All DIS kinematic variables can be determined from a measurement of the scattered electron energy E' and angle θ with respect to the incident beam. In particular, show that for fixed-target experiments (proton at rest and the electron coming in from the z direction) we have the relations

$$\begin{aligned} Q^2 &= 4EE' \sin^2(\theta/2) \\ \nu &= E - E' \\ x &= Q^2/(2M\nu) \\ y &= \nu/E \\ W^2 &= M^2 - Q^2 + 2M\nu \\ s &= M(M + 2E) \approx 2ME \end{aligned}$$

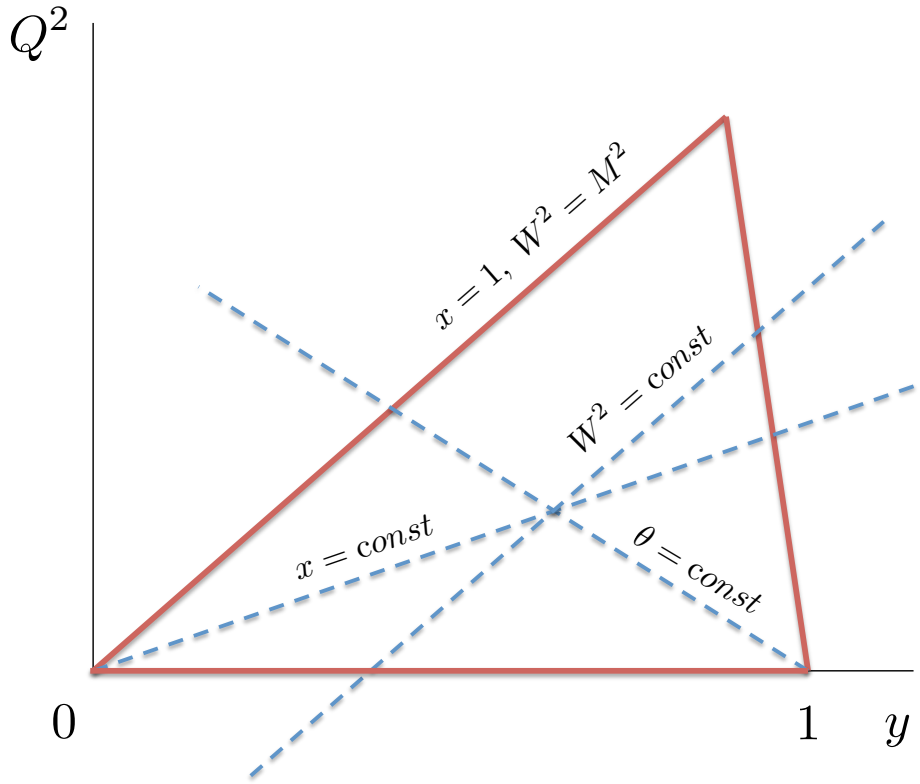
Hint: For the expression of Q^2 use the half-angle formula

$$\cos\theta = 1 - 2\sin^2(\theta/2).$$

- It is convenient to plot the allowed kinematical region in the y-Q^2 plane ($\rightarrow$ **Fig**.)

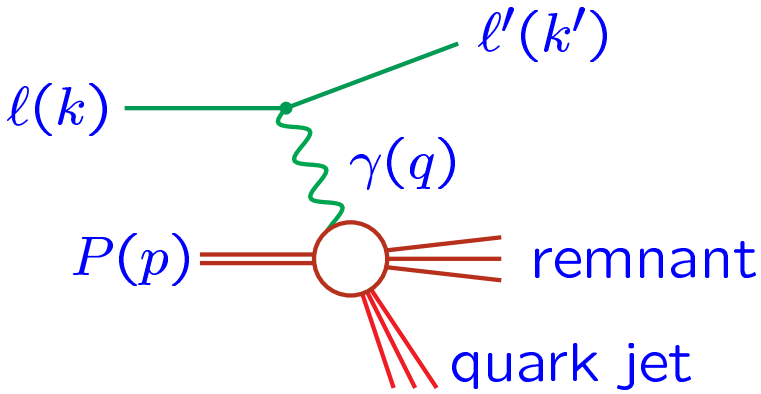
[51] We will see later that the cross-section drops like Q^{-4} so that DIS events at very large Q^2 are rare. It is thus difficult to collect data in this kinematic region.

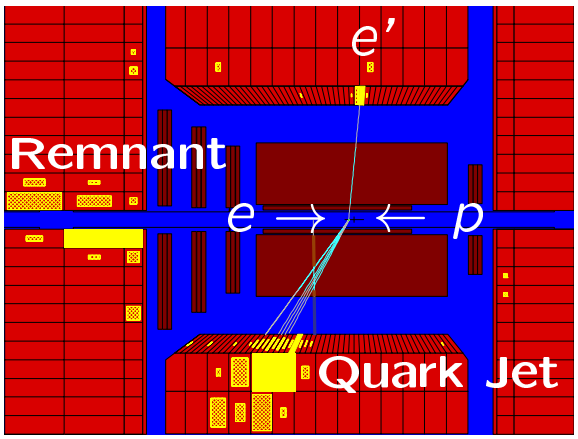
DIS kinematic plane

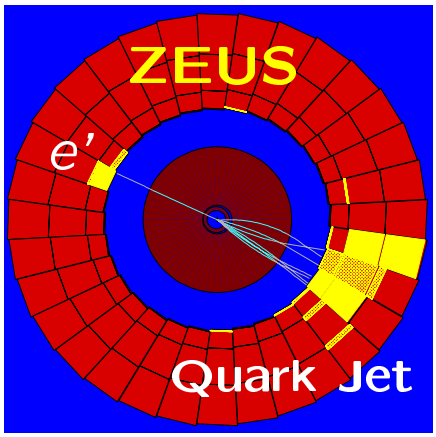


$$\begin{aligned} Q^2 &= 2MEy + M^2 - W^2 \\ &= 2MExy \\ &= 4(1-y)E^2 \sin^2 \theta/2 \end{aligned}$$

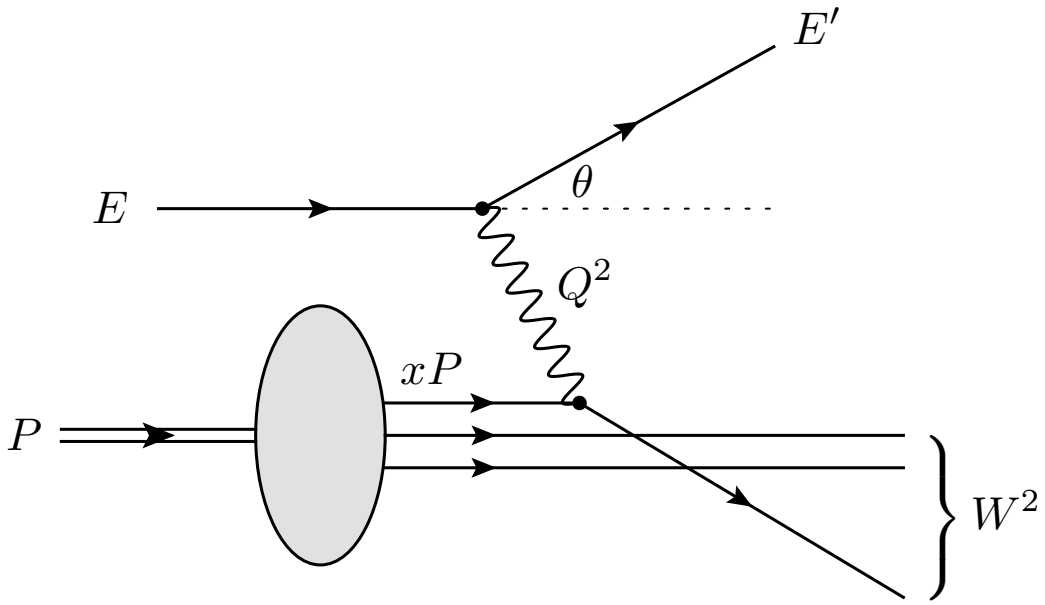
DIS kinematics in real life







The quark-parton model



- To explain the DIS measurements at SLAC, Feynman, Bjorken, and others (1969) proposed the so-called **parton model** which states that

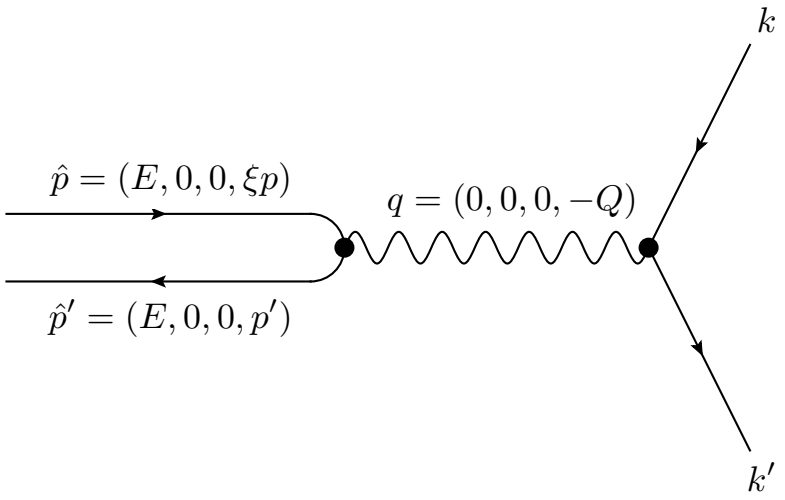
  Assumption I: A fast moving hadron appears as a jet of partons (quarks and gluons) moving in more or less the same direction as the parent hadron and sharing its 3-momentum.

  Assumption II: The reaction cross-section is the incoherent sum of partonic cross-sections, as calculated with free partons.[52]

- We will now use the quark-parton model and results from the PP-I course to derive the DIS cross-section. The kinematics is best understood in the so-called Breit-, or infinite-momentum frame.

[52] By 'incoherent sum' we mean that cross-sections are added, instead of amplitudes.

The Breit frame I



- Because the virtual photon is space-like ($q^2 < 0$) it follows that we can boost the photon along its direction of propagation (which points to the proton) such that q^0 vanishes. This frame is called the **Breit frame** or **infinite momentum frame** since the proton then moves with very large momentum towards the virtual photon.
- In this frame the incoming quark moves with a 3-momentum ξp_z along the z axis, where ξ is the fraction of the proton 3-momentum p_z. The virtual photon moves with a 3-momentum Q along $-z$.
- We take the incoming quark to be point-like, so that the scattering is necessarily elastic:[53]

$$\hat{p}^2 = (\hat{p} + q)^2 \;\rightarrow\; \hat{p}^2 = \hat{p}^2 + 2\hat{p} \cdot q - Q^2 \;\rightarrow\; Q^2 = 2\hat{p} \cdot q$$

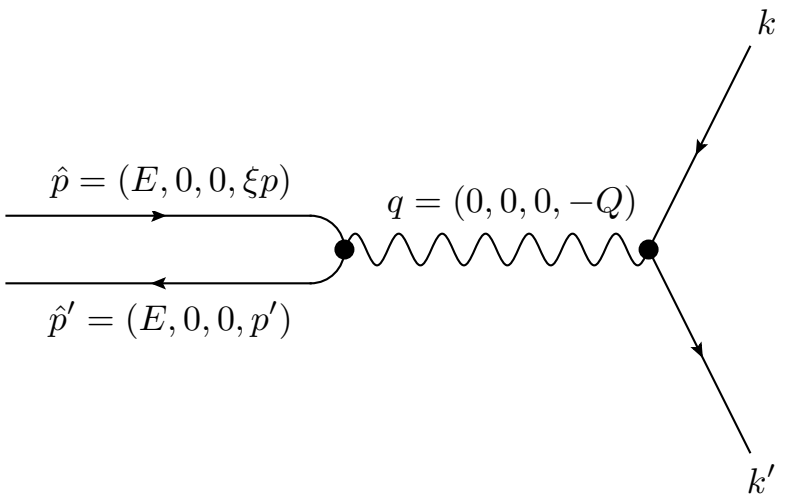
- If we denote the proton 4-momentum by p then, in the Breit frame,

$$\begin{aligned} \hat{p} \cdot q &= (E, 0, 0, \xi p_z) \cdot (0, 0, 0, -Q) = \xi p_z Q \\ \xi p \cdot q &= \xi (E_p, 0, 0, p_z) \cdot (0, 0, 0, -Q) = \xi p_z Q \end{aligned}$$

  Thus $\hat{p} \cdot q = \xi \, p \cdot q$ but remember that this is *only* true in the Breit frame where the virtual photon does not transfer energy.

