

Einstein's Derivation of the Lorentz Transformations in the1905 Paper is Internally Inconsistent

Jon C. Freeman

Abstract— The general consensus in literature concerning Einstein's 1905 paper on special relativity, is that he independently derived the Lorentz transformations using a kinematic method. That is he was apparently unaware of the latter work of Lorentz and others while composing the paper. Here it is demonstrated that the derivation is mathematically inconsistent. There are several errors in the development and the main one is the inconsistent steps of requiring a partial derivative to be zero at one point, but necessarily finite at the last step. This derivative can only equal zero if the relative velocity between the frames goes to zero. It is also shown that the scheme of following a light beam reflecting between two moving mirrors (the kinematic method) cannot correctly arrive at the transformations. A study of many textbooks on relativity will reveal that this derivation is almost never used to introduce the Lorentz transformations. Only one source was found that discussed this particular method; and it was in an appendix.

Index Terms—Einstein, History of Science, Special Theory of Relativity.

THE first derivation of the Lorentz transformations given by Einstein was in the classic paper published in 1905. To the best of our knowledge this method was never given again in any of his subsequent publications. An up-to-date discussion on the paper by A. Martinez [1] on the concepts and technique is illuminating. Briefly, he says the derivation contains ambiguities due to both imprecise notation and possible multiple definitions. Further investigation here reveals inconsistent math steps that invalidate the procedure. It is remarkable that the inconsistencies have not been noted in the literature. One may justifiably ask "How can there be any errors noted now, after 110 years since the publication"? It is speculated that most readers of the paper apparently did not attempt to fill in the omitted steps where the problems arise. There has been uncertainty among historians and biographers about the question if Einstein knew the form of the equations he wanted, and sought to develop them; rather than obtaining them directly from the derivation itself without prior knowledge. The results here answer the question; yes, he must have known the final form, but the steps used to get them were imprecise. The equations had been published prior to 1905 so it is not surprising that Einstein was aware of various forms for the transformations. In his paperback [2] Max Born, a Nobel Laureate and personal friend of

Einstein, gave the following quote in the Introduction. "Relativity actually ought not be connected with a single name or with a single date. It was in the air about 1900 and several great mathematicians and physicists- Larmor, Fitzgerald, Lorentz, Poincare', to mention a few- were in possession of many of its contents. In 1905 Albert Einstein based the theory on very general principles of a philosophical character, and a few years later Hermann Minkowski gave it final logical and mathematical expression." In chapter VI Born developed Einstein's kinematics and did not use the technique given in the 1905 paper. Instead he used a different approach to develop the Lorentz transformations.

MATH

The Lorentz transformations are as follows:

$$\tau(x,t) = \beta(t - vx/c^{2})$$

$$\xi(x,t) = \beta(x - vt)$$

$$\beta = \frac{1}{\sqrt{1 - (v/c)^{2}}}$$
(1)

The constant β is given the symbol γ in modern notation; but we are adhering to Einstein's notation. The rest frame uses coordinates (x, y, z, t) while the frame moving at speed 'v' to the right uses (ξ, η, ζ, τ) . By definition x and t are independent variables as are ξ and τ . The scheme considers two mirrors stationary in the moving frame. They are parallel to the η -axis and separated by an unspecified distance (which is chosen as 'd' for the moment). The lower edge of the left mirror is at the origin of the (ξ, η) plane. Now consider three successive events. They are: emission of a photon from the left mirror, reflection at the right mirror, and then returning to the left one. Assume the time instants recorded in the moving frame are τ_0 , τ_1 , τ_2 . For an observer in the moving frame the time intervals for the photon to traverse the distance between the mirrors are equal, thus $\tau_1 - \tau_0 = \tau_2 - \tau_1$, then rearrange to $(1/2)[\tau_0 + \tau_2] = \tau_1$. Next assume the time instants for τ may be expressed as a function in terms of x and t. That is: at Event 0, $\tau_0 = \tau(x_0, t_0)$ where x_0 and t_0 are coordinates of Event 0 as measured in the rest frame. Similarly $\tau_1 = \tau(x_1, t_1)$, etc. Next he states instead of solving for $\tau(x, t)$, solve for $\tau(x', t)$ where x' will be prescribed later.

