Lecture notes Particle Physics II

Quantum Chromo Dynamics

9. The QCD improved Parton Model

Michiel Botje
Nikhef, Science Park, Amsterdam

December 5, 2013

The QCD factorisation theorem

- In Section 7 we have seen that QCD suffers from infrared singularities when two daughter partons cannot be resolved because they become collinear, or because one of them becomes soft. We have also seen that these singularities are associated with 'long-distance' physics which takes place a long time after the initial hard scattering. So-called infrared-safe observables are still calculable in perturbative QCD, but since this is quite restrictive we have to look for ways to extend the predictive power of the theory. This way-out is provided by the QCD **factorisation theorem**.
- For hadron-hadron scattering the factorisation theorem states that the singular long-distance pieces can be removed from the partonic cross section and factored into the parton distributions of the incoming hadrons, and that this can be done consistently at all orders in the perturbative expansion.
- The partonic cross section is then calculable in perturbation theory, and does not depend on the type of incoming hadron.
- The parton distributions, on the other hand, are a property of the incoming hadrons but are universal in the sense that they do not depend on the hard scattering process.[57] Parton distributions are nonperturbative and have to be obtained from experiment.
- Factorisation is a fundamental property of QCD. It turns perturbative QCD into a reliable calculation tool, unlike the naive parton model that does not take the parton dynamics into account.

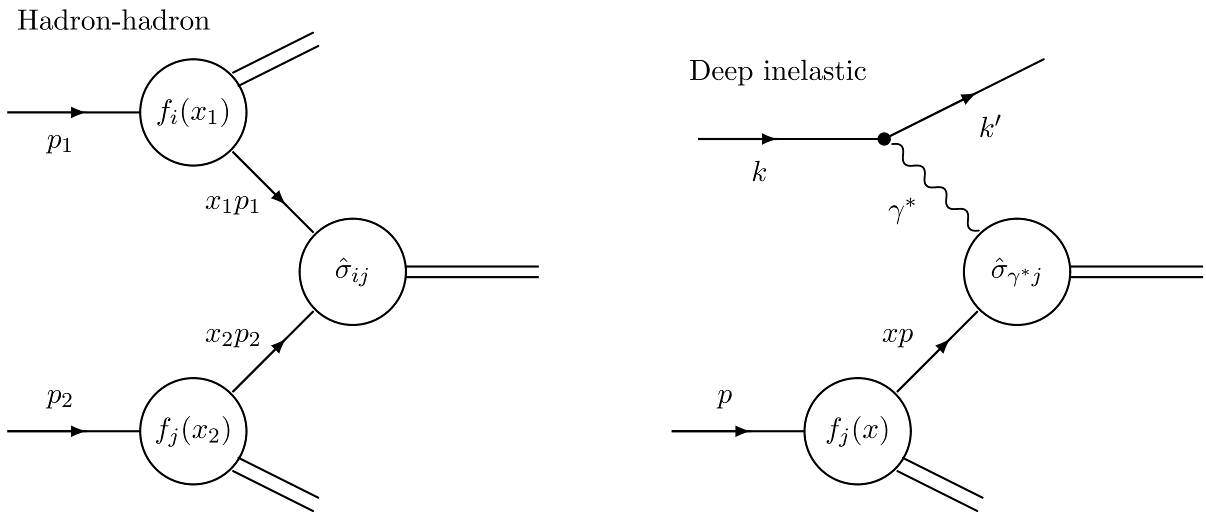
[57] See also Exercise 8.7.

Hadron-hadron cross sections I

- Schematically, a hadron-hadron cross section can be written as

$$\sigma^{\text{h-h}} = \sum_{i,j} \int \mathrm{d}x_1 \mathrm{d}x_2\, f_i(x_1,\mu^2) f_j(x_2,\mu^2)\, \hat{\sigma}_{ij}(x_1, x_2, Q^2/\mu^2, \cdots)$$

and can be depicted by (left diagram):



- Here the (arbitrary) **factorisation scale** μ can be thought of as the scale which separates the long and short-distance physics. Roughly speaking, a parton with a transverse momentum less than μ is then considered to be part of the hadron structure and is absorbed in the parton distribution. Partons with larger transverse momenta participate in the hard scattering process with a short-distance partonic cross-section $\hat{\sigma}$.

Hadron-hadron cross sections II

- What is taken for the hard scale Q^2 depends on the scattering process we are interested in. In jet production, for instance, one usually takes the transverse momentum of the jet as the hard scale, in deep inelastic scattering one takes the square of the four momentum transfer from the electron to the proton, and in $\mathrm{e}^+\mathrm{e}^-$ scattering one takes the centre-of-mass energy, and so on. Often the simplifying assumption is made that the factorisation scale is equal to the hard scale: $\mu^2 = Q^2$.

- The factorisation theorem also applies to deep inelastic scattering, with one of the parton distributions replaced by an $\mathrm{ee}'\gamma^*$ vertex as is shown in the right-hand side diagram on page 9–4:

$$\sigma^{\mathrm{DIS}} = \sum_j \int \mathrm{d}x \, f_j(x, \mu^2) \, \hat{\sigma}_{\gamma^* j}(x, Q^2/\mu^2, \cdots)$$

- We will use DIS to show how the infrared singularities are absorbed in the parton distributions. The QCD evolution equations of the parton densities are then derived from the **renormalisation group equation**.

