Lecture notes Particle Physics II

Quantum Chromo Dynamics

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Preliminary

This section is not part of the lectures, but a small collection of material that should be familiar from special relativity, electrodynamics, quantum mechanics and the lecture series Particle Physics I.

Also included is a summary of group theory, but still very incomplete.

Units and conversion factors

In particle physics, energy is measured in units of GeV $= 10^6$ eV, where 1 eV $= 1.6 \times 10^{-19}$ J is the change in kinetic energy of an electron when it traverses a potential difference of one volt. From the relation $E^2 = p^2c^2 + m^2c^4$ it follows that the units of momentum and mass are GeV$/c$ and GeV$/c^2$, respectively. The dimension of $\hbar$ is energy$\times$time so that the unit of time is $\hbar$/GeV; $\hbar c$ has dimension energy$\times$length so that length has unit $\hbar c$/GeV.

One often works in a system of units where $\hbar$ and c have a numerical value of one, so that these constants can be omitted in expressions, as in $E^2 = p^2 + m^2$. A disadvantage is that the dimensions carried by $\hbar$ (energy$\times$time) and c (length/time) also disappear but these can always be restored, if necessary, by a dimensional analysis afterward.

Here are some useful conversions.

	Conversion	$\hbar = c = 1$ units	Natural units
Mass	$1\ \text{kg} = 5.61 \times 10^{26}$	GeV	GeV$/c^2$
Length	$1\ \text{m} = 5.07 \times 10^{15}$	GeV^{-1}	$\hbar c$/GeV
Time	$1\ \text{s} = 1.52 \times 10^{24}$	GeV^{-1}	$\hbar$/GeV
Charge	$e = \sqrt{4\pi\alpha}$	dimensionless	$\sqrt{\hbar c}$

$$1\ \text{TeV} = 10^3\ \text{GeV} = 10^6\ \text{MeV} = 10^9\ \text{KeV} = 10^{12}\ \text{eV}$$

$$1\ \text{fm} = 10^{-15}\ \text{m} = 10^{-13}\ \text{cm} = 5.07\ \text{GeV}^{-1}$$

$$1\ \text{barn} = 10^{-28}\ \text{m}^2 = 10^{-24}\ \text{cm}^2$$

$$1\ \text{fm}^2 = 10\ \text{mb} = 10^4\ \mu\text{b} = 10^7\ \text{nb} = 10^{10}\ \text{pb}$$

$$1\ \text{GeV}^{-2} = 0.389\ \text{mb}$$

$$\hbar c = 197\ \text{MeV fm}$$

$$(\hbar c)^2 = 0.389\ \text{GeV}^2\ \text{mb}$$

$$\alpha = e^2/(4\pi\hbar c) \approx 1/137$$

Covariant notation ($c = 1$)

- Contravariant space-time coordinate: $x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x})$
- Covariant space-time coordinate: $x_\mu = (x_0, x_1, x_2, x_3) = (t, -\boldsymbol{x})$
- Contravariant derivative: $\partial_\mu \equiv \partial/\partial x^\mu = (\partial_t, +\nabla)$
- Covariant derivative: $\partial^\mu \equiv \partial/\partial x_\mu = (\partial_t, -\nabla)$
- Metric tensor: $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- Index raising/lowering: $a_\mu = g_{\mu\nu}\, a^\nu, \quad a^\mu = g^{\mu\nu}\, a_\nu$
- Lorentz boost along x-axis:[2] $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

  We have also: $x'_\mu = \Lambda_\mu{}^\nu\, x_\nu$ with $\Lambda_\mu{}^\nu(\beta) = \Lambda^\mu{}_\nu(-\beta) \equiv (\Lambda^\mu{}_\nu)^{-1}$

- Inproduct (Lorentz scalar): $a \cdot b = a^\mu\, b_\mu = a^0\, b_0 - \boldsymbol{a} \cdot \boldsymbol{b} = a_\mu\, b^\mu$
- $a^2 > 0$ time-like 4-vector $\rightarrow$ possible causal connection
  $a^2 = 0$ light-like 4-vector
  $a^2 < 0$ space-like 4-vector $\rightarrow$ no causal connection
- 4-momentum: $p^\mu = (E, \boldsymbol{p}), \quad p_\mu = (E, -\boldsymbol{p})$
- Invariant mass: $p^2 = p^\mu\, p_\mu = p_\mu\, p^\mu = E^2 - \boldsymbol{p}^2 = m^2$
- Particle velocity: $\gamma = E/m, \quad \beta = |\boldsymbol{p}|\,/E$

[2]This is the relation between the coordinates x^μ of an event observed in a system S and the coordinates x'^μ of that *same* event observed in a system S' that moves with a velocity $+\beta$ along the x-axis of S.

