Lecture notes Particle Physics II

Quantum Chromo Dynamics

7. Soft and Collinear Singularities

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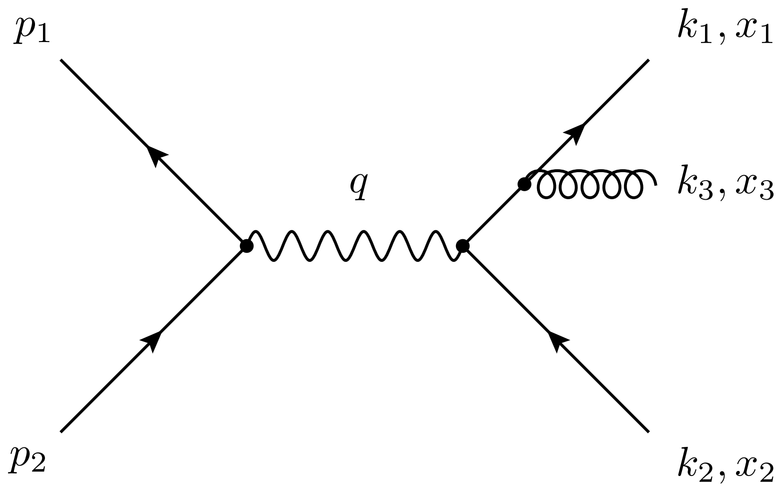
Can perturbative QCD predict anything?

- We have seen that asymptotic freedom allows us to use perturbation theory to calculate quark and gluon interactions at short distances. But is this enough to arrive at predictions for experimental observables?
- The answer is 'no', because the detectors in an experiment can only observe hadrons and not the constituent quarks and gluons.
- We will see that we need two more things, if we want to make the connection between theory and experiment:
  1. either infrared safety,
  2. or factorisation.
- These concepts are intimately related to the separation of the short and long distance aspects of the strong interaction.
  - **Infrared Safety**: There is a class of observables that do not depend on long distance physics and are therefore calculable in perturbative QCD.
  - **Factorisation**: There is a wide class of processes that can be factorised in a universal long distance piece (nonperturbative, but process independent) and a short distance piece that is calculable in perturbative QCD.
- To understand these ideas we will, in this section, study the lowest order QCD correction to the process $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}$.

The process $e^+e^- \to q\bar{q}g$



- Consider the process $e^+e^- \to q\bar{q}g$. We have the following kinematic variables:

  1. The four-momentum $q = p_1 + p_2$ of the virtual photon. The square of the centre-of-mass energy is $s = q^2 = q^\mu q_\mu$.
  2. The outgoing four-momenta k_1, k_2 and k_3. The energies of the outgoing partons[45] in the centre-of-mass frame are $E_i = k_i^0$.

- We define the energy fractions by

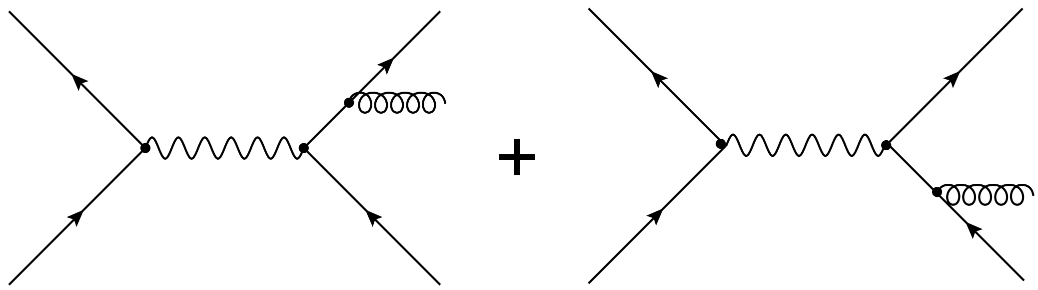
$$x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2q \cdot k_i}{s}$$

Exercise 7.1: [0.5] Show that $q \cdot k_i = E_i \sqrt{s}$ and that $\sum_i x_i = 2$.

From $\sum_i x_i = 2$ it follows that only two of the x_i are independent.

[45] We use the name 'parton' for both quark and gluon.