[53] We indicate the unobservable partonic kinematic variables by a hat, like $\hat{p}$ for a partonic 4-momentum.

The Breit frame II



- The elastic scattering condition now becomes

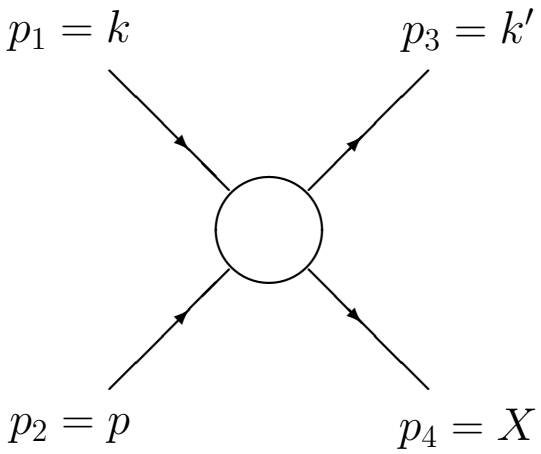
$$Q^2 = 2\,\hat{p}\cdot q = 2\xi\, p\cdot q \quad \rightarrow \quad \xi = \frac{Q^2}{2\,p\cdot q} = x$$

- So we can identify the Bjorken-x variable as the 3-momentum fraction of the struck quark in the Breit frame.

- Let us at this point introduce the notion of a **quark distribution** $f_i(x)\mathrm{d}x$, which gives the number of quarks of flavour i which carry a 3-momentum fraction (in the Breit frame) between x and $x+\mathrm{d}x$.

- Remark: note that in the Breit frame the proton moves very fast towards the photon, and is therefore Lorentz contracted to a kind of pancake. The interaction then takes place on the very short time scale when the photon passes that pancake. On the other hand, in the rest frame of the proton, the inter-quark interactions take place on time scales of the order of r_{p}/c but because of time dilatation these interactions are like 'frozen' the Breit frame. During the short interaction time, the struck quark thus does not interact with the spectator quarks and can be regarded as a free parton.

Intermezzo: Mandelstam variables



$$
\begin{aligned}
s &= (p_1 + p_2)^2 \approx 2 p_1 \cdot p_2 \\
t &= (p_1 - p_3)^2 = (p_4 - p_2)^2 \approx -2\, p_1 \cdot p_3 \approx -2\, p_2 \cdot p_4 \\
u &= (p_1 - p_4)^2 = (p_3 - p_2)^2 \approx -2\, p_1 \cdot p_4 \approx -2\, p_3 \cdot p_2
\end{aligned}
$$

- **Exercise 8.3**: [0.5] Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- **Exercise 8.4**: [0.5] Show that, if we neglect the electron and proton mass,

$$
\begin{aligned}
Q^2 &= -t \\
x &= -t/(s+u) \\
y &= (s+u)/s \\
W^2 &= s + t + u
\end{aligned}
$$

Note that we immediately get $Q^2 = xys$ (if we ignore the masses).

Parton density

- On page 8–13 we have introduced the **parton density function** $f_i(x)\,\mathrm{d}x$ which gives the number of quarks of flavour $i = \mathrm{d}, \mathrm{u}, \mathrm{s}, \ldots, \bar{\mathrm{d}}, \bar{\mathrm{u}}, \bar{\mathrm{s}}, \ldots$ between momentum fraction x and $x + \mathrm{d}x$.

- Huh? But does not the proton consist of *three* quarks? No, not in a dynamical picture: inside the proton there are also gluons from the QCD splitting $q \to q\mathrm{g}$ and quark-antiquark pairs from the splitting $\mathrm{g} \to q\bar{q}$ ($\to$**fig**). What is true is that there is a *net excess* of three quarks that carry the quantum numbers of the proton.

- Now we use the second assumption of the parton model and write the cross section as an incoherent sum of partonic cross sections

$$\mathrm{d}\sigma = \sum_i \mathrm{d}\hat{\sigma}(\hat{s}, \hat{t}, \hat{u})\; f_i(x)\mathrm{d}x$$

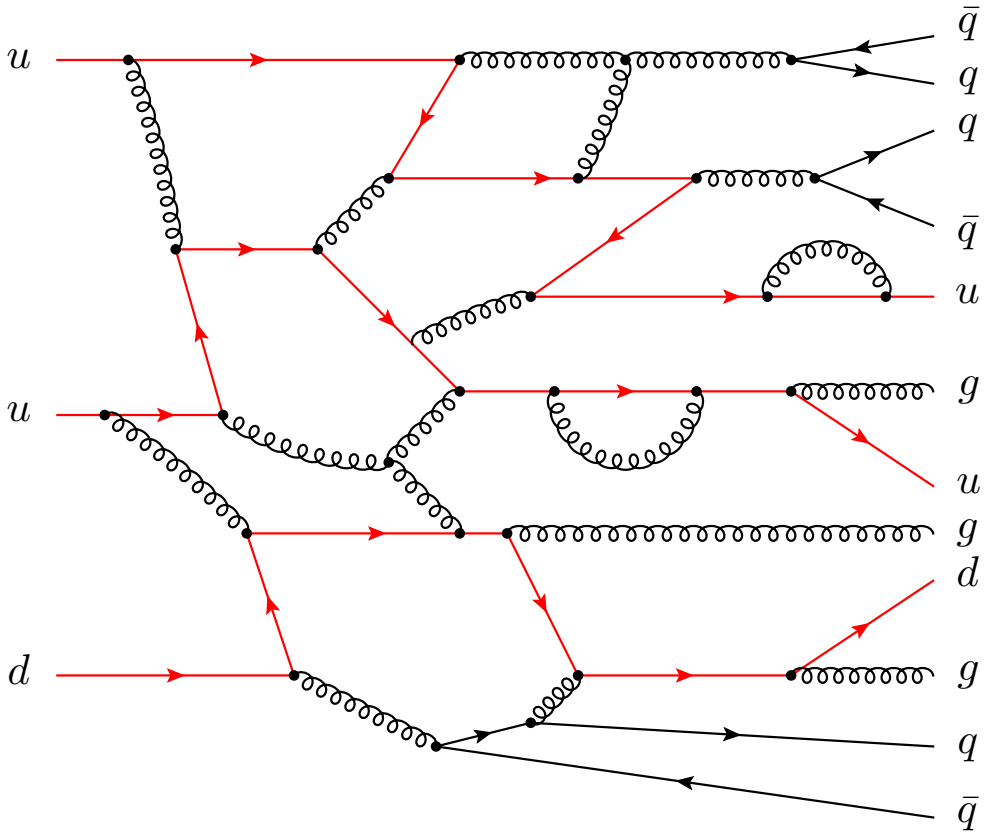
  Here we have introduced the parton kinematic variables

$$\hat{s} \approx 2\,xp \cdot k = xs, \quad \hat{t} = (k - k')^2 = t, \quad \hat{u} \approx -2\,xp \cdot k' = xu$$

- For the partonic cross section we just take $\sigma(\mathrm{e}\mu \to \mathrm{e}\mu)$ as calculated in PP-I (lecture 8), with the muon charge replaced by the quark charge.

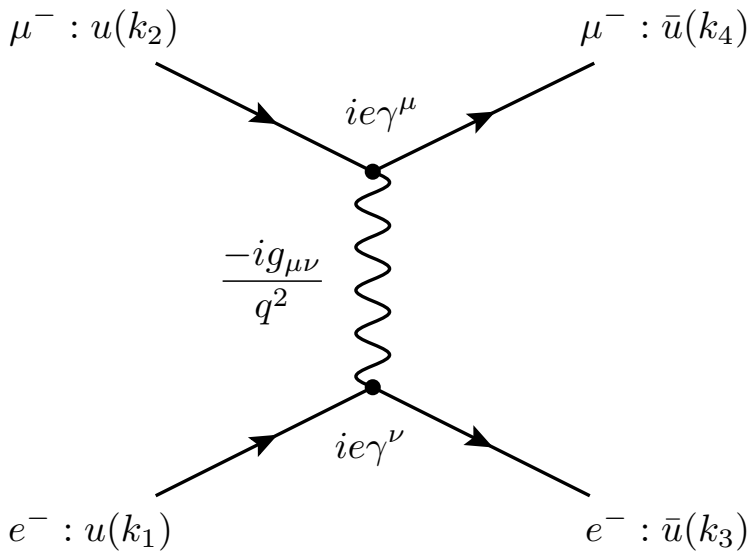
  Exercise 8.5: [0.5] Why do we take the cross section for $\mathrm{e}\mu \to \mathrm{e}\mu$ as our reference, and not that of $\mathrm{ee} \to \mathrm{ee}$?

Dynamical picture of the proton



Schematic picture of the QCD proton structure. The uud **valence quarks** that carry the quantum numbers of the proton enter the diagram on the left. This corresponds to a low-resolution 3-quark picture of the proton that only accounts for its quantum numbers. At the right of the diagram we see a high-resolution picture (at large Q^2) of the proton where the valence quarks are dressed with gluons and a **sea** of $q\bar{q}$ pairs. Note that the valence quarks can zig-zag through the diagram but will never disappear so that the proton quantum numbers are the same in both the low- and high-resolution pictures.

Cross section for e-μ scattering



- In PP-I Section 8.2 the e-μ cross section is calculated as

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{com}} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2}\right)$$

- COM frame with momentum k_1 along x and scattering angle θ

$$\begin{aligned} k_1 &= (k, \quad k, 0, 0) & \qquad k_3 &= (k, \quad k\cos\theta, \quad k\sin\theta, 0) \\ k_2 &= (k, -k, 0, 0) & \qquad k_4 &= (k, -k\cos\theta, -k\sin\theta, 0) \end{aligned}$$

$$s = 4k^2, \quad t = -2k^2(1-\cos\theta), \quad u = -2k^2(1+\cos\theta)$$

- Cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\sin\theta \mathrm{d}\theta \mathrm{d}\phi} = \frac{\mathrm{d}\sigma}{\sin\theta \mathrm{d}\phi \mathrm{d}t}\left|\frac{\mathrm{d}t}{\mathrm{d}\theta}\right| = 2k^2\frac{\mathrm{d}\sigma}{\mathrm{d}\phi \mathrm{d}t} = \frac{s}{2}\frac{\mathrm{d}\sigma}{\mathrm{d}\phi \mathrm{d}t}$$

- Integrating over ϕ then gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{4\pi}{s}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi\alpha^2}{s^2}\left(\frac{s^2+u^2}{t^2}\right)$$

DIS cross section

- Using the partonic variables, and multiplying the charge at the muon vertex by the fractional quark charge e_i, we get the partonic cross section

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}\hat{t}} = \frac{2\pi\alpha^2 e_i^2}{\hat{s}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right)$$

  It is a simple matter to re-write this in terms of the DIS kinematic variables (see page 8–14).