Vector calculus

$$
\begin{aligned}
\nabla \times (\nabla \psi) &= \mathbf{0} \\
\nabla \cdot (\nabla \times \boldsymbol{A}) &= 0 \\
\nabla \times (\nabla \times \boldsymbol{A}) &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\
\int_V \nabla \cdot \boldsymbol{A} \, \mathrm{d}V &= \int_S \boldsymbol{A} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S & \text{(Divergence theorem)} \\
\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, \mathrm{d}V &= \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{\boldsymbol{n}} \, dS & \text{(Green's theorem)} \\
\int_S (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S &= \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} & \text{(Stokes' theorem)}
\end{aligned}
$$

- In the above, S is a closed surface bounding V, with $\hat{\boldsymbol{n}}$ the outward normal unit vector at the surface element $\mathrm{d}S$.
- In Stokes' theorem, the direction of $\hat{\boldsymbol{n}}$ is related by the right-hand rule to the sense of the contour integral around C.

Maxwell's equations in vacuum

- Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \rho \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t = 0 \qquad \nabla \times \boldsymbol{B} - \partial \boldsymbol{E}/\partial t = \boldsymbol{j}$$

- Continuity equation

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$$

- The potentials V and $\boldsymbol{A}$ are defined such that the second and third of Maxwell's equations are automatically satisfied

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \rightarrow \quad \nabla \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \nabla V \quad \rightarrow \quad \nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$$

- Gauge transformations leave the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields invariant

$$V' = V + \frac{\partial \lambda}{\partial t} \qquad \text{and} \qquad \boldsymbol{A}' = \boldsymbol{A} - \nabla \lambda$$

- Maxwell's equations in 4-vector notation

4-vector potential	$A^\mu = (V, \boldsymbol{A})$
4-vector current	$j^\mu = (\rho, \boldsymbol{j})$
Electromagnetic tensor	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
Maxwell's equations	$\partial_\mu F^{\mu\nu} = j^\nu$
Continuity equation	$\partial_\mu j^\mu = 0$
Gauge transformation	$A^\mu \rightarrow A^\mu + \partial^\mu \lambda$

- Lorentz gauge and Coulomb condition

Lorentz gauge	$\partial_\mu A^\mu = 0 \quad \rightarrow \quad \partial_\mu \partial^\mu A^\nu = j^\nu$
Coulomb condition	$A^0 = 0$ or equivalently $\nabla \boldsymbol{A} = 0$

The Lagrangian in classical mechanics

In classical mechanics, the Lagrangian is the difference between the kinetic and potential energy: $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \equiv T - V$. The coordinates $\boldsymbol{q}(t) = \{q_1(t), \ldots, q_N(t)\}$ fully describe the system at any given instant t. The number N of coordinates is called the **number of degrees of freedom** of the system.

Let the system move from $A(t_1)$ to $B(t_2)$ along some given path. The **action** $S[\text{path}]$ is defined by the integral of the Lagrangian along the path:

$$S[\text{path}] = \int_{t_1}^{t_2} \mathrm{d}t \; L(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

The action S assigns a number to each path and is thus a function of the path. In mathematics, S is called a **functional**.

The **principle of least action** states that the system will evolve along the path that minimises the action.

Let $q(t)$ be a path and $q(t) + \delta q(t)$ be some deviating path between the *same* points $A(t_1)$ and $B(t_2)$. That is, $\delta q(t_1) = \delta q(t_2) = 0$. The variation in the action is then given by

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \; \delta L(q, \dot{q}) = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \underset{\text{Iwant}}{=} 0.$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q},$$

we find, by partial integration,

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \int_{t_1}^{t_2} \mathrm{d} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \underset{\text{Iwant}}{=} 0.$$

The second integral vanishes because $\delta q(t_1) = \delta q(t_2) = 0$.

The first integral vanishes for all δq if and only if the term in brackets vanishes, leading to the **Euler-Lagrange equations**, for N degrees of freedom:

$$\boxed{\frac{\delta S}{\delta q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, N}$$

Solving the EL equations for a given Lagrangian lead to the **equations of motion** of the system.

