Lecture notes Particle Physics II

Quantum Chromo Dynamics

2. SU(2) and SU(3) Symmetry

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Symmetry in (particle) physics

- If the Lagrangian of the world would be fully known we could derive the equations of motion from it, and the symmetries of nature and the conservation laws would automatically follow.
- For instance the Maxwell Lagrangian yields, via the Maxwell equations, all the symmetries and conservation laws of electrodynamics.
- In subatomic physics the Lagrangians are not so obvious, and symmetry considerations provide essential clues to construct them.
- It can be shown that an invariance of the Lagrangian under a symmetry operation leads to a conserved quantity (Noether's theorem). Thus, if a symmetry is found, the hunt is open for the related conservation law, and if a conservation law is found, the hunt is open for the related symmetry. For instance we know that electric charge is conserved in all reactions of elementary particles, but what symmetry is responsible for this charge conservation? (The answer will be given in the next lecture.)
- As will become clear later, it turns out that discrete symmetries lead to *multiplicative* conserved quantum numbers (*e.g.* reflection symmetry $\rightarrow$ parity conservation $\rightarrow$ multiplication of parities) while continuous symmetries lead to *additive* conserved quantum numbers (*e.g.* rotation invariance $\rightarrow$ angular momentum conservation $\rightarrow$ addition of angular momentum quantum numbers).
- We will now use some elementary non-relativistic quantum mechanics to establish the relation between symmetries and constants of motion.

When is an observable conserved?

- The **expectation value** of a quantum mechanical operator F is

$$\langle F\rangle \equiv \langle\psi|F|\psi\rangle \quad \text{with Hermitian conjugate} \quad \langle F\rangle^* \equiv \langle\psi|F^\dagger|\psi\rangle$$

- The expectation value of an **observable** is a real number so that the operator of an observable should be **Hermitian**

$$F = F^\dagger \qquad \text{if } \langle F\rangle \text{ is observable}$$

- Because energy is an observable the Hamiltonian H is Hermitian. We have for the Schrödinger equation and its Hermitian conjugate

$$i\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \qquad \text{and} \qquad -i\frac{\partial\langle\psi|}{\partial t} = \langle\psi|H^\dagger = \langle\psi|H$$

- This immediately leads to

$$\frac{\partial\langle F\rangle}{\partial t} = i\,\langle\psi|HF - FH|\psi\rangle = 0 \quad \Leftrightarrow \quad HF - FH = 0$$

An observable constant of motion F is
Hermitian and commutes with the Hamiltonian

- When H is known, we can find observable constants of motion by searching for Hermitian operators that commute with H.
- However, when H is *not* fully known, it is sufficient to establish (or postulate) the invariance of H, or the Lagrangian, under a **symmetry operation**, as we will now show.

Symmetry operators

- A **transformation operator** U transforms one wave function into another

$$|\psi'\rangle = U|\psi\rangle$$

- Wave functions are always **normalized** so that we must have

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1$$

- It follows that the transformation operator must be **unitary**

$$U^\dagger U = UU^\dagger = I$$

- We call U a **symmetry operator** when $|\psi'\rangle$ obeys the same Schrödinger equation as $|\psi\rangle$. Then, with U time independent,

$$i\frac{\partial U|\psi\rangle}{\partial t} = HU|\psi\rangle \quad \rightarrow \quad i\frac{\partial|\psi\rangle}{\partial t} = U^{-1}HU|\psi\rangle \underset{\text{Iwant}}{=} H|\psi\rangle$$

and thus

$$U^{-1}HU = H \quad \text{or} \quad [H,U] = 0$$

> A symmetry operator U is unitary and commutes with the Hamiltonian

- Thus U commutes with the Hamiltonian, as does a constant of motion. However, we cannot identify U with an observable since it is unitary, and not necessarily Hermitian.

Discrete symmetries

- There is a class of unitary transformations with the property

$$U^2 = I$$

  Multiplying from the right with $U^\dagger$ and using $UU^\dagger = I$ we find that $U = U^\dagger$: the operator is both unitary and Hermitian.

- Thus if U is a symmetry of (commutes with) the Hamiltonian we can directly conclude that it is an observable constant of motion.

- Examples of this the are the charge conjugation operator C (exchange of particles and antiparticles) and the parity operator P (reflection of the spatial coordinates).[12]

- Remark: C and P are not the only operators that are both unitary and Hermitian. This is, for instance, also true for the Pauli spin matrices, as is straight-forward to check.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- If $|\psi\rangle$ is an eigenvector of both U_1 and U_2 then

$$U_{1,2}|\psi\rangle = \lambda_{1,2}|\psi\rangle \qquad \text{and} \qquad U_1 U_2|\psi\rangle = U_2 U_1|\psi\rangle = \lambda_1\lambda_2|\psi\rangle$$

  The quantum numbers of a discrete symmetry are *multiplicative.*

- In these lectures we are not so much interested in **discrete** transformations (like C, P, T) but, instead, in **continuous** transformations. These transformations are unitary (by definition), but not necessarily Hermitian. But the **generator** of a unitary continuous transformation is Hermitian, as we will see.

[12] The time reversal operator T also has $T^2 = I$ but it is antiunitary, and not unitary.

Continuous transformations

- There is a large class of **continuous transformations** that depend on one or more continuous parameters, say α

$$|\psi'\rangle = U(\alpha)|\psi\rangle$$

An example is the transformation induced by a rotation over an angle α of the coordinate system (passive rotation), or of the wave function (active rotation).

- Such transformations have the property that they can be written as a succession of infinitesimal deviations from the identity

$$U(\alpha) = \lim_{n\to\infty}\left(I + \frac{i\alpha}{n}F\right)^n = \exp(i\alpha F)$$

The factor 'i' is a matter of definition but important (see below). In the above, F is called the **generator** of U.[13]

- Now if U is unitary we have, to first order in α,

$$U^\dagger U = (I - i\alpha F^\dagger)(I + i\alpha F) = I + i\alpha(F - F^\dagger) = I$$

so that $F = F^\dagger$. In other words,

The generator of a unitary operator is Hermitian

- Now we also understand the factor 'i' in the definition of a generator: without it the generator $G \equiv iF$ of a unitary operator would not be Hermitian but **anti-Hermitian**:

$$G = -G^\dagger$$

[13]Exponentiation of an operator F should be interpreted as $\exp(i\alpha F) = I + i\alpha F + \frac{1}{2!}(i\alpha F)^2 + \cdots$ But watch out, the familiar relation $e^A e^B = e^{A+B}$ is *only* true when A and B commute.