- **Exercise 8.6**: [1.0] Show that

$$\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}\hat{t}} \;\rightarrow \frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}Q^2} = \frac{2\pi\alpha^2 e_i^2}{Q^4}\left[1+(1-y)^2\right]$$

- Now we can put this result in our master formula on page 8–15

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \sum_i \frac{2\pi\alpha^2 e_i^2}{Q^4}\left[1+(1-y)^2\right] f_i(x)\mathrm{d}x$$

  or

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{2\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\sum_i e_i^2 f_i(x)$$

- The F_2 **structure function** is defined as the charge weighted sum of the parton **momentum densities** $xf_i(x)$

$$\boxed{F_2(x) = \sum_i e_i^2 x f_i(x)}$$

  so that the DIS cross section can be written as[54]

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\,\frac{\left[1+(1-y)^2\right]}{2x}\,F_2(x)}$$

[54] One can think of the y dependence as being an angular dependence through the relation $1-y = \frac{1}{2}(1+\cos\theta^*)$.

The Gallan-Gross relation

- We have given a rather simple derivation of the parton model cross section and established the relation between parton density functions and the F_2 structure function.
- However, a more formal derivation (see H&M Section 8), which does not use the parton model, leads to the result

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{xQ^4}\ \left[(1-y)F_2(x,Q^2) + xy^2F_1(x,Q^2)\right]$$

  Here another structure function shows up, which turns out to be proportional to the absorption cross-section of *transversely polarised* photons: $2xF_1 \propto \sigma_\mathrm{T}$. Because the exchanged photon is virtual, is also has a *longitudinally polarised* component. The F_2 structure function is proportional to the sum of the transverse and longitudinal absorption cross sections: $F_2 \propto \sigma_\mathrm{T} + \sigma_\mathrm{L}$.
- In the Breit frame, where the quarks are highly relativistic without transverse momenta, the quark spins will be aligned parallel or antiparallel to the direction of motion (z axis) so that it can absorb a head-on photon with helicity ± 1, just by flipping the spin.
- However the quarks cannot absorb a longitudinally polarised photon because for this the quark spin must have a non-vanishing transverse component. Thus, in the parton model, $\sigma_\mathrm{L} = 0$ and $F_2 = 2xF_1$. This is called the **Gallan-Gross relation**. Setting $F_2 = 2xF_1$ above, we recover the formula on page 8–18.[55]
- In the QCD improved parton model, gluon radiation imparts small transverse momenta to the quarks so that now $\sigma_\mathrm{L} \neq 0$ and another (small) structure function shows up, $F_\mathrm{L} \equiv F_2 - 2xF_1$, with its own characteristic y-dependence.

[55] Historically, experimental verification of the Gallan-Gross relation was a proof that quarks carry spin $\frac{1}{2}$.

Bjorken scaling

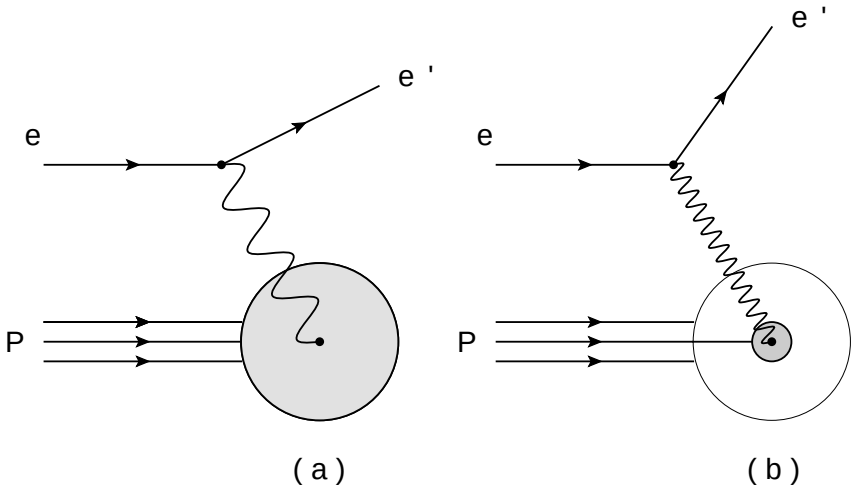
- In principle, the structure functions depend on two variables, x and Q^2 say, but in the parton model derivation on page 8–18 we have defined

$$F_2(x) = \sum_i e_i^2 x f_i(x)$$

which depends on x only. This Q^2 independence is called **Bjorken scaling** and is formally stated as follows (in terms of lab variables):

$$\text{If} \quad \left.\begin{array}{r} Q^2 \to \infty \\ \nu \to \infty \end{array}\right\} \text{ with } x = \frac{Q^2}{2M\nu} \text{ finite, then } \; F_2(x, Q^2) \to F_2(x)$$

- This scaling behaviour is easy to understand by noticing that the wavelength of the virtual photon $\lambda \sim 1/Q$. But the resolving power is irrelevant when we scatter on *point-like* objects, hence the independence on Q^2. In short, scaling $\to$ point-like scattering.



- Indeed, measurements of F_2 at different Q^2 values seem to fall on a universal curve ($\to$ **fig**) but close inspection reveals a characteristic scale-breaking pattern. This Q^2 dependence is caused by the QCD processes of gluon radiation and $q\bar{q}$ formation, as we will see later.

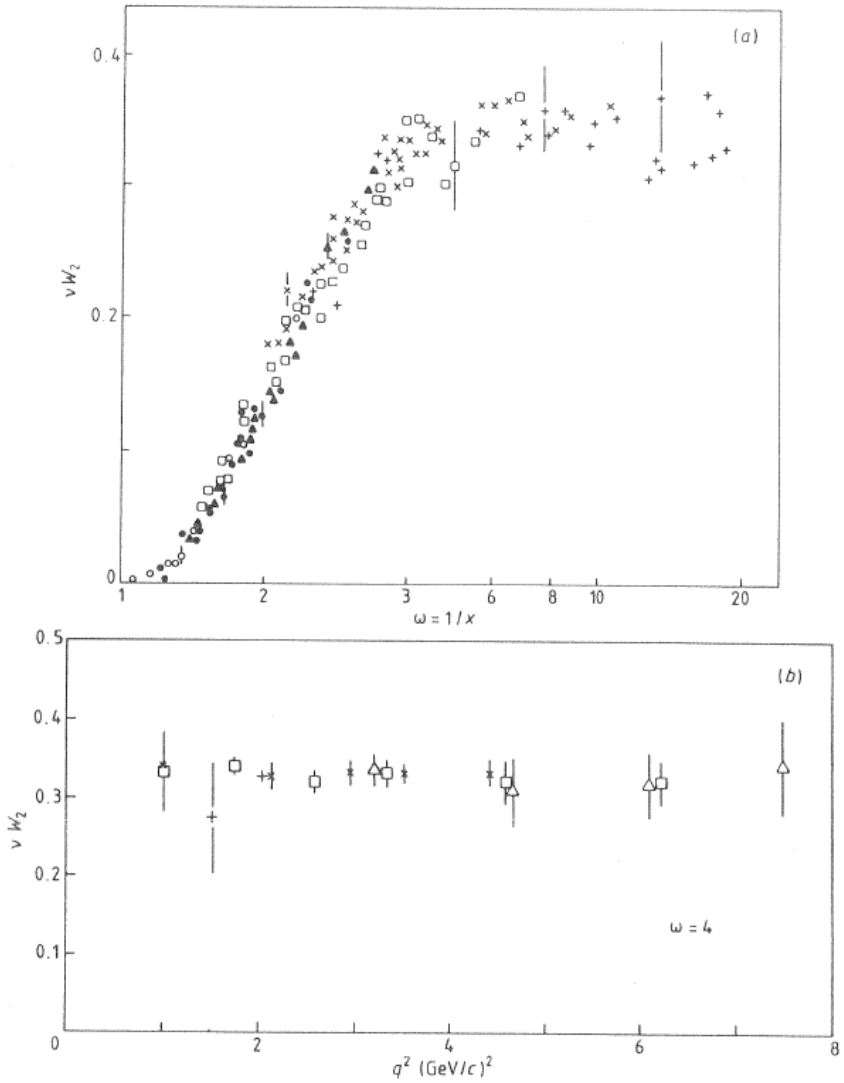
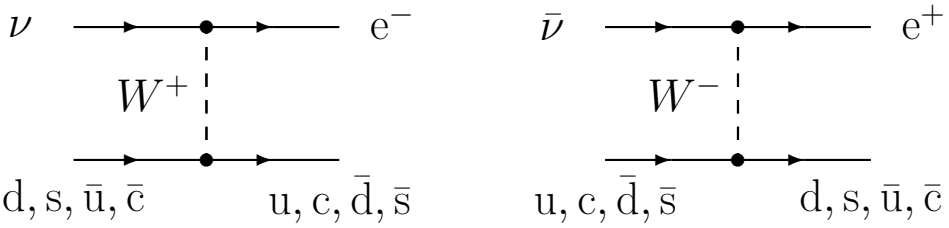


Figure 4.2 Bjorken scaling: the structure function νW_2 (*a*) plotted against $\omega = 1/x$ for different q^2 values (Miller *et al* 1972) (*b*) plotted against q^2 for a single value of $x = 0.25$ ($\omega = 4$) (Friedman and Kendall 1972).

Neutrino scattering



- Much information on the proton structure comes from (anti-)neutrino deep inelastic scattering. We will not derive here the expressions for the cross-sections (see H&M) but simply list the result for $\nu\mathrm{p} \to \mathrm{e}^-\mathrm{X}$ scattering and for $\bar{\nu}\mathrm{p} \to \mathrm{e}^+\mathrm{X}$ scattering[56]

$$\frac{\mathrm{d}^2\sigma^{\nu\mathrm{p}}}{\mathrm{d}x\mathrm{d}Q^2} = \sigma_0 \left\{ \left[1 + (1-y)^2\right] F_2^{\nu\mathrm{p}} + \left[1 - (1-y)^2\right] x F_3^{\nu\mathrm{p}} \right\}$$

$$\frac{\mathrm{d}^2\sigma^{\bar{\nu}\mathrm{p}}}{\mathrm{d}x\mathrm{d}Q^2} = \sigma_0 \left\{ \left[1 + (1-y)^2\right] F_2^{\bar{\nu}\mathrm{p}} - \left[1 - (1-y)^2\right] x F_3^{\bar{\nu}\mathrm{p}} \right\}$$

