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Group theory factors for Feynman diagrams

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Abstract

We present algorithms for the group independent reduction of group theory factors of Feynman diagrams. We also give formulas and values for a large number of group invariants in which the group theory factors are expressed. This includes formulas for various contractions of symmetric invariant tensors, formulas and algorithms for the computation of characters and generalized Dynkin indices and trace identities. Tables of all Dynkin indices for all exceptional algebras are presented, as well as all trace identities to order equal to the dual Coxeter number. Further results are available through efficient computer algorithms.

1 Introduction

As the number of loops to which perturbative field theories are evaluated increases, the group[1] structure of the individual diagrams becomes more and more complicated. This problem has been recognized many years ago and on a group-by-group basis some very compact algorithms were proposed [1] for their computation. Especially for the defining and the adjoint representations of the classical groups $SU(N)$, $SO(N)$ and $Sp(N)$ these algorithms can be implemented rather easily in a symbolic program that will then give the color trace of a diagram as a function of the parameter N [2]. The disadvantage of these algorithms is however that these results give no information about group invariants and hence it is only possible for very simple diagrams to generalize the results such that they are valid for arbitrary groups and arbitrary representations. Hence a different type of algorithm is needed, if one would like a more general answer. That such information is useful can be seen from some recent calculations in QCD [3] in which the representation in terms of invariants could show immediately why extrapolations of lower orders in perturbation theory could not be successful. In addition the presentation in terms of group invariants is more general and needs hardly any new work when one needs to apply it for different groups or representations. The need for this kind of generality is clear, for example, from grand unification and string theory, where all semi-simple Lie groups may occur.

We consider non-abelian gauge theories based on simple compact Lie groups. The extension to semi-simple algebras and additional $U(1)$ factors is then straightforward. The gauge bosons are assumed to couple to matter in some irreducible representation R of the gauge group. The generalization to reducible representations is also straightforward. The group-theoretical quantities that appear in the initial expressions are the structure constants f^{abc} (appearing in gauge self-couplings and ghost couplings) and the Lie-algebra generators T_R^a in the representation R, appearing in the coupling of the gauge bosons to matter. In this paper we consider only "vacuum bubbles", *i.e.* diagrams without external lines. As far as the group theoretical factor is concerned, our results are relevant for any diagram whose external lines carry no gauge quantum number, or for the absolute value squared of any amplitude if one sums over the gauge quantum numbers of all external lines. The group theoretical factor of other diagrams can be obtained by multiplying the diagram by projection operators.

The group theory factor of a vacuum bubble diagram consists of traces of a certain number of matrices T_R^a, whose indices are fully contracted among each other and with some combination of structure constants. Our goal is to obtain an expression for this factor that is minimally representation- or group-dependent. In principle, this goal can be achieved as follows.

1. Express the traces in terms of symmetrized traces. This can always be done at the expense of some additional factors f^{abc}

   Now one may simplify the resulting expression further by observing that the structure constants can be viewed as representation matrices in the adjoint representation. This allows us to

2. Eliminate all closed loops of structure constants f^{abc}.

   This amounts to performing step 1. on traces of adjoint matrices T_A^a. Step 2. can also be performed in an algorithmic way to arbitrary order. However, the algorithm is not identical to that of step 1 because of the special properties of the adjoint representation.

[1] Since we are dealing with perturbation theory we only encounter Lie algebras, and we are insensitive to the global properties of the Lie group. Nevertheless, following standard practice, we will often use the word "group" rather than "algebra".

3. Express the symmetrized traces in terms of a standard basis of symmetric invariant tensors. A Lie algebra of rank r has precisely r such tensors [4, 5].

   At this point we have succeeded in expressing every group theory factor in terms of $r+1$ representation-independent quantities, namely the symmetric tensors and f^{abc}. The representation dependence is encapsulated in terms of (generalized) indices [6]. We show how these indices can be computed for any representation of any Lie-algebra to any desired order. This algorithm requires a convenient choice for the basis of tensors, which is *not* the mathematically more elegant "orthogonal" basis advocated in [6]. The result is also to a reasonable extent group-independent. The only way group-dependence enters is trough the (non)-existence of certain invariant tensors, but one may simply take all possible tensors into account, and only eliminate them at the end. The only problem is that the group $SO(4N)$ has *two* distinct tensors of rank $2N$. This case can rather easily be dealt with explicitly.

   Although our main goal has now been achieved, the result is expressed in terms of many combinations of symmetric tensors and structure constants that are not all independent. Unfortunately there do not seem to exist many mathematical results regarding these invariants. In particular, we are not aware of any theorem regarding the minimal number of invariant combinations. For this reason the rest of our program is limited to finite orders, and is not guaranteed to yield the optimal answer in all cases. As a first step we

4. Eliminate as much as possible the remaining structure constants f^{abc}.

   We do not know of a proof that this is always possible, and in fact we have only been able to do this explicitly up to a certain order. The first object where we were unable to perform step 4 is built out of two structure constants and three rank 4 symmetric tensors.

5. If step 4 is completed, one is left with a fully contracted combination of symmetric invariant tensors. We derive formulas expressing many such contractions in terms of a few basic ones.

   The last step is essentially group dependent (and therefore somewhat outside our main interest):

6. Compute a formula for the basic invariants for each group.

   Here "basic invariant" is any of the contracted combinations of symmetric tensors that could not be expressed in terms of others, and any combination involving additional structure constants that could not be eliminated.

None of these steps is new in itself, but we believe that in all cases we are going considerably beyond previous results (see *e.g* [7-25]). Since the application we have in mind is to Feynman diagrams, it is essential not just to develop an algorithm, but also to make sure it can be carried out efficiently. Complicated group theory factors appear only at higher orders in perturbation theory, which implies that one must be able to deal with a very large number of diagrams.

The organization of this paper is as follows. In the next section we give some definitions and conventions, and present some well-known general results on invariant tensors. In section 3 we present the algorithm to perform step 1. Although this is in principle straightforward, without proper care such an algorithm may quickly get out of control. The same is true for step 2, which is presented in section 4. In section 5 we present the character method for computing indices and symmetrized traces. This section is based on results presented in [26] and [6] , the main novelty being the extension to all higher indices of exceptional algebras. In section 6 we discuss steps 4 and 5. Section 7 contains some remarks regarding the advantages and disadvantages of the two basis choices for the symmetric tensors.

In appendix A we present explicit results for indices and trace identities of the exceptional algebras; appendix B contains a description of the computer methods used, and in appendix C we give a few examples to demonstrate the efficiency of the algorithm. Appendix D contains many explicit formulas for invariants (step 6). In appendix E we discuss chiral traces in $SO(2N)$

2 Generalities

2.1 Definitions

We consider simple Lie-algebras whose generators satisfy the commutation relation

$$[T^a, T^b] = if^{abc}T^c \tag{1}$$

Our conventions is to use hermitean generators T^a and to choose the Killing form proportional to δ^{ab}:

$$\text{Tr } T^a T^b \propto \delta^{ab} \tag{2}$$

with a positive and representation dependent proportionality constant that will be fixed later. With this convention the structure constants f_{abc} are real and completely anti-symmetric.

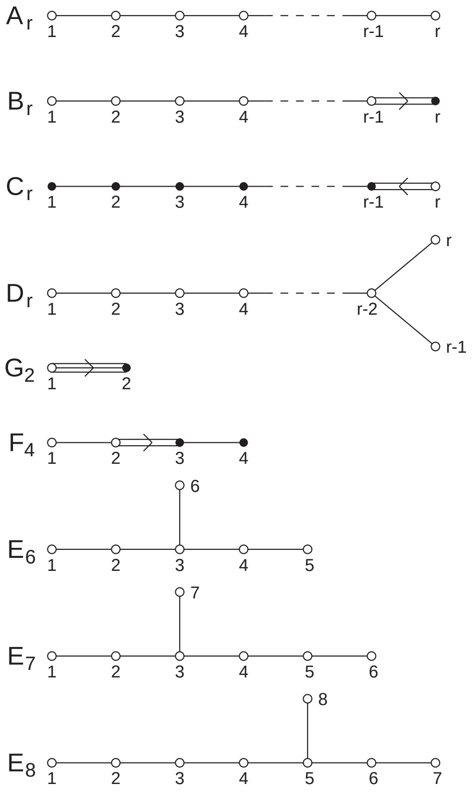


Figure 1: *Dynkin diagrams and labelling conventions*

Representations are denoted either by their dimensions (if no confusion is possible), or by their Dynkin labels $(a_1, \ldots, a_r)$, where r is the rank. Our labelling convention is indicated in fig 1. A bar denotes the complex conjugate representation; "A" denotes the adjoint representation, "R" a generic representation and "r" the reference representation to be defined later. The dimensions of these representations are denoted by N_A, N_R or N_r; the quadratic Casimir eigenvalues (defined more precisely below) by C_A, C_R and C_r respectively.

The generators of the adjoint representation are related to the structure constants

$$(T_A)^a_{bc} = -i\ f^{abc}\ . \tag{3}$$

2.2 Invariant Tensors

We will encounter traces

$$\text{Tr}\ T_R^{a_1} \dots T_R^{a_n} \tag{4}$$

in any representation R of any simple Lie algebra. We wish to express the result in the minimal number of quantities.

Every trace defines an invariant tensor M:

$$\text{Tr}\ T_R^{a_1} \dots T_R^{a_n} = M_R^{a_1 \dots a_n}\ . \tag{5}$$

This tensor is invariant because the trace is invariant under the replacement

$$T_R^a \to U_R T_R^a U_R^{-1} = U_A^{ab} T_R^b\ , \tag{6}$$

where U_R is an element of the group in the representation R; U_A is the same group element in the adjoint representation. Hence we have

$$M_R^{a_1 \dots a_n} = U_A^{a_1 b_1} \dots U_A^{a_n b_n} M_R^{b_1 \dots b_n}\ , \tag{7}$$

which implies, in infinitesimal form

$$\sum_i f^{cba_i} M_R^{a_1 \dots b \dots a_n} = 0\ , \tag{8}$$

with b inserted at position i. This "generalized Jacobi-identity" may be taken as the definition of an invariant tensor.

2.3 Casimir Operators

Every invariant tensor M defines a Casimir operator $C(M)$

$$C_R(M) = \sum_{a_1,\dots,a_n} T_R^{a_1} \dots T_R^{a_n} M^{a_1 \dots a_n}\ . \tag{9}$$

It follows from equation (8) that $C_R(M)$ commutes with all generators T_R in the representation R. If R is irreducible Schur's lemma implies that $C_R(M)$ is constant on the representation space R. Note that this is true independent of the symmetry of M, and irrespective of any concrete definition of M in terms of traces. All that is used is the Jacobi-identity (equation (8)).

Of special interest are the quadratic Casimir operators C_R, which we define as

$$(T_R^a T_R^a)_{ij} = C_R \delta_{ij}\ . \tag{10}$$

As a special case of this identity we can write, using equation (3)

$$f^{acd} f^{bcd} = C_A \delta^{ab}\ . \tag{11}$$

2.4 Symmetrized traces

Not all invariant tensors and Casimir operators constructed so far are independent. We would like to express all traces in terms of a minimal set of invariant tensors. As a first step one may use the commutation relations to express the trace in a completely symmetric trace plus terms of lower order in the generators, which in their turn can also be expressed in terms of symmetrized traces. An efficient algorithm for doing this will be discussed in the next section. After this step we only need to consider symmetrized traces

$$\mathrm{Str}\ T^{a_1} \ldots T^{a_n} \equiv \frac{1}{n!} \sum_{\pi} \mathrm{Tr}\ T^{a_{\pi(1)}} \ldots T^{a_{\pi(n)}} \ , \tag{12}$$

where the sum is over all permutations of the indices (the cyclic permutation may of course be factored out using the cyclic property of the trace).