Generators as conserved observables

- We have seen that a symmetry operator U commutes with the Hamiltonian so it remains to show that its generator will then also commute with H. The proof is very simple:
- First, if $U(\alpha)$ is a symmetry operator then the infinitesimal transformation $U(\epsilon)$ will also be a symmetry operator. Expanding to the first order in ϵ obtains

$$[H, U] \doteq [H, I + i\epsilon F] = \underbrace{[H, I]}_{0} + i\epsilon\,[H, F] = 0 \quad \rightarrow \quad [H, F] = 0$$

> If U is a unitary operator that commutes with the Hamiltonian then its generator F is a Hermitian operator that also commutes with the Hamiltonian

- We now have the work plan to find the relation between a continuous symmetry of H and the corresponding conserved observable:
  1. Find the generator F of the symmetry transformation U.
  2. The expectation value of F is a constant of motion
- Clearly a multiplication of continuous symmetry operators corresponds to the addition of their generators in the exponent. The conserved quantum numbers, which are related to F and not to U, are therefore *additive*.
- We will now proceed with the introduction of some concepts of **group theory** which is the mathematical framework to systematically describe and classify symmetry operations.

Exercise 2.1:

Show that (consult a quantum mechanics book if necessary)

(a) [0.5] Invariance for translations in space leads to the conservation of momentum.

(b) [0.5] Invariance for translations in time leads to the conservation of energy.

(c) [0.5] Rotational invariance leads to the conservation of angular momentum.

Group theory

- It is clear that a combination of two symmetry operations—that each leaves the system unchanged—is again a symmetry operation. And there is of course the trivial symmetry operation, namely, 'do nothing'. Furthermore, we can assume that each symmetry operation can be undone. We say, in fact, that symmetry operations form a **group**.

- What is a group? It is a set of elements $\{g_i\}$,
  - with a composition law $g_i \cdot g_j = g_k$
  - that is associative $(g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$
  - with a unit element e such that $e \cdot g_i = g_i \cdot e = g_i$
  - and with an inverse g_i^{-1} such that $g_i \cdot g_i^{-1} = g_i^{-1} \cdot g_i = e$

- Examples:

  The set $\{1, i, -1, -i\}$ under multiplication (discrete, 4 elements)

  The set of integers under addition (discrete, infinite # elements)

  Rotations in 3 dimensions (continuous, 3 parameters)

  Lorentz transformations (continuous, 6 parameters: which ones?)

- A group is called **Abelian** when the group operation is commutative $g_i \cdot g_j = g_j \cdot g_i$ (*e.g.* 2-dim rotations). Non-commutative groups are called non-Abelian (*e.g.* 3-dim rotations).

- A systematic study of symmetries is provided by a branch of mathematics called **group theory**. We will not present group theory in these lectures, but only a few basic concepts.[14]

[14] A nice summary of group theory can be found in A&H-II, Appendix M.

Representation of a group

- In these lectures, we will be concerned with groups of *matrices.*
- It may be the case, of course, that the group '*is*' a set of matrices. For instance, the group SO(2) of orthogonal 2×2 matrices with determinant 1, that describe 2-dimensional rotations.
- But a matrix representation may also come from mapping each element g_i of some group to an $n \times n$ matrix M_i (why must M be square?), such that the multiplication structure is preserved

$$g_1 \cdot g_2 = g_3 \qquad \to \qquad M_1 M_2 = M_3$$

  This is called an n-dimensional **representation** of the group $\{g\}$. Thus, SO(2) is *defined* by $2{\times}2$ matrices, (the **fundamental representation**) but it has also representations in higher dimensions.

- Two groups with the same multiplication structure are said to be **isomorphic ($\cong$)** if the elements map one-to-one. If the mapping is not one-to-one, they are called **homomorphic** ($\sim$).
- **Exercise 2.2**: [0.5] Show that

$$\{1, i, -1, -i\} \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

- From an n and an m-dimensional representation we can always construct an $(n+m)$-dimensional representation through

$$M_i^{(n+m)} = \begin{bmatrix} M_i^{(n)} & 0 \\ 0 & M_i^{(m)} \end{bmatrix} \equiv \boldsymbol{n} \oplus \boldsymbol{m}$$

  but this does not classify as a new representation. The relevant representations are the so-called **irreducible** ones which cannot be decomposed in block diagonal form. It is a (non-trivial) task of group theory to find all the irreducible representations of a group.

Lie groups

- On page 2–10 we have encountered discrete groups (elements labelled by an index, or a set of indices) and continuous groups where the elements are labelled by a set of continuous parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. Important groups of transformations $U(\boldsymbol{\alpha})$ are those which can be written as a succession of infinitesimal deviations from the identity transformation (see also page 2–7):

$$U(\boldsymbol{\alpha}) = \lim_{n \to \infty} [\, 1 + i(\boldsymbol{\alpha}/n) \cdot \boldsymbol{T}\,]^n = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{T})$$

  Such a group is called a **Lie group**,[15] and the matrices $\boldsymbol{T}$ are called the **generators** of the group.[16] The number of generators is equal to the number of parameters that label the group elements.

  Example: Rotations are a Lie group but reflections are not since these are not continuously connected to the identity.

- There is a theorem which states that the commutator of two generators is always a linear combination of the generators

$$[T_i, T_j] = f_{ij}^k \, T_k \qquad \text{(summation over } k \text{ implied)}$$

  These commutation relations are called the **algebra**, and the (complex) numbers f_{ij}^k are called the **structure constants** of the group. It can be shown that these structure constants fully characterise the multiplication structure of a Lie group.

- On page 2–7 we have shown that if U is unitary then $T_i = T_i^\dagger$. In other words, the generators of a unitary operator are Hermitian.

[15] The formal definition of a Lie group states first of all that the number of parameters is finite, and furthermore that $U(\alpha_1) \cdot U(\alpha_2) = U(\alpha_3)$, with α_3 an *analytic* function of α_1 and α_2.

[16] Discrete groups also have generators: *e.g.* repeated rotation over $2\pi/n$ generates the cyclic group $\mathcal{Z}_n$.