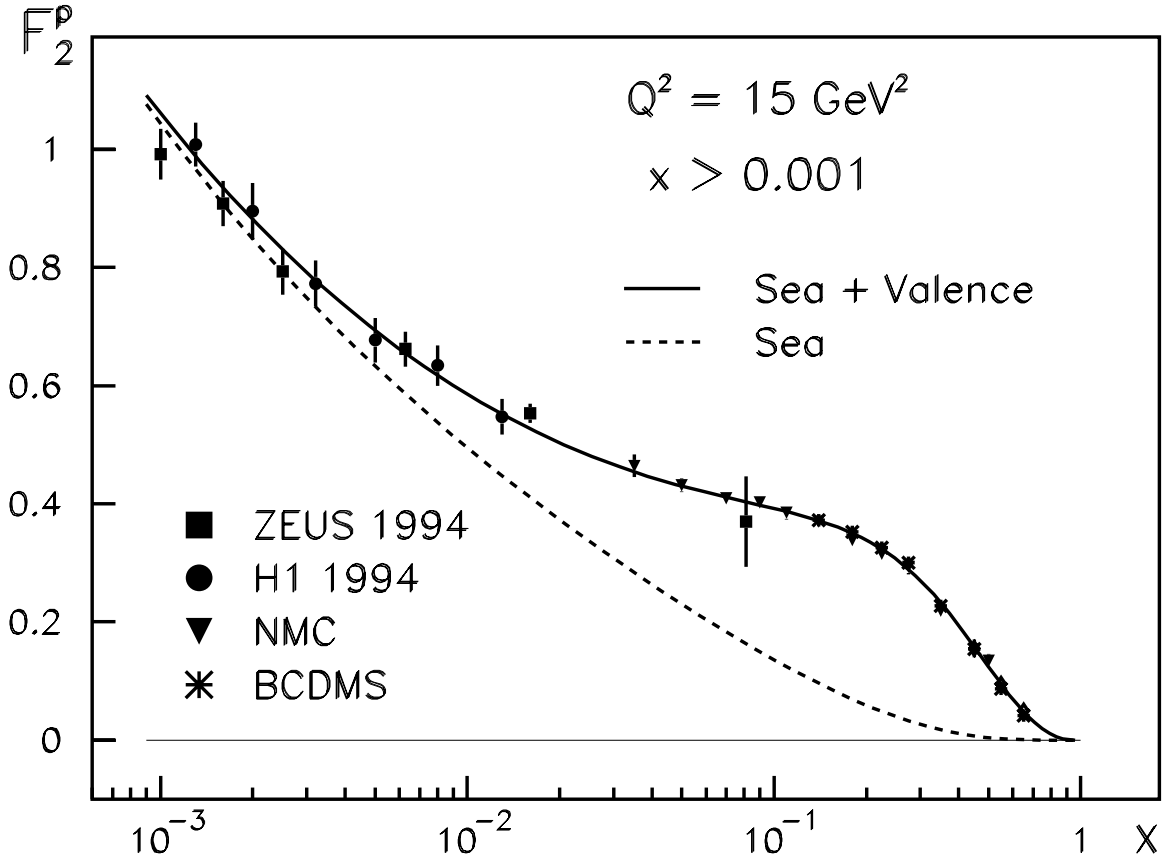
- Here we encounter a new structure function, xF_3, which is sensitive to the *difference* of quark and anti-quark distributions.

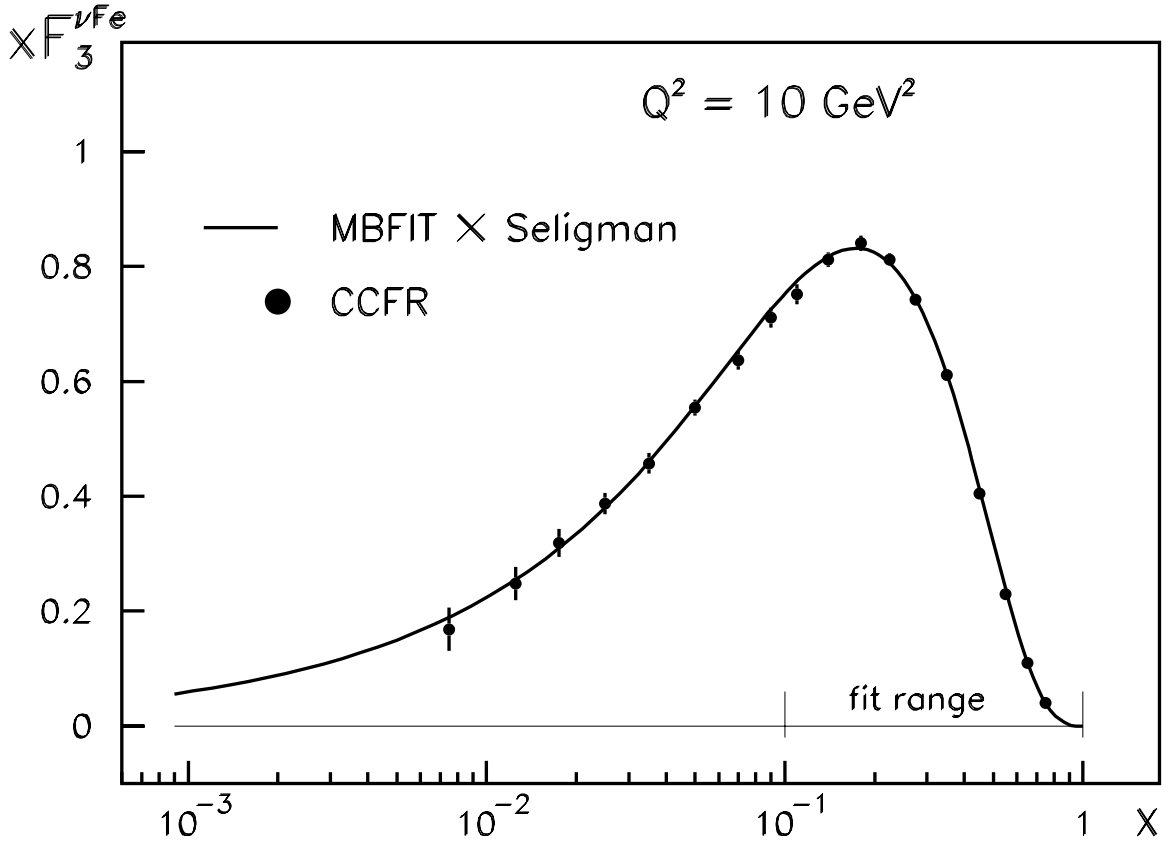
- In the parton model, the neutrino structure functions are

$$\begin{aligned} F_2^{\nu\mathrm{p}} &= 2x(d + s + \bar{u} + \bar{c}) \\ xF_3^{\nu\mathrm{p}} &= 2x(d + s - \bar{u} - \bar{c}) \end{aligned}$$

$$\begin{aligned} F_2^{\bar{\nu}\mathrm{p}} &= 2x(u + c + \bar{d} + \bar{s}) \\ xF_3^{\bar{\nu}\mathrm{p}} &= 2x(u + c - \bar{d} - \bar{s}) \end{aligned}$$

[56] The factor in front is $\sigma_0 = \frac{G_\mathrm{F}^2}{4\pi x}\left(\frac{M_\mathrm{W}^2}{Q^2 + M_\mathrm{W}^2}\right)^2$


F_2^p
$Q^2 = 15$ GeV^2
$x > 0.001$
Sea + Valence
Sea
ZEUS 1994
H1 1994
NMC
BCDMS
1
0.8
0.6
0.4
0.2
0
10^{-3}
10^{-2}
10^{-1}
1
X


$xF_3^{\nu Fe}$
$Q^2 = 10$ GeV^2
MBFIT × Seligman
CCFR
fit range
1
0.8
0.6
0.4
0.2
0
10^{-3}
10^{-2}
10^{-1}
1
X

Valence and momentum sum rules

- We have introduced the quark number distributions $f_i(x)$ which we will write as $u(x), \bar{u}(x), d(x), \bar{d}(x)$, *etc.* It is convenient to define the **valence** and **sea** distributions by

$$\begin{array}{llll} u_{\rm v} = u - \bar{u}, & d_{\rm v} = d - \bar{d}, & s_{\rm v} = s - \bar{s} = 0, & \cdots \\ u_{\rm s} = 2\,\bar{u}, & d_{\rm s} = 2\,\bar{d}, & s_{\rm s} = 2\,\bar{s}, & \cdots \end{array}$$

  so that $u + \bar{u} = u_{\rm v} + u_{\rm s}$, *etc.* See also the diagram on page 8–16.

- Because the quantum numbers of the proton must be carried by the surplus of quarks over antiquarks, we get the counting rules

$$\int_0^1 u_{\rm v}(x)\,{\rm d}x = 2 \quad \text{and} \quad \int_0^1 d_{\rm v}(x)\,{\rm d}x = 1$$

- The momentum distributions $x f_i(x){\rm d}x$ give the **probability** that a quark carries a momentum fraction between x and $x+{\rm d}x$.[57] Thus if all quarks carry the momentum of the proton we should have

$$\sum_i \int_0^1 x f_i(x){\rm d}x = 1$$

- But integration of the quark distributions obtained from deep inelastic charged lepton and neutrino scattering gives

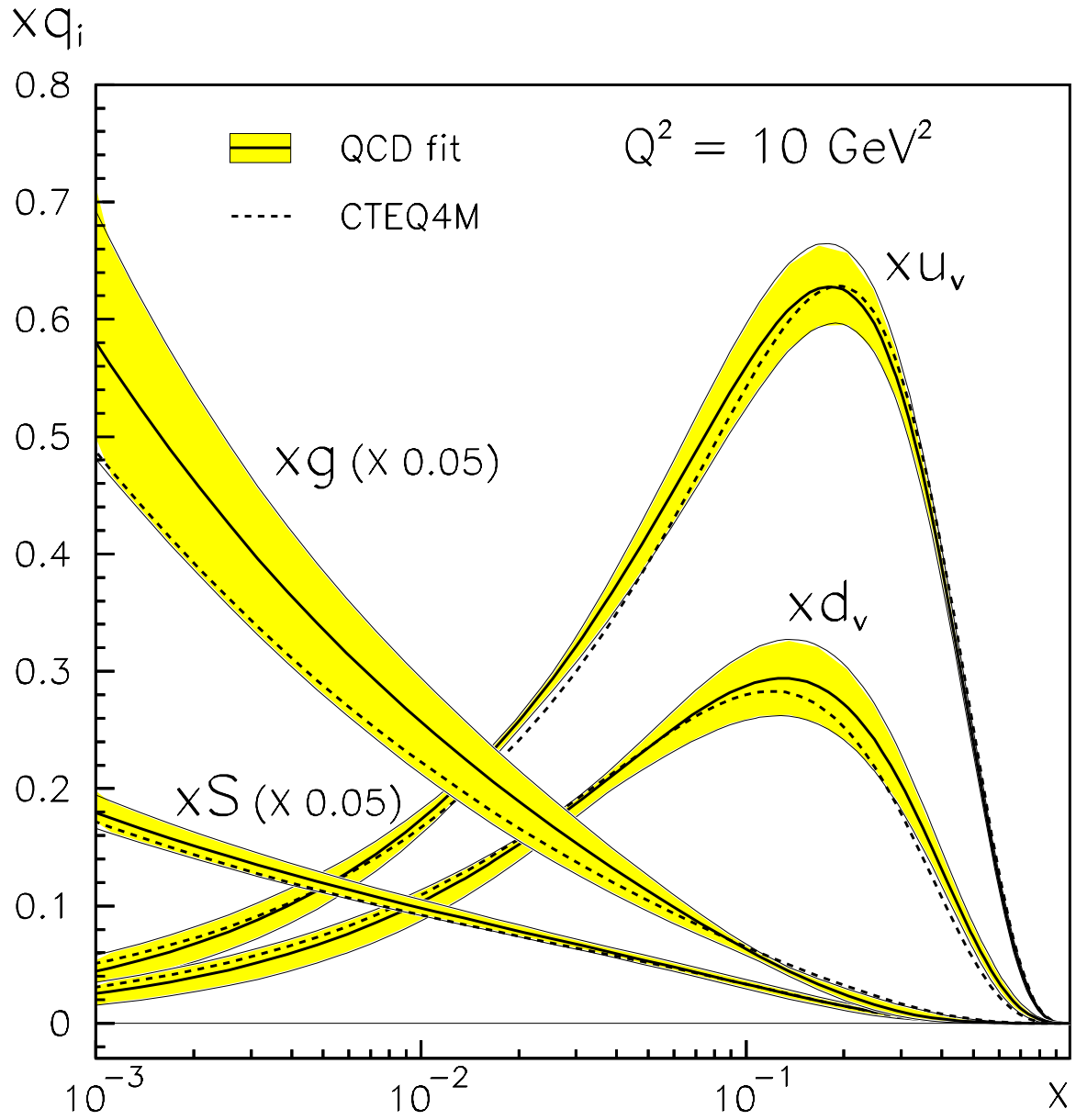
$$\sum_i \int_0^1 x f_i(x){\rm d}x \approx 0.5$$

- Where is the missing momentum? The answer is that it is carried by gluons. Introducing a gluon momentum distribution $xg(x)$, the correct momentum sum rule is

$$\sum_i \int_0^1 x f_i(x){\rm d}x + \int_0^1 x g(x){\rm d}x = 1$$

[57] If $f(x){\rm d}x$ is the *number* of quarks carrying a fraction x of the proton momentum P, then the total momentum carried by these partons is $p = xPf(x){\rm d}x$. The *probability* to carry a fraction x is thus $p/P = xf(x){\rm d}x$.