For each representation one may define a symmetric invariant tensor d_R with

$$d_R^{a_1 \ldots a_n} \equiv \mathrm{Str}\ T_R^{a_1} \ldots T_R^{a_n} \ , \tag{13}$$

but this still vastly overparametrizes the problem, because a new tensor is defined for every order n and for every representation R.

2.5 Basic Casimir invariants

It is well-known that the number of independent symmetric invariant tensors is equal to the rank of the algebra. This can be seen as follows. For each invariant symmetric tensor d of order n define a polynomial

$$P_d(F) = F^{a_1} \ldots F^{a_n} d^{a_1 \ldots a_n} \ , \tag{14}$$

where F^a is a real vector of dimension equal to the dimension of the algebra, N_A. The tensor d can be derived from $P_d(F)$ by differentiating with respect to F:

$$d^{a_1 \ldots a_n} = \frac{1}{n!} \frac{\partial}{\partial F^{a_1}} \cdots \frac{\partial}{\partial F^{a_n}} P_d(F) \ . \tag{15}$$

Although *a priori* $P_d(F)$ is a polynomial in N_A variables, the fact that d is an invariant tensor implies that P_d depends in fact only on r variables, where r is the rank of the algebra. This is true because the polynomial is invariant under

$$F^a \to U_A^{ac} F^c \ , \tag{16}$$

and it is well-known that for every F^a one can find a transformation U_A that rotates F^a into the Cartan subalgebra. Hence $P_d(F)$ depends only on as many parameters as the dimension of the Cartan subalgebra, *i.e.* r. Therefore it is not surprising that any such polynomial can be expressed in terms of r basic ones, although the precise details (*e.g.* the orders of the basic polynomials) don't follow from this simple argument.

The orders of the basic polynomials for each group are known [4, 5], and are given in the following table (for future purposes this table also gives the "dual Coxeter number" g). As explained above, each basic polynomial corresponds to an invariant tensor, which in its turn corresponds to a Casimir invariant. The implication of table 1 is that for any given algebra the polynomials

$$\mathrm{Tr}\ F_R^n \equiv \mathrm{Tr}\ (\sum_a F^a T_R^a)^n \tag{17}$$

Algebra	g	Invariant tensor ranks
A_r	$r+1$	$2,3,4,\ldots,r,r+1$
B_r	$2r-1$	$2,4,6,\ldots,2r$
C_r	$r+1$	$2,4,6,\ldots,2r$
D_r	$2r-2$	$2,4,6,\ldots,2r-2;r$
G_2	4	$2,6$
F_4	9	$2,6,8,12$
E_6	12	$2,5,6,8,9,12$
E_7	18	$2,6,8,10,12,14,18$
E_8	30	$2,8,12,14,18,20,24,30$

Table 1: *Ranks of basic invariant tensors*

can be expressed in terms of r basic polynomials of degrees as indicated above. In section 5 we will show how to obtain such expressions for any irreducible representation (irrep) of any (semi)-simple Lie algebra. The ranks of the invariant tensors – or more accurately those[2] that can be written as traces over some representation R – are in fact an outcome of these calculations.

If we can express a polynomial corresponding to some invariant tensor d in terms of basic polynomials, we can also express the invariant tensors into basic ones. Namely, suppose

$$P_d(F) = \sum \prod_i P_{d_i}(F) \tag{18}$$

where the sum is over various terms of this type, with coefficients. Then the differentiation (eq.15) of a term on the right hand side yields precisely the fully symmetrized combination of the tensors d_i, with weight 1; this means that the overall combinatorial factor equals the number of terms. For example

$$\frac{1}{4!}\frac{\partial}{\partial F^a}\frac{\partial}{\partial F^b}\frac{\partial}{\partial F^c}\frac{\partial}{\partial F^d}(\sum_e (F^e)^2)^2 = \frac{1}{3}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}) \ . \tag{19}$$

In practice we will express all higher traces in terms of r basic ones, but we do not obtain the full dependence on F of the basic traces, and consequently we cannot say anything about the explicit form of the basic invariant tensors. Given an explicit basis for the Lie-algebra one may compute the full F-dependence, but that is the same as computing the invariant tensor directly by computing a trace.

2.6 Indices

Since a Casimir operator is constant on a irrep, its value can be computed by taking the symmetrized trace over this irrep. We will see explicitly how to expand a trace in terms of fundamental symmetric invariant tensors. In general one has

$$\text{Str } T_R^{a_1}\ldots T_R^{a_n} = I_n(R) d^{a_1\ldots a_n} + \text{products of lower orders} \ . \tag{20}$$

The invariant tensors can be chosen in some representation-independent way, for example by computing it for one given reference representation. Then all symmetrized traces can be expressed in terms of this basis of tensors. The leading term necessarily has the indicated form, with a computable representation dependent coefficient $I_n(R)$. This coefficient is called the n^{th} index of

[2] We are not aware of others, but also not of a proof that they do not exist.

the representation. If there is no fundamental invariant tensor of order n the indices $I_n(R)$ are obviously zero for any representation.[3]

The extra terms in equation (20) are symmetrized products of lower order tensors such that the total order is n, without contracted indices. The coefficients of these terms will be called *sub-indices.*

There is a lot of freedom in defining $d^{a_1 \ldots a_n}$ since we could have modified it by any combination of lower order terms in equation (20). Note that modifying the tensors by lower order terms does not affect the indices, but does change the sub-indices. This freedom can be used to impose the conditions [6, 24]

$$d_{\perp}^{a_1 \ldots a_l \ldots a_n} d_{\perp}^{a_1 \ldots a_l} = 0 \quad l < n \tag{21}$$

This then defines the symmetrized tensors up to an overall normalization. The normalization can be fixed by fixing a normalization for the indices. This basis will be referred to as the *orthogonal basis.* It is the most elegant one from a mathematical point of view, but, as we will see, not the most convenient one for our purposes. In the following we will use the notation $d_{\perp}^{a_1 \ldots a_l \ldots a_n}$ for tensors in the orthogonal basis.

As mentioned before, tensors defined in any basis can be used to define Casimir invariants, but using the orthogonal basis has a clear advantage because it leads to a simple relation with the indices (N_R is the dimension of R) :

$$I_n(R) = \frac{N_R}{\mathcal{N}_n} C_p(R) \ , \tag{22}$$

where

$$\mathcal{N}_n = d_{\perp}^{a_1 \ldots a_n} d_{\perp}^{a_1 \ldots a_n} \ . \tag{23}$$

This relation holds because when contracting with an orthogonal tensor only the leading terms survive. Note that this is true even if we do not expand equation (20) in terms of the orthogonal basis, but in terms of any other basis. Hence the Casimir eigenvalues are determined up to a representation independent factor once the indices are known.

The indices are of interest in their own right, as was in particular emphasized in [6]. In some cases they have a topological interpretation via index theorems. Furthermore they satisfy a useful tensor product sum rule. If

$$R_1 \otimes R_1 = \sum_i \oplus R_i \tag{24}$$

then

$$N_{R_1} I_p(R_2) + N_{R_2} I_p(R_1) = \sum_i I_p(R_i) \ . \tag{25}$$

In subgroup embeddings $H \subset G$ there is also such a sum rule for branching rules: if $R \to \sum_i \oplus r_i$ then

$$I_p^G(R) = I_p(G/H) \sum_i I_p^H(r_i) \ , \tag{26}$$

where $I_p(G/H)$ is the embedding index (here we assume that both G and H have precisely one index of order p; other cases require just slightly more discussion).

[3] For D_r, r even, there are *two* indices of order r. The additional one will be denoted as $\tilde{I}_r$. We will deal with this case in more detail below.

2.7 Normalization

To arrive at a universal normalization we make use of the following general formula for the quadratic Casimir invariant

$$C_R = \frac{\eta}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(a_i+2)G_{ij}a_j \tag{27}$$

Here a_i are the Dynkin labels of the representation R, and G_{ij} is the inverse Cartan matrix. The advantage of this formula is that the Dynkin labels as well as the Cartan matrix have a fixed normalization that is not subject to conventions. Only the overall normalization is convention dependent. The factor η is introduced to allow the reader to fix the normalization according to taste. The dependence on η will be shown explicitly in all formulas. Given the universality of (27) it is natural to choose η in a group-independent way. Using (27) get for C_A:

$$C_A = \eta g \ ,$$

where g is the dual Coxeter number.

This convention defines the normalization of the generators once we have fixed the rank 2 symmetric tensor. The natural definition is

$$d^{ab} = d_{\perp}^{ab} = \delta^{ab}.$$

Then $\mathcal{N}_2 \equiv d_{\perp}^{ab} d_{\perp}^{ab} = N_A$. Now the second index $I_2(R)$ is also fixed via (22):

$$I_2(R) = \frac{N_R}{N_A} C_R \ .$$

For the vector representations V of the classical Lie algebras we find then $I_2(V) = \frac{1}{2}\eta$ for $SU(N)$ and $Sp(N)$, and $I_2(V) = \eta$ for $SO(N)$.

There are (at least) two considerations that might lead to a choice for η. First of all it is possible to fix the conventions in such a way that $I_2(R)$ is always an integer. This leads to the choice $\eta = 2$. On the other hand there are standard choices for the generators of $SU(2)$ namely $T^a = \frac{1}{2}\sigma^a$ (where σ^a are the Pauli-matrices), and for $SO(N)$, namely $T^{\mu\nu}_{ij} = i(\delta^{\mu}_i \delta^{\nu}_j - \delta^{\mu}_j \delta^{\nu}_i)$, where the pair $\mu\nu$ with $\mu < \nu$ represents an adjoint index. Unfortunately these two choices correspond to different values of η, namely $\eta = 1$ for $SU(2)$ and $\eta = 2$ for $SO(N)$.

2.8 Indices versus Casimir invariants

We conclude this section with a few historical remarks.

A vast amount of literature exists on the computation and properties of Casimir invariants. Most of these papers, [7-19], give more or less explicit expressions for the Casimir eigenvalues of the classical Lie Algebras A_n, B_n, C_n and D_n and in one case, [17], also for G_2. In [22, 23] formulas for G_2 and F_4 are obtained, whereas E_8 was considered, up to order 14, in [25]. The issue of completeness of a set of Casimir operators was studied in [20, 21].

In applications to Feynman diagrams indices are more important than Casimir invariants, because traces over matter loops yield indices and sub-indices, and not Casimir invariants. Indices have been discussed most frequently in relation to chiral anomalies. The second index was introduced by Dynkin, [27], and generalized to higher order in [28]. Shortly afterwards [6] it was realized that the definition of the indices could be improved by imposing the orthogonality constraint (21). In [6] formulas are given for the indices of classical Lie algebras. Indices of exceptional algebras have been studied up to sixth order, mainly for the purpose of anomaly cancellation in ten dimensions, relevant for string theory. Sub-indices, when defined in the orthogonal basis, can be expressed in terms of indices. Unfortunately these relations are difficult to obtain, and become very complicated at higher orders unless some lower indices vanish. In [6] formulas for $SU(n)$ have been given up to fifth order. We have computed the sixth order formula, but the result is rather awkward and does not encourage extension to higher orders.

Although, as explained above, indices are closely related to Casimir invariants, the available formulas for the latter are of little use to us since they do not use orthogonal tensors for the definition of the Casimirs. Even if they did, one would still need the normalization factor $\mathcal{N}_n$, (23). The computation of this factor for all Lie-groups and all values of n is a difficult problem, related to the even more difficult problem of determining the tensors $d_\perp^{a_1 \cdots a_n}$ explicitly. For recent progress on the latter problem for the classical Lie algebras see [24]. We do not present explicit expressions for the symmetric tensors here. Since they always appear in contracted form, we never need them explicitly.

Furthermore the Casimir eigenvalues give no information on sub-indices.