The 2-state nucleon system

- After the discovery of the neutron by Chadwick in 1932, the near equality of its mass (939.5 MeV) to that of the proton (938.3 MeV) suggested to Heisenberg that, as far as the strong interactions are concerned, these are two nearly degenerate states of one particle: the **nucleon**.
- This 'isospin symmetry' of the strong force is further supported by, for instance, the observation of very similar energy levels in **mirror nuclei** (the number of protons in one, is equal to number of neutrons in the other, and *vice versa*, like in ${}^{13}_{7}$N and ${}^{13}_{6}$C).
- In addition, apart from the p-n doublet, there are other particles that are nearly degenerate in mass, like the pion triplet ($\sim$140 MeV) and the quadruplet of Δ resonances ($\sim$1.23 GeV) $\rightarrow$ **Fig**. This looks like the doublet, triplet and quadruplet structure of spin-$\frac{1}{2}$, spin-1 and spin-$\frac{3}{2}$ systems built from spin-$\frac{1}{2}$ states, and is thus strongly suggestive of hadronic substructure.
- We know today that hadrons are built up from quarks and we can explain isospin symmetry from the fact that the strong interaction is insensitive to the quark flavour. The mass differences within the nucleon, π and Δ multiplets are, after electromagnetic correction, believed to be due to the difference in the u and d quark masses.
- The invariance for p to n transitions obeys the mathematics of ordinary spin, hence the term '**isospin**'. The reason is that transitions in *any* 2-state quantum mechanical system are described by the special unitary group SU(2), as will become clear next.

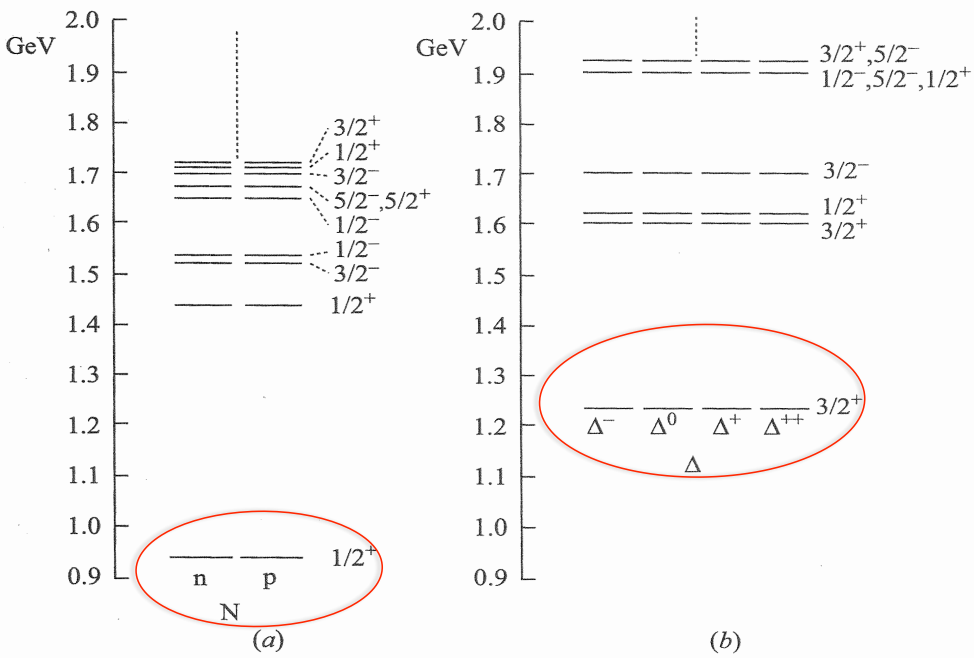


Figure 1.10. Baryon energy levels: (*a*) doublets (N); (*b*) quartets (Δ).

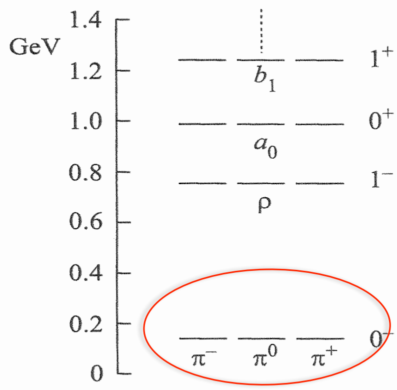


Figure 1.11. Meson triplets.

Isospin symmetry

- We work in a 2-dim Hilbert space spanned by the basis vectors[17]

$$|\mathrm{p}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\mathrm{n}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  The Hermitian conjugates are $\langle \mathrm{p}| = (1, 0)$ and $\langle \mathrm{n}| = (0, 1)$. An arbitrary state is written as the linear combination

$$|\psi\rangle = \alpha\, |\mathrm{p}\rangle + \beta\, |\mathrm{n}\rangle$$

  Because $|\alpha|^2$ is the probability to find the system in a $|p\rangle$ state and $|\beta|^2$ the same for the $|n\rangle$ state we must have, for any state $|\psi\rangle$,

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

- We have seen already that a transformation $|\psi'\rangle = U|\psi\rangle$ must preserve the norm so that U must be unitary: $U^\dagger U = 1$.

- Taking determinants we find

$$\det(U^\dagger U) = \det(U^\dagger)\ \det(U) = \det(U)^*\ \det(U) = 1$$

  Therefore $\det(U) = e^{i\phi}$ with ϕ some arbitrary phase factor.

- So we may set $U = e^{i\phi} V$ with $\det(V) = 1$. Invariance for phase shifts is called a U(1) invariance and leads to charge conservation, as we will see later. The charge conserved in the p-n case here is not electrical charge, but **baryon number**

$$A = (N_\mathrm{p} - N_{\bar{\mathrm{p}}}) + (N_\mathrm{n} - N_{\bar{\mathrm{n}}})$$

- Putting U(1) invariance aside, we have to deal with unitary 2×2 matrices V with unit determinant, that is, with the group SU(2).

[17] When we talk about quarks we will use the notation $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ instead.

The group SU(2)

- The mathematics of SU(2) is well known from the treatment of ordinary spin in quantum mechanics. A transformation can be written as $U = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{I})$ with the three generators $\boldsymbol{I} \equiv \boldsymbol{\tau}/2$ given by the Pauli matrices
$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
These generators are clearly Hermitian ($\tau_i^\dagger = \tau_i$), as they should be, since U is unitary. It can be shown (Exercise 2.3) that, quite in general, $\det[\exp(A)] = \exp[\mathrm{Tr}(A)]$ so that the traces of the τ_i vanish because the SU(2) transformations have unit determinant.

> The generators of a unitary matrix group with unit determinant are Hermitian and traceless

- By matrix multiplication you may check the commutation relations
$$[I_i, I_j] = i\,\epsilon_{ijk} I_k$$
with ϵ_{ijk} the antisymmetric tensor (+1 for cyclic permutations of 123 and −1 for cyclic permutations of 213, zero otherwise).