Singularities in the cross section



- To calculate the cross section $\sigma(\mathrm{e^+e^-} \to q\bar{q}\mathrm{g})$ two Feynman graphs have to be taken into account, one where the gluon is radiated from the quark and another where it is radiated from the antiquark. The calculation of the cross section is rather lengthy so we will not give it here; you can find it in H&M Section 11.5.
- The result is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = \sigma_0 \, \frac{2\alpha_\mathrm{s}}{3\pi} \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

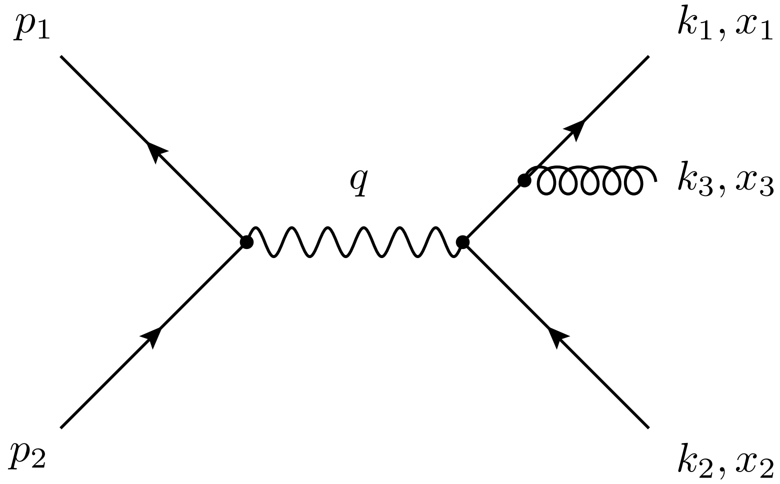
Here $\sigma_0 = \sigma(\mathrm{e^+e^-} \to \text{hadrons}) = (4\pi\alpha^2/s)\sum e_i^2$, see page 2–30.

- There are three singularities in this cross section

1. $(1 - x_1) = 0$
2. $(1 - x_2) = 0$
3. $(1 - x_1) = (1 - x_2) = 0$

We will now have a look where these singularities come from.

More kinematics



- In the following we will neglect the quark masses ($k_i^2 = 0$) so that

$$(k_i + k_j)^2 = k_i^2 + k_j^2 + 2k_i \cdot k_j = 2k_i \cdot k_j$$

- Denote by θ_{ij} the angle between the momenta of partons i and j. Then we can relate these angles to the energy fractions as follows

$$2k_1 \cdot k_2 = (k_1 + k_2)^2 = (q - k_3)^2 = s - 2q \cdot k_3 \quad \rightarrow$$

$$2E_1E_2(1 - \cos\theta_{12}) = s(1 - x_3)$$

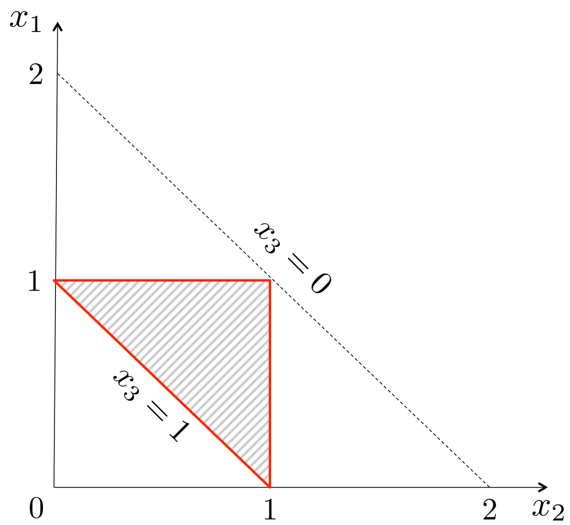
Exercise 7.2: [$\times$] Show that $k_i \cdot k_j = E_iE_j(1 - \cos\theta_{ij})$.