3 Reduction to symmetrized traces

In this section we will discuss the reduction of traces as they occur in Feynman diagrams into the invariants of the previous section. This is by no means a trivial affair because the necessary symmetrizations make that the algorithms typically involve $\mathcal{O}(n!)$ terms when there are n generators in the trace. It is therefore important to choose the method carefully. We will have to distinguish two cases. In this section we will make the reduction of traces of the type Str $T_R^{a_1} \dots T_R^{a_n}$ in which R can be any representation with the exception of the adjoint representation. In the next section we will consider such traces for the adjoint representation. The special role of the adjoint representation lies in the fact that, because of the equation (3), the commutation relation

$$[T_A^a, T_A^b] = i\ f^{abc} T_A^c \tag{28}$$

does not really diminish the number of generators of the adjoint representation. It is actually just a different way of writing the Jacobi identity. A related reason for considering the adjoint representation separately is that the reduction of the other traces generates new structure constants.

We will continuously keep in mind that the algorithms we derive are for implementation in a symbolic computer program. This means that in many cases a recursion type algorithm may suffice, even though it may not be very practical for hand calculations.

3.1 First stage elimination

The first part of the reduction is dedicated to the replacement of the traces over the generators T_R by the invariants d_R. For all representations except for the adjoint this can be done in a general algorithm. One should realize however that for very complicated traces the results may not be very short.

In general a trace is not symmetrized. Therefore the introduction of the tensors d_R needs some work with commutation relations to make it symmetric. On the other hand, computer algebra needs algorithms that work from a formula, rather than towards one. Hence one can use the substitution

$$\mathrm{Tr}[T_R^{a_1} \cdots T_R^{a_n}] \quad = \quad \mathrm{Tr}[T_R^{a_1} \cdots T_R^{a_n}] - \mathrm{Str}\ T_R^{a_1} \cdots T_R^{a_n} + d_R^{a_1 \cdots a_n} \tag{29}$$

Writing out the symmetrized trace will of course give $n!$ terms, each with a factor $1/n!$. Then we can commute the various $T_R^{a_i} T_R^{a_j}$ till they are all in the order of the original trace after which the $n!$

terms with n generators will cancel the original trace. At this point we are left with the symmetric tensor d_R and $\mathcal{O}(n\ n!)$ terms which all have $n-1$ generators. As a recursion it will eventually result in terms with only two generators for which we know the trace. This algorithm is however rather costly when the number of generators inside the trace is large.

The above formula has as its main benefit that it proves that one can express a trace of generators T of any representation $R \neq A$ in terms of symmetrized traces and structure constants f. For practical purposes we have a better algorithm. It is based on the formula:

$$T_R^{\{a_1} \cdots T_R^{a_n\}} T_R^b \quad = \quad \sum_{j=0}^{n} \frac{(-1)^j}{j!} B_j E_j^{a_1 \cdots a_n b} \tag{30}$$

in which B_j is the j-th Bernoulli number. ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$ and $B_4 = -1/30$) and the function E_j is defined by the recursion

$$E_0^{a_1 \cdots a_n} \quad = \quad T_R^{\{a_1} \cdots T_R^{a_n\}} \tag{31}$$

$$E_j^{a_1 \cdots a_n b} \quad = \quad \sum_{i=1}^{n} E_{j-1}^{a_1 \cdots a_{i-1} a_{i+1} \cdots a_n k}\ i f^{a_i b k} \tag{32}$$

Basically in the E_j one extracts a string of j structure constants f. By writing out the functions E_j one can show with some work that the proof of this formula is equivalent to proving the relation

$$\begin{aligned} R_0 \quad = \quad & \frac{1}{n+1} B_0 \sum_{i=0}^{n} R_i \\ & + \sum_{i=1}^{n} \frac{n!}{(n+1-i)!} \frac{(-1)^i}{i!} B_i \sum_{j=0}^{i-1} (-1)^j \frac{(i-1)!}{j!(i-1-j)!} (R_j + (-1)^i R_{n-j}) \end{aligned} \tag{33}$$

This equation should hold for any positive value of n and any choice of the R_i (The R_i represent the symmetric combination of T^{i_1} to T^{i_n} with a T^b inserted at i places from the right. Hence R_0 is the left hand side term of equation (30)). Actually the R_i are independent objects and hence we have a set of equations each of which is characterized by a value for n and the index j of R_j. To prove the whole formula we have to prove that all of these equations are valid. We use the following approach: One can test their validity for any small value of n and all allowed values of j (computer algebra lets one check this easily up to $n = 100$). Next one takes the special case of only R_0 not equal to zero and the case of only R_n not equal to zero. These are rather easy to prove. Then one takes the case for R_1 not equal to zero. This is only a little bit more complicated. Finally one can express the case for other values of n and j in terms of the equation for $n-1, j-1$ and $n, j-1$. This then combines into a proper induction proof. The above formula can give some interesting summations when one selects special values for the R_i like $R_i = x^i$ after which the inner sum can be done.

An important part in the application of equation (30) is how to terminate the recursion. We note that due to the cyclic property of traces

$$\mathrm{Tr}\left[T_R^{\{a_1} \cdots T_R^{a_n\}} T_R^b\right] \quad = \quad d_R^{a_1 \cdots a_n b} \tag{34}$$

Additionally we can terminate two cases in which there are still two generators outside the symmetrization:

$$\mathrm{Tr}\left[T_R^{a_1} T_R^{a_2} T_R^{a_3}\right] \quad = \quad d_R^{a_1 a_2 a_3} + \frac{i}{2} f^{a_1 a_2 a_3} I_2(R) \tag{35}$$

$$\begin{aligned}
\mathrm{Tr}\left[T_R^{\{a_1}T_R^{a_2\}}T_R^{a_3}T_R^{a_4}\right] &= d_R^{a_1a_2a_3a_4}+\frac{i}{2}f^{a_3a_4k}d_R^{a_1a_2k}\\
&\quad +\frac{1}{12}I_2(R)(f^{a_1a_3k}f^{a_2a_4k}+f^{a_1a_4k}f^{a_2a_3k})
\end{aligned} \tag{36}$$

All together this algorithm is far superior. For a trace of 7 generators T_R all with different indices it is about 50 times faster than the one based on equation (29).

Just as with traces of γ-matrices one can make the algorithms much faster with a number of supporting tricks for contracted indices. This is however not quite as simple as with γ-matrices. When the contracted indices are close to each other, we use

$$T_R^bT_R^aT_R^aT_R^c = C_RT_R^bT_R^c \tag{37}$$

$$T_R^aT_R^bT_R^a = (C_R-\frac{1}{2}C_A)T_R^b\ . \tag{38}$$

The relation between $I_2(R)$ and C_R is given by $N_RC_R=I_2(R)N_A$. Additionally we have

$$T_R^{a_1}T_R^{a_2}T_R^b\cdots T_R^cT_R^{a_1}T_R^{a_2} = T_R^{a_1}T_R^{a_2}T_R^b\cdots T_R^cT_R^{a_2}T_R^{a_1}-\frac{1}{2}C_AT_R^{a_1}T_R^b\cdots T_R^cT_R^{a_1} \tag{39}$$

And then for contracted indices that are not very close to each other we can use:

$$\begin{aligned}
T_R^j\cdots T_R^aT_R^bT_R^j &= T_R^j\cdots T_R^aT_R^jT_R^b+if^{bjc}T_R^j\cdots T_R^aT_R^c\\
&= T_R^j\cdots T_R^aT_R^jT_R^b+if^{bjc}T_R^j\cdots T_R^cT_R^a-f^{bjc}f^{acd}T_R^j\cdots T_R^d
\end{aligned} \tag{40}$$

We commute the T_R^c matrix also towards the T_R^j matrix because

$$\begin{aligned}
if^{bjc}T_R^jT_R^c &= \frac{i}{2}f^{bjc}(T_R^jT_R^c-T_R^cT_R^j)\\
&= -\frac{1}{2}f^{bjc}f^{jcd}T_R^d\\
&= -\frac{1}{2}C_AT_R^b
\end{aligned} \tag{41}$$

Hence we can always eliminate two generators T_R when there is a pair of contracted indices. Some terms may however obtain two f-matrices in exchange. The above equation shows also that the trace of a string of generators of which two generators are contracted with the same structure constant f, will lead to simplifications by commuting the two generators towards each other. In this case many of the terms that come from a commutator have just one generator fewer than the original term.

By now it should be clear why the adjoint representation cannot be treated in exactly the same way. Eliminating two generators at the cost of introducing two structure constants f leaves us with exactly the same number of generators of the adjoint representation.

We can use a few extra shortcuts for simple cases to avoid the use of equation (30) in those cases:

$$\begin{aligned}
\mathrm{Tr}[T_R^{a_1}T_R^{a_2}T_R^{a_3}] &= d_R^{a_1a_2a_3}+\frac{i}{2}I_2(R)f^{a_1a_2a_3}\\
\mathrm{Tr}[T_R^{a_1}T_R^{a_2}T_R^{a_3}T_R^{a_4}] &= d_R^{a_1a_2a_3a_4}+\frac{i}{2}(d_R^{a_1a_4n}f^{a_2a_3n}-d_R^{a_2a_3n}f^{a_1a_4n})\\
&\quad +\frac{1}{6}I_2(R)(f^{a_1a_4n}f^{a_2a_3n}-f^{a_1a_2n}f^{a_3a_4n})
\end{aligned} \tag{42}$$

To get the last equation into its minimal form we have used the Jacobi identities:

$$\begin{aligned} 0 &= f^{i_1 i_2 j} f^{i_3 i_4 j} + f^{i_2 i_3 j} f^{i_1 i_4 j} + f^{i_3 i_1 j} f^{i_2 i_4 j} \\ 0 &= d^{i_1 i_2 j} f^{i_3 i_4 j} + d^{i_2 i_3 j} f^{i_1 i_4 j} + d^{i_3 i_1 j} f^{i_2 i_4 j} \end{aligned} \tag{43}$$

It is possible to create similar shortcuts for the higher traces. This serves however not much purpose. The majority of cases involves short traces and these expressions are rather lengthy.

4 Reduction of adjoint traces

At this stage we have only tensors of the type d_R and structure constants f left as objects with indices. All these indices are indices in the adjoint space and hence all have N_A dimensions. Additionally there can be various constants like the second order Casimir's and the second order indices $I_2(R)$, but these do not play a role in the following.

For the other representations the loops that define the trace were rather easy to find. For the adjoint representation this is more complicated: all three indices of the structure constant f can play a role and hence there are more possibilities. The advantage is that very often one can find 'smaller' loops. If for instance we have a diagram that consists of only vertices in the adjoint representation and there are no loose ends we have the results of table 2 in which 'girth' is the size

girth	2	3	4	5	6	7	8	9
$n_{\min}$	2	4	6	10	14	24	30	54

Table 2: *Minimum number of vertices needed for a diagram with a given smallest loop.*

of the smallest loop in the diagram and $n_{\min}$ is the minimal number of vertices needed to construct such a diagram. Of course in mixed diagrams in which other representations are involved one can get loops in the adjoint representation that have up to $n/2$ vertices if n is the total number of vertices in the diagram. Such would be the case of there is one loop of $n/2$ vertices in a representation R and a parallel loop of adjoint vertices. But in that case the loop is easy to find, and the symmetry of the invariant d_R that is present already makes the introduction of the d_A into a triviality: due to its contraction with d_R the trace over the adjoint generators has already been symmetrized.

Let us first have a look at the canonical reduction algorithm of equation (29). At first one might be worried that it will not terminate for the adjoint representation. After all it does not diminish the number of structure constants f. One can see quickly however that the commutator terms have a simpler loop structure. Hence each term will end up having a number of f's grouped into an invariant or have a loop with fewer f's even though the total number of f's is still the same. Once we have a loop with at most three structure constants it can be reduced with the equation

$$f^{i_1 i_2 a_1} f^{i_2 i_3 a_2} f^{i_3 i_1 a_3} = \frac{1}{2} C_A f^{a_1 a_2 a_3} \tag{44}$$

which can be derived from the Jacobi identity. Hence in all cases the number of structure constants f will become less. Similarly the algorithm of equation (30) will reduce a number of generators of the adjoint representation to a symmetrized trace and a smaller number of structure constants f. Hence also this algorithm will terminate.