- SU(2) has one so-called **Casimir operator** that commutes with all the generators, and is always some non-linear function of the generators. For SU(2) this is the total isospin operator:
$$I^2 = I_1^2 + I_2^2 + I_3^2$$
A state can then be a simultaneous eigenstate[18] of I^2 with eigenvalue $i(i+1)$, $i = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and of I_3 with eigenvalue $m = -i, \ldots, +i$. The eigenvalues label the state, like $|\psi\rangle = |i, m\rangle$.

[18] A Hermitian matrix has the property that it can always be diagonalised by a unitary transformation. Hermitian matrices can be *simultaneously* diagonalised by a single transformation if they commute.

Exercise 2.3:

In this exercise we will review a few easy-to-prove properties of matrices and of **matrix transforms** (also called **similarity transforms**) defined by

$$A' = SAS^{-1},$$

where S is a non-singular transformation matrix. Such transforms can come in very handy in a calculation because they allow you to transform matrices to convenient forms, such as a transformation to diagonal form which is used for the proof in (e) below.

(a) [0.1] Show that $\text{Tr}(AB) = \text{Tr}(BA)$.

(b) [0.2] Show that a matrix transform preserves the algebra of a Lie group. Representations that are related by similarity transformations are therefore called **equivalent**.

(c) [0.2] Show that a matrix transform preserves the product, determinant and trace, that is,

$$(AB)' = A'B', \quad \det(A') = \det(A) \quad \text{and} \quad \text{Tr}(A') = \text{Tr}(A).$$

What about Hermitian conjugation: $(A')^\dagger \stackrel{?}{=} (A^\dagger)'$.

(d) [0.2] Show that a matrix transform preserves the terms in a power series, that is,

$$(A^n)' = (A')^n \quad \rightarrow \quad (\exp A)' = \exp(A').$$

(e) [0.3] Now show that

$$\det[\exp(A)] = \exp[\text{Tr}(A)]$$

for all matrices A that can be brought into diagonal form.

Exercise 2.4:

(a) [0.5] Show that $\tau_i \tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k$. Together with the fact that the τ are Hermitian, we thus have $\tau_i^\dagger = \tau_i = \tau_i^{-1}$.

(b) [0.5] Now show that $(\boldsymbol{a} \cdot \boldsymbol{\tau})(\boldsymbol{b} \cdot \boldsymbol{\tau}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\tau} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ and, from this, that $(\boldsymbol{\theta} \cdot \boldsymbol{\tau})^2 = |\boldsymbol{\theta}|^2$.

(c) [0.5] Use the above, and the Taylor expansions of exp(), sin() and cos(), to show that $\exp(i\boldsymbol{\theta} \cdot \boldsymbol{\tau}) = \cos|\boldsymbol{\theta}| + i(\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\tau}) \sin|\boldsymbol{\theta}|$. Here $\hat{\boldsymbol{\theta}}$ is the unit vector along $\boldsymbol{\theta}$.

(d) [0.25] Instead of $|\mathrm{p}\rangle$ and $|\mathrm{n}\rangle$ we will write $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ to reflect isospin symmetry on the quark level. Verify that

$$I_3\,|\mathrm{u}\rangle = \tfrac{1}{2}|\mathrm{u}\rangle, \quad I_3\,|\mathrm{d}\rangle = -\tfrac{1}{2}|\mathrm{d}\rangle$$

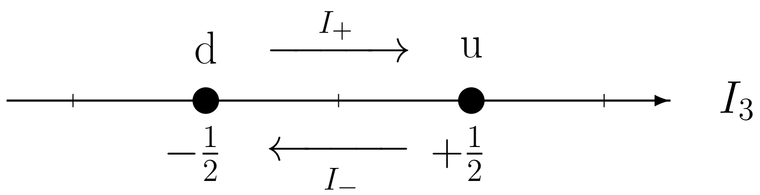
and that the Casimir operator $I^2 = I_1^2 + I_2^2 + I_3^2$ is a multiple of the unit operator, with

$$I^2\,|\mathrm{u}\rangle = \tfrac{3}{4}\,|\mathrm{u}\rangle, \qquad I^2\,|\mathrm{d}\rangle = \tfrac{3}{4}\,|\mathrm{d}\rangle$$

(e) [0.25] Define the step operators $I_\pm = I_1 \pm i\,I_2$ and verify that

$$I_+|\mathrm{u}\rangle = 0, \qquad I_+|\mathrm{d}\rangle = |\mathrm{u}\rangle, \qquad I_-|\mathrm{u}\rangle = |\mathrm{d}\rangle, \qquad I_-|\mathrm{d}\rangle = 0$$

We can now draw a, kind of trivial, **weight diagram** like



Composite states

- The rules for addition of angular momenta from quantum mechanics carry straight over to the addition of isospins. We will not derive here the mathematics but will only indicate how it works.

- Addition of two states $|i_1, m_1\rangle$ and $|i_2, m_2\rangle$, results in $(2i_1+1) \times (2i_2+1)$ different states which can be classified according to the eigenvalue label i of the Casimir operator I^2 which ranges from $|i_1 - i_2|$ to $i_1 + i_2$, and the eigenvalues m of the I_3 operator that, for each state i, range from $-i$ to $+i$. Here $m = m_1 + m_2$. Formally, the combined state can be written as

$$|i, m\rangle = \sum \langle i_1, i_2, m_1, m_2 | i, m\rangle |i_1, m_1\rangle |i_2, m_2\rangle$$