The Hamiltonian in classical mechanics

If L does not explicitly depend on time we have for the time derivative

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial \dot{q}}\,\ddot{q} + \frac{\partial L}{\partial q}\,\dot{q}$$

Substituting $\partial L/\partial q$ from the Euler-Lagrange equations gives

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial \dot{q}}\,\ddot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right) \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q} - L\right) = 0$$

The term in brackets is the **Legendre transform** of L and is called the **Hamiltonian**:

$$H \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}}\,\dot{q} - L = p\dot{q} - L \qquad \text{with} \qquad p \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}},$$

where we have also introduced the **canonical momentum** p. The Hamiltonian is identified with the total energy $E = T + V$ which is thus conserved in the time evolution of the system. This is an example of a conservation law.

In the Lagrangian, the dependence on $\dot{q}$ resides in the kinetic energy term T while the dependence on q is contained in the potential energy V. Thus if $V = 0$ (or a constant) we have in the EL equations

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\mathrm{d}p}{\mathrm{d}t} = 0$$

Thus the momentum p is conserved in a system that is not under the influence of an external potential. This is another example of a conservation law.

The Hamiltonian equations of motion are

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial q}}$$

This can be derived as follows. Consider the total differential

$$\mathrm{d}L = \frac{\partial L}{\partial q}\mathrm{d}q + \frac{\partial L}{\partial \dot{q}}\mathrm{d}\dot{q}$$

Now

$$\frac{\partial L}{\partial q} = \dot{p} \text{ (from EL)}, \qquad \frac{\partial L}{\partial \dot{q}} = p \text{ (by definition)},$$

and thus, using $p\mathrm{d}\dot{q} = d(p\dot{q}) - \dot{q}\mathrm{d}p$, we obtain

$$\mathrm{d}L = \dot{p}\mathrm{d}q + \mathrm{d}(p\dot{q}) - \dot{q}\mathrm{d}p \quad \rightarrow \quad \mathrm{d}(p\dot{q} - L) = \mathrm{d}H = \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q,$$

from which the Hamiltonian equations immediately follow.

Dirac $\boldsymbol{\delta}$-function

- The Dirac δ-function can be defined by[3]

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases} \qquad \text{with} \qquad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

- Generalisation to more dimensions is trivial, like $\delta(\boldsymbol{r}) \equiv \delta(x)\delta(y)\delta(z)$.

- For $x \to 0$ we may write $f(x)\delta(x) = f(0)\delta(x)$ so that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- For a linear transformation $y = k(x-a)$ we have

$$\delta(y) = \frac{1}{|k|}\,\delta(x-a)$$

  This is straight-forward to prove by showing that $\delta(y)$ satisfies the definition of the δ-function given above.

- Likewise, if $\{x_i\}$ is the set of points for which $f(x_i) = 0$, then it is easy to show by Taylor expansion around the x_i that

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$$

- There exist many representations of the δ-function, for instance,

$$\delta(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{k} \qquad \text{or} \qquad \delta(x) = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x},$$

$$\text{with} \quad \theta(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \geq 0 \end{cases} \qquad \text{(Heaviside step function).}$$

[3] A more rigorous mathematical definition is usually in terms of a limiting sequence of functions.

Green functions

- Let Ω be some linear differential operator. A **Green function** of the operator Ω is a solution of the differential equation

$$\Omega\, G(\boldsymbol{r}) = \delta(\boldsymbol{r})$$

These Green functions can be viewed as some potential caused by a point source at $\boldsymbol{r}$.

- Once we have the Green function we can immediately solve the differential equation for any source density $s(\boldsymbol{r})$

$$\Omega\, \psi(\boldsymbol{r}) = s(\boldsymbol{r})$$

By substitution it is easy to see that (ψ_0 is the solution of $\Omega\, \psi_0 = 0$)

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \int G(\boldsymbol{r} - \boldsymbol{r'}) s(\boldsymbol{r'})\, \mathrm{d}\boldsymbol{r'}$$

Here it is clearly seen that $G(\boldsymbol{r} - \boldsymbol{r'})$ 'propagates' the contribution from the source element $s(\boldsymbol{r'})\mathrm{d}\boldsymbol{r'}$ to the potential $\psi(\boldsymbol{r})$.