It is however possible to be more efficient about the reductions inside the adjoint representation. Our first observation is that for loops with an odd number of vertices in the adjoint representation reverting the order of the vertices gives a minus sign. Hence the fully symmetric object with an odd number of indices must be zero. And because of this we do not need a full symmetrization to express loops with an odd number of indices. For example for five indices we write

$$\begin{aligned} F^{i_1i_2i_3i_4i_5} &= -F^{i_5i_4i_3i_2i_1} \\ &= -F^{i_1i_5i_4i_3i_2} \\ &= -F^{i_1i_2i_3i_4i_5} + F^{ki_4i_3i_2} f^{i_1i_5k} \\ &\quad +F^{i_1i_4ki_5} f^{i_3i_2k} + F^{i_1i_3ki_5} f^{i_4i_2k} + F^{i_1i_2ki_5} f^{i_4i_3k} \end{aligned} \tag{45}$$

in which F represents the trace over a number of f's. Its difference with a trace over the T_A-generators (3) is just powers of i. Similarly one can use the reversal symmetry for traces of an even number of f's to reduce the amount of work by a factor two.

Because of the above simplifications the efficiency of the two algorithms ((29) and (30)) is not very different for the adjoint representation.

The reduction algorithm should be clear now. One looks for the smallest loop among the various f's. Such a loop will not have contracted indices, because otherwise there would be a smaller loop. Hence we do not have to worry about contracted indices as we had to do for the other representations. If the loop has only two or three f's, we can eliminate it with either equation (11) or equation (44). Otherwise we can use a simplified version of the canonical reduction algorithm of equation (29) to obtain an invariant and terms with a simpler loop. Actually the fastest way here is to tabulate this reduction all the way up to loops with 7 f's. For loops of 8 or more vertices in the adjoint representation we use an adapted version of equation (30).

5 Computation of symmetrized traces

At this point our group theory factors consist of combinations of structure constants, symmetrized traces $d_A^{a_1,\ldots,a_n}$ over the adjoint representation, and symmetrized traces over one or more other irreducible representations. We will now show how such traces can be expressed in terms of r traces over a single representation, where r is the rank of the algebra.

As explained in section 2, in principle there are three quantities one might be interested in: Casimir invariants, indices and symmetric tensors.

The results presented here amount to a computation of the coefficients of combinations of fundamental traces appearing in the expansion of a trace in an arbitrary representation. In other words, we compute indices and sub-indices (but, as explained in section 2, the latter are basis-choice dependent).

With our method these quantities can rather straightforwardly be computed to any desired order, and for any representation of any Lie algebra. To demonstrate this we will compute all the indices for the lowest-dimensional representations of the exceptional algebras, including the 30$^{\text{th}}$ index of E_8.

The method we follow here is an extension of results of [26] (where it was used to obtain the "elliptic genus" in string theory), which in its turn was an extension of results presented in [6] (where it was used for computing the indices of the classical algebras).

5.1 Characters

An extremely useful tool for computing traces are the characters

$$\mathrm{Ch}_R(F) = \mathrm{Tr}\; e^{F_R} \tag{46}$$

where $F_R = F^a T^a_R$. Hence the expansion of the exponential gives us all symmetrized traces in terms of the polynomials defined in equation (14). What makes the characters especially useful is their tensor property

$$\mathrm{Ch}_{R_1 \otimes R_1}(F) = \mathrm{Ch}_{R_1}(F)\mathrm{Ch}_{R_2}(F) \ , \tag{47}$$

which follows directly from its definition. In addition characters are combinations of traces and therefore also have nice properties on direct sums

$$\mathrm{Ch}_{R_1 \oplus R_1}(F) = \mathrm{Ch}_{R_1}(F) + \mathrm{Ch}_{R_2}(F) \ . \tag{48}$$

With a little more effort one can also derive a formula for characters of symmetrized and anti-symmetrized tensor products [29]. These formulas can be derived from the following generating functions

$$\sum_{k=0}^{\infty} x^k \mathrm{Ch}_{[k]*R}(F) = \det(1 + x e^F_R) = \prod_{l=1}^{\infty} \exp(-(-x)^l \mathrm{Ch}_R(lF)) \tag{49}$$

$$\sum_{k=0}^{\infty} x^k \mathrm{Ch}_{(k)*R}(F) = \det(1 - x e^F_R)^{-1} = \prod_{l=1}^{\infty} \exp((x)^l \mathrm{Ch}_R(lF)) \ . \tag{50}$$

Here $[k]$ denotes the order k anti-symmetric tensor product of some representation R, and (k) the order k symmetric product. We use the notation $[k] * R$ or $(k) * R$ to denote the anti-symmetrized or symmetrized tensor product of the representation R. Note that the sum in equation (49) is in fact always finite.

The generating functions can be expanded explicitly to obtain

$$\mathrm{Ch}_{[k]*R}(F) = -\sum_{\substack{\{n_i, m_i\} \\ k = n_i m_i}} \prod_i \frac{1}{m_i!} \left(-\frac{\mathrm{Ch}_R(n_i F)}{n_i} \right)^{m_i} \tag{51}$$

$$\mathrm{Ch}_{(k)*R} = \sum_{\substack{n_i, m_i \\ k = n_i m_i}} \prod_i \frac{1}{m_i!} \left(\frac{\mathrm{Ch}_R(n_i F)}{n_i} \right)^{m_i} \ , \tag{52}$$

where the sum is over all partitions of the integer k into different integers n_i, each appearing with multiplicity m_i.

5.2 Character computation method

Our method for computing the characters is as follows. We begin by choosing a reference representation, which in all cases is the one of smallest dimension. The reference representations we choose for the simple Lie algebras are shown in table 3. (the last column of this table is explained later) Note that for $SO(N)$ the reference representation is the vector representation for $N \geq 7$, but for lower values of N it is a spinor representation. Another way of saying this is that we treat $SO(N), N \leq 6$ according to the Lie-algebra isomorphisms $D_3 \sim A_3$, $B_2 \sim C_2$ and $B_1 \sim A_1$. The last column of table 3 is discussed below. The algebras B_3, B_4, D_4 and D_5 are listed separately because, although they have the "standard" reference representation, they have non-standard index normalizations.

For the reference representation the character is left in the form

$$\mathrm{Ch}(F) = \mathrm{Tr}\ e^F \ , \tag{53}$$

All traces whose order does not appear in table 1 can be expressed in terms of lower traces. Hence equation (53) must be supplemented by trace identities for those traces. These trace identities will

Algebra	Reference representation	Dimension	Indices
A_r	$(1,0,\ldots,0)$	$r+1$	$1,\ldots,1$
B_1	(1)	2	1
B_2	(0,1)	4	1,1
B_3	(1,0,0)	7	2,2,1
B_4	(1,0,0,0)	9	2,1,1,2
$B_r,\ \ r \geq 5$	$(1,0,\ldots,0)$	$2r+1$	$2,1,\ldots,1$
C_r	$(1,0,\ldots,0)$	$2r$	$1,\ldots,1$
D_3	$(0,0,1)$	4	1,1,1
D_4	$(1,0,0,0)$	8	2,2,1,0
D_5	$(1,0,0,0,0)$	10	2,1,1,2,0
$D_r,\ \ r \geq 3$	$(1,0,\ldots,0)$	$2r$	$2,1,\ldots,1,0$
G_2	$(0,1)$	7	2,1
F_4	$(0,0,0,1)$	26	6,1,1,1
E_6	$(1,0,0,0,0,0)$	27	6,1,1,1,1,1
E_7	$(0,0,0,0,0,1,0)$	56	12,1,1,1,1,29,1229
E_8	$(1,0,0,0,0,0,0,0)$	248	60,1,1,1,1,41,199,61

Table 3: *Reference representations, dimensions and indices.*

be derived below. The remaining traces will be called "fundamental" and equation (53) is taken to be the definition of the corresponding polynomials and symmetric tensors. This then defines a set of reference tensors:

$$d_r^{a_1\ldots,a_n} = \text{Str } T_r^{a_1} \ldots T_r^{a_n} \tag{54}$$

where n is the order of a fundamental Casimir operator. The precise form of this tensor, or equivalently the precise form of the fundamental polynomials Tr F^n depends on the details of the Lie algebra basis choice, but will never be needed.

Any other character is now written in terms of traces of F^n over the reference representation, using all available trace identities. By differentiating with respect to F (*c.f.* eq. (19)) one can then read off the expression of any d_R in terms of reference tensors.

This fails if the reference representation has an index that is zero. This happens only for the n^{th} index of the algebra D_n, and we will deal with that case separately.

The next step is to express the characters of all "basic" representations in terms of the reference character. The i^{th} basic representation is defined by Dynkin labels $a_j = \delta_{ij},\ j = 1,\ldots,r$. The most important tool for obtaining these characters is equation (51). This yields all fundamental representations of the algebras A_r and C_r, whereas for the orthogonal groups only the spinor representations are still missing. The spinors, as well as the basic representations of the exceptional algebras, require some extra work, and are discussed below.

Finally one can compute the characters of all other representations by using in a systematic way the sum rule for tensor products. It can be proved that for any simple Lie-algebra this allows one to relate the characters of all other irreps linearly to those of the basic ones. In principle this still allows for the possibility that complicated linear equations need to be solved. We find however, that one can organize the tensor products in such a way that only one unknown character appears at every step. This can be proved for the classical Lie algebras (see below), and we have checked it empirically for the exceptional ones.

Let us contrast this procedure with the computation via Weyl's character formula

$$\mathrm{Ch}_\Lambda(h) = \frac{\sum_{w\in W} \epsilon_w \exp(w(\Lambda+\rho), h)}{\sum_{w\in W} \epsilon_w \exp(w(\rho), h)} \tag{55}$$

where Λ is the highest weight of a representation, the summation is over all elements w in the Weyl group W, ϵ_w is the determinant of w, ρ is the Weyl vector (with Dynkin labels all equal to 1), and h is a vector in weight space, which plays the rôle of F in the foregoing discussion. One obvious disadvantage of this formula is the summation over all elements of the Weyl group, although this is still manageable in most cases of interest. A less obvious disadvantage is that numerator and denominator both have a zero of order N_+, the number of positive roots, in h. For example, to obtain the highest non-trivial Casimir eigenvalue of E_8, which is of order 30, one needs to expand numerator and denominator to order $N_+ + 30 = 150$. This is an impossible task. The method sketched above, and worked out below, *does* allow an expansion of the character to order 30, even for E_8.

An important ingredient in our procedure is obviously the computation of tensor products. Conceptually this is certainly not easier than the computation of characters, but nowadays computer programs exist that can do this very efficiently.[4] Rather than using characters (and in particular index sum rules) to compute tensor products, it is then more efficient to use tensor products to compute characters. The procedure described here requires just a small effort to compute the characters of the basic representations up to a certain desired order. The computation of the character of any other representation is then just a matter of simply polynomial operations (multiplications, additions and subtractions which can be efficiently performed by any symbolic manipulation program, such as `FORM`) guided by the output of a program that computes tensor products.[5]

We will now discuss the various types of algebras in more detail.

5.3 A_r characters

Let us now apply these tools first of all to Lie algebras of type A_r ($SU(r+1)$). For the reference representation we choose the vector representation $(r+1)$. Using equation (49) we can then immediately write down the characters for all the anti-symmetric tensor product representations $[k]$. In terms of Dynkin labels these are all the representations with labels $(0,\ldots,0,1,0,\ldots,0)$, *i.e.* a single entry 1. These are precisely the basic representations.