Recap of the F_2 structure function

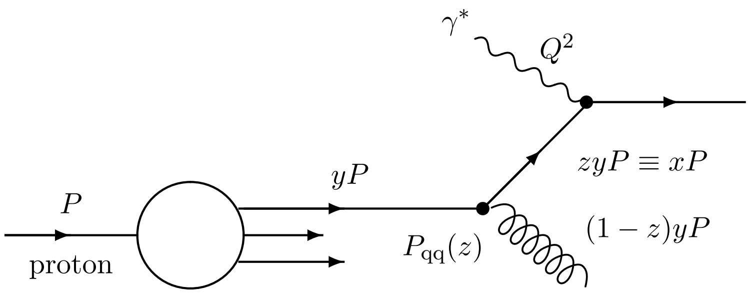
- We have seen that the F_2 structure function measured in deep inelastic electron-proton scattering is, to first order, independent of Q^2, the negative square of the momentum transfer from the electron to the proton. This implies that DIS does not depend on the resolution $1/Q$ with which the proton is probed. This is explained in the naive parton model by assuming that the electron scatters incoherently off *pointlike* quarks in the proton. The F_2 structure function can then be written as the charge weighted sum of quark momentum distributions

$$F_2^{\mathrm{ep}}(x) = \sum_i e_i^2 x f_i(x).$$

  Here e_i is the charge of the quark, and $f_i(x)\mathrm{d}x$ is the number of quarks that carry a fraction between x and $x + \mathrm{d}x$ of the proton momentum. The probability that the parton carries a momentum fraction x is then given by $xf_i(x)$. The index i denotes the quark flavour $\mathrm{d}, \mathrm{u}, \mathrm{s}, \ldots, \bar{\mathrm{d}}, \bar{\mathrm{u}}, \bar{\mathrm{s}}, \ldots$

- Although gluons show up in the naive parton model as missing momentum, they are not treated as dynamical entities. We will now incorporate the effect of gluon radiation by quarks, which leads to the so-called **QCD improved parton model**.

Scaling violation I



- Consider a quark that carries a fraction y of the proton momentum and radiates a gluon with a fraction $1 - z$ of its momentum. After radiating the gluon, the quark with momentum fraction zy scatters off the virtual photon. The momentum fraction seen by the photon is thus $x = zy$ which implies that $z = x/y$.

- Taking gluon radiation into account, the F_2 structure function is found to be (see H&M Section 10.1–5 for a lengthy derivation):

$$\frac{F_2(x, Q^2)}{x} = \sum_i e_i^2 \int_x^1 \frac{\mathrm{d}y}{y} f_i(y) \left[\delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_{\rm s}}{2\pi} P_{\rm qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{m^2} \right]$$

Here m^2 is a lower transverse momentum cut-off to regularise the divergence when the gluon becomes collinear with the quark.

- In the above, the **splitting function** $P_{\rm qq}$ is given by

$$P_{\rm qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right).$$

It represents the probability that a parent quark emits a gluon with the daughter quark retaining a fraction z of the parent momentum. Note that an infrared divergence shows up when $(1 - z) \to 0$ where the gluon becomes soft so that daughter and parent cannot be resolved anymore.

Scaling violation II

$$\boxed{\frac{F_2(x,Q^2)}{x} = e^2 \int_x^1 \frac{\mathrm{d}y}{y} f(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_\mathrm{s}}{2\pi} P_\mathrm{qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{m^2} \right]}$$

- **<u>Exercise 9.1</u>**: [1.0] Carry out the integral on the first term and check that it corresponds to the parton model expression for F_2, as is given on page 8–18 (note that for clarity we have suppressed the flavour index i and the summation over flavours).

- The expression above depends on the cutoff parameter m and diverges when $m \to 0$. To simplify the notation we set

$$I_\mathrm{qq}(x) = \frac{\alpha_\mathrm{s}}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y) P_\mathrm{qq}\left(\frac{x}{y}\right)$$

and write

$$\frac{F_2(x,Q^2)}{x} = e^2 \underbrace{\left[\underbrace{f(x) + I_\mathrm{qq}(x)\ln\frac{\mu^2}{m^2}}_{f(x,\mu^2)} + I_\mathrm{qq}(x)\ln\frac{Q^2}{\mu^2} \right]}_{f(x,Q^2)}$$

- Here we have defined the *renormalised* quark distribution $f(x,\mu^2)$ at the so-called **factorisation scale** μ where we separate the singular factor, which depends on m but not on Q^2, from the calculable factor which depends on Q^2 but not on m.