Now the procedure for developing x'; start with
$$\tau(x,t) = \beta(t - vx/c^2)$$
, replace x with $(x' + vt)$, then
 $\tau(x,t) = \beta[t - v(x' + vt)/c^2] =$
 $\beta t - \beta(v/c^2)x' - \beta t(v/c)^2$
 $= \beta t - \beta t(v/c)^2 - \beta(v/c^2)x'$
 $= \beta t\{1 - (v/c)^2\} - \beta(v/c^2)x'$
 $= \beta t/\beta^2 - \beta(v/c^2)x' = t/\beta - \beta(v/c^2)x'$
 $= (1/\beta)\{t - \beta^2(v/c^2)x'\}$
 $= (1/\beta)\{t - [1/(1 - (v/c)^2)](v/c^2)x'\}$
 $= (1/\beta)\{t - [c^2/(c^2 - v^2)](v/c^2)x'\}$
 $= (1/\beta)\{t - [vx'/(c^2 - v^2)]\}$
(2)

These manipulations have rearranged things so that only x' and t appear on the right side. Can we say this is $\tau(x',t)$? If we do, can we also say x' and t can then be treated as independent variables? Since x'= x-vt, it appears x' must depend on t. However if one assumes the values for x are only those that keep x' a constant, then x' (being a constant) is independent of t. Thus we attempt to justify the independence of x' and t by stating: we are now following a point fixed in the moving system. Let $a = (1/\beta)$, then the last line of (2) is exactly the expression for $\tau(x',t)$ as given by Einstein. Thus $\tau(x,t) = \tau(x',t)$, where a different interpretation must be given for each side of the equation. On the left x and t are independent whereas on the right x' and t are treated as independent as we are following a fixed point in the moving system. The function $\tau(x',t)$ was given as the solution of the following equation.

$$\partial \tau(x',t) / \partial x' + \left[v / (c^2 - v^2) \right] \partial \tau(x',t) / \partial t = 0$$
(3)

The steps to arrive at (3) were not given and an attempt will be made to provide them later.

Now the times of flight for a photon moving between the mirrors as perceived by the rest observer will be developed. For the rest observer the mirror separation is not assumed to be 'd' but some value to be determined. For now let the separation be 'L' as measured in the rest system. At the instant of emission the right mirror is L units away. During the flight it moves the distance $v\delta t_r$ where δt_r is the flight time. The photon must travel the total distance $L + v\delta t_r$. The photon always moves at speed c, so the total distance traveled is $c\delta t_r$. Equating distances we find $\delta t_r = L/(c-v)$. For the return trip after reflection, the total distance is now L - $v\delta t_L$. Again the photon's speed is c, thus the total flight time while moving to the left is $\delta t_L = L/(c+v)$. The denominators in the previous expressions are the closing speeds between a moving mirror and a photon.

Next Einstein fills out the equation for the three time instants

$$(1/2)\{\tau(0,0,0,t) + \tau[0,0,0,t+x'/(c-v) + x'/(c+v)]\} = \tau[x',0,0,t+x'/(c-v)]$$
(4)

A self-consistent interpretation is as follows. The right side indicates the first slot is for x[']. The

last slot is for the t as measured by the rest observer. The first term on the left means the photon emission occurs at some arbitrary time, which is event 0. The right side is event 1, and the photon is the distance x' from the left mirror and the time is now t + x'/(c-v), where the second term is δt_r found earlier. The second term on the left is the return with the time now being the previous value with δt_L added. The photon is again at the left mirror so its distance from that mirror is zero (the first slot). Thus at events 0 and 1 the photon's distance from the first mirror is the appropriate value for x'. Therefore x' must be the photon's distance from the left mirror at different instants as measured in the rest frame. Thus the mirrors are separated the distance x' in the rest frame. The next step is to show the steps to move from (4) to (3). These steps were not given by Einstein, and only one source has been found that shows some of them [3]. In his appendix Prokhovnik says take the partial with respect to x' of (4), and he gives

$$(1/2)(1/(c-v)+1/(c+v))[\partial \tau / \partial t] = [\partial \tau / \partial x'] + [1/(c-v)][\partial \tau / \partial t$$
(5)

This equation may be developed as follows. Start with the first term on the left side of (4), $\tau_0 = \tau(0,0,0,t) = \tau(x'=0,0,0,t)$. Its partial is $(\partial \tau / \partial x')_{x'=0}$ where the subscript means evaluate the partial derivative at x'=0. The second term on the left of (4) is

$$\tau_2 = \tau \left[x' = 0, 0, 0, t + x' / (c - v) + x' / (c + v) \right]$$
(6)