- Dividing by $s/2$ and repeating the above for the angles between other pairs of particles gives

$$x_1x_2(1 - \cos\theta_{12}) = 2(1 - x_3)$$
$$x_2x_3(1 - \cos\theta_{23}) = 2(1 - x_1)$$
$$x_3x_1(1 - \cos\theta_{31}) = 2(1 - x_2)$$

Phase space

- From the above it follows that $0 \leq x \leq 1$. Together with the constraint $x_3 = 2 - x_1 - x_2$ this implies that the allowed region for (x_1, x_2) is the triangle shown below.



- From

$$x_1 x_2 (1 - \cos\theta_{12}) = 2(1 - x_3)$$
$$x_2 x_3 (1 - \cos\theta_{23}) = 2(1 - x_1)$$
$$x_3 x_1 (1 - \cos\theta_{31}) = 2(1 - x_2)$$

we find that the collinear configurations are related to the x_i by

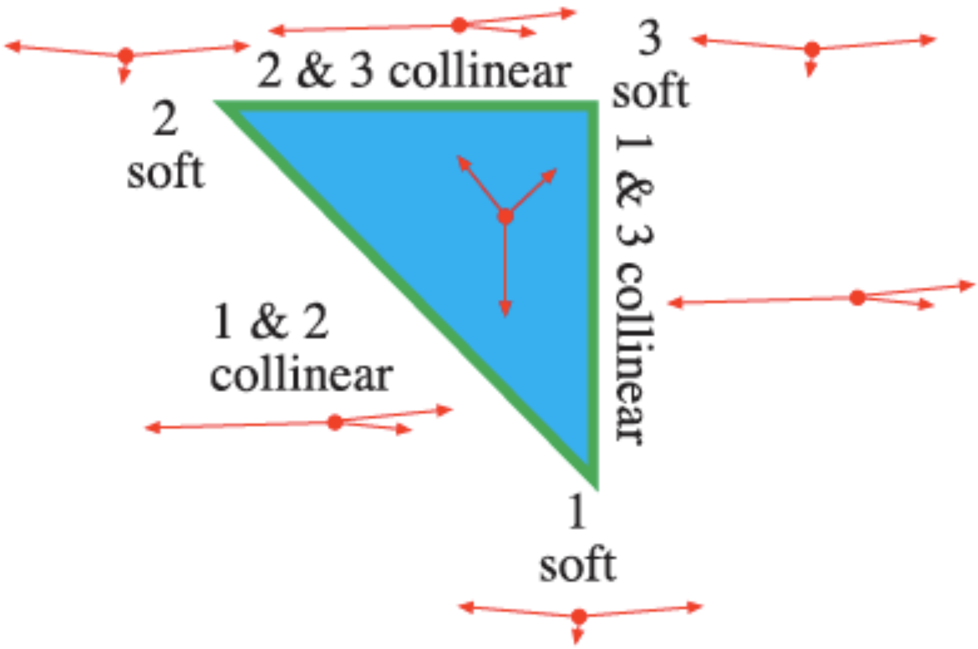
$$\theta_{12} \to 0 \quad \Leftrightarrow \quad x_3 \to 1$$
$$\theta_{23} \to 0 \quad \Leftrightarrow \quad x_1 \to 1$$
$$\theta_{31} \to 0 \quad \Leftrightarrow \quad x_2 \to 1$$

Thus when $x_i \to 1$ then $\theta_{jk} \to 0$, that is, j and k are collinear.

Exercise 7.3: [0.5] Show that when $x_i \to 1$ then i is back-to-back with both j and k. Also show that $x_i \to 0$ implies $E_i \to 0$: particle i becomes soft. What can you say about the relative directions of the particles j and k in this case?

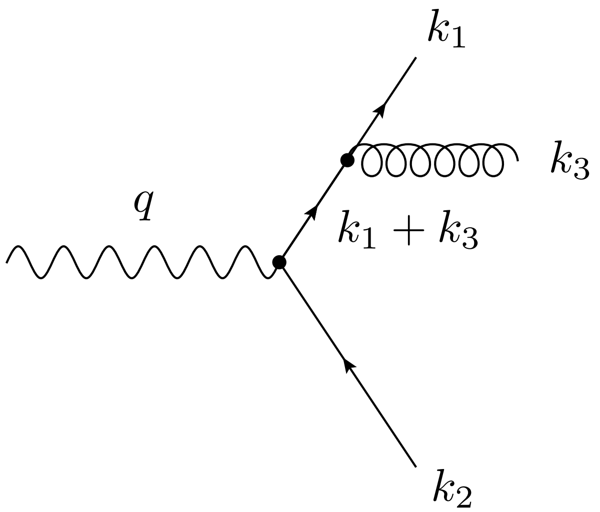
Three-parton configurations

- This plot shows the three-parton configurations at the boundaries of phase space.