- If we substitute the renormalised distribution for the bare distribution in I_qq we obtain, neglecting terms beyond $\mathrm{O}(\alpha_\mathrm{s})$,

$$f(x,Q^2) = f(x,\mu^2) + \frac{\alpha_\mathrm{s}}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_\mathrm{qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} + \mathrm{O}(\alpha_\mathrm{s}^2)$$

DGLAP evolution

- The expression for F_2 now becomes, up to $\mathrm{O}(\alpha_\mathrm{s}^2)$,

$$\frac{F_2(x,Q^2)}{x} = e^2\left[f(x,\mu^2) + \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_\mathrm{qq}\left(\frac{x}{y}\right)\ln\frac{Q^2}{\mu^2}\right]$$

- Clearly F_2 should not depend on the choice of factorisation scale which leads to the following renormalisation group equation:

$$\frac{1}{e^2 x}\frac{\partial F_2(x,Q^2)}{\partial \ln\mu^2} = \frac{\partial f(x,\mu^2)}{\partial\ln\mu^2} + \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\left[\frac{\partial f(y,\mu^2)}{\partial\ln\mu^2}\ln\left(\frac{Q^2}{\mu^2}\right) - f(y,\mu^2)\right] P_\mathrm{qq}\left(\frac{x}{y}\right) = 0$$

- From this equation it is seen that $(\partial f/\partial\ln\mu^2)$ is of order α_s so that the first term in the integral above is of order α_s^2. Neglecting this term we obtain an evolution equation for the quark distribution[58]

$$\boxed{\frac{\partial f(x,\mu^2)}{\partial\ln\mu^2} = \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1\frac{\mathrm{d}y}{y} f(y,\mu^2) P_\mathrm{qq}\left(\frac{x}{y}\right) + \mathrm{O}(\alpha_\mathrm{s}^2)}$$

  This is, together with the evolution equation for α_s (page 6–16), the most famous equation in QCD. It describes the evolution of a quark distribution due to gluon radiation and is called the **DGLAP evolution equation** after several authors who claim eternal fame: Dokshitzer, Gribov, Lipatov, Altarelli and Parisi.

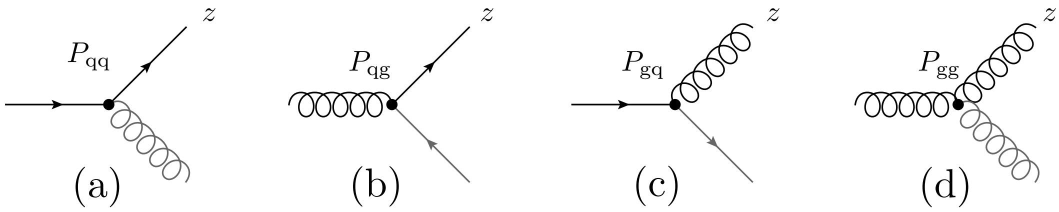
- This equation can be solved (numerically) once $f(x,\mu_0^2)$ is given as an input at some starting scale μ_0^2. This is similar to the running coupling constant α_s where also an input has to be given at some scale (usually taken to be m_Z^2, as we have seen).

[58] In our derivation we have assumed that α_s is a constant. Taking the running of α_s into account is somewhat subtle, but leads to the same evolution equation; see the comment in H&M exercise 10.7 on page 218.

Quark and gluon evolution

$$\boxed{\frac{\partial f(x,\mu^2)}{\partial \ln \mu^2} = \frac{\alpha_\mathrm{s}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_\mathrm{qq}\left(\frac{x}{y}\right) + \mathrm{O}(\alpha_\mathrm{s}^2)}$$

- We have seen the DGLAP evolution of quark distributions with splitting function P_qq but when we introduce the gluon distribution, more splitting graphs have to be included.



(a) A daughter quark from the splitting of a parent quark into a quark and a gluon. When the gluon becomes soft $(1-z) \to 0$, the distinction between daughter and parent vanishes, and a singularity develops.

(b) A daughter quark from a parent gluon which splits into a quark-antiquark pair. Here no singularity develops since daughter and parent can always be distinguished.

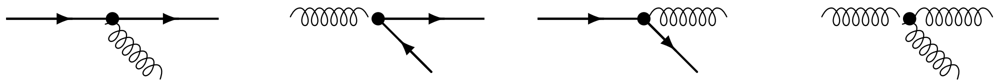
(c) A daughter gluon from a quark parent. Also here no singularity.

(d) A daughter gluon from a parent gluon. Like in $q \to q$g a singularity develops in the soft limit $(1-z) \to 0$.

The leading order splitting functions

- Here are the leading order splitting functions

P_{qq} $\quad$ P_{qg} $\quad$ P_{gq} $\quad$ P_{gg}



$$
\begin{aligned}
P_{\mathrm{qq}}^{(0)}(z) &= \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right] \\
P_{\mathrm{qg}}^{(0)}(z) &= \frac{1}{2}\left[z^2 + (1-z)^2\right] \\
P_{\mathrm{gq}}^{(0)}(z) &= \frac{4}{3}\left[\frac{1+(1-z)^2}{z}\right] \\
P_{\mathrm{gg}}^{(0)}(z) &= 6\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right]
\end{aligned}
$$

- The singularities showing up in P_{qq} and P_{gg} at $(1-z) \to 0$ are regulated by a so-called ‘plus’ prescription which guarantees that the integral $\int_x^1$ exists of the splitting function multiplied by a parton density function (provided that the pdf $\to 0$ when $x \to 1$).