For notational convenience write this as $\tau(\mathbf{x} = 0, \mathbf{u})$ where \mathbf{u} is the argument in the last slot. Then $\partial \tau_2(\mathbf{x}', \mathbf{u}) / \partial \mathbf{x}' = (\partial \tau / \partial \mathbf{x}')_{\mathbf{x}=0} + [\partial \tau / \partial \mathbf{u}] [\partial \mathbf{u} / \partial \mathbf{x}']$ where we have applied the chain rule. From (6)

$$u = t + x'/(c - v) + x'/(c + v)$$

so $\partial u / \partial x' = 1/(c-v) + 1/(c+v)$, since $\partial t / \partial x' = 0$, as x' and t are assumed to be independent. Then the left hand side is

$$\partial \tau / \partial x'|_{x'=0} + (1/2) \{ [\partial \tau / \partial u] [1/(c-v) + 1/(c+v)] \}$$
(7)

Working on the right side of (4); here let w = t + x'/(c-v), then using the chain rule again we find $\partial \tau / \partial x' + [\partial \tau / \partial w] [1/(c-v)]$. Assembling all terms we have

$$\frac{\partial \tau}{\partial x'}|_{x'=0} + (1/2) \left[\frac{\partial \tau}{\partial u} \left[\frac{1}{(c-v)} + \frac{1}{(c+v)} \right] \right] = \frac{\partial \tau}{\partial x'} + \left[\frac{\partial \tau}{\partial w} \right] \left[\frac{1}{(c-v)} \right]$$
(8)

Inspection of (5) and (8) shows they can be equal under the following conditions. First u = w = t. This will be the case as x' approaches zero. However, the second necessary condition

$$\partial \tau / \partial x'|_{x'=0} = 0 \tag{9}$$

causes a major problem. By inspection of (2), using the form where $(1/\beta) = a$,

$$\tau(x',t) = a\{t - [vx'/(c^2 - v^2)]\}$$
(10)

We see that (9) can only be satisfied if v = 0. This is the fatal error which shows the entire derivation is invalid. It is interesting that v = 0 will allow x' and t to be independent, for then x' is just x.

By inspection, rearrangement of (5) results in (3). Since the previous steps do not get us to the desired result without internal inconsistencies, let us try another path. For x' small, perhaps a Taylor expansion might work. Start with $\tau_0 = \tau(0,0,0,t) = \tau(x'=0,0,0,t) = \tau(0,0,0,t)_{x'=0}$

Write τ_2 as $\tau_2 \approx \tau(0,0,0,t) + [x'/(c-v) + x'/(c+v) - t][\partial \tau / \partial t]$

Recall for a Taylor series

 $f(x, y) \cong f(a, b) + (x - a)(\partial f / \partial x)_{a, b} + (y - b)(\partial f / \partial y)_{a, b}$ Now write τ_1 as $\tau_1 \approx \tau(0, 0, 0, t) + [x' - 0][\partial \tau / \partial x'] + [x' / (c - v) - t][\partial \tau / \partial t]$

Collecting all terms for (4) yields

 $(1/2)\{\tau(0,0,0,t) + \tau(0,0,0,t) + [x'/(c-v) + x'/(c+v) - t][\partial \tau / \partial t]\} = \tau(0,0,0,t) + x'(\partial \tau / \partial x') + [x'/(c-v) - t][\partial \tau / \partial t]$

Observe $\tau(0,0,0,t)$ cancels which leaves

$$(1/2)\{[x'/(c-v)+x'/(c+v)-t][\partial \tau/\partial t]\}$$

= $x'(\partial \tau/\partial x')+[x'/(c-v)-t][\partial \tau/\partial t]$

Since t is arbitrary, choose it as zero, then

$$(1/2)\{[x'/(c-v)+x'/(c+v)][\partial \tau / \partial t]\}$$

= $x'(\partial \tau / \partial x')+[x'/(c-v)][\partial \tau / \partial t]$

Therefore we are expanding about the point (x' = 0, t = 0). Then if we restrict $x' \neq 0$, then we may divide it out of the above and arrive at

(11)

$$(1/2)\{[1/(c-v)+1/(c+v)-t][\partial \tau / \partial t]\}$$

= $(\partial \tau / \partial x') + [1/(c-v)][\partial \tau / \partial t]$

Which we observe is (5) of Prokhovnik. Therefore the same problem mentioned earlier is still

present. This approach also shows an additional problem. Here we must expand about x' = 0 but we also must divide by x'! Even if this problem can be argued away by a limiting explanation, the issue of requiring v = 0 cannot be overcome. The primary conclusion is therefore; one cannot arrive at (3) starting from (4). This is the path stated in the paper; thus the derivation is invalid.

If, however, one just starts with (10) and takes partials assuming x' and t to be independent, $\partial \tau / \partial t = a$ then

 $\partial \tau / \partial x' = -av/(c^2 - v^2)$

Multiply the first line by $v/(c^2 - v^2)$ and then add this line to the second one and arrive at (3). However, (2) and (10) were developed by starting with the known expression for τ .