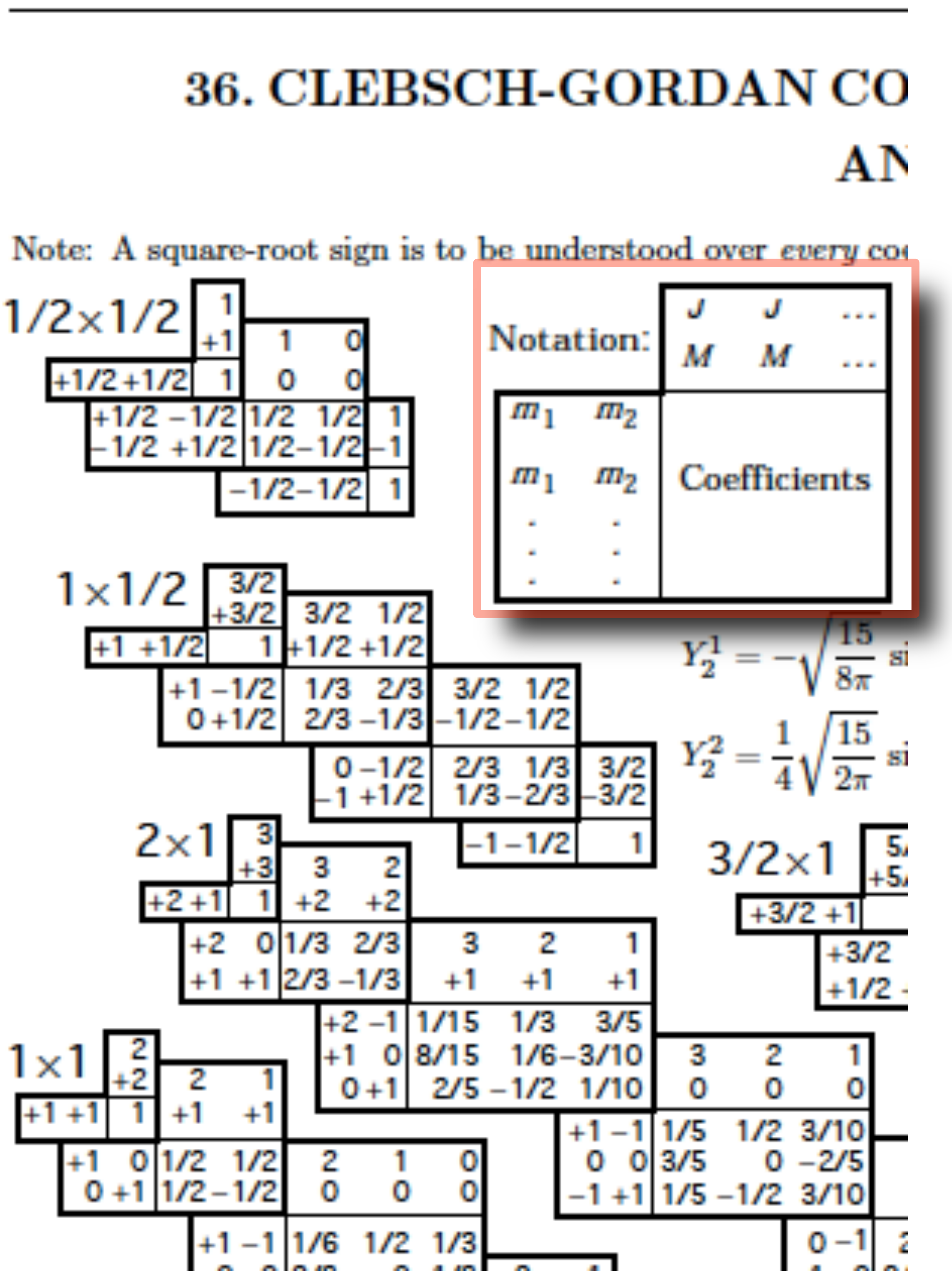
  The Clebsch-Gordan coefficients $\langle \cdot | \cdot \rangle$ can be found in the Particle Data Book → **Fig**. For a nucleon-nucleon system we get

$$\begin{aligned} |I, I_3\rangle &= |0, \ \ 0\rangle = (\text{pn} - \text{np})/\sqrt{2} \\ &= |1, \ \ 1\rangle = \text{pp} \\ &= |1, \ \ 0\rangle = (\text{pn} + \text{np})/\sqrt{2} \\ &= |1, -1\rangle = \text{nn} \end{aligned}$$

- **Exercise 2.5**: [1.0] Use exchange symmetry arguments or the step operators $I_\pm \equiv I_\pm^{(1)} + I_\pm^{(2)}$ to justify the decomposition above.[19] Hint: See H&M Exercise 2.1.

- This splitting of the combination of two 2-component states into a singlet and a triplet state is often written as $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. The significance of such a decomposition is that under a SU(2) transformation the substates of the **1** and **3** representation will transform among themselves.

[19] In full, the step operator is defined by $I_\pm |i, m\rangle = \sqrt{i(i+1) - m(m \pm 1)}\ |i, m \pm 1\rangle$.



36. CLEBSCH-GORDAN CO
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Note: A square-root sign is to be understood over every co

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			

$1/2 \times 1/2$

m_1	m_2	J=1, M=+1	J=1, M=0	J=0, M=0	J=1, M=−1
+1/2	+1/2	1			
+1/2	−1/2		1/2	1/2	
−1/2	+1/2		1/2	−1/2	
−1/2	−1/2				1

$1 \times 1/2$

m_1	m_2	3/2, +3/2	3/2, +1/2	1/2, +1/2	3/2, −1/2	1/2, −1/2	3/2, −3/2
+1	+1/2	1					
+1	−1/2		1/3	2/3			
0	+1/2		2/3	−1/3			
0	−1/2				2/3	1/3	
−1	+1/2				1/3	−2/3	
−1	−1/2						1

$Y_2^1 = -\sqrt{\frac{15}{8\pi}}$ si

$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}$ si

2×1

m_1	m_2	3, +3	3, +2	2, +2	3, +1	2, +1	1, +1	3, 0	2, 0	1, 0
+2	+1	1								
+2	0		1/3	2/3						
+1	+1		2/3	−1/3						
+2	−1				1/15	1/3	3/5			
+1	0				8/15	1/6	−3/10			
0	+1				2/5	−1/2	1/10			
+1	−1							1/5	1/2	3/10
0	0							3/5	0	−2/5
−1	+1							1/5	−1/2	3/10
0	−1									

$3/2 \times 1$

m_1	m_2	5/2, +5/2
+3/2	+1	
+3/2		
+1/2		

1×1

m_1	m_2	2, +2	2, +1	1, +1	2, 0	1, 0	0, 0
+1	+1	1					
+1	0		1/2	1/2			
0	+1		1/2	−1/2			
+1	−1				1/6	1/2	1/3

Clebsch-Gordan coefficients from the Particle Data Book. Given in the tables is the *square* of the coefficients, so you should take the square root.