- A few well-known Green functions are ...

$$\nabla^2\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G(\boldsymbol{r}) = -1/(4\pi r)$$

$$(\nabla^2 + \boldsymbol{k}^2)\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G^{\pm}(\boldsymbol{r}) = -\exp(\pm ikr)/(4\pi r)$$

$$(\nabla^2 - m^2)\, G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \qquad G(\boldsymbol{r}) = -\exp(-mr)/(4\pi r)$$

- ... and here are their Fourier transforms

$$G(\boldsymbol{r}) = -1/(4\pi r) \qquad \tilde{G}(\boldsymbol{q}) = -1/q^2$$

$$G^{+}(\boldsymbol{r}) = -\exp(ikr)/(4\pi r) \qquad \tilde{G}^{+}(\boldsymbol{q}) = 1/(k^2 - q^2 + i\varepsilon)$$

$$G(\boldsymbol{r}) = -\exp(-mr)/(4\pi r) \qquad \tilde{G}(\boldsymbol{q}) = -1/(q^2 + m^2)$$

Non-relativistic scattering theory I

- Classical relation $E = p^2/2m + V$ with substitution $E \to i\partial/\partial t$ and $\boldsymbol{p} = -i\nabla$ gives the Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\left[\frac{\nabla^2}{2m} - V(\boldsymbol{r},t)\right]\psi(\boldsymbol{r},t)$$

- Separate $\psi(\boldsymbol{r},t) = \phi(t)\psi(\boldsymbol{r})$. Dividing through by $\phi\psi$ gives

$$\frac{i\partial_t\phi(t)}{\phi(t)} = -\frac{[\nabla^2 - 2mV(\boldsymbol{r},t)]\,\psi(\boldsymbol{r})}{2m\psi(\boldsymbol{r})}$$

Assume now that V does not depend on t. The left and right-hand side must then be equal to a constant, say E, and we have

$$\frac{\partial\phi(t)}{\partial t} = -iE\,\phi(t) \quad \to \quad \phi(t) = e^{-iEt}$$

$$[\nabla^2 + k^2]\,\psi(\boldsymbol{r}) = 2mV(\boldsymbol{r})\psi(\boldsymbol{r})$$

where we have set $k^2 = 2mE$. Using Green functions we get

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

- For large $r \gg r'$ we have $|\boldsymbol{r}-\boldsymbol{r}'| \approx r - \hat{\boldsymbol{r}}\boldsymbol{r}'$ so that

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\pi}\frac{e^{ikr}}{r}\int e^{-ik\hat{\boldsymbol{r}}\boldsymbol{r}'}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

- We set $\boldsymbol{k}' \equiv k\hat{\boldsymbol{r}}$ and write, formally,

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + f(\boldsymbol{k}')\frac{e^{ikr}}{r} \quad \text{with} \quad f(\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{-i\boldsymbol{k}'\boldsymbol{r}'}V(\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

The function $f(\boldsymbol{k}')$ is called the **scattering amplitude**.

Non-relativistic scattering theory II

- An incoming plane wave $\psi_{\rm in} = Be^{ikz}$ describes beam particles moving along the z axis with momentum k. The wave function is normalised such that $\rho = \psi^*\psi = |B|^2$ is the particle density (number of particles per unit volume). The current density is
$$\boldsymbol{j}_{\rm in} = \frac{1}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = |B|^2\frac{\boldsymbol{k}}{m} = \rho\,\frac{\boldsymbol{k}}{m} = \rho\boldsymbol{v}$$
with $\boldsymbol{v}$ the velocity of the particle. The number of beam particles passing per second through an area A is $R_{\rm in} = \rho v A = |\boldsymbol{j}_{\rm in}|A$. Likewise, the number of scattered particles that pass per second through an area $r^2{\rm d}\Omega$ is $R_{\rm sc} = |\boldsymbol{j}_{\rm sc}|r^2{\rm d}\Omega$.

- We now imagine a hypothetical area ${\rm d}\sigma$ such that the number of beam particles that pass through that area is equal to the number of particles that scatter in the solid angle ${\rm d}\Omega$. We then have, by definition, $|\boldsymbol{j}_{\rm in}|\,{\rm d}\sigma = |\boldsymbol{j}_{\rm sc}|\,r^2{\rm d}\Omega$, or
$$\boxed{\frac{{\rm d}\sigma}{{\rm d}\Omega} = \frac{r^2\,|\boldsymbol{j}_{\rm sc}|}{|\boldsymbol{j}_{\rm in}|}}$$
The quantity ${\rm d}\sigma/{\rm d}\Omega$ is called a **differential cross section**.