- Edges: two partons collinear: $\theta_{ij} \to 0 \;\Leftrightarrow\; x_k \to 1$.
- Corners: one parton soft $x_i \to 0 \;\Leftrightarrow\; E_i \to 0$ (other two partons are back-to-back).
- Note that at the boundaries of phase space $2 \to 3$ kinematics goes over to $2 \to 2$ kinematics.

Origin of the singularities



- Where do the singularities actually come from? This is easy to see by noting that internal quark momentum is $(k_1 + k_3)$, giving a propagator term $\sim 1/(k_1 + k_3)^2$ in the cross section. Now

$$(k_1 + k_3)^2 = 2k_1 \cdot k_3 = 2E_1E_3\,(1 - \cos\theta_{31})$$

  so that the propagator term evidently is singular when $\theta_{31} \to 0$ and when $E_3 \to 0$.

- The collinear singularity at $\theta_{31} \to 0$ and $E_3 \to 0$ can be made explicit by rewriting the cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_3\,\mathrm{d}\cos\theta_{31}} = \sigma_0\,\frac{2\alpha_\mathrm{s}}{3\pi}\,\frac{f(E_3, \theta_{31})}{E_3\,(1 - \cos\theta_{31})}.$$

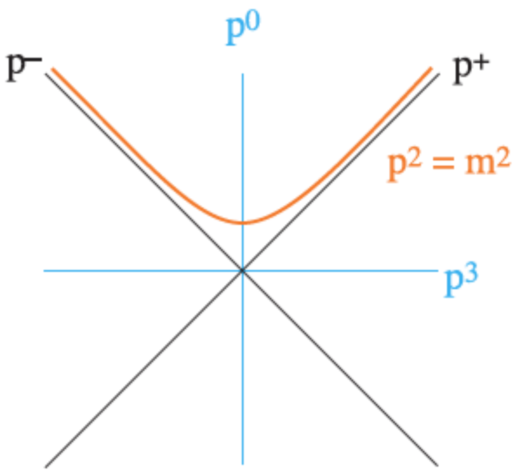
  Here $f(E_3, \theta_{31})$ is a rather complicated function that turns out to be finite when $E_3 \to 0$ or $\theta_{31} \to 0$.

- Clearly we get a logarithmic divergence when we attempt to integrate over θ_{31} with E_3 kept fixed or over E_3 with θ_{31} kept fixed.

Infrared singularities

- Are we seeing here a breakdown of perturbative QCD? The answer is no: the problem is that we are trying to work with cross sections on the parton level that are not **infrared safe**.
- These infrared problems always show up when $2 \to 3$ kinematics becomes $2 \to 2$ kinematics. We have seen that this happens at the edges of phase space when two partons become collinear or one parton becomes soft. Another way of stating this is that the internal propagator goes on shell: $(k_1 + k_3)^2 \to 0$.
- Please note that infrared divergences are omnipresent in QCD (and also in QED) and are by no means limited to $\mathrm{e}^+\mathrm{e}^- \to q\bar{q}\mathrm{g}$.
- It is useful to get a space-time picture with the help of **light cone coordinates**. We will then see that the divergences are caused by long distance interactions.