Now we can systematically use the tensor product rule (47) and the sum rule (48) to obtain character formulas for all other irreducible representations. If

$$R_1 \otimes R_2 = \sum_i \oplus n_i R_i \tag{56}$$

then

$$\mathrm{Ch}_{R_1}(F)\mathrm{Ch}_{R_2}(F) = \sum_i n_i \mathrm{Ch}_{R_i}(F) \tag{57}$$

By computing the product of two known characters and subtracting the known characters on the right hand side one is left with the character of some (in general reducible) representation, which is thereby determined.

[4]We have used the programs `LiE` [30] that computes tensor products of Lie-algebra representations directly and `Kac` [31] that uses the Verlinde formula to compute fusion rules of Kac-Moody algebras. A subset of these fusion rules coincides with tensor product rules, and it turns out that this precisely includes the tensor product rules we need.

[5]We have implemented this idea in the program `Kac`. The results in the appendix were produced in that way. At a given Kac-Moody level, the fusion rules that coincide with tensor products are found to be sufficient to obtain the characters of all representations at that level. For examples and software see http://norma.nikhef.nl/pub/~t58

To show how this works we label the $SU(r+1)$ irreps by Young tableaux and assign a partial ordering to them. We use Young tableaux because for A_r they provide a convenient description of the tensor product rule. A Young tableau is ordered above another one if it has more columns; if the number of columns is the same the one with the largest last column is ordered above the other one. Suppose now that we know the characters of all representations ordered below a representation R with Young tableau $[k_1,\ldots,k_l]$. Consider then the tensor product $[k_1,\ldots,k_{l-1}]\otimes[k_l]$. Both are ordered below R and hence their characters are known according to our assumption. The tensor product yields $[k_1,\ldots,k_l]$ plus representations ordered below $[k_1,\ldots,k_l]$, and hence we can now determine the character of R. Proceeding like this we can systematically compute all characters.

Not only the characters, but also the trace identities for A_r were obtained in [26]

$$\sum_{\substack{\{n_i,m_i\}\\ k=n_i m_i}} \prod_i \frac{1}{m_i!}\left(-\frac{\mathrm{Tr}\ F^{n_i}}{n_i}\right)^{m_i} = 0 \quad (k > r+1)\ , \tag{58}$$

where the summation is as in equation (51). This result was obtained from equation (51) using the fact that for A_r anti-symmetric tensors of rank larger than the rank of the algebra are trivial.

To illustrate this let us return to the Weyl formula, equation(55). For A_1 this yields a very simple result for a representation of spin j:

$$\mathrm{Ch}_j(h) = \frac{\sinh((2j+1)h)}{\sinh(h)} \tag{59}$$

Expanding this for the spin-$\frac{1}{2}$ representation ($j=\frac{1}{2}$) we get

$$\mathrm{Ch}_{\frac{1}{2}}(h) = 2 + h^2 + \frac{1}{12}h^4 + \frac{1}{360}h^6 + \ldots \tag{60}$$

The spin-$\frac{1}{2}$ representation serves as the reference representation in our method. Hence its character is

$$\mathrm{Tr}\ e^F = 2 + \frac{1}{2}(TrF^2) + \frac{1}{24}(TrF^4) + \frac{1}{720}(TrF^6) + \ldots \tag{61}$$

Using the $SU(2)$ trace identities (58) Tr $F^4 = \frac{1}{2}(TrF^2)^2$ and

$$\begin{aligned}\mathrm{Tr}\ F^6 &= 6(-\frac{1}{48}((TrF^2)^3) + \frac{1}{8}(TrF^2)(TrF^4)\\ &= \frac{1}{4}(TrF^2)^3\end{aligned} \tag{62}$$

we arrive at the answer

$$\mathrm{Tr}\ e^F = 2 + \frac{1}{2}(2h^2) + \frac{1}{24}(2h^4) + \frac{1}{720}(2h^6) + \ldots \tag{63}$$

were we substituted $F = h\sigma^3$, so that Tr $F^2 = 2h^2$ (Obviously we could have substituted the diagonal form of F directly in equation (61), but the use of trace identities is far more convenient for larger algebras).

Clearly the equations (59) and (63) agree, as expected. However, the way the agreement comes out is not entirely trivial (although it can easily be derived). Note in particular that the Weyl formula is *a priori* expressed in terms of only r variables, so that all trace identities are already built in. On the other hand, in writing down the formal expression Tr e^F there is no need to specify the number of variables, and indeed the formula is the same for any algebra. The non-trivial group structure is thus encapsulated in the trace identities. It is instructive to compare the two formulations also for other representations.

In this case the Weyl formula is superior in elegance and simplicity, although it is somewhat more difficult to expand to higher orders due to its denominator. For higher rank groups the Weyl formula becomes extremely cumbersome, as explained earlier, while our method does not grow in complexity.

5.4 B_r characters

The basic representations are the anti-symmetric tensors of rank $1\ldots,r-1$ plus the spinor representation. The characters of the anti-symmetric tensors are related to the vector character as in the case of A_r. The spinor character can be expressed in terms of traces of the vector representation by explicit computation. The result is [26]

$$\mathrm{Ch}_{(0,\ldots,0,1)}(F) = 2^r \exp\left[\sum_{n=1}^{\infty} \frac{(2^{2n}-1)B_{2n}}{4n(2n)!}\mathrm{Tr}\ F^{2n}\right] , \tag{64}$$

where B_{2n} are the Bernoulli numbers.

For algebras of type B_r the same trace identity as for A_r holds, but with order $k > 2r+1$. This is true because of the embedding $B_r \subset A_{2r}$. All traces of odd order vanish trivially.

The demonstration that the other characters can be obtained recursively from the tensor product rule is similar as for A_r, with some complications due to the spinors. We will omit the details (and the same holds for C_r and D_r).

5.5 C_r characters

The fundamental representations are the anti-symmetric tensors of rank $l = 1\ldots,r$ with a symplectic trace removed. The character of the fundamental representation l is equal to the l^{th} anti-symmetric tensor power of the vector character minus the $(l-2)^{\mathrm{th}}$ anti-symmetric power of the vector character (if $l \geq 2$).

The C_r trace identities can be derived using the embedding $C_r \subset A_{2r-1}$, which leads to trace identities for traces of order $k > 2r$. Just as for B_r, the odd traces vanish.

5.6 D_r characters

The fundamental representations are the anti-symmetric tensors of rank $l = 1\ldots,r-2$ plus the two conjugate spinor representations. The anti-symmetric tensor characters are computed as for B_r, but the spinor characters cannot be expressed completely in terms of traces over the vector representation. This is because there exists a symmetric tensor of rank r which never appears in traces over the vector representation, namely the Levi-Civita tensor. This tensor is an anti-symmetric tensor of rank $2r$ with vector indices. Combining the $2r$ vector indices in pairs, with each pair labelling an element of the adjoint representation, we can view the Levi-Civita tensor also as a symmetric tensor of rank r with adjoint indices.

Using this new invariant, we can write down the spinor character:

$$\begin{aligned}\mathrm{Ch}_{(0,\ldots,1,0)}(F) &= 2^{r-1} \exp\left[\sum_{n=1}^{\infty} \frac{(2^{2n}-1)B_{2n}}{4n(2n)!}\mathrm{Tr}\ F^{2n}\right] \\ &\quad + \frac{1}{r!}\chi_r(F) \exp\left[\sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!}\mathrm{Tr}\ F^{2n}\right]\end{aligned} \tag{65}$$

$$\begin{aligned}\mathrm{Ch}_{(0,\ldots,0,1)}(F) &= 2^{r-1}\exp\left[\sum_{n=1}^{\infty}\frac{(2^{2n}-1)B_{2n}}{4n(2n)!}\mathrm{Tr}\ F^{2n}\right] \\ &\quad -\frac{1}{r!}\chi_r(F)\exp\left[\sum_{n=1}^{\infty}\frac{B_{2n}}{4n(2n)!}\mathrm{Tr}\ F^{2n}\right] . \end{aligned} \tag{66}$$

where $\frac{1}{r!}\chi_r(F)$ is a polynomial of order r in F defined by the leading term in the difference of these expressions. It is proportional to the Levi-Civita tensor with indices pairwise contracted with F^a. The precise definition of the tensor is given in appendix E.

The trace identities for D_r are as those for B_r for $k > 2r$. However, due to the extra fundamental trace of order r, there must be an additional trace identity to reduce the number of independent ones back to r. Indeed, it turns out the the trace of order $2r$ can be eliminated using the identity

$$\sum_{\substack{n_i,m_i \\ n_i m_i = 2r}} \prod_i \frac{1}{m_i!}\left(-\frac{\mathrm{Tr}\ F^{n_i}}{n_i}\right)^{m_i} = 4(-1)^r[\frac{1}{r!}\chi_r(F)]^2 \tag{67}$$

This identity was also obtained in [26] (the coefficient on the right hand side is incorrect in [26]).

5.7 G_2 characters

Exceptional group characters can be computed by expressing them in terms of characters of a regular subalgebra. Since the subalgebra has the same rank, one gets polynomials in the same number of variables and hence no information is lost. For G_2 the only option is the subalgebra A_2. We have

$$\mathrm{Ch}_{G_2,7} = \mathrm{Ch}_{A_2,3} + \mathrm{Ch}_{A_2,\bar{3}} + \mathrm{Ch}_{A_2,1} , \tag{68}$$

denoting representations by their dimension and omitting the argument F. Since all G_2 representations are real, the third order invariant of A_2 is always cancelled out, and all other odd invariants vanish as well. The fourth order invariant can be expressed in terms of second order ones using the A_2 trace identity. The sixth order invariant of A_2 can be expressed in terms of lower ones, but this expression involves the third order invariant which doesn't exist in G_2. Hence in G_2 the sixth order invariant is new. After a little algebra we can write the reference character of G_2 as

$$\mathrm{Ch}_{G_2,7} = 7+\frac{1}{2}\mathrm{Tr}\ F^2+\frac{1}{4!}\frac{1}{4}(\mathrm{Tr}\ F^2)^2+\frac{1}{6!}\mathrm{Tr}\ F^6+\frac{1}{8!}[\frac{2}{3}(\mathrm{Tr}\ F^2)(\mathrm{Tr}\ F^6)-\frac{5}{192}(\mathrm{Tr}\ F^2)^4]+\ldots \tag{69}$$

Here all explicit traces are over the reference representation (0,1) of dimension 7. The character of the other fundamental representation, (1,0) of dimension 14 is easily computed from the anti-symmetric tensor product $(7\otimes 7)_A = (7) + (14)$. Explicitly:

$$\begin{aligned}\mathrm{Ch}_{G_2,14} &= 14+\frac{1}{2!}4\mathrm{Tr}\ F^2+\frac{1}{4!}\frac{5}{2}(\mathrm{Tr}\ F^2)^2+\frac{1}{6!}[-26\mathrm{Tr}\ F^6+\frac{15}{4}(\mathrm{Tr}\ F^2)^3] \\ &\quad +\frac{1}{8!}[-\frac{160}{3}(\mathrm{Tr}\ F^2)(\mathrm{Tr}\ F^6)-\frac{515}{96}(\mathrm{Tr}\ F^2)^4]+\ldots\end{aligned} \tag{70}$$

Note that all traces here are over the reference representation. From this expression we read off the second and sixth indices of the representation (14): they are 4 and -26 respectively.

Since the (7) of G_2 can be embedded in the vector representation of $SO(7)$, G_2 inherits all B_3 trace identities for traces of order 8 and higher. There is an additional trace identity for the fourth order trace, which can be read off directly from $\mathrm{Ch}_{G_2,7}$:

$$\mathrm{Tr}\ F^4 = \frac{1}{4}(\mathrm{Tr}\ F^2)^2 \tag{71}$$

This exhausts the set of trace identities for G_2.