- For our scattered wave $\psi_{\rm sc} = f(\boldsymbol{k}')\,e^{ikr}/r$ we find
$$j_{\rm sc} = \frac{1}{2mi}\left(\psi^*\frac{\partial\psi}{\partial r} - \psi\frac{\partial\psi^*}{\partial r}\right) = |f(\boldsymbol{k}')|^2\frac{k}{mr^2}$$
and thus
$$\boxed{\frac{{\rm d}\sigma}{{\rm d}\Omega} = |f(\boldsymbol{k}')|^2}$$
where we have assumed $\rho = 1$ and $k_{\rm sc} = k_{\rm in}$ (elastic scattering).

Non-relativistic scattering theory III

- Recall that for scattering on a potential, the outgoing wave is

$$\psi_{\text{out}}(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + f(\boldsymbol{k}')\frac{e^{ikr}}{r}$$

with

$$f(\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{-i\boldsymbol{k}'\boldsymbol{r}'}V(\boldsymbol{r}')\psi_{\text{out}}(\boldsymbol{r}')\text{d}\boldsymbol{r}'$$

Here $\boldsymbol{k}'$ is the momentum vector of the scattered particle.

- The problem now is that ψ_{out} occurs on both sides of the equation above. A first order approximation is achieved by setting in the scattering amplitude $\psi_{\text{out}} \approx \psi_{\text{in}} = e^{ikz} = e^{i\boldsymbol{k}\boldsymbol{r}}$. This gives

$$f(\boldsymbol{k},\boldsymbol{k}') \equiv -\frac{m}{2\pi}\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}'}V(\boldsymbol{r}')\text{d}\boldsymbol{r}' = -\frac{m}{2\pi}\int e^{-i\boldsymbol{q}\boldsymbol{r}'}V(\boldsymbol{r}')\text{d}\boldsymbol{r}'$$

where we have set the momentum transfer $\boldsymbol{q} \equiv \boldsymbol{k}' - \boldsymbol{k}$. In this so-called **Born approximation**, the scattering amplitude $f(\boldsymbol{k},\boldsymbol{k}') \equiv f(\boldsymbol{q})$ is thus the Fourier transform of the potential.

- Example: Yukawa potential $V(r) = Q_1Q_2\, e^{-ar}/r$

$$f(\boldsymbol{q}) = -\frac{mQ_1Q_2}{2\pi}\int \frac{e^{-ar'}}{r'}\, e^{-i\boldsymbol{q}\boldsymbol{r}'}\text{d}\boldsymbol{r}' = \cdots = \frac{2mQ_1Q_2}{q^2+a^2}$$

$$\frac{\text{d}\sigma}{\text{d}\Omega} = |f(\boldsymbol{q})|^2 = \left[\frac{2mQ_1Q_2}{q^2+a^2}\right]^2$$

- Example: Coulomb potential $V(r) = Q_1Q_2/r$ set $a = 0$ above:

$$\frac{\text{d}\sigma}{\text{d}\Omega} = \left[\frac{2mQ_1Q_2}{q^2}\right]^2 = \left[\frac{Q_1Q_2}{2mv^2\sin^2(\theta/2)}\right]^2$$

This is the famous formula for **Rutherford scattering**.

Dirac's bra-ket notation I

- A state vector ψ_α can be represented by a column vector of complex numbers in a Hilbert space and is denoted by the ket $|\alpha\rangle$. To each ket is associated a bra vector $\langle\alpha|$ in a dual Hilbert space. This bra is represented by the conjugate transpose $\psi_\alpha^\dagger$, that is, by the row vector of complex conjugates. The operation of **Hermitian conjugation** turns a bra into a ket and *vice versa*

  $$|\alpha\rangle^\dagger = \langle\alpha| \quad \text{and} \quad (c|\alpha\rangle)^\dagger = \langle\alpha|c^* \qquad (c \text{ any complex number})$$

  Note that the Hermitian conjugate of a c-number is the complex conjugate. The inproduct $\psi_\alpha^\dagger \cdot \psi_\beta$ is denoted by $\langle\alpha|\beta\rangle$ and is a c-number so that

  $$\langle\beta|\alpha\rangle \equiv \langle\alpha|\beta\rangle^\dagger = c^\dagger = c^* = \langle\alpha|\beta\rangle^*$$