- For reference, we give here the definition of the plus prescription

$$[f(x)]_+ = f(x) - \delta(1-x)\int_0^1 f(z)\,\mathrm{d}z$$

or, equivalently,

$$\int_x^1 f(z)[g(z)]_+\,\mathrm{d}z = \int_x^1 [f(z) - f(1)]\,g(z)\,\mathrm{d}z - f(1)\int_0^x g(z)\,\mathrm{d}z.$$

The coupled DGLAP equations

- The qq, qg, gq and gg transitions lead to a set of $2n_f + 1$ coupled evolution equations that can be written as[59]

$$\frac{\partial f_i(x,\mu^2)}{\partial \ln \mu^2} = \sum_{j=-n_f}^{n_f} \frac{\alpha_{\rm s}}{2\pi} \int_x^1 \frac{{\rm d}y}{y} P_{ij}\left(\frac{x}{y}\right) f_j(y,\mu^2),$$

where the splitting function $P_{ij}(z)$ represents the probability that a daughter parton i with momentum fraction z splits from a parent parton j.[60] Here the indexing is as follows

$$i,j = \begin{cases} -1\,,\ldots,-n_f & \text{antiquarks} \\ \ \ 0 & \text{gluon} \\ \ \ 1\,,\ldots,\ n_f & \text{quarks} \end{cases}$$

- To simplify the expressions for the evolution equations we write the Mellin convolution in short-hand notation as

$$P \otimes f \equiv \int_x^1 \frac{{\rm d}y}{y} P\left(\frac{x}{y}\right) f(y,\mu^2)$$

With this notation the set of coupled equations reads

$$\frac{\partial f_i}{\partial \ln \mu^2} = \sum_{j=-n_f}^{n_f} \frac{\alpha_{\rm s}}{2\pi} P_{ij} \otimes f_j$$

- In leading order QCD we can write for the splitting functions:[61]

$$P_{\bar{\rm q}_i\bar{\rm q}_j} = P_{{\rm q}_i{\rm q}_j} \equiv P_{\rm qq}\,\delta_{ij}, \quad P_{\bar{\rm q}_i{\rm g}} = P_{{\rm q}_i{\rm g}} \equiv P_{\rm qg}, \quad P_{{\rm g}\bar{\rm q}_i} = P_{{\rm g}{\rm q}_i} \equiv P_{\rm gq}$$

---

[59] Here n_f is the number of 'active' quark flavours. Usually a quark species is considered to be active (*i.e.* it participates in the QCD dynamics) when its mass $m < \mu$.

[60] The conventional index notation for splitting functions is thus $P_{\text{daughter-parent}}$.

[61] The splitting functions are flavour independent since the strong interaction is flavour independent. Furthermore, leading order splitting cannot change the flavour of a quark, as is expressed by the delta function.

Singlet/gluon and non-singlet evolution

- Exploiting the symmetries in the splitting functions (previous page), the set of $2n_f+1$ coupled equations can to a large extent be decoupled by defining the **singlet distribution** q_{s}, which is the sum over all flavours of the quark and antiquark distributions,

$$q_{\mathrm{s}} \equiv \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

- It is easy to show that the evolution of this distribution is coupled to that of the gluon

$$\begin{aligned} \frac{\partial q_{\mathrm{s}}}{\partial \ln \mu^2} &= \frac{\alpha_{\mathrm{s}}}{2\pi} \left[P_{\mathrm{qq}} \otimes q_{\mathrm{s}} + 2n_f P_{\mathrm{qg}} \otimes g \right] \\ \frac{\partial g}{\partial \ln \mu^2} &= \frac{\alpha_{\mathrm{s}}}{2\pi} \left[P_{\mathrm{gq}} \otimes q_{\mathrm{s}} + P_{\mathrm{gg}} \otimes g \right] \end{aligned}$$

  In compact matrix notation, this is often written as

$$\frac{\partial}{\partial \ln \mu^2} \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \frac{\alpha_{\mathrm{s}}}{2\pi} \begin{pmatrix} P_{\mathrm{qq}} & 2n_f P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

- Likewise it is easy to show that **non-singlet distributions**

$$q_{\mathrm{ns}} \equiv \sum_{i=1}^{n_f} (C_i\, q_i + D_i\, \bar{q}_i) \quad \text{with} \quad \sum_{i=1}^{n_f} (C_i + D_i) = 0$$