$\boxed{\text{SU(2)}_\text{f} \text{ for antiquarks}}$

- If $|\psi\rangle$ is a particle state then the complex conjugate is identified with the corresponding antiparticle state:[20] $|\bar{\psi}\rangle \equiv |\psi\rangle^*$. An antiquark state therefore transforms in the complex conjugate representation of SU(2), denoted by $\mathbf{2}^*$ or $\bar{\mathbf{2}}$.

$$|\bar{\psi}'\rangle = U^*|\bar{\psi}\rangle = \exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}^*/2)\,|\bar{\psi}\rangle \equiv \exp(i\boldsymbol{\alpha}\cdot\bar{\boldsymbol{\tau}}/2)\,|\bar{\psi}\rangle$$

  The two representations are thus related by $\bar{\boldsymbol{\tau}} = -\boldsymbol{\tau}^*$.

- To combine a quark with an antiquark we could calculate from scratch the Clebsch-Gordan coefficients of $\mathbf{2}\otimes\bar{\mathbf{2}}$ but we can save us the effort by using a trick that, by the way, only works for SU(2).

- Just replace $\bar{\text{u}}$ by $-\bar{\text{d}}$ and $\bar{\text{d}}$ by $\bar{\text{u}}$ in $|\bar{\psi}\rangle$, that is, define

$$|\tilde{\psi}\rangle \equiv C|\bar{\psi}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\text{u}} \\ \bar{\text{d}} \end{pmatrix} = \begin{pmatrix} -\bar{\text{d}} \\ \bar{\text{u}} \end{pmatrix}$$

  It is now straight-forward to show (Exercise 2.7) that $|\tilde{\psi}\rangle$ transforms as a quark state $|\tilde{\psi}'\rangle = U|\tilde{\psi}\rangle$ so that we just can use the Clebsch-Gordans of the $\mathbf{2}$ representation.[21]

- <u>**Exercise 2.6**</u>: [×] Take the $|qq\rangle$ states given on page 2–19 (substitute u for p and d for n), to arrive at $|q\bar{q}\rangle$ meson states that properly transform under SU(2):

$$\begin{aligned}
\omega &= |0,\ \ 0\rangle = (\text{u}\bar{\text{u}} + \text{d}\bar{\text{d}})/\sqrt{2} \\[1em]
\pi^+ &= |1,\ \ 1\rangle = -\text{u}\bar{\text{d}} \\
\pi^0 &= |1,\ \ 0\rangle = (\text{u}\bar{\text{u}} - \text{d}\bar{\text{d}})/\sqrt{2} \\
\pi^- &= |1,-1\rangle = \text{d}\bar{\text{u}}
\end{aligned}$$

[20] We use here $|\bar{\psi}\rangle$ to indicate an antiparticle; please do not confuse it with a conjugate Dirac spinor $\overline{\psi}$.

[21] In fact, for SU(2) the generators $\bar{\tau}_i$ and τ_i are related by the similarity transformation $\bar{\tau}_i = C^{-1}\tau_i C$ so that they are equivalent, that is, they are not regarded as different representations, see also Exercise 2.3.

Exercise 2.7:

(a) [1.0] Use isospin invariance to show that the ratio

$$\frac{\sigma(\mathrm{pp} \to \pi^+\mathrm{d})}{\sigma(\mathrm{pn} \to \pi^0\mathrm{d})} = 2$$

Here the deuteron has isospin $I = 0$ and the pion isospin $I = 1$. You may assume that the cross section is

$$\sigma \sim |\mathrm{amplitude}|^2 = \sum_I |\langle I', I_3'|A|I, I_3\rangle|^2 = A^2 \sum_I |\langle I', I_3'|I, I_3\rangle|^2 .$$

Hint: See H&M Exercise 2.3.

(b) [0.2] Show that the generators $\bar{\boldsymbol{\tau}}$ are a representation of SU(2).

(c) [×] Verify that $I_3(\bar{\mathrm{u}}) = -\frac{1}{2}$ and $I_3(\bar{\mathrm{d}}) = +\frac{1}{2}$.

(d) [0.3] Show that

$$|\tilde{\psi}\rangle = C|\bar{\psi}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\mathrm{d}} \end{pmatrix}$$

transforms as a particle state.

The group $\mathbf{SU(3)_f}$ I

- To accommodate strange quarks, our space has to be extended

$$\text{from} \quad \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \\ \mathrm{s} \end{pmatrix}$$

- Like in the (iso)spin case we can write a unitary transformation as

$$|\psi'\rangle = U|\psi\rangle = \exp(i\boldsymbol{a} \cdot \boldsymbol{\lambda}/2)\,|\psi\rangle \equiv \exp(i\boldsymbol{a} \cdot \boldsymbol{T})\,|\psi\rangle$$

but the generators $\boldsymbol{\lambda}$ are now Hermitian 3×3 matrices. A complex 3×3 matrix is characterised by 18 numbers but only 8 are independent because the matrices are Hermitian, and traceless since $\det U = 1$. Thus there are 8 independent generators.

- The 8 **Gell-Mann matrices** (with Pauli matrices inside!) are

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_1} \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\lambda_3} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{\lambda_4}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}}_{\lambda_5} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\lambda_6} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{\lambda_7} \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\lambda_8}$$

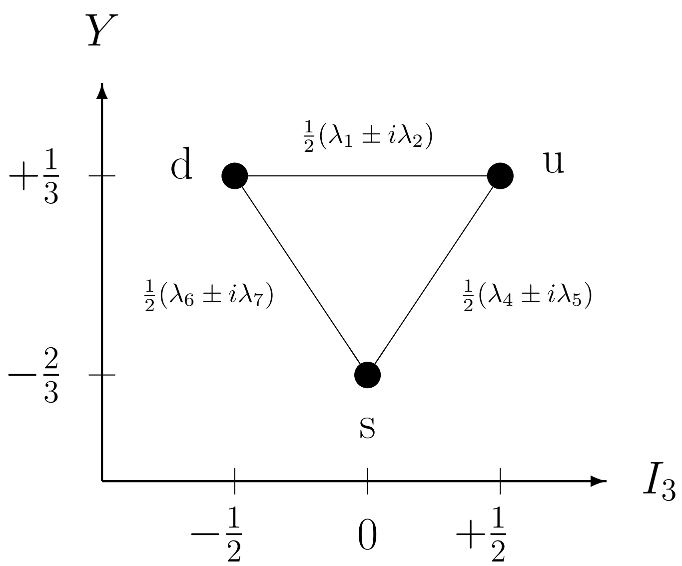
- The algebra of the SU(3) group is given by the commutation relation of the matrices $T_a = \lambda_a/2$: $\quad [T_a, T_b] = i f_{ab}^{c} T_c$