Example of a pdf set



Remark: The widely used abbreviation 'pdf' stands for 'parton density function'. Usually, but not always, it is clear from the context or notation (xf or f) if a *momentum* or a *number* density is meant.

Exercise 8.7: Universality of pdfs

- The **isospin symmetry** assumption says that the u (anti)quark distribution in the proton is equal to the d (anti)quark distribution in the neutron, and *vice versa*. Thus we have

$$\begin{aligned} F_2^{\mathrm{ep}} &= \tfrac{1}{9}x(d+\bar{d}) + \tfrac{4}{9}x(u+\bar{u}) + \tfrac{1}{9}x(s+\bar{s}) + \cdots \\ F_2^{\mathrm{en}} &= \tfrac{1}{9}x(u+\bar{u}) + \tfrac{4}{9}x(d+\bar{d}) + \tfrac{1}{9}x(s+\bar{s}) + \cdots \end{aligned}$$

  The same applies to (anti)neutrino scattering: $F_2^{\nu\mathrm{n}} = F_2^{\nu\mathrm{p}}(u \leftrightarrow d)$. Note that the parton distributions, by convention, always refer to those of the proton.

  (a) [0.5] Use isospin symmetry to write down the parton model expressions for $F_2^{\nu\mathrm{n}}$ and $F_2^{\bar{\nu}\mathrm{n}}$ (F_2^{ep} and F_2^{en} are already given above). Define the *nucleon* structure function $F_2^{\mathrm{e}N}$ by averaging the proton and neutron F_2. Likewise define $F_2^{\nu N}$ by averaging over proton and neutron and *also* over ν and $\bar{\nu}$.

  (b) [0.5] Neglect charm and assume 3 flavours (d, u, s). Show that

$$\frac{F_2^{\mathrm{e}N}}{F_2^{\nu N}} = \frac{5}{18}\left[\, 1 - \frac{3}{5}\,\frac{(s+\bar{s})}{\sum(q+\bar{q})}\,\right] \approx \frac{5}{18}$$

  Because the (anti)strangeness content of the nucleon turns out to be small, it follows that the strangeness correction term above is also small; correction for charm would be even smaller.

- The plot on the next page shows an early experimental verification of $F_2^{\nu N} \approx \frac{18}{5}F_2^{\mathrm{e}N}$. This not only tests the (fractional) quark charges, but also that electron and neutrino DIS probe the *same* parton distribution: parton distributions are a process independent property of the nucleon (universality of pdfs).

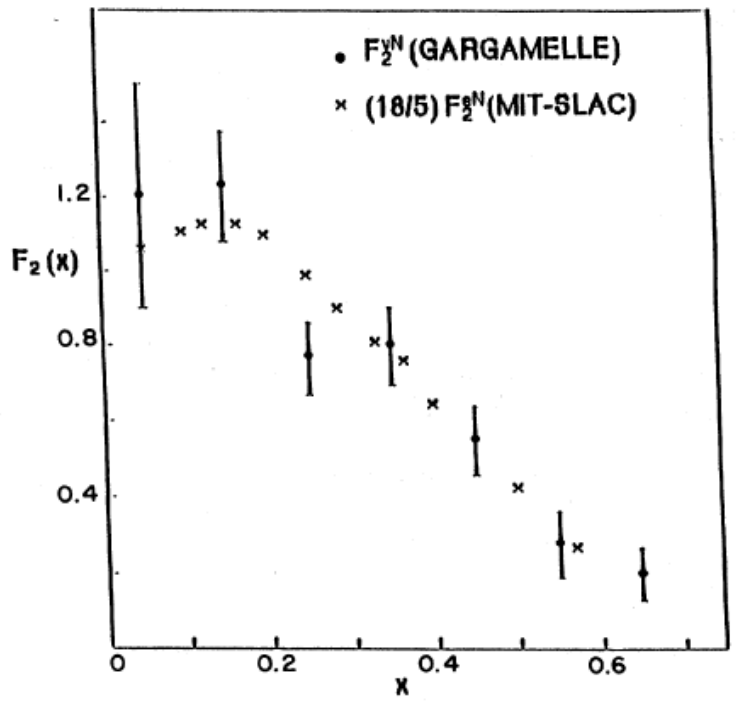


FIG. 10. Early Gargamelle measurements of $F_2^{\nu N}$ compared with $(18/5)F_2^{eN}$ calculated from the MIT-SLAC results.

Early verification that $F_2^{\nu N} \approx \frac{18}{5} F_2^{\mathrm{e}N}$. Figure taken from Jerome Friedman Nobel lecture, Rev. Mod. Phys. 63(1991)615.

Lecture notes Particle Physics II

Quantum Chromo Dynamics

9. The QCD improved Parton Model

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December 5, 2013

The QCD factorisation theorem

- In Section 7 we have seen that QCD suffers from infrared singularities when two daughter partons cannot be resolved because they become collinear, or because one of them becomes soft. We have also seen that these singularities are associated with 'long-distance' physics which takes place a long time after the initial hard scattering. So-called infrared-safe observables are still calculable in perturbative QCD, but since this is quite restrictive we have to look for ways to extend the predictive power of the theory. This way-out is provided by the QCD **factorisation theorem**.
- For hadron-hadron scattering the factorisation theorem states that the singular long-distance pieces can be removed from the partonic cross section and factored into the parton distributions of the incoming hadrons, and that this can be done consistently at all orders in the perturbative expansion.
- The partonic cross section is then calculable in perturbation theory, and does not depend on the type of incoming hadron.
- The parton distributions, on the other hand, are a property of the incoming hadrons but are universal in the sense that they do not depend on the hard scattering process.[58] Parton distributions are nonperturbative and have to be obtained from experiment.
- Factorisation is a fundamental property of QCD. It turns perturbative QCD into a reliable calculation tool, unlike the naive parton model that does not take the parton dynamics into account.

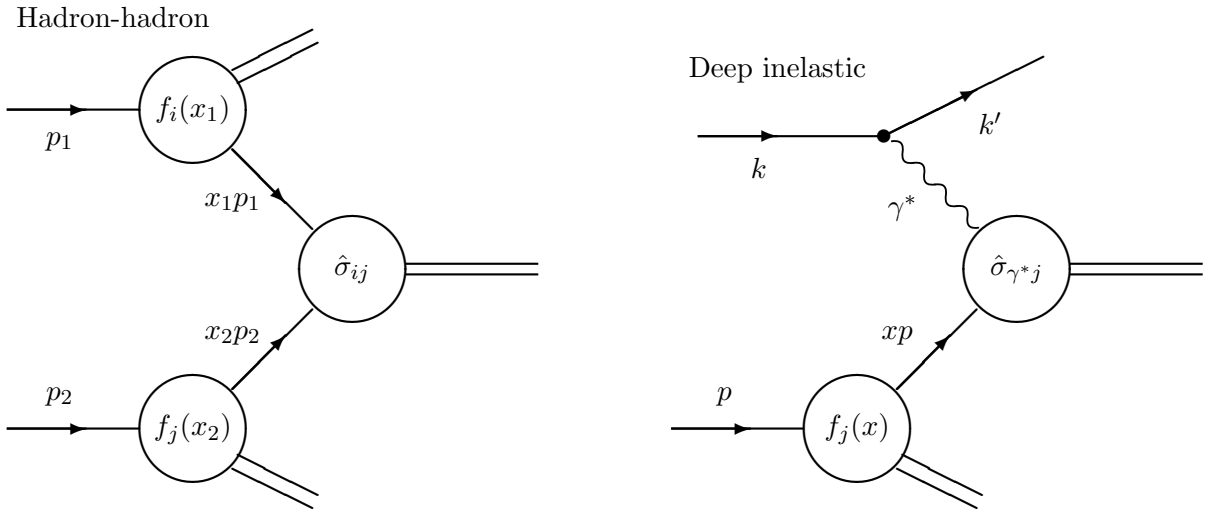
[58] See also Exercise 8.7.

Hadron-hadron cross sections I

- Schematically, a hadron-hadron cross section can be written as

$$\sigma^{\text{h-h}} = \sum_{i,j} \int \mathrm{d}x_1 \mathrm{d}x_2 \, f_i(x_1, \mu^2) f_j(x_2, \mu^2) \, \hat{\sigma}_{ij}(x_1, x_2, Q^2/\mu^2, \cdots)$$

  and can be depicted by (left diagram):



- Here the (arbitrary) **factorisation scale** μ can be thought of as the scale which separates the long and short-distance physics. Roughly speaking, a parton with a transverse momentum less than μ is then considered to be part of the hadron structure and is absorbed in the parton distribution. Partons with larger transverse momenta participate in the hard scattering process with a short-distance partonic cross-section $\hat{\sigma}$.

Hadron-hadron cross sections II

- What is taken for the hard scale Q^2 depends on the scattering process we are interested in. In jet production, for instance, one usually takes the transverse momentum of the jet as the hard scale, in deep inelastic scattering one takes the square of the four momentum transfer from the electron to the proton, and in e^+e^- scattering one takes the centre-of-mass energy, and so on. Often the simplifying assumption is made that the factorisation scale is equal to the hard scale: $\mu^2 = Q^2$.
- The factorisation theorem also applies to deep inelastic scattering, with one of the parton distributions replaced by an $ee'\gamma^*$ vertex as is shown in the right-hand side diagram on page 9–4:

$$\sigma^{\mathrm{DIS}} = \sum_j \int \mathrm{d}x\, f_j(x,\mu^2)\, \hat{\sigma}_{\gamma^* j}(x, Q^2/\mu^2, \cdots)$$

- We will use DIS to show how the infrared singularities are absorbed in the parton distributions. The QCD evolution equations of the parton densities are then derived from the **renormalisation group equation**.