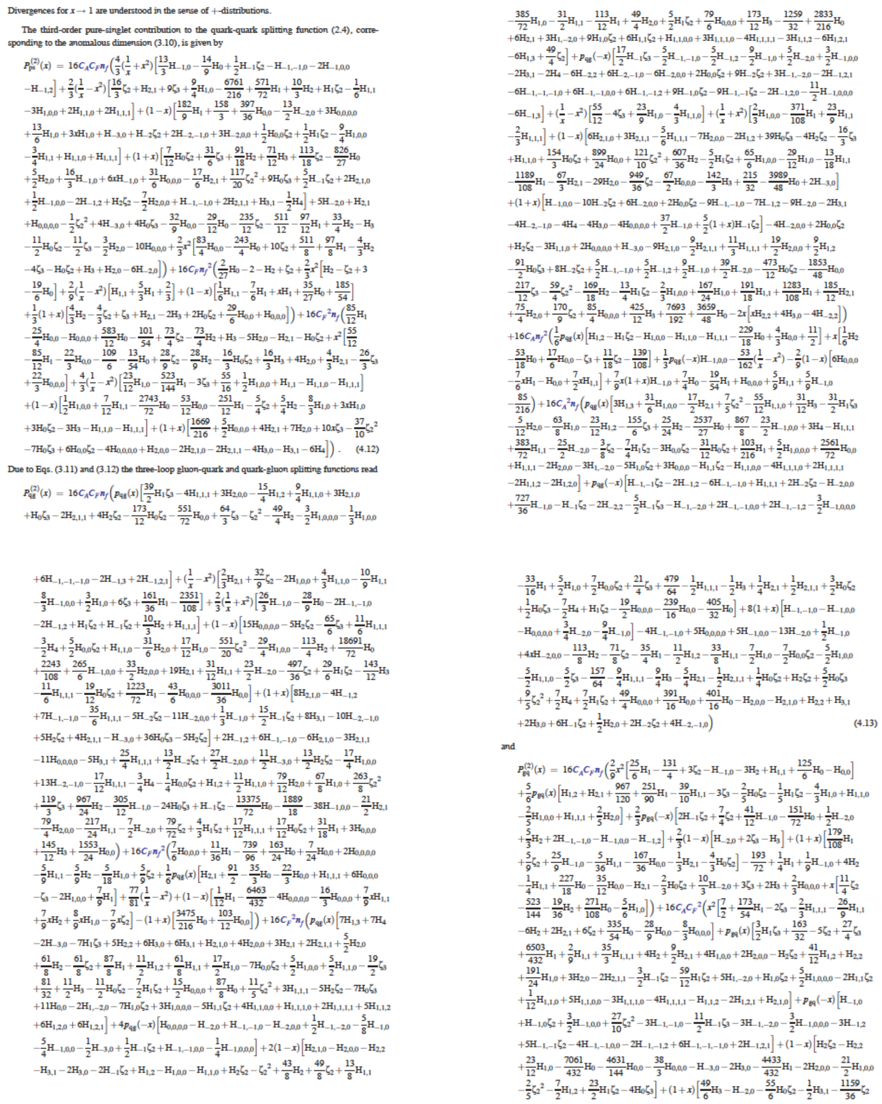
  evolve independent from the gluon and from each other:

$$\frac{\partial q_{\mathrm{ns}}}{\partial \ln \mu^2} = \frac{\alpha_{\mathrm{s}}}{2\pi} P_{\mathrm{qq}} \otimes q_{\mathrm{ns}}$$

  An example of a non-singlet is the valence distribution $q_i - \bar{q}_i$.

- Thus, in practice we do not evolve the quark distributions u, $\bar{u}$, d, $\bar{d}$, ... but, instead, the singlet distribution (coupled to the gluon) and a well chosen set of $2n_f - 1$ non-singlet distributions.

Higher orders ...


Divergences for $x \to 1$ are understood in the sense of +-distributions.

The third-order pure-singlet contribution to the quark-quark splitting function (2.4), corresponding to the anomalous dimension (3.10), is given by

Due to Eqs. (3.11) and (3.12) the three-loop gluon-quark and quark-gluon splitting functions read

and

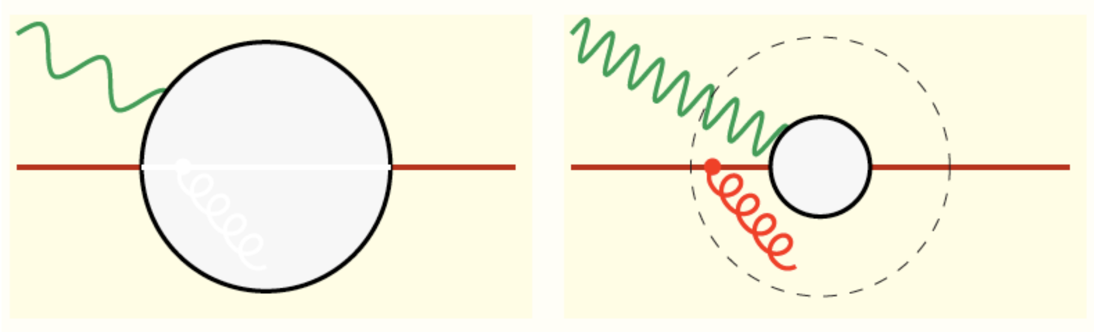
- The LO splitting functions presented on page 9–11 can be seen as the first term of a power series in α_s

$$P_{ij} = P_{ij}^{(0)} + (\alpha_\mathrm{s}/2\pi)P_{ij}^{(1)} + (\alpha_\mathrm{s}/2\pi)^2 P_{ij}^{(2)} + \cdots$$

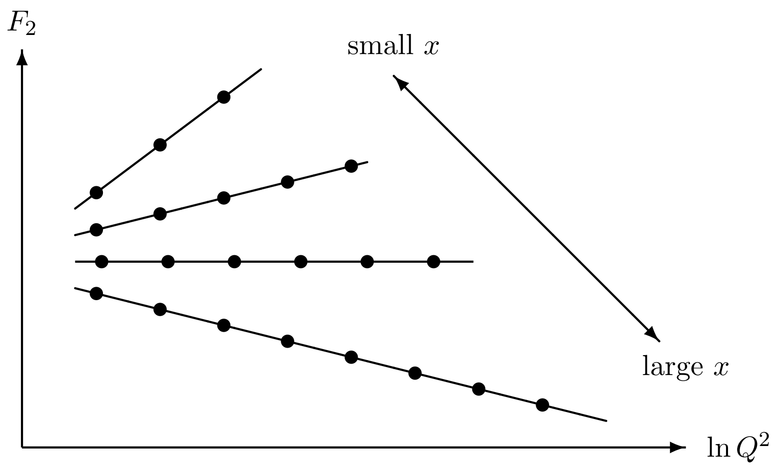
  Presently the splitting functions are known up to next-to-next-to-leading order (NNLO), that is, up to $P_{ij}^{(2)}$. Such a calculation (done at Nikhef) in no sinecure as the expression above shows. It goes on for many more pages...[62]

[62] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. **B691**, 129 (2004), hep-ph/0404111.