5.8 F_4 characters

The computation is similar to the previous case, now using the sub-algebra B_4. We have

$$\mathrm{Ch}_{F_4,26} = \mathrm{Ch}_{B_4,16} + \mathrm{Ch}_{B_4,9} + \mathrm{Ch}_{B_4,1} \tag{72}$$

The vanishing of the fourth order invariant is not obvious in this case, but follows easily. The sixth and eight order polynomials are directly related to those of B_4. The tenth order one vanishes again by inspection (*i.e.* the tenth order trace can be expressed in terms of B_4 traces of order 2,6 and 8, but not 4), and the twelfth order trace involves the third power of the fourth order polynomial of B_4, which did not occur before. It is absorbed in the definition of Tr F^{12}, the 12$^{\text{th}}$ order term in $\mathrm{Ch}_{F_4,26}$ (up to a factor $\frac{1}{12!}$). To obtain the character of $(1,0,0,0)$ (dimension (52)) we use

$$\mathrm{Ch}_{F_4,52} = \mathrm{Ch}_{B_4,36} + \mathrm{Ch}_{B_4,16} \tag{73}$$

and we express all B_4 traces into F_4 traces using the definitions introduced when computing 72. The other characters of basic representations can be obtained from the anti-symmetric tensor products

$$\mathrm{Ch}_{F_4,273} = \mathrm{Ch}_{F_4,[2]*26} - \mathrm{Ch}_{F_4,52} \tag{74}$$

$$\mathrm{Ch}_{F_4,1274} = \mathrm{Ch}_{F_4,[3]*26} - \mathrm{Ch}_{F_4,52}\mathrm{Ch}_{F_4,26} + \mathrm{Ch}_{F_4,26} \tag{75}$$

The last identity involves a little algebra. In the third order anti-symmetric tensor power of (26) occurs, in addition to (1274) also the representations (273) and (1053) (with Dynkin labels (1,0,0,1)). The former character is known, the latter can be computed using the tensor product $(26) \otimes (52)$.

Of course the characters of (273) and (1274) can also be computed using the B_4 embedding. We have used this as a check.

Just as for G_2 one may read off the Dynkin indices from the characters. They are shown in the appendix. Furthermore there are trace relations for traces of fourth and tenth order which are read off from $\mathrm{Ch}_{(26)}$. By expanding the characters to sufficiently high order one obtains trace identities for traces of order 14 and higher. The embedding $F_4 \subset D_{13}$ gives trace identities for all traces of order 26 and higher, namely precisely those of D_{13}. We will only present the identities for orders lower than that of the maximal Casimir operator.

5.9 E_6 characters

Here we used the sub-algebra $A1 \oplus A5$, and the decompositions

$$\mathrm{Ch}_{E_6,27} = \mathrm{Ch}_{A_1,2}\mathrm{Ch}_{A_5,\bar{6}} + \mathrm{Ch}_{A_5,15} \tag{76}$$

$$\mathrm{Ch}_{E_6,78} = \mathrm{Ch}_{A_1,2} + \mathrm{Ch}_{A_5,35} + \mathrm{Ch}_{A_1,2}\mathrm{Ch}_{A_5,20} \tag{77}$$

The computation is very similar to the previous cases. We get another basic representation, the $(\overline{27})$, by conjugation:

$$\mathrm{Ch}_{E_6,\overline{27}}(F) = \mathrm{Ch}_{E_6,27}(-F) \ . \tag{78}$$

Furthermore the anti-symmetric tensor power of order 2 gives us the representations (0,1,0,0,0,0) (351) and (0,0,0,1,0,0) $(\overline{351})$ and the order three anti-symmetric power yields precisely the representation (0,0,1,0,0,0) (2925).

The indices and trace identities for orders up to 12 are listed in the appendix.

5.10 E_7 characters

Here we used the sub-algebra A_7, and the decompositions or anti-symmetric tensor products

$$\begin{aligned}
\mathrm{Ch}_{E_7,56} &= \mathrm{Ch}_{A_7,28} + \mathrm{Ch}_{A_7,\overline{28}} && (79)\\
\mathrm{Ch}_{E_7,133} &= \mathrm{Ch}_{A_7,70} + \mathrm{Ch}_{A_7,\overline{63}} && (80)\\
\mathrm{Ch}_{E_7,912} &= \mathrm{Ch}_{A_7,420} + \mathrm{Ch}_{A_7,\overline{420}} + \mathrm{Ch}_{A_7,36} + \mathrm{Ch}_{A_7,\overline{36}} && (81)\\
\mathrm{Ch}_{E_7,1539} &= \mathrm{Ch}_{E_7,[2]*56} - \mathrm{Ch}_{E_7,1} && (82)\\
\mathrm{Ch}_{E_7,8645} &= \mathrm{Ch}_{E_7,[2]*133} - \mathrm{Ch}_{E_7,133} && (83)\\
\mathrm{Ch}_{E_7,27664} &= \mathrm{Ch}_{E_7,[3]*56} - \mathrm{Ch}_{E_7,56} && (84)\\
\mathrm{Ch}_{E_7,365750} &= \mathrm{Ch}_{E_7,[4]*56} - \mathrm{Ch}_{E_7,1539} - \mathrm{Ch}_{E_7,1} && (85)
\end{aligned}$$

5.11 E_8 characters

Here we used the sub-algebra D_8, and

$$\begin{aligned}
\mathrm{Ch}_{E_8,248} &= \mathrm{Ch}_{D_8,128} + \mathrm{Ch}_{D_8,120} && (86)\\
\mathrm{Ch}_{E_8,3875} &= \mathrm{Ch}_{D_8,1920} + \mathrm{Ch}_{D_8,1820} + \mathrm{Ch}_{D_8,135} && (87)\\
\mathrm{Ch}_{E_8,147250} &= \mathrm{Ch}_{D_8,60060} + \mathrm{Ch}_{D_8,56320} + \mathrm{Ch}_{D_8,15360} \\
&\quad + \mathrm{Ch}_{D_8,7020} + \mathrm{Ch}_{D_8,6435} + \mathrm{Ch}_{D_8,1920} + \mathrm{Ch}_{D_8,135} && (88)\\
\mathrm{Ch}_{E_8,30380} &= \mathrm{Ch}_{E_8,[2]*248} - \mathrm{Ch}_{E_8,248} && (89)\\
\mathrm{Ch}_{E_8,2450240} &= \mathrm{Ch}_{E_8,[3]*248} - (\mathrm{Ch}_{E_8,248})^2 + \mathrm{Ch}_{E_8,248} && (90)\\
\mathrm{Ch}_{E_8,6696000} &= \mathrm{Ch}_{E_8,[2]*3875} - \mathrm{Ch}_{E_8,3875}(\mathrm{Ch}_{E_8,248} - 1) + \mathrm{Ch}_{E_8,147250} && (91)\\
\mathrm{Ch}_{E_8,146325270} &= \mathrm{Ch}_{E_8,[4]*248} - (\mathrm{Ch}_{E_8,[2]*248} - \mathrm{Ch}_{E_8,248})(\mathrm{Ch}_{E_8,248} - 1) && (92)\\
\mathrm{Ch}_{E_8,6899079264} &= \mathrm{Ch}_{E_8,[5]*248} - \mathrm{Ch}_{E_8,248}(\mathrm{Ch}_{E_8,[3]*248} - 2\mathrm{Ch}_{E_8,[2]*248} + \mathrm{Ch}_{E_8,248} - 1) && (93)
\end{aligned}$$

In the first three lines all D_8 spinor representations must be from the same conjugacy class, which is fixed by the decomposition one chooses for the (248). Since the choice one makes for the class is irrelevant, there is no need for a label to distinguish conjugate spinors. The representation (6435) is an (anti-)selfdual tensor. It belongs to the trivial conjugacy class, but it does carry a non-trivial chirality. If for the representation denoted (128) we choose the one with Dynkin labels $(0,0,0,0,0,0,0,1)$, then the correct set of Dynkin labels for the 6345 is $(0,0,0,0,0,0,2,0)$.

5.12 Normalization of indices

The normalization of the symmetric tensors is fixed by fixing a normalization for the indices. We will do this in such a way that they are always integers, as the word "index" suggests. For the second index there is a natural normalization in terms of the Atiyah-Singer index theorem for instantons on S_3. For any representation of any algebra we can choose the second index equal to the net number of zero modes of a Weyl fermion in that representation in an instanton field of minimal non-trivial topological charge (where "net" means the difference between the two chiralities). Then the second index is equal to 1 for the reference representations of A_r and C_r, 2 for those of B_r and C_r, and 2,6,6,12,60 respectively for the reference representations of G_2, F_4, E_6, E_7 and E_8. For the adjoint representation the second index is always equal to twice the dual Coxeter number g listed in table 1. This choice corresponds to setting $\eta = 2$ in (27). This value of η was used in the last column of table 3.

For the higher indices there is a similar topological interpretation in terms of gauge bundles on higher dimensional manifolds, but we will not explore that here in detail. One may however follow the spirit of such an interpretation and define all higher indices in such a way that they are integers. This is automatically true if they are integers for the basic representations, because the characters of all representations are polynomials with integer coefficients in terms of the characters of the basic representations. Furthermore, within the set of basic representations the ones obtained by means of anti-symmetric tensor products of a given representation R, have indices that are an integer multiple of those of R. Then only the spinor representations of $SO(N)$, one representation of F_4, E_6 and E_7 and two of E_8 require special attention.

For the reference representations we choose all higher indices equal to 1, except when a larger integer is required to make all indices integral. The $2n^{\text{th}}$ index of a spinor representation of $SO(N)$ follows directly from the character equations (64), (65) and (66):

$$\dim(S)\frac{(2^{2n}-1)B_{2n}}{4n} , \tag{94}$$

where $\dim(S)$ is the dimension of a spinor representation, *i.e.* $\dim(S) = 2^r$ for algebras of type $B_r(SO(2r+1)$ and $\dim(S) = 2^{r-1}$ for type $D_r(SO(2r))$. This assumes that the $(2n)^{\text{th}}$ index of the vector representation is set to 1. By inspection, this expression is an integer except for the fourth index for $SO(N), N \leq 8$ and the eighth index for $N \leq 10$. We have checked that the spinor index is an integer in all other cases for $2n < 100$. Table 4 gives the indices for the $SO(N)$ vector and spinor representations, according to our normalization (the index $\tilde{I}_r$ is not listed here; its value is 0 for the vector representation and chosen ± 1 for the two fundamental spinors of $SO(N)$, N even). Note that for $N = 3, \ldots, 6$ the spinor representation, and not the vector is the reference representation, which automatically leads to the entries in the table.

N	I_2	I_4	I_6	I_8	I_{10}	I_{12}	I_{14}
3	4,1	—	—	—	—	—	—
4	4,1	—	—	—	—	—	—
5	2,1	-4,1	—	—	—	—	—
6	2,1	-4,1	—	—	—	—	—
7	2,2	2,–1	1,1	—	—	—	—
8	2,2	2,–1	1,1	—	—	—	—
9	2,4	1,–1	1,2	2,–17	—	—	—
10	2,4	1,–1	1,2	2,–17	—	—	—
11	2,8	1,–2	1,4	1,–17	1,124	—	—
12	2,8	1,–2	1,4	1,–17	1,124	—	—
13	2,16	1,–2	1,8	1,–34	1,248	1,–2764	—
14	2,16	1,–4	1,8	1,–34	1,248	1,–2764	—
15	2,32	1,–4	1,16	1,–68	1,496	1,–5528	1,87376
16	2,32	1,–8	1,16	1,–68	1,496	1,–5528	1,87376

Table 4: Indices for the $SO(N)$ vector and spinor representations. The chiral index $\tilde{I}_r$ is not listed here; we choose it equal to $+1$ for the fundamental spinor $(0,\ldots,1,0)$ and equal to -1 for $(0,\ldots,0,1)$.