Recap of the F_2 structure function

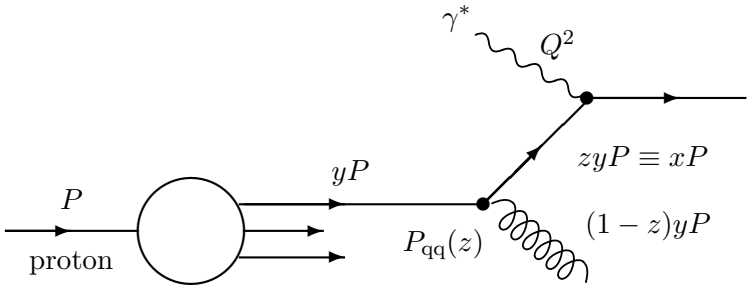
- We have seen that the F_2 structure function measured in deep inelastic electron-proton scattering is, to first order, independent of Q^2, the negative square of the momentum transfer from the electron to the proton. This implies that DIS does not depend on the resolution $1/Q$ with which the proton is probed. This is explained in the naive parton model by assuming that the electron scatters incoherently off *pointlike* quarks in the proton. The F_2 structure function can then be written as the charge weighted sum of quark momentum distributions

$$F_2^{\mathrm{ep}}(x) = \sum_i e_i^2 x f_i(x).$$

  Here e_i is the charge of the quark, and $f_i(x)\mathrm{d}x$ is the number of quarks that carry a fraction between x and $x + \mathrm{d}x$ of the proton momentum. The probability that the parton carries a momentum fraction x is then given by $x f_i(x)$. The index i denotes the quark flavour $\mathrm{d}, \mathrm{u}, \mathrm{s}, \ldots, \bar{\mathrm{d}}, \bar{\mathrm{u}}, \bar{\mathrm{s}}, \ldots$

- Although gluons show up in the naive parton model as missing momentum, they are not treated as dynamical entities. We will now incorporate the effect of gluon radiation by quarks, which leads to the so-called **QCD improved parton model**.

Scaling violation I



- Consider a quark that carries a fraction y of the proton momentum and radiates a gluon with a fraction $1-z$ of its momentum. After radiating the gluon, the quark with momentum fraction zy scatters off the virtual photon. The momentum fraction seen by the photon is thus $x = zy$ which implies that $z = x/y$.

- Taking gluon radiation into account, the F_2 structure function is found to be (see H&M Section 10.1–5 for a lengthy derivation):

$$\frac{F_2(x,Q^2)}{x} = \sum_i e_i^2 \int_x^1 \frac{\mathrm{d}y}{y} f_i(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_{\rm s}}{2\pi} P_{\rm qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{m^2} \right]$$

  Here m^2 is a lower transverse momentum cut-off to regularise the divergence when the gluon becomes collinear with the quark.

- In the above, the **splitting function** $P_{\rm qq}$ is given by

$$P_{\rm qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right).$$

  It represents the probability that a parent quark emits a gluon with the daughter quark retaining a fraction z of the parent momentum. Note that an infrared divergence shows up when $(1-z) \to 0$ where the gluon becomes soft so that daughter and parent cannot be resolved anymore.

Scaling violation II

$$\frac{F_2(x,Q^2)}{x} = e^2 \int_x^1 \frac{\mathrm{d}y}{y} f(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_\mathrm{s}}{2\pi} P_{\mathrm{qq}}\left(\frac{x}{y}\right) \ln\frac{Q^2}{m^2} \right]$$

- **Exercise 9.1**: [1.0] Carry out the integral on the first term and check that it corresponds to the parton model expression for F_2, as is given on page 8–18 (note that for clarity we have suppressed the flavour index i and the summation over flavours).
- The expression above depends on the cutoff parameter m and diverges when $m \to 0$. To simplify the notation we set

  $$I_{\mathrm{qq}}(x) = \frac{\alpha_\mathrm{s}}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y) P_{\mathrm{qq}}\left(\frac{x}{y}\right)$$

  and write

  $$\frac{F_2(x,Q^2)}{x} = e^2 \underbrace{\left[\underbrace{f(x) + I_{\mathrm{qq}}(x) \ln\frac{\mu^2}{m^2}}_{f(x,\mu^2)} + I_{\mathrm{qq}}(x) \ln\frac{Q^2}{\mu^2} \right]}_{f(x,Q^2)}$$

- Here we have defined the *renormalised* quark distribution $f(x,\mu^2)$ at the so-called **factorisation scale** μ where we separate the singular factor, which depends on m but not on Q^2, from the calculable factor which depends on Q^2 but not on m.
- If we substitute the renormalised distribution for the bare distribution in I_{qq} we obtain, neglecting terms beyond $\mathrm{O}(\alpha_\mathrm{s})$,

  $$f(x,Q^2) = f(x,\mu^2) + \frac{\alpha_\mathrm{s}}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_{\mathrm{qq}}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} + \mathrm{O}(\alpha_\mathrm{s}^2)$$

DGLAP evolution

- The expression for F_2 now becomes, up to $\mathrm{O}(\alpha_\mathrm{s}^2)$,

$$\frac{F_2(x,Q^2)}{x} = e^2 \left[f(x,\mu^2) + \frac{\alpha_\mathrm{s}}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_{\mathrm{qq}}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} \right]$$

- Clearly F_2 should not depend on the choice of factorisation scale which leads to the following renormalisation group equation:

$$\frac{1}{e^2 x}\frac{\partial F_2(x,Q^2)}{\partial \ln\mu^2} = \frac{\partial f(x,\mu^2)}{\partial \ln\mu^2} + \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\left[\frac{\partial f(y,\mu^2)}{\partial \ln\mu^2}\ln\left(\frac{Q^2}{\mu^2}\right) - f(y,\mu^2)\right] P_{\mathrm{qq}}\left(\frac{x}{y}\right) = 0$$

- From this equation it is seen that $(\partial f/\partial \ln\mu^2)$ is of order α_s so that the first term in the integral above is of order α_s^2. Neglecting this term we obtain an evolution equation for the quark distribution[59]

$$\boxed{\frac{\partial f(x,\mu^2)}{\partial \ln\mu^2} = \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_{\mathrm{qq}}\left(\frac{x}{y}\right) + \mathrm{O}(\alpha_\mathrm{s}^2)}$$

  This is, together with the evolution equation for α_s (page 6–16), the most famous equation in QCD. It describes the evolution of a quark distribution due to gluon radiation and is called the **DGLAP evolution equation** after several authors who claim eternal fame: Dokshitzer, Gribov, Lipatov, Altarelli and Parisi.

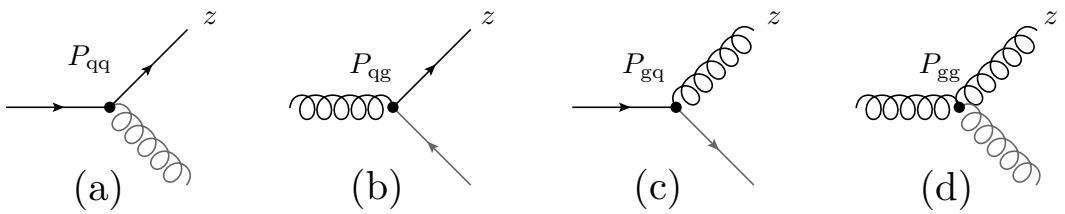
- This equation can be solved (numerically) once $f(x,\mu_0^2)$ is given as an input at some starting scale μ_0^2. This is similar to the running coupling constant α_s where also an input has to be given at some scale (usually taken to be m_Z^2, as we have seen).

[59] In our derivation we have assumed that α_s is a constant. Taking the running of α_s into account is somewhat subtle, but leads to the same evolution equation; see the comment in H&M exercise 10.7 on page 218.

Quark and gluon evolution

$$\frac{\partial f(x,\mu^2)}{\partial \ln \mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} f(y,\mu^2) P_{qq}\left(\frac{x}{y}\right) + O(\alpha_s^2)$$

- We have seen the DGLAP evolution of quark distributions with splitting function P_{qq} but when we introduce the gluon distribution, more splitting graphs have to be included.



(a) A daughter quark from the splitting of a parent quark into a quark and a gluon. When the gluon becomes soft $(1-z) \to 0$, the distinction between daughter and parent vanishes, and a singularity develops.

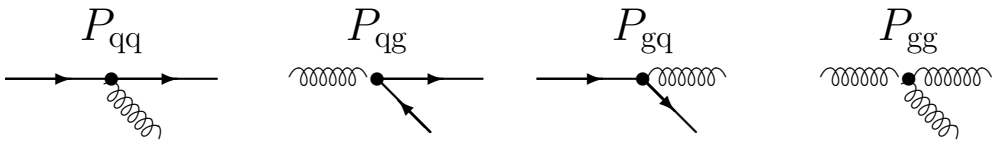
(b) A daughter quark from a parent gluon which splits into a quark-antiquark pair. Here no singularity develops since daughter and parent can always be distinguished.

(c) A daughter gluon from a quark parent. Also here no singularity.

(d) A daughter gluon from a parent gluon. Like in $q \to qg$ a singularity develops in the soft limit $(1-z) \to 0$.

The leading order splitting functions

- Here are the leading order splitting functions



$$
\begin{aligned}
P^{(0)}_{\rm qq}(z) &= \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right] \\
P^{(0)}_{\rm qg}(z) &= \frac{1}{2}\left[z^2 + (1-z)^2\right] \\
P^{(0)}_{\rm gq}(z) &= \frac{4}{3}\left[\frac{1+(1-z)^2}{z}\right] \\
P^{(0)}_{\rm gg}(z) &= 6\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right]
\end{aligned}
$$

- The singularities showing up in $P_{\rm qq}$ and $P_{\rm gg}$ at $(1-z) \to 0$ are regulated by a so-called 'plus' prescription which guarantees that the integral $\int_x^1$ exists of the splitting function multiplied by a parton density function (provided that the pdf $\to 0$ when $x \to 1$).
- For reference, we give here the definition of the plus prescription

$$[f(x)]_+ = f(x) - \delta(1-x)\int_0^1 f(z)\,{\rm d}z$$

or, equivalently,

$$\int_x^1 f(z)[g(z)]_+\,{\rm d}z = \int_x^1 \left[f(z)-f(1)\right]g(z)\,{\rm d}z - f(1)\int_0^x g(z)\,{\rm d}z.$$

The coupled DGLAP equations

- The qq, qg, gq and gg transitions lead to a set of $2n_f + 1$ coupled evolution equations that can be written as[60]

$$\frac{\partial f_i(x,\mu^2)}{\partial \ln \mu^2} = \sum_{j=-n_f}^{n_f} \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} P_{ij}\left(\frac{x}{y}\right) f_j(y,\mu^2),$$

  where the splitting function $P_{ij}(z)$ represents the probability that a daughter parton i with momentum fraction z splits from a parent parton j.[61] Here the indexing is as follows

$$i,j = \begin{cases} -1\,,\ldots,-n_f & \text{antiquarks} \\ 0 & \text{gluon} \\ 1\,,\ldots,\,n_f & \text{quarks} \end{cases}$$

- To simplify the expressions for the evolution equations we write the Mellin convolution in short-hand notation as

$$P \otimes f \equiv \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) f(y,\mu^2)$$

  With this notation the set of coupled equations reads

$$\frac{\partial f_i}{\partial \ln \mu^2} = \sum_{j=-n_f}^{n_f} \frac{\alpha_s}{2\pi} P_{ij} \otimes f_j$$

- In leading order QCD we can write for the splitting functions:[62]

$$P_{\bar{q}_i\bar{q}_j} = P_{q_iq_j} \equiv P_{qq}\,\delta_{ij}, \quad P_{\bar{q}_ig} = P_{q_ig} \equiv P_{qg}, \quad P_{g\bar{q}_i} = P_{gq_i} \equiv P_{gq}$$

[60] Here n_f is the number of 'active' quark flavours. Usually a quark species is considered to be active (*i.e.* it participates in the QCD dynamics) when its mass $m < \mu$.