Intuitive picture

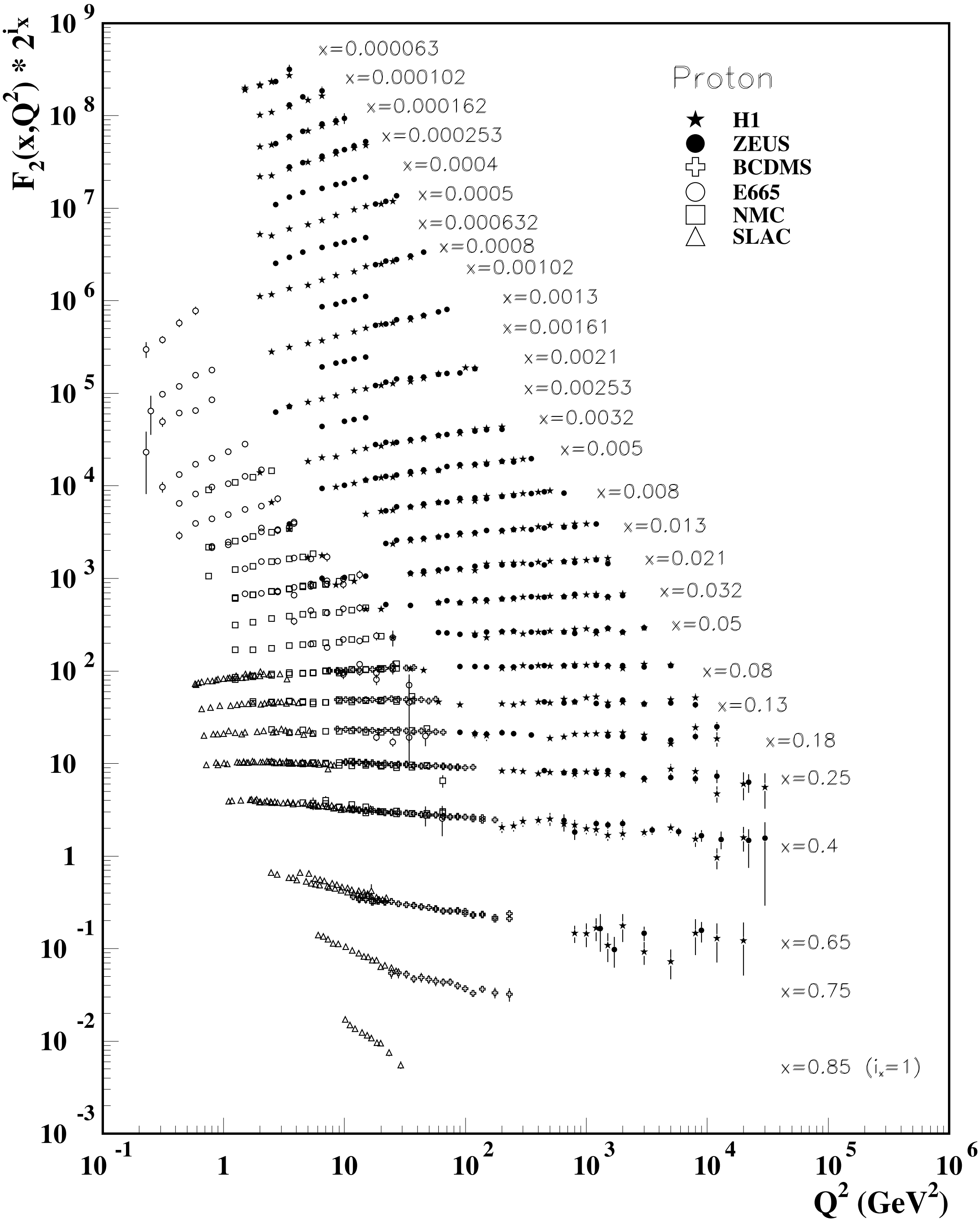


- The gross features of the evolution can be easily understood as follows. In the left plot above we indicate by the blob the resolution $1/Q$ of a photon with virtuality Q^2. Increasing Q^2 will resolve a quark into a quark-gluon pair of lower momentum (right plot). Thus when Q^2 increases, more and more quarks are seen that have split into low momentum quarks. As a consequence, the quark distribution will shift to lower values of x with increasing Q^2. This results in the characteristic scale breaking pattern of F_2, when plotted versus Q^2 for several values of x ($\rightarrow$ **fig.**)