Conclusion

The results of the analysis show that the derivation given in the 1905 paper is invalid. It is flawed for several reasons. The most serious is the contradictory requirement that the partial of τ with respect to x' must be zero to arrive at the partial differential equation for determining τ . But the solution for τ from that equation does not allow this to be the case. The only way it could be satisfied is for v to be zero. Another problem is that with a Taylor expansion attempting to get the defining partial differential equation, one must divide by x' while starting with x' equal to zero. A third problem is the inconsistent reassignment of independence and dependence between the three variables x, t, and x'. The derivation starts with x and t as independent, then defines x' as being dependent on both. Then later treats x' and t as independent so x is then dependent. Later after determining $\tau(x',t)$ he writes x' in its original form and goes back to x and t being independent. The reason for this switching back and forth is supposedly justified by stating that in some places in the derivation one is following a fixed point, while in others one is no longer doing that. Another problem is the interpretation of x'. If it is indeed the distance from the left mirror as perceived by the rest observer, then we have apparently set β to unity (since $\xi = \beta x'$). It (β) is then brought back in the undetermined constant 'a' after tacitly assuming directions perpendicular to the translational motion are the same for both observers. Then for a photon moving vertically along the surface of the left mirror in the moving frame, the rest observer perceives x' = 0. Also the photon is perceived to move in a straight line sloped upward and to the right. The length is ct, while the horizontal displacement is vt. The vertical distance is $c\tau$ as both observers agree on this length to be the same. Using this right triangle one finds the relationship between t and τ for the case when x' = 0. This then shows $\beta = 1/a$, which we knew already after (2).

It is hoped that the history of the development of special relativity will be somewhat clarified by considering the above results. In [4] it was stated that the definitive history of special relativity was yet to be written; so some new information here may be helpful. It seems that Einstein knew the Lorentz transformations and was attempting to develop them using a kinematic derivation. However, the attempt was flawed. As the paper gave no references, many subsequent authors have incorrectly assumed he probably was not aware of many of the ideas that were "in the air" as stated by Born.

APPENDIX

Professor A. A. Martinez has developed an equation that will replace (4) when only x and t are used (x' not used).

$$\begin{aligned} &(1/2)\{\tau(0,0,0,t) + \\ &\tau[vx/(c-v)+vx/(c+v),0,0,t+x/(c-v)+x/(c+v)]\} \\ &= \tau[x+vx/(c-v),0,0,t+x/(c-v)] \end{aligned}$$
(A-1)

Notice that the function $\tau(x, t)$ is assumed to be linear in both variables, and in the above equation the function is that of variables in terms of both x and t. Under the linear assumption the following relationships are true.

$$\tau(x,t) = jx + kt$$

$$\partial \tau(x,t) / \partial x = j$$

$$\partial \tau(x,t) / \partial t = k$$

$$\tau(ax,bt) = j(ax) + k(bt)$$

$$\partial \tau(ax,bt) / \partial x = ja = a\partial \tau(x,t) / \partial x$$

$$\partial \tau(ax,bt) / \partial t = kb = b\partial \tau(x,t) / \partial t$$

Where j and k are constants. After expanding (A-1) in Taylor series, a long but straightforward calculation using the above relationships will show that one must assume $\partial \tau / \partial x|_{x=0} = 0$ to arrive at the final equation

$$\partial \tau / \partial x + \left(v / c^2 \right) \left[\partial \tau / \partial t \right] = 0 \tag{A-2}$$

Which one can easily show to be the equation satisfied by (1). Inspection of (1) shows $\partial \tau / \partial x|_{x=0} = -\beta v / c^2$. This necessitates again that one needs v = 0. Thus the method of following the photon between moving mirrors cannot develop the Lorentz transformations.

ACKNOWLEDGMENT

Thanks are due to Professor A. A. Martinez's article [1] for enlightenment on the ambiguities in the 1905 paper. His paper started the inquiry on the subject that lead to this effort.

References

- [1] A. A. Martinez, "Kinematic subtleties in Einstein's first derivation of the Lorentz transformations," Am. J. Phys., vol. 72, no. 5, pp. 1-9, May, 2004.
- [2] M. Born, *Einstein's Theory of Relativity*, 1965 ed. New York, Dover, Introduction, p.1.
- [3] S. J. Prokhovnik, The Logic of Special Relativity, New South Wales University Press, 1978.
- [4] H. M. Schwartz, Introduction to Special Relativity, McGraw-Hill, 1968, p. 23.

Jon C. Freeman is retired from the NASA-Glenn Research Center, in Cleveland, OH 44135. He lives in Beachwood, OH. 44122. USA (e-mail: freem231@umn.edu)