For the exceptional algebras the two highest indices of E_7 and the three highest ones of E_8 come out fractional unless we choose a different normalization for the reference representation.

Our preferred index normalization for the reference representations is summarized in the last column of table 3. For $SO(2N)$ the last entry indicates the index $\tilde{I}_N$ of the spinor representation

$(0,\ldots,1,0)$. Only the second index is affected by the choice of η. For all higher order traces we fix the normalization of the index, and then the η-dependence goes into the normalization of the symmetric tensor.

This also fixes the normalization of all symmetric tensors as

$$\mathrm{Str}\ T_r^{a_1}\ldots T_r^{a_n} = I_n(r) d^{a_1\ldots a_n}\ . \tag{95}$$

This defines the tensors given a choice of generators in the reference representation.

In Appendix D we present some results for contractions of these tensors.

6 Reduction identities

At this stage we have terms that contain combinations of the invariants d_r and the structure constant f. Our task is now to eliminate f from the terms as much as possible, and reduce the total number of invariants in the final answer.

Unlike the results obtained so far, for these reductions we cannot give a general algorithm. In fact, we do not even know what the desirable outcome is, since we are not aware of a mathematical theorem that gives us a basic set of invariants in terms of which all others can be expressed. At any given order we can derive large numbers of identities among the various invariants, but there will be new relations at every order. In practice this is not a major problem. First of all we have obtained results relevant for vacuum bubble Feynman diagrams of up to nine loops, and secondly the number of invariants we are left with is small, although possibly not minimal.

The most useful identities for doing this are the Jacobi identities. Thus far we have seen two of them in equation (43). The second equation there can be generalized into

$$0 \;=\; \sum_{\text{cyclic permutations of } i_1\cdots i_n} d_R^{i_1\cdots i_{n-1}a} f^{i_n ba} \tag{96}$$

for all representations R, including the adjoint representation. In fact, there is an advantage to postponing the replacement of the adjoint symmetric tensors by reference tensors, since they satisfy additional identities. Furthermore some identities produce new tensors d_A. For this reason we present the results in terms of d_R and d_A rather than d_r.

The first identity that can be derived from this is one for invariants with three indices:

$$d_R^{abi} f^{ajc} f^{blc} \;=\; \frac{1}{2} C_A\, d_R^{ijk} \tag{97}$$

It is actually the simplest identity in a class of triangle reductions that involve one or two invariant tensors d_R. We have also

$$d_{R_1}^{i_1 j_1\cdots j_n k_1} d_{R_2}^{i_2 j_1\cdots j_n k_2} f^{k_1 k_2 i_3} \;=\; \frac{1}{N_A}\frac{1}{n+1} d_{R_1}^{j_1\cdots j_{n+2}} d_{R_2}^{j_1\cdots j_{n+2}} f^{i_1 i_2 i_3} \tag{98}$$

$$d_{R_1}^{i_1 j_1\cdots j_n k_1} d_{R_2}^{i_3\cdots i_m j_1\cdots j_n k_2} f^{k_1 k_2 i_2} \;=\; \frac{-1}{n+1} d_{R_1}^{j_1\cdots j_{n+1} k} d_{R_2}^{i_3\cdots i_m j_1\cdots j_{n+1}} f^{k i_1 i_2} \tag{99}$$

These identities are very powerful when a large number of invariants is involved.

For showing further reduction identities we will use a special notation which corresponds closely to a notation that can be used inside a computer program. We will represent a (symmetric) invariant by a product of vectors:

$$d_{R_1}^{i_1\cdots i_n} \;=\; p_1^{i_1}\cdots p_1^{i_n} \tag{100}$$

The lower index on the vector refers to the particular invariant. In this notation we have no problems with the symmetric property of the invariants. Of course we are not implying that each invariant can be mathematically written this way. It is just notation.

Additionally we will use Schoonschip notation on contracted indices. That is: if the index of a vector is contracted with an index of a tensor we put the vector in the place of this index. Hence

$$f^{p_1p_2i}f^{p_1p_2i}(p_1\cdot p_2)^n = d_{R_1}^{i_1\cdots i_nj_1j_2}d_{R_2}^{i_1\cdots i_nk_1k_2}f^{j_1k_1i}f^{j_2k_2i} \tag{101}$$

Furthermore we can add a weight to each formula. This weight is basically the number of vertices (assuming that all vertices are three-point vertices) in the diagram before we started the elimination procedures. For our current algorithms the weight is the total number of vectors p_i plus the number of structure constants f. Hence the weight of the above formula is $2n+6$. We will present all reduction identities that are relevant for weights up to 12. This corresponds to 7-loop vacuum bubbles or 6-loop propagator diagrams. The derivation of all these identities involves the use of the generalized identity of equation (96).

$$f^{p_1p_2i}f^{p_1p_2i}(p_1\cdot p_2)^n = \frac{1}{n+1}\ C_A\ (p_1\cdot p_2)^{n+2} \tag{102}$$

$$f^{p_1p_2i_1}f^{p_1p_2i_2}f^{p_1i_1i_3}f^{p_1i_2i_3}(p_1\cdot p_2)^n = \frac{1}{n+1}\ d_A^{p_1p_1p_1p_2}(p_1\cdot p_2)^{n+1} \tag{103}$$

$$\begin{aligned} f^{p_1p_2i_1}f^{p_1i_1i_2}f^{p_2i_2i_3}f^{p_1p_2i_3}(p_1\cdot p_2)^n &= \frac{5}{6(n+1)(n+2)}\ C_A^2\ (p_1\cdot p_2)^{n+3} \\ &\quad -\frac{1}{n+1}\ d_A^{p_1p_1p_2p_2}(p_1\cdot p_2)^{n+1} \end{aligned} \tag{104}$$

$$f^{p_1p_2i_1}f^{p_1p_2i_2}f^{p_1p_2i_3}f^{i_1i_2i_3}(p_1\cdot p_2)^n = 0 \tag{105}$$

$$\begin{aligned} f^{p_1p_2i_1}f^{p_1p_2i_1}f^{p_1p_2i_2}f^{p_1p_2i_2}(p_1\cdot p_2)^n &= \frac{2\ n!}{(n+2)!}\ d_A^{p_1p_1p_2p_2}(p_1\cdot p_2)^{n+2} \\ &\quad +\frac{(3n+1)\ n!}{3\ (n+3)!}\ C_A^2\ (p_1\cdot p_2)^{n+4} \end{aligned} \tag{106}$$

$$f^{p_1p_2i}f^{p_1p_2i}(p_1\cdot p_3)(p_2\cdot p_3)^n = \frac{1}{2}\ C_A\ (p_1\cdot p_2)^2(p_1\cdot p_3)(p_2\cdot p_3)^n \tag{107}$$

$$f^{p_1p_2i}f^{p_1p_3i}(p_1\cdot p_2)(p_2\cdot p_3)^n = \frac{1}{2}\ C_A\ (p_1\cdot p_2)^2(p_1\cdot p_3)(p_2\cdot p_3)^n \tag{108}$$

$$f^{p_1p_2i}f^{p_1p_3i}(p_1\cdot p_2)(p_2\cdot p_3)(p_1\cdot p_3)^n = \frac{C_A}{2(n+1)}\ (p_1\cdot p_2)^2(p_2\cdot p_3)(p_1\cdot p_3)^{n+1} \tag{109}$$

$$f^{p_1p_2p_3}f^{p_1p_2p_3}(p_1\cdot p_2)(p_1\cdot p_3) = \frac{1}{4}\ C_A\ (p_1\cdot p_2)^2(p_1\cdot p_3)^2(p_2\cdot p_3) \tag{110}$$

The equations with an odd number of f's that are relevant are all zero. This has been explicitly shown, but for many of them one can see this already on the basis of symmetry principles.

We are not going to present the identities that would be needed for diagrams of weight 14 or 16. There would be too many of them and moreover, this is not how we have constructed the computer program. In the program we have found a way to apply equation (96) recursively in such a way that it reduces all combinations with the exception of one (at weight 14). The program has to guess at what combination of invariants and f's to take for the application of the formula and we let it guess several times. In the end this covers all cases except for the one that we cannot do by these methods anyway.

The first object that causes some real problems because the above algorithms are not sufficient to handle them, occurs at weight 14. This object can either be written as

$$f^{p_1p_2p_3}f^{p_1p_2p_3}(p_1\cdot p_2)(p_1\cdot p_3)(p_2\cdot p_3) \tag{111}$$

or with some rewriting (and omitting trivial terms that are a byproduct of the rewriting):

$$f^{p_1p_2i}f^{p_1p_2i}(p_1\cdot p_3)^2(p_2\cdot p_3)^2 \tag{112}$$

If at least two of the three invariants are in the adjoint representation this object can be reduced with the same technique that we use below to simplify some combinations of invariants only (see equation (119)).

6.1 Combinations of invariants

Here we will consider combinations of invariants only. The easiest combinations are full contractions between two invariants as in

$$d_{R_1}^{i_1\cdots i_n}d_{R_2}^{i_1\cdots i_n} \tag{113}$$

Unfortunately, when the weight of the diagrams increases, the complexity of the combinations increases correspondingly. In some cases one can make reductions. For instance:

$$d_{R_1}^{j\ i_1\cdots i_n}d_{R_2}^{k\ i_1\cdots i_n} \;=\; \frac{1}{N_A}\ \delta^{jk}d_{R_1}^{i_0i_1\cdots i_n}d_{R_2}^{i_0i_1\cdots i_n} \tag{114}$$

and hence

$$d_{R_1}^{a\ i_1\cdots i_n}d_{R_2}^{b\ i_1\cdots i_n}d_{R_3}^{a\ j_1\cdots j_m}d_{R_4}^{b\ j_1\cdots j_m} \;=\; \frac{1}{N_A}\ d_{R_1}^{i_0i_1\cdots i_n}d_{R_2}^{i_0i_1\cdots i_n}d_{R_3}^{j_0j_1\cdots j_m}d_{R_4}^{j_0j_1\cdots j_m} \tag{115}$$

but for objects of the type

$$d_{R_1}^{i_1i_2i_3i_4}d_{R_2}^{i_1i_2i_5}d_{R_3}^{i_3i_4i_5} \tag{116}$$

or

$$d_{R_1}^{i_1i_2i_3i_4i_5}d_{R_2}^{i_1i_2i_3i_6}d_{R_3}^{i_4i_5i_6} \tag{117}$$

there does not seem to be a general simplification of this type. In the case of all invariants belonging to the adjoint representation we can still do things as we see in the next formula:

$$d_A^{abcdef}d_A^{abcdef} - \frac{5}{8}d_A^{abcd}d_A^{cdef}d_A^{efab} + \frac{7}{240}C_A^2d_A^{abcd}d_A^{abcd} + \frac{1}{864}C_A^6N_A \;=\; 0 \tag{118}$$

The derivation of this formula is a matter of evaluating a circular ladder with 6 rungs in two different ways. In the first way one sees it as two loops with 6 vertices, and in the second way as three loops with 4 vertices. The algorithms of the previous sections are then sufficient to obtain this formula. Note however that such derivations usually need the use of a computer program: the intermediate stages can contain large numbers of terms. A similar technique can be used for the object in equation (112). We look at a circular ladder with 7 rungs. If we see this as three loops with 4 vertices (and two f's left) we get the form of the equation, and if we see it as two loops with 7 vertices, we get a representation involving two d_A invariants with 6 indices (the ones with 7 indices are zero for the adjoint representation). The result is (after a rather lengthy calculation, applying equation (118) and normalizing):

$$\begin{aligned} & d_R^{abcd}d_A^{cdef}d_A^{efgh}f^{agi}f^{bhi} + \frac{2}{27}C_A^3d_R^{abcd}d_A^{abcd} \\ & -\frac{19}{15}C_Ad_R^{abcd}d_A^{cdef}d_A^{efab} + \frac{8}{9}d_A^{abcdef}d_R^{abcg}d_A^{defg} \;=\; 0 \end{aligned} \tag{119}$$

with R replaced by A. Because a similar exchange of the order of evaluation in one of the examples below gives the same equation with a slightly extended generality we have already presented this more general form with the representation R. One can also derive this equation from equation (148), but the derivation of that formula uses basically the same technique.