[61] The conventional index notation for splitting functions is thus $P_{\text{daughter-parent}}$.

[62] The splitting functions are flavour independent since the strong interaction is flavour independent. Furthermore, leading order splitting cannot change the flavour of a quark, as is expressed by the delta function.

Singlet/gluon and non-singlet evolution

- Exploiting the symmetries in the splitting functions (previous page), the set of $2n_f+1$ coupled equations can to a large extent be decoupled by defining the **singlet distribution** $q_{\rm s}$, which is the sum over all flavours of the quark and antiquark distributions,

$$q_{\rm s} \equiv \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

- It is easy to show that the evolution of this distribution is coupled to that of the gluon

$$\begin{aligned} \frac{\partial q_{\rm s}}{\partial \ln \mu^2} &= \frac{\alpha_{\rm s}}{2\pi} \left[P_{\rm qq} \otimes q_{\rm s} + 2n_f P_{\rm qg} \otimes g \right] \\ \frac{\partial g}{\partial \ln \mu^2} &= \frac{\alpha_{\rm s}}{2\pi} \left[P_{\rm gq} \otimes q_{\rm s} + P_{\rm gg} \otimes g \right] \end{aligned}$$

  In compact matrix notation, this is often written as

$$\frac{\partial}{\partial \ln \mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \frac{\alpha_{\rm s}}{2\pi} \begin{pmatrix} P_{\rm qq} & 2n_f P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Likewise it is easy to show that **non-singlet distributions**

$$q_{\rm ns} \equiv \sum_{i=1}^{n_f} (C_i\, q_i + D_i\, \bar{q}_i) \quad \text{with} \quad \sum_{i=1}^{n_f} (C_i + D_i) = 0$$

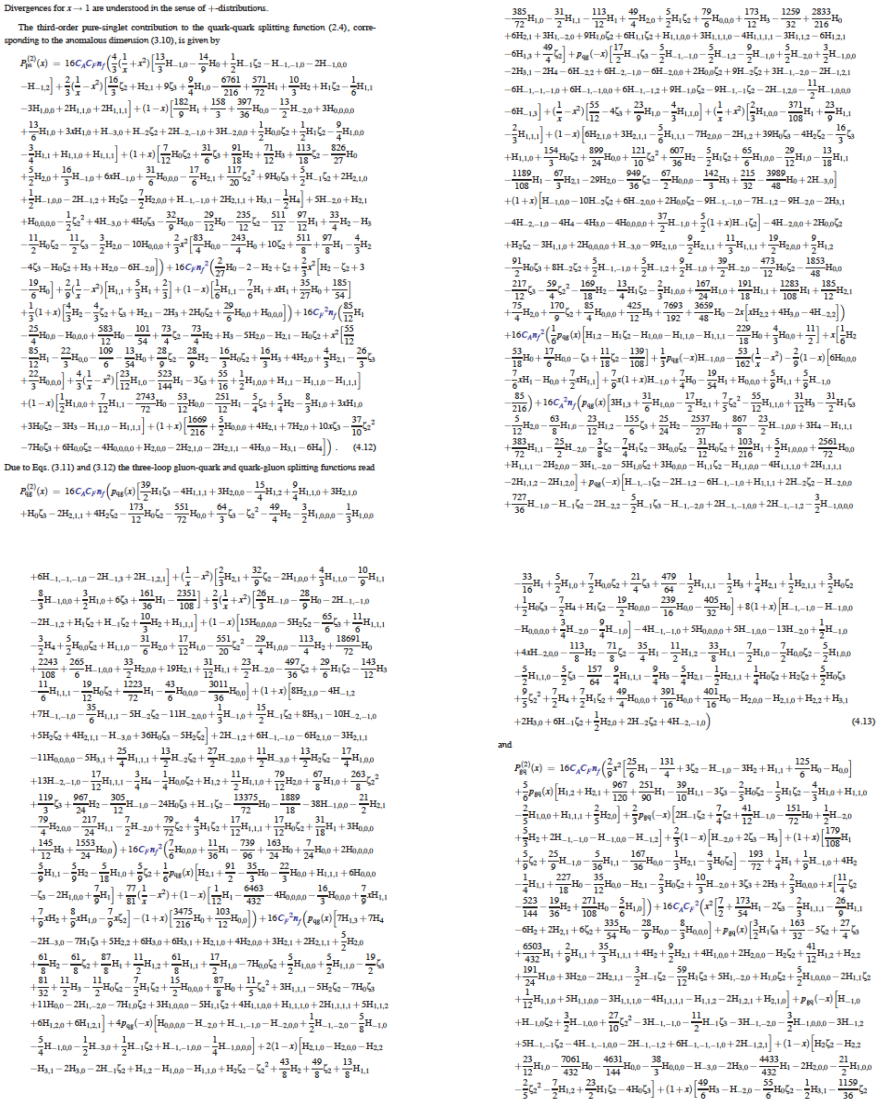
  evolve independent from the gluon and from each other:

$$\frac{\partial q_{\rm ns}}{\partial \ln \mu^2} = \frac{\alpha_{\rm s}}{2\pi}\, P_{\rm qq} \otimes q_{\rm ns}$$

  An example of a non-singlet is the valence distribution $q_i - \bar{q}_i$.

- Thus, in practice we do not evolve the quark distributions u, $\bar{u}$, d, $\bar{d}$, ... but, instead, the singlet distribution (coupled to the gluon) and a well chosen set of $2n_f - 1$ non-singlet distributions.

Higher orders ...



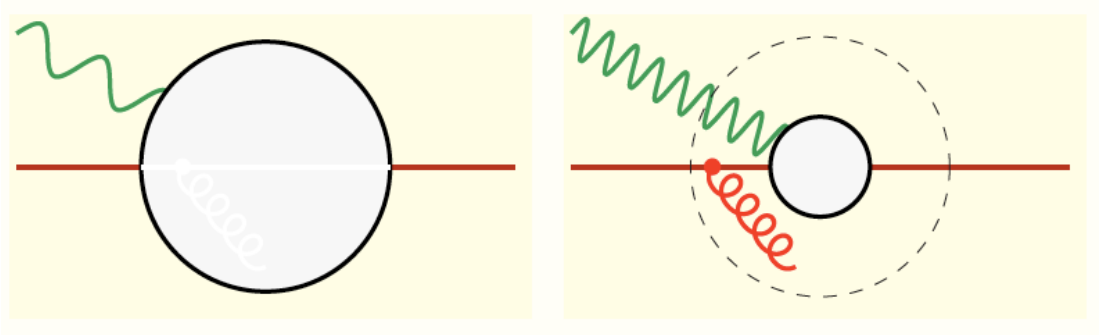
- The LO splitting functions presented on page 9–11 can be seen as the first term of a power series in α_s

$$P_{ij} = P_{ij}^{(0)} + (\alpha_s/2\pi)P_{ij}^{(1)} + (\alpha_s/2\pi)^2 P_{ij}^{(2)} + \cdots$$

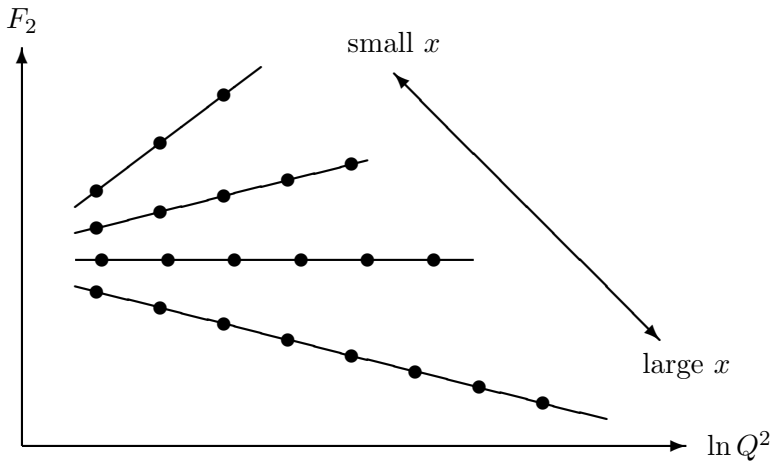
  Presently the splitting functions are known up to next-to-next-to-leading order (NNLO), that is, up to $P_{ij}^{(2)}$. Such a calculation (done at Nikhef) in no sinecure as the expression above shows. It goes on for many more pages...[63]

[63] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. **B691**, 129 (2004), hep-ph/0404111.

Intuitive picture

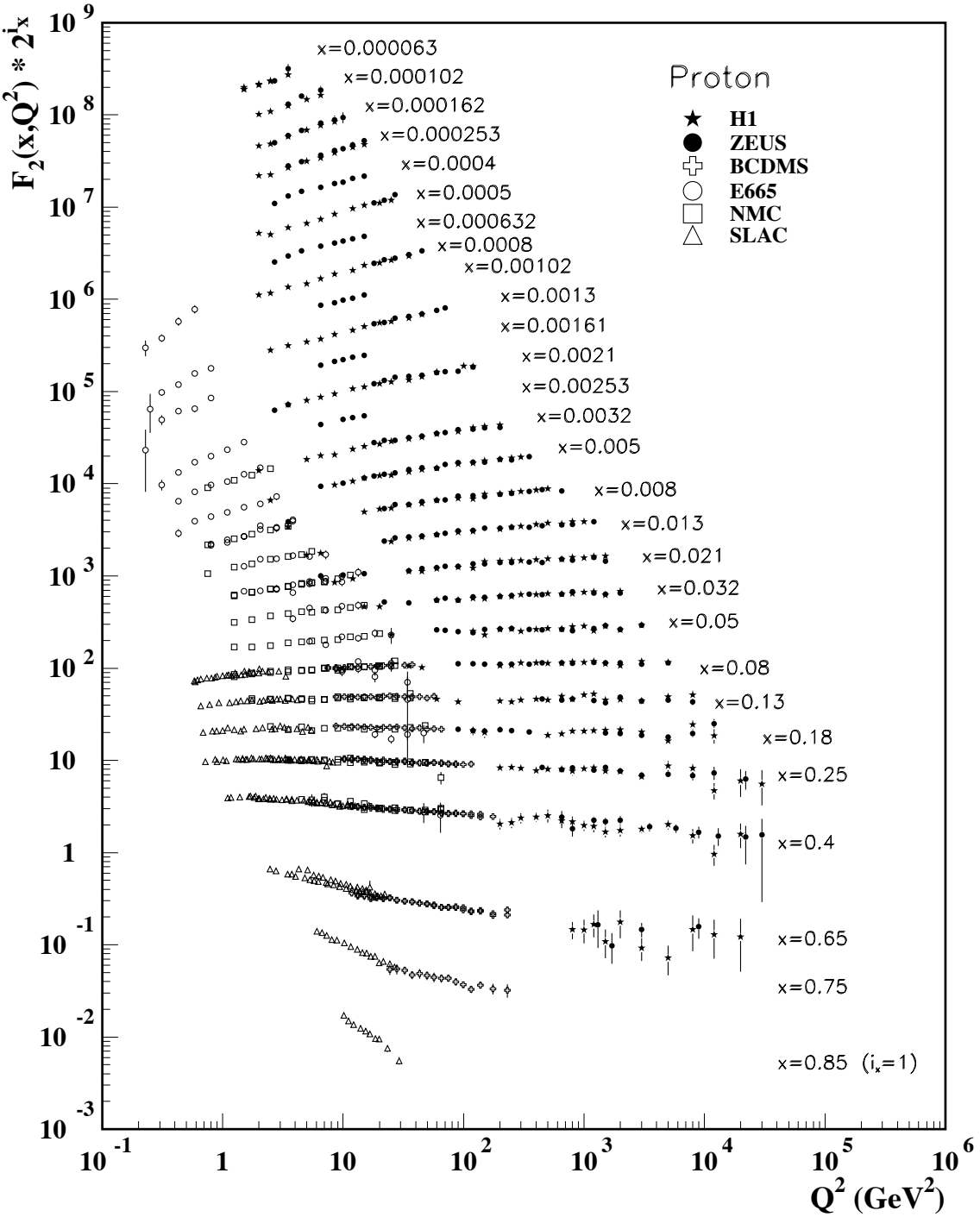


- The gross features of the evolution can be easily understood as follows. In the left plot above we indicate by the blob the resolution $1/Q$ of a photon with virtuality Q^2. Increasing Q^2 will resolve a quark into a quark-gluon pair of lower momentum (right plot). Thus when Q^2 increases, more and more quarks are seen that have split into low momentum quarks. As a consequence, the quark distribution will shift to lower values of x with increasing Q^2. This results in the characteristic scale breaking pattern of F_2, when plotted versus Q^2 for several values of x ($\rightarrow$ **fig**.)