The group SU(3)$_\mathrm{f}$ II

- The structure constants f_{ab}^c are antisymmetric in the exchange of two indices (see Exercise 2.8); the non-zero ones are

$$\begin{array}{l} f_{12}^3 = 1 \\ f_{14}^7 = f_{16}^5 = f_{24}^6 = f_{25}^7 = f_{34}^5 = f_{37}^6 = \frac{1}{2} \\ f_{45}^8 = f_{67}^8 = \frac{1}{2}\sqrt{3} \end{array}$$

- It is seen that λ_3 and λ_8 are simultaneously diagonal so that we can label quark states by the simultaneous eigenvalues of the **isospin** operator $T_3 = \lambda_3/2$ and the **hypercharge** operator $Y = 2T_8/\sqrt{3} = \lambda_8/\sqrt{3}$. This gives rise to following weight diagram for the quark states (see Exercise 2.8 for antiquarks):



- As mentioned on page 2–16 there is one Casimir operator for SU(2), but there are two Casimirs for SU(3). By definition, these commute with all the λ_i. One of them is is the total 'isospin' operator $\sum \lambda_i^2$ while the other is a rather complicated trilinear function of the λ_i which can be found in A&H-II, Appendix M.5.

Exercise 2.8:

(a) [0.5] The λ matrices are normalised such that $\text{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. Check this for a few matrices λ_a and λ_b.

(b) [0.5] Use $\text{Tr}(AB) = \text{Tr}(BA)$ to show that $\text{Tr}(\lambda_c[\lambda_a, \lambda_b]) = 4if^c_{ab}$ and, by changing the order of the λ, that the structure constants f^c_{ab} are antisymmetric in the exchange of two indices.

(c) [0.5] Plot the eigenvalues of the isospin and hypercharge operator for the u, d and s quarks in an I_3-Y diagram. Check the Gell-Mann Nishijima formula $Q = I_3 + \frac{1}{2}Y$ and also that $Y = S + B$. Repeat the exercise for antiquarks in the $\bar{\mathbf{3}}$ representation.

(d) [0.5] Write down the matrices for the step operators $\frac{1}{2}(\lambda_1 \pm i\lambda_2)$, $\frac{1}{2}(\lambda_4 \pm i\lambda_5)$ and $\frac{1}{2}(\lambda_6 \pm i\lambda_7)$ and justify their position in the weight diagram on page 2–24.

Exercise 2.9: The adjoint representation of SU(3)

- We have encountered the algebra of the groups SU(2) and SU(3) in terms of the two-dimensional Pauli matrices and the three-dimensional Gell-Mann matrices, respectively. These matrices are, together with the 2- or 3-dim vectors on which they act, called the **fundamental representation** of SU(2) or SU(3).
- However, the structure constants of a Lie group automatically generate a representation with a dimension that is equal to the number of generators, *e.g.* 8×8 for SU(3). This is called the **adjoint representation**. Below we let you find out how this works.

(a) [×] Verify the **Jacobi identity** for matrices A, B and C:

$$[[A,B],C]+[[B,C],A]+[[C,A],B]=0$$

(b) [×] Now show that in terms of the SU(3) structure constants the Jacobi identity reads

$$f_{ij}^{m}\,f_{mk}^{n}+f_{jk}^{m}\,f_{mi}^{n}+f_{ki}^{m}\,f_{mj}^{n}=0$$

(c) [×] Verify that $f_{ij}^{k}=-f_{ji}^{k}$

(d) [1.0] Define the 8×8 matrices C_i with elements

$$(C_i)_j^k=-f_{ij}^{k}$$

and show that the C_i obey the SU(3) algebra

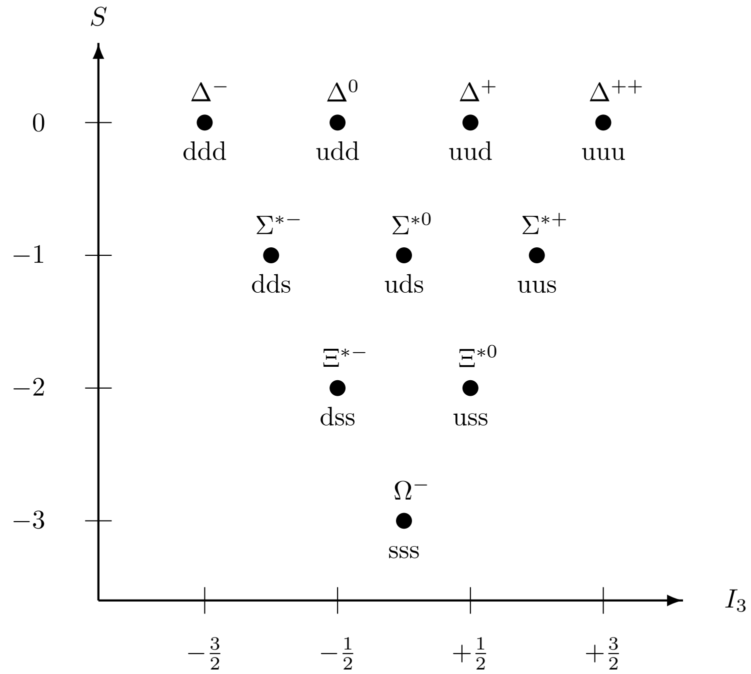
$$[C_i,C_j]=f_{ij}^{k}C_k$$

In this way, we have constructed the adjoint representation of SU(3) from its structure constants. We will see later that coloured quarks are described by the fundamental representation of SU(3), of dimension 3, and gluons by the adjoint representation, of dimension 8.

The Eightfold Way

- Because our interest in SU(3) lies in the fact that it is an exact (colour) symmetry of QCD, we will not present here how $\mathrm{SU}(3)_{\mathrm{f}}$ is used to classify the hadrons (the Eightfold Way). This is treated in great detail in H&M Chapter 2, and also in Griffiths Chapter 5.
- We just mention that the mesons $|q\bar{q}\rangle$ can be grouped into octets and singlets ($\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$) and baryons $|qqq\rangle$ can be grouped into decuplets, octets and singlets ($\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$).
- Nevertheless, let us have a look at the spin 3/2 baryon decuplet, because it provides us with an argument to introduce the colour quantum number.