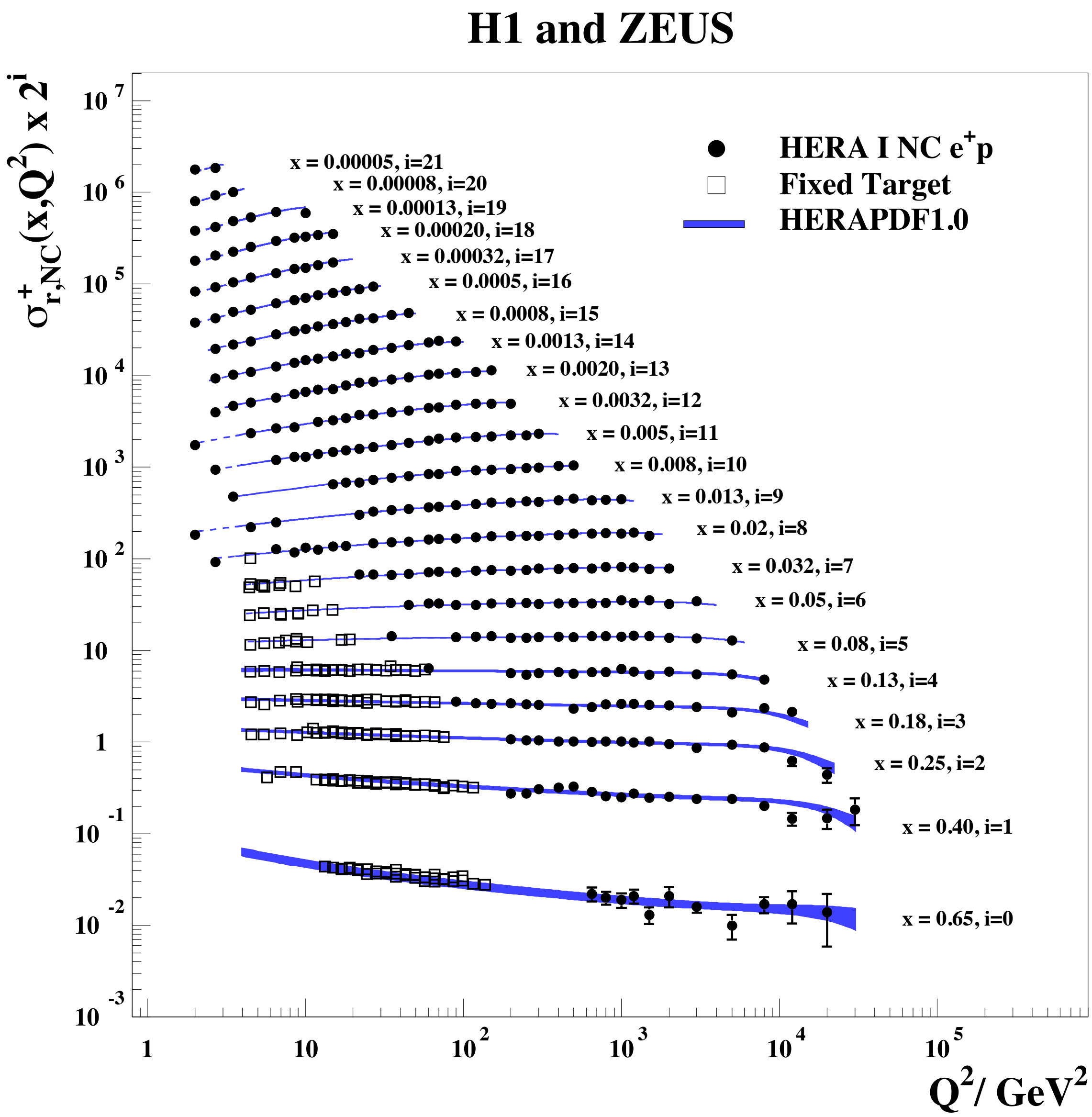


Scale breaking pattern of the F_2 structure function

As expected!