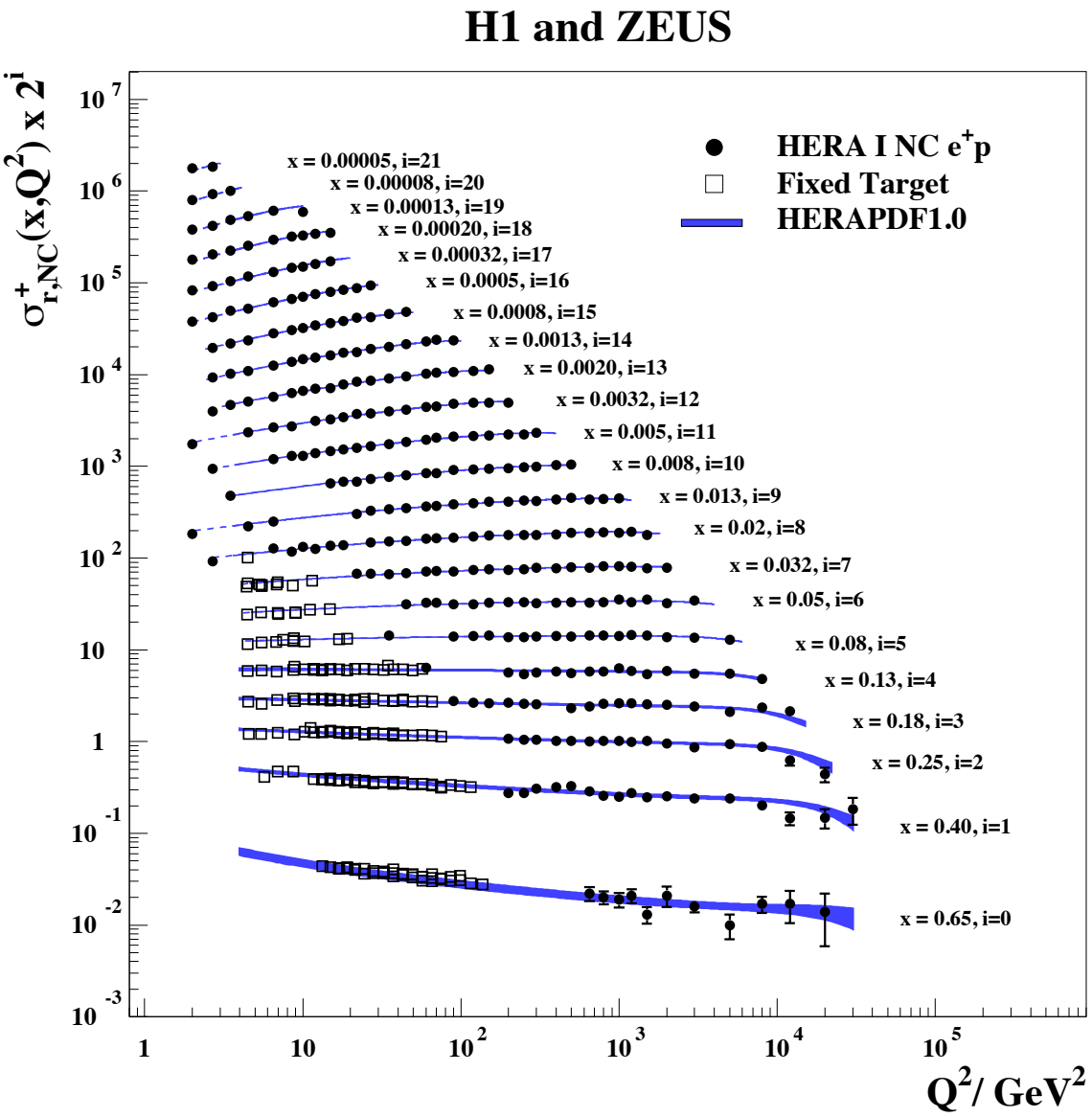


Scale breaking pattern of the F_2 structure function

As expected!

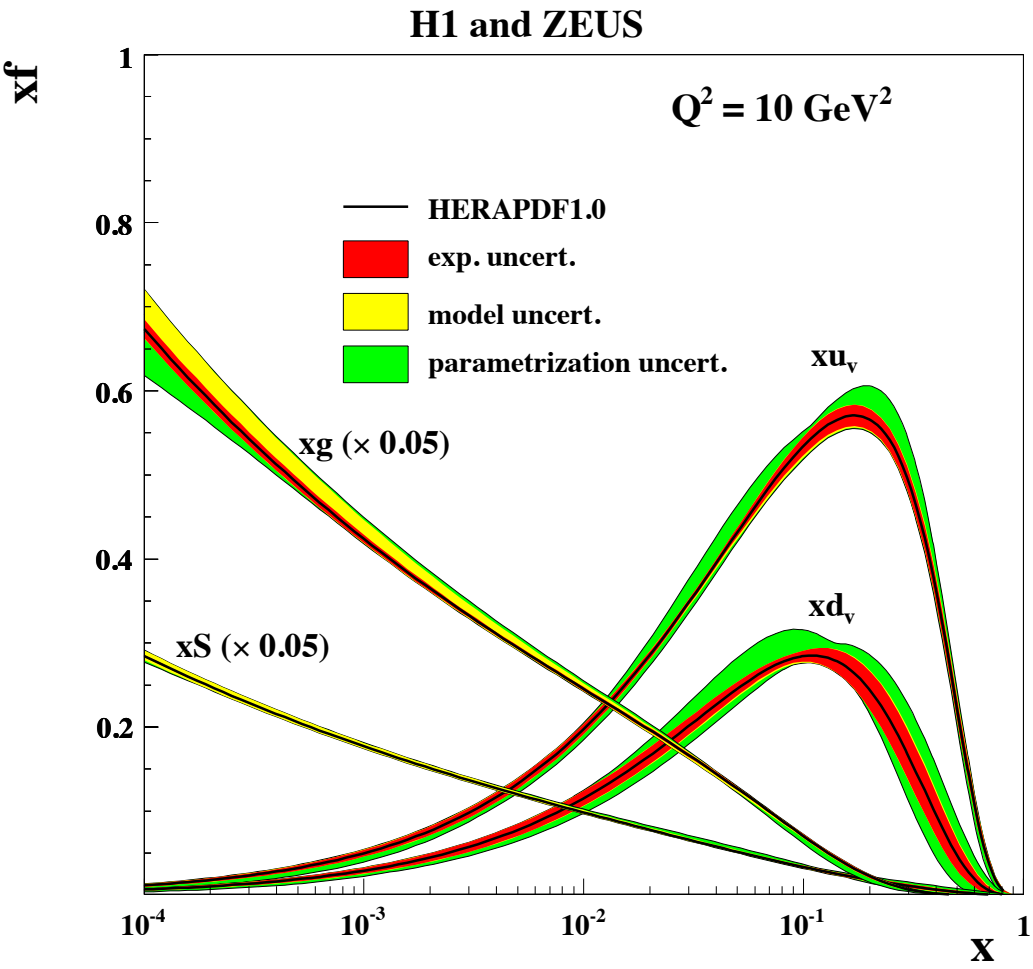


Comparison of the F_2 data with the QCD prediction



- This plot shows a recent QCD analysis (up to third order) of HERA F_2 structure function data. In such an analysis the quark and the gluon distributions are parameterised at an input scale of about 2 GeV^2 and evolved over the whole Q^2 range. The parameters of the input distributions are then obtained from a least squares fit. There is an impressive agreement between data and theory.

The pdf set from the HERA QCD analysis



- Parton distributions obtained from the HERA QCD analysis. The sea ($xS = 2x\bar{q}(x)$, see page 8–24) and the gluon are divided by a factor of 20. The parton distributions are parameterised at an input scale of $\mu_0^2 = 1.9$ GeV^2 and evolved to 10 GeV^2 for this plot. The bands indicate various sources of uncertainty.
- In DIS the electrons only scatter off the charged quarks in the proton and not off the gluons. However, we still have *indirect* access to the gluon distribution via the coupled singlet/gluon evolution.

Scale dependence

- At this point we have introduced *three* different scales:
  1. The factorization scale $\mu_{\rm F}^2$ where we have separated the short and long distance physics, and on which the pdfs evolve.
  2. The renormalisation scale $\mu_{\rm R}^2$ (called Q^2 in Section 6) on which the strong coupling constant $\alpha_{\rm s}$ evolves.
  3. The hard scattering scale Q^2 which, in DIS, is the square of the 4-momentum transfer from the electron to the proton.
- Exposing the different scales, we write the (non-singlet) evolution equation, and the leading order expression for F_2 as

$$\frac{\partial q_{\rm ns}(x,\mu_{\rm F}^2)}{\partial \ln \mu_{\rm F}^2} = \frac{\alpha_{\rm s}(\mu_{\rm R}^2)}{2\pi}\int_x^1 \frac{{\rm d}y}{y} P_{\rm qq}\left(\frac{x}{y}\right) q_{\rm ns}(y,\mu_{\rm F}^2)$$

$$F_2(x,Q^2) = \sum_{i=1}^{n_f} e_i^2 x\left[q_i(x,\mu_{\rm F}^2)+\bar{q}_i(x,\mu_{\rm F}^2)\right] + {\rm O}(\alpha_{\rm s})$$

- Usually one sets $\mu_{\rm R}^2 = \mu_{\rm F}^2 = Q^2$. The sensitivity to this choice is then quantified by varying the scales in the range, typically,

$$\tfrac{1}{4}\mu_{\rm F}^2 \le \mu_{\rm R}^2 \le 4\mu_{\rm F}^2 \quad \text{and} \quad \tfrac{1}{4}Q^2 \le \mu_{\rm F}^2 \le 4Q^2$$

- But note, however, that $F_2(x,Q^2)$ above depends only on $\mu_{\rm F}^2$ which, for a given Q^2, is arbitrary. It follows that the leading order perturbative stability is poor, and that LO perturbative QCD has little predictive power. This defect is remedied when higher order terms are included that are functions of both Q^2 and $\mu_{\rm F}^2$.
- Fortunately, the scale dependence rapidly decreases when higher order corrections are included, and this is of course the motivation behind that huge effort, at Nikhef, to calculate the splitting functions and the F_2 correction terms up to NNLO.