Intermezzo: light cone coordinates



- The light cone components of a four-vector a are defined by

$$a^{\pm} = (a^0 \pm a^3)/\sqrt{2}$$

The vector is then be written as $a = (a^+, a^-, a^1, a^2) = (a^+, a^-, \boldsymbol{a}_\mathrm{T})$.[46]

- **<u>Exercise 7.4</u>**: [1.0] Show that

$$a \cdot b = a^+ b^- + a^- b^+ - \boldsymbol{a}_\mathrm{T} \cdot \boldsymbol{b}_\mathrm{T}, \quad \text{and} \quad a^2 = 2a^+ a^- - \boldsymbol{a}_\mathrm{T}^2$$

Show that the vector transforms under boosts along the z axis like

$$a'^{+} = a^+ e^{\psi}, \quad a'^{-} = a^- e^{-\psi}, \quad \boldsymbol{a}'_\mathrm{T} = \boldsymbol{a}_\mathrm{T}$$

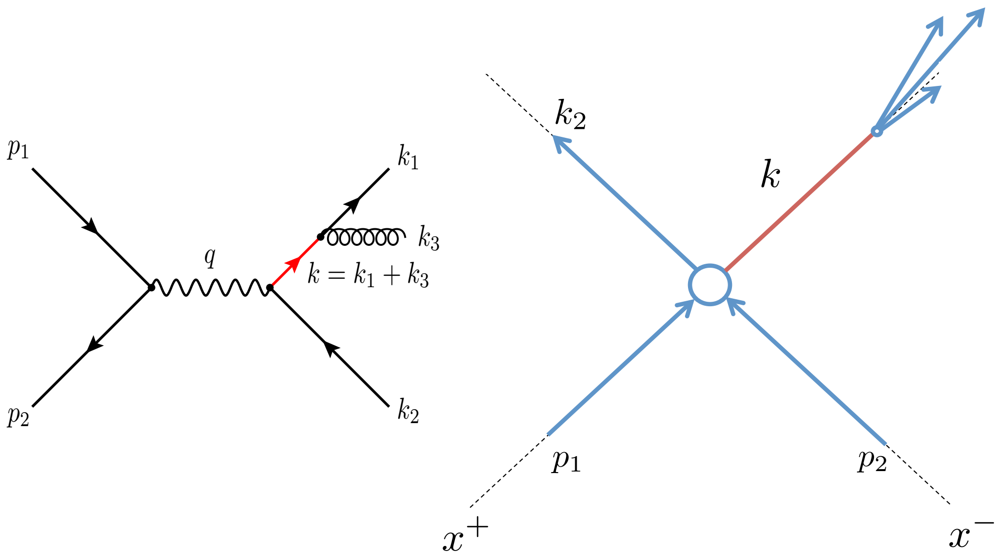
with $\psi = \frac{1}{2}\ln[(1-\beta)/(1+\beta)]$. How does a transform under two successive boosts β_1 and β_2?

- One often chooses the z axis such that, perhaps after a boost, a particle or a group of particles have large momenta along that axis. For these particles p^+ is large and (since they are on the mass shell)

$$p^- = \frac{m^2 + \boldsymbol{p}_\mathrm{T}^2}{2p^+} \quad \text{is small.}$$

[46]Note that a^+ and a^- are *not* 4-vector components.

Space-time picture of the singularities



- To see what happens when $k^2 = (k_1 + k_3)^2$ becomes small (goes on-shell), we choose the z axis along k with k^+ large and $\boldsymbol{k}_\mathrm{T} = 0$. Thus $k^2 = 2k^+k^- \to 0$ when k^- becomes small.
- In QFT, the Green functions (propagators) in momentum space are related to those in coordinate space by a Fourier transform:

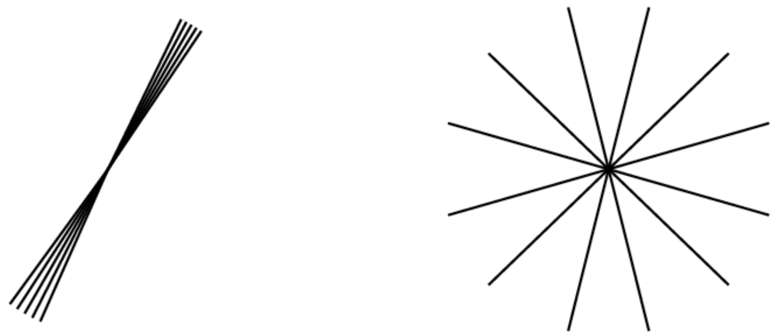
$$\begin{aligned} S_F(x) &= \int \mathrm{d}^4k \, \exp(-ikx) S_F(k) \\ &= \int \mathrm{d}^4x \exp[-i(k^+x^- + k^-x^+ - \boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{x}_\mathrm{T})] \, S_F(k) \end{aligned}$$

  Because k^+ is large and k^- is small, the contributing values of x have small x^- and large x^+. This means that the quark propagates a long distance in the x^+ direction before decaying in a quark-gluon pair, as is indicated in the space-time diagram above.
- It follows that the singularities that can lead to divergent perturbative cross-sections arise from interactions that happen a long time after the creation of a quark-antiquark pair.