The need for a colour quantum number



- In this spin 3/2 baryon decuplet, the flavour wave functions at the corners are obviously symmetric under the exchange of two quarks. Although this is not apparent from the labels, all wave functions of the decuplet are symmetric, as you will discover in Exercise 2.10.
- But now we have a problem: the *total* wave function

$$\psi = \psi_{\text{space}}(L=0) \;\times\; \psi_{\text{spin}}(\uparrow\uparrow\uparrow) \;\times\; \psi_{\text{flavour}}(q_1 q_2 q_3)$$

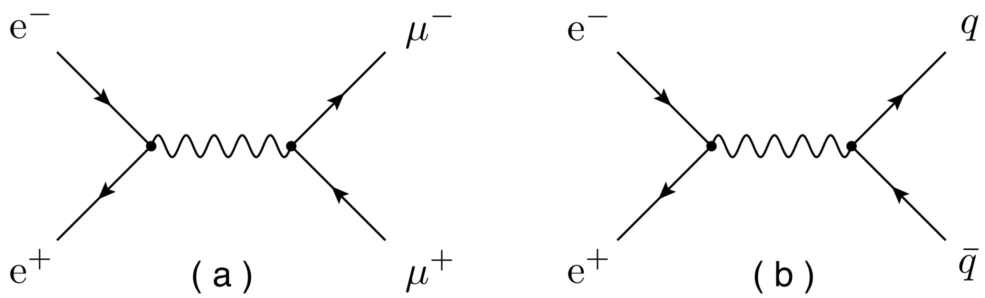
  is symmetric under the exchange of two quarks, while it should be anti-symmetric, since baryons are fermions (half-integer spin).
- The solution is to assign a 'colour' quantum number (r, g, b) to each quark so that the quarks can be distinguished by their colour, provided, of course, that we do not allow two quarks in a baryon to have the same colour. Thus the *three* colours are always present and we say that baryons are 'white', or **colour singlets** (= invariant under $\mathrm{SU}(3)_\mathrm{c}$ transformations). By anti-symmetrising the wave function in colour space, over-all anti-symmetry is established.

Exercise 2.10:

(a) [0.5] Use the step operators defined in the weight diagram on page 2–24 (and also in Exercise 2.8d) to generate all quark states of the baryon decuplet, starting from one of the corner states (ddd), (uuu) or (sss). You will not obtain the correct normalisation in this way, but that is not so important here (you can always normalise the wave functions afterwards, if you wish). The point of this exercise is to note that all wave functions that you obtain by stepping through the diagram are *symmetric* in the exchange of two quarks.

(b) [0.5] Construct a wave function $\psi_{\text{colour}}(c_1, c_2, c_3)$ that is fully anti-symmetric in the exchange of two colours.

Experimental evidence for colour I



- The cross section for the left diagram is given in PP-I section 8.3:

$$\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

Here particle masses are neglected and if we do the same for the right diagram, we obtain the cross section for $q\bar{q}$ production simply by putting the correct charge at the $\gamma q\bar{q}$ vertex

$$\sigma(\mathrm{e}^+\mathrm{e}^- \to q_i\ \bar{q}_i) = \frac{4\pi\alpha^2 e_i^2}{3s}$$

- Because quarks fragment with 100% probability into hadrons, we can sum over all available quark species to get the observable

$$\sigma(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons}) = N_\mathrm{c} \sum_i \frac{4\pi\alpha^2 e_i^2}{3s}$$

- Here the sum runs over all quark flavours that can be produced at a given energy $\sqrt{s}$, and N_c counts the number of coloured duplicates of each quark. Thus $N_\mathrm{c} = 3$ for the quark colours q_r, q_g and q_b.

Experimental evidence for colour II

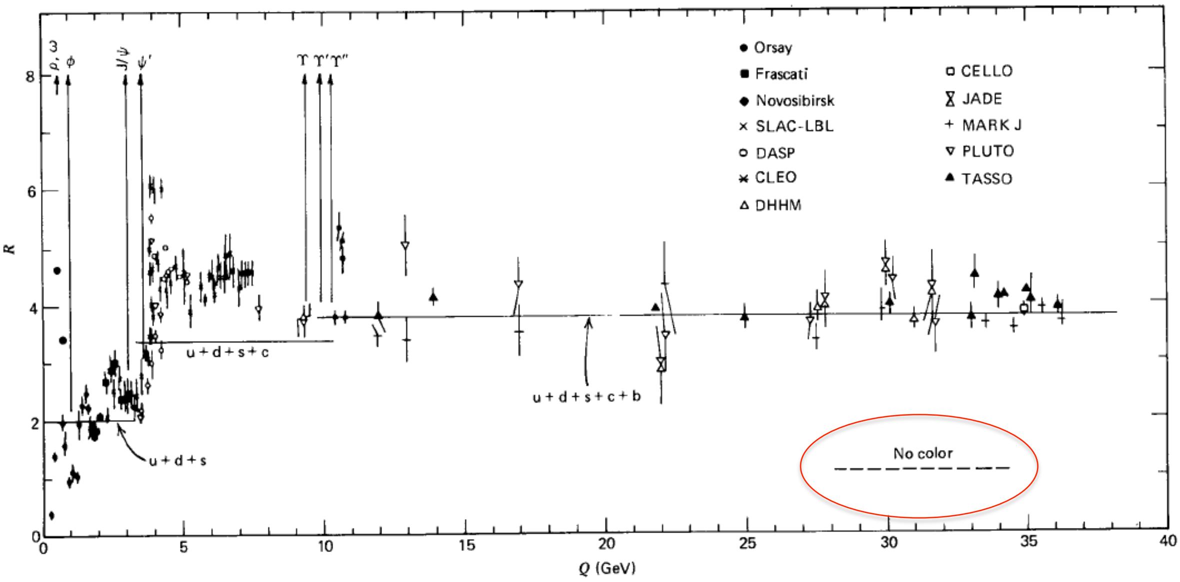


Fig. 11.3 Ratio R of (11.6) as a function of the total e^-e^+ center-of-mass energy. (The sharp peaks correspond to the production of narrow 1^- resonances just below or near the flavor thresholds.)

- This plot shows, as a function of $\sqrt{s}$, measurements of the ratio

$$R = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)} = N_\mathrm{c} \sum_i e_i^2 = 3 \sum_i e_i^2$$

- The data are consistent with $N_\mathrm{c} = 3$ and certainly exclude $N_\mathrm{c} = 1$.
- Remark: There is quite some structure in this plot, in particular around the thresholds of heavy quark production where $q\bar{q}$ pairs are produced with little relative momentum so that they can form bound states, like the J/ψ family ($\mathrm{c}\bar{\mathrm{c}}$) at about 3 GeV, and the Υ family ($\mathrm{b}\bar{\mathrm{b}}$) at about 10 GeV.