- An operator O transforms a ket $|\alpha\rangle$ into another ket, say $|\gamma\rangle$. The operator and its Hermitian conjugate are then defined by

  $$O|\alpha\rangle = |\gamma\rangle \qquad \text{and} \qquad \langle\alpha|O^\dagger = \langle\gamma|$$

  Multiplying from the left with $\langle\beta|$ and from the right with $|\beta\rangle$ we find the relation between the **matrix elements** of O and $O^\dagger$

  $$O_{\beta\alpha} \equiv \langle\beta|O|\alpha\rangle = \langle\beta|\gamma\rangle$$
  $$O^\dagger_{\alpha\beta} \equiv \langle\alpha|O^\dagger|\beta\rangle = \langle\gamma|\beta\rangle = \langle\beta|\gamma\rangle^* = \langle\beta|O|\alpha\rangle^* = O^*_{\beta\alpha}$$

- An operator for which $O = O^\dagger$ is called **self-adjoint** or **Hermitian**. Observable quantities are always represented by Hermitian operators. Indeed, the **expectation value** $\langle\alpha|O|\alpha\rangle$ is then real, as it should be, since

  $$\langle\alpha|O|\alpha\rangle \equiv \langle\alpha|O^\dagger|\alpha\rangle = \langle\alpha|O|\alpha\rangle^*$$

Dirac's bra-ket notation II

- An orthonormal basis is written as $|e_i\rangle$ with $\langle e_i|e_j\rangle = \delta_{ij}$. On this basis, a state $|\alpha\rangle$ is given by the linear combination

$$|\alpha\rangle = \sum_i |e_i\rangle\langle e_i|\alpha\rangle$$

  The operator $|e_i\rangle\langle e_i|$ is called a **projection operator**, for obvious reasons. The **closure relation** reads $\sum_i |e_i\rangle\langle e_i| = 1$

- We denote the wave function $\psi_\alpha(\boldsymbol{r})$ by $\langle \boldsymbol{r}|\alpha\rangle$ and its Hermitian conjugate $\psi^\dagger_\alpha(\boldsymbol{r})$ by $\langle\alpha|\boldsymbol{r}\rangle$. In particular, the wave function of a momentum eigenstate is $\langle\boldsymbol{r}|\boldsymbol{k}\rangle \propto e^{i\boldsymbol{k}\boldsymbol{r}}$.

- For the complete set of states $|\boldsymbol{r}\rangle$ the closure relation reads

$$\int |\boldsymbol{r}\rangle\langle\boldsymbol{r}|\, \mathrm{d}\boldsymbol{r} = 1$$

  From this, we nicely recover the expression for the inproduct of two wave functions

$$\langle\alpha|\beta\rangle = \int \langle\alpha|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\beta\rangle\, \mathrm{d}\boldsymbol{r} = \int \psi^*_\alpha(\boldsymbol{r})\psi_\beta(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

  that of the delta function

$$\delta(\boldsymbol{k}-\boldsymbol{k}') = \langle\boldsymbol{k}'|\boldsymbol{k}\rangle = \int \langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}\rangle \mathrm{d}\boldsymbol{r} \propto \frac{1}{(2\pi)^3}\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

  and also that of Fourier transforms

$$\psi(\boldsymbol{k}) = \langle\boldsymbol{k}|\psi\rangle = \int \langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\psi\rangle \mathrm{d}\boldsymbol{r} \propto \int e^{-i\boldsymbol{k}\boldsymbol{r}}\psi(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$$

Dirac equation

- Dirac equation:

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

$$\underbrace{(\not{p} - m)u = 0}_{\text{particle in}}, \quad \underbrace{\bar{u}(\not{p} - m) = 0}_{\text{particle out}}, \quad \underbrace{(\not{p} + m)v = 0}_{\text{antiparticle out}}, \quad \underbrace{\bar{v}(\not{p} + m) = 0}_{\text{antiparticle in}}$$

$$\overline{\psi} = \psi^\dagger\gamma^0, \qquad \not{a} = \gamma^\mu a_\mu$$

- Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad \sigma_i^\dagger = \sigma_i = \sigma_i^{-1}, \quad [\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k$$

$$(\boldsymbol{a}\cdot\boldsymbol{\sigma})(\boldsymbol{b}\cdot\boldsymbol{\sigma}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$\exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}) = \cos|\boldsymbol{\theta}| + i(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})\sin|\boldsymbol{\theta}|$$

- Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i, \quad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad \left(\gamma^5\right)^2 = 1$$