Infrared safe observables

- We have seen that soft/collinear singularities arise from interactions that happen a long time after the creation of the quark-antiquark pair and that perturbation theory cannot handle this long-time physics. But a detector is a long distance away from the interaction so we must somehow take long-time physics into account in our theory.
- Fortunately there are measurements that are *insensitive* to long-time physics. These are called **infrared safe** observables. We have seen that soft/collinear singularities appear when 2→3 kinematics reduces to 2→2 kinematics at the boundaries of phase space. Therefore a meaningful infrared safe observable must be insensitive to the indistinguishable 2→2, 2→3 origin of the long-distance interactions.
- The most well known example of an infrared safe observable is the total cross section $\sigma(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons})$, see page 2–30. This cross section is infrared safe because it is a totally *inclusive* quantity (we sum over all particles in the final state and don't care how many there are) and the transition from partons to the hadronic final state in a given event always occurs with unit probability, whatever the details of the long-time hadronisation process.
- As an example of another infrared safe variable used in the analysis of $\mathrm{e}^+\mathrm{e}^-$ collisions, we mention the thrust event shape variable.

Thrust



pencil-like: $T \lesssim 1$ spherical: $T \gtrsim 1/2$

- Thrust is an event shape variable, used to discriminate between pencil-like and spherical events.
- Thrust is defined by

$$T = \max_{\boldsymbol{u}} \frac{\sum_i |\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}|}{\sum_i |\boldsymbol{p}_i|}$$

  Here the sum runs over all particles i in the event, and the unit vector $\hat{\boldsymbol{u}}$ is varied to maximise the sum of the projections of the 3-momenta $\boldsymbol{p}_i$ on $\hat{\boldsymbol{u}}$.

- So why is thrust infrared safe?

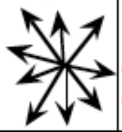
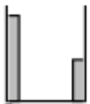
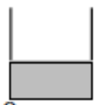
  1. Zero-momentum particles do not contribute to T.
  2. A collinear splitting does not change the trust:

$$|(1-\lambda)\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}| + |\lambda \boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}| = |\boldsymbol{p}_i \cdot \hat{\boldsymbol{u}}|$$

$$|(1-\lambda)\boldsymbol{p}_i| + |\lambda \boldsymbol{p}_i| = |\boldsymbol{p}_i|$$

IR safe observables used in e^+e^- physics

Here is a list of more infrared safe observables.

Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\Sigma_i \lvert \vec{p}_i \vec{n} \rvert}{\Sigma_i \lvert \vec{p}_i \rvert} \right)$	1	$\geq 2/3$	$\geq 1/2$	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	$\leq 1/3$	$\leq 1/\sqrt{2}$	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	$\leq 1/2$	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	$\leq 1/3$	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq \ldots \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\Sigma_i p_i^\alpha p_i^\beta}{\Sigma_i p_i^2}$	0	$\leq 3/4$	≤ 1	none (not infrared safe)
Aplanarity	$A = 1.5\, Q_1$	0	0	$\leq 1/2$	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = \left(\Sigma_i E_i^2 - \Sigma_i \vec{p}_i^{\,2}\right)_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	$\leq 1/3$ $\leq 1/3$	$\leq 1/2$ 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\Sigma_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2\, \Sigma_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	$\leq 1/(2\sqrt{3})$ $\leq 1/(2\sqrt{3})$	$\leq 1/(2\sqrt{2})$ $\leq 1/(2\sqrt{3})$	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi + \frac{\Delta\chi}{2}}^{\chi - \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$	 0 π	 0 π	 0 π	(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$	 0 $\pi/2$	 0 $\pi/2$	 0 $\pi/2$	$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$