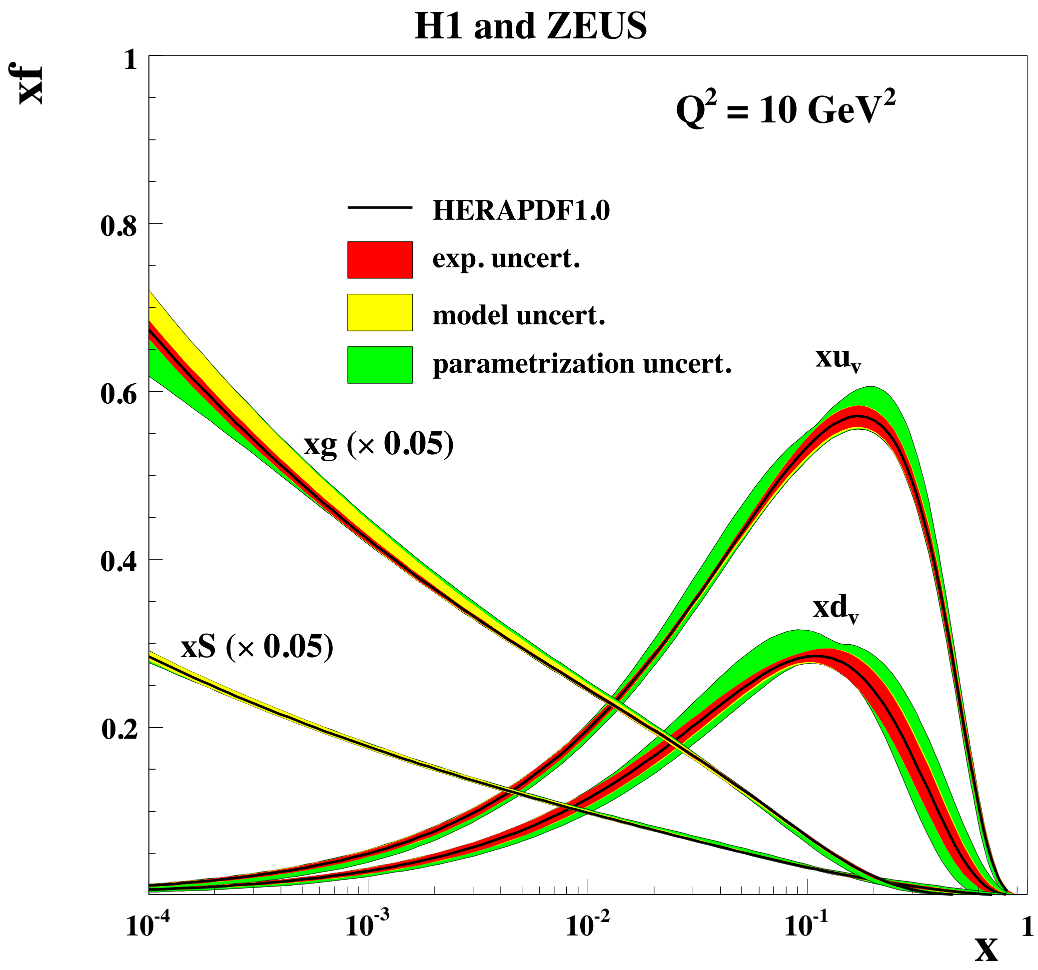


Comparison of the F_2 data with the QCD prediction



- This plot shows a recent QCD analysis (up to third order) of HERA F_2 structure function data. In such an analysis the quark and the gluon distributions are parameterised at an input scale of about 2 GeV^2 and evolved over the whole Q^2 range. The parameters of the input distributions are then obtained from a least squares fit. There is an impressive agreement between data and theory.

The pdf set from the HERA QCD analysis



- Parton distributions obtained from the HERA QCD analysis. The sea ($xS = 2x\bar{q}(x)$, see page 8–24) and the gluon are divided by a factor of 20. The parton distributions are parameterised at an input scale of $\mu_0^2 = 1.9\ \text{GeV}^2$ and evolved to $10\ \text{GeV}^2$ for this plot. The bands indicate various sources of uncertainty.
- In DIS the electrons only scatter off the charged quarks in the proton and not off the gluons. However, we still have *indirect* access to the gluon distribution via the coupled singlet/gluon evolution.

Scale dependence

- At this point we have introduced *three* different scales:
  1. The factorization scale μ_{F}^2 where we have separated the short and long distance physics, and on which the pdfs evolve.
  2. The renormalisation scale μ_{R}^2 (called Q^2 in Section 6) on which the strong coupling constant α_{s} evolves.
  3. The hard scattering scale Q^2 which, in DIS, is the square of the 4-momentum transfer from the electron to the proton.
- Exposing the different scales, we write the (non-singlet) evolution equation, and the leading order expression for F_2 as

$$\begin{aligned}\frac{\partial q_{\mathrm{ns}}(x,\mu_{\mathrm{F}}^2)}{\partial \ln \mu_{\mathrm{F}}^2} &= \frac{\alpha_{\mathrm{s}}(\mu_{\mathrm{R}}^2)}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\,P_{\mathrm{qq}}\left(\frac{x}{y}\right) q_{\mathrm{ns}}(y,\mu_{\mathrm{F}}^2)\\ F_2(x,Q^2) &= \sum_{i=1}^{n_f} e_i^2 x\left[q_i(x,\mu_{\mathrm{F}}^2)+\bar{q}_i(x,\mu_{\mathrm{F}}^2)\right] + \mathrm{O}(\alpha_{\mathrm{s}})\end{aligned}$$

- Usually one sets $\mu_{\mathrm{R}}^2 = \mu_{\mathrm{F}}^2 = Q^2$. The sensitivity to this choice is then quantified by varying the scales in the range, typically,

$$\tfrac{1}{4}\mu_{\mathrm{F}}^2 \le \mu_{\mathrm{R}}^2 \le 4\mu_{\mathrm{F}}^2 \quad \text{and} \quad \tfrac{1}{4}Q^2 \le \mu_{\mathrm{F}}^2 \le 4Q^2$$

- But note, however, that $F_2(x,Q^2)$ above depends only on μ_{F}^2 which, for a given Q^2, is arbitrary. It follows that the leading order perturbative stability is poor, and that LO perturbative QCD has little predictive power. This defect is remedied when higher order terms are included that are functions of both Q^2 and μ_{F}^2.
- Fortunately, the scale dependence rapidly decreases when higher order corrections are included, and this is of course the motivation behind that huge effort, at Nikhef, to calculate the splitting functions and the F_2 correction terms up to NNLO.