Considering that the algorithms we have presented can reduce all combinations of invariants and f's, with the exception of the above combination with at least two invariants not in the adjoint representation, up to weight 14 we have only a limited number of topologies (contractions between invariants) left. We will show them graphically, omitting the very trivial ones that can be reduced with equation 114 and objects of the type $d_R^{aai_1\cdots i_n}$. The elements of this pictorial language are

 $= d_R^{i_1i_2i_3}$

 $= d_R^{i_1i_2i_3i_4}$

 $= d_A^{i_1i_2i_3i_4}$

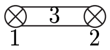
 $= d_\perp^{i_1i_2i_3i_4}$

 $= f^{i_1i_2i_3}$

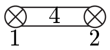
 $= f^{i_3i_2i_1}$ (120)

in which we assume the indices of f to run counterclockwise in the diagram. For weight 6 we have

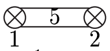
$$d_{33}(p_1,p_2) = d_{R_1}^{ijk}d_{R_2}^{ijk}$$

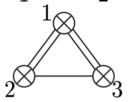
$=$  (121)

For weight 8 there is also only one topology:

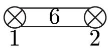
$d_{44}(p_1,p_2) =$  (122)

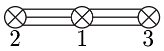
For weight 10 there are two topologies:

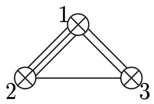
$d_{55}(p_1,p_2) =$  (123)

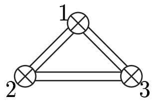
$d_{433}(p_1,p_2,p_3) =$  (124)

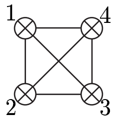
For weight 12 we have 5 topologies:

$d_{66}(p_1,p_2) =$  (125)

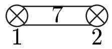
$d_{633}(p_1,p_2,p_3) =$  (126)

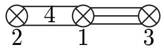
$d_{543}(p_1,p_2,p_3) =$  (127)

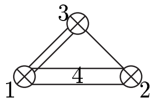
$d_{444}(p_1,p_2,p_3) =$  (128)

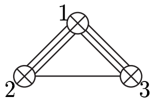
$d_{3333}(p_1,p_2,p_3,p_4) =$  (129)

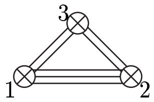
Finally for weight 14 we have 9 topologies:

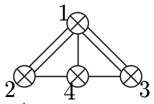
$d_{77}(p_1,p_2) =$  (130)

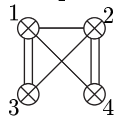
$d_{743}(p_1,p_2,p_3) =$  (131)

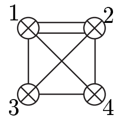
$d_{653}(p_1,p_2,p_3) =$  (132)

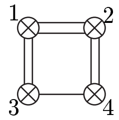
$d_{644}(p_1,p_2,p_3) =$  (133)

$d_{554}(p_1,p_2,p_3) =$  (134)

$d_{5333}(p_1,p_2,p_3,p_4) =$  (135)

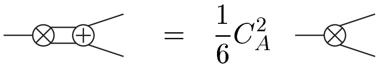
$d_{4433a}(p_1,p_2,p_3,p_4) =$  (136)

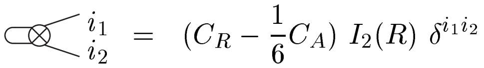
$d_{4433b}(p_1,p_2,p_3,p_4) =$  (137)

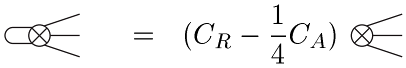
$d_{4433c}(p_1,p_2,p_3,p_4) =$  (138)

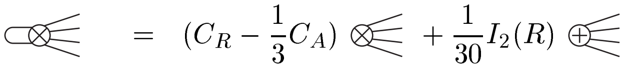
For weight 16 there are more than 20 non-trivial topologies. We will not show them here.

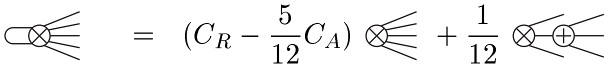
In terms of these diagrams of invariants we can compose a few extra equations that can be very useful:

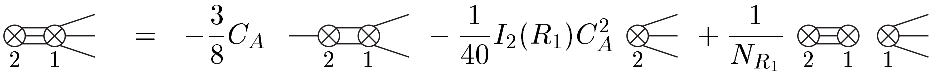
 (139)

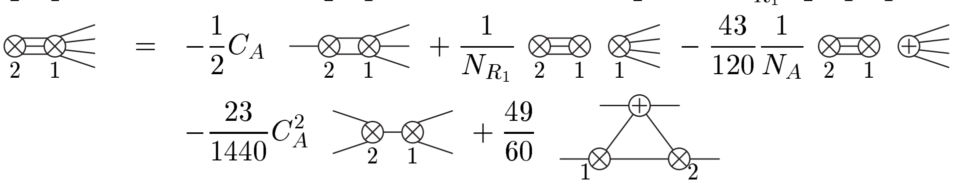
 (140)

 (141)

 (142)

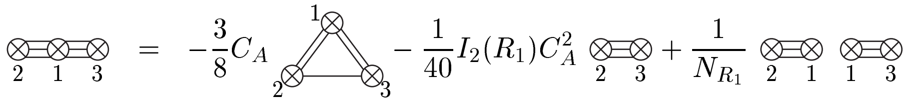
 (143)

 (144)

 (145)

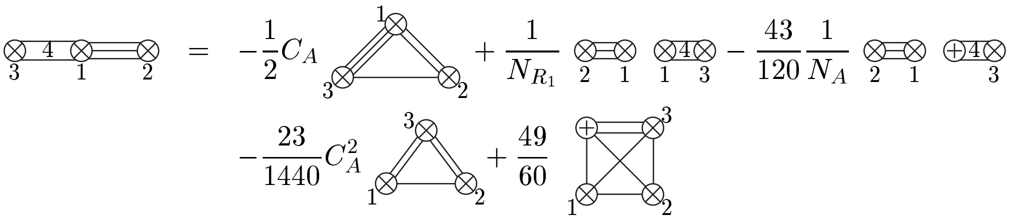
We assume symmetrization over the external legs in the terms of the right hand sides.

One of the spin-offs of equation (144) is



(146)

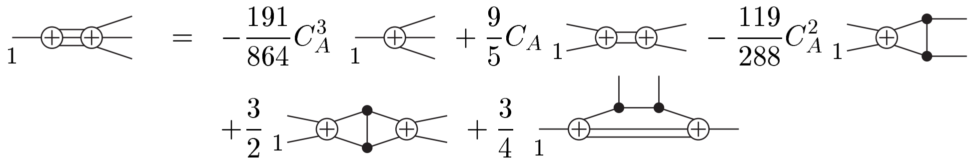
This allows us to eliminate the topology d_{633} completely. For groups for which d^{ijk} exists and $I_4(A)$ is not zero we can use the techniques of the next section to also eliminate the topologies d_{433}, d_{4433a} and d_{4433c}. Similarly equation (145) gives



(147)

and therefore also the topology d_{743} can be eliminated.

An equation that is also rather interesting is



(148)

It can be used to derive equation (119), but it can also be useful for bigger diagrams. It is derived by writing a diagram with 10 f's into loops in two different ways. After that the application of the reduction algorithms and some rewriting leads to this formula.

7 Representation independent invariants

The invariants that we have used thus far were only symmetrized traces. It is possible to define a new set of invariants that is not only symmetric, but also orthogonal (see equation (21)). Because the invariant with two indices is proportional to the Kronecker delta in the adjoint space, this means for instance that these invariants have a zero trace (a contraction of any two indices gives zero). But also contractions with all the indices of invariants with fewer indices than the invariant under study should give zero. For most algebras we can define (the exceptions are certain $SO(4N)$ algebras with *two* independent tensors of order $2N$):

$$d_R^{i_1i_2i_3} = I_3(R)\ d_\perp^{i_1i_2i_3} \tag{149}$$

$$d_R^{i_1i_2i_3i_4} = I_4(R)\ d_\perp^{i_1i_2i_3i_4} + I_{2,2}(R)\ (\delta^{i_1i_2}\delta^{i_3i_4} + \delta^{i_1i_3}\delta^{i_2i_4} + \delta^{i_1i_4}\delta^{i_2i_3})/3 \tag{150}$$

$$d_R^{i_1i_2i_3i_4i_5} = I_5(R)\ d_\perp^{i_1i_2i_3i_4i_5} + I_{3,2}(R)\ (d_\perp^{i_1i_2i_3}\delta^{i_4i_5} + \cdots)/10 \tag{151}$$

$$\begin{aligned} d_R^{i_1i_2i_3i_4i_5i_6} &= I_6(R)\ d_\perp^{i_1i_2i_3i_4i_5i_6} + I_{4,2}(R)\ (d_\perp^{i_1i_2i_3i_4}\delta^{i_5i_6} + \cdots)/15 \\ &\quad + I_{3,3}(R)\ (d_\perp^{i_1i_2i_3}d_\perp^{i_4i_5i_6} + \cdots)/10 + I_{2,2,2}(R)\ (\delta^{i_1i_2}\delta^{i_3i_4}\delta^{i_5i_6} + \cdots)/15 \end{aligned} \tag{152}$$

$$\begin{aligned} d_R^{i_1i_2i_3i_4i_5i_6i_7} &= I_7(R)\ d_\perp^{i_1i_2i_3i_4i_5i_6i_7} + I_{5,2}(R)\ (d_\perp^{i_1i_2i_3i_4i_5}\delta^{i_6i_7} + \cdots)/21 \\ &\quad + I_{4,3}(R)\ (d_\perp^{i_1i_2i_3i_4}d_\perp^{i_5i_6i_7} + \cdots)/35 + I_{3,2,2}(R)\ (d_\perp^{i_1i_2i_3}\delta^{i_4i_5}\delta^{i_6i_7} + \cdots)/105 \end{aligned} \tag{153}$$

and higher ones are defined analogously. The composite constants like $I_{2,2}(R)$ can now be derived by the orthogonality conditions. When we multiply equation (150) by $\delta^{i_3i_4}$ we obtain

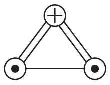
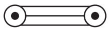
$$\frac{N_A+2}{3} I_{2,2}(R)\delta^{i_1i_2} = d_R^{i_1i_2i_3i_3}$$

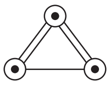
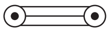
$$
\begin{aligned}
&= (C_R - C_A/6) d_R^{i_1 i_2} \\
&= (C_R - C_A/6) I_2(R) \delta^{i_1 i_2} \\
\frac{N_A+2}{3} I_{2,2}(R) &= (I_2(R)\frac{N_A}{N_R} - \frac{1}{6} I_2(A)) I_2(R) \qquad (154)
\end{aligned}
$$

Similarly we can derive

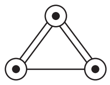
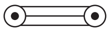
$$
I_{3,2}(R) = \frac{10}{N_A+6}(C_R - \frac{1}{4}C_A) I_3(R) \qquad (155)
$$

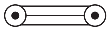
For the next level of constants we get coupled equations. The easiest way to derive them is with the use of the equations (142) and (146). If we contract the first equation once with $d_\perp^{i_1 i_2 i_3 i_4}$ and once with $\delta^{i_1 i_2}\delta^{i_3 i_4}$ and substitute equation (152) in all three of the equations that we obtain this way we have enough to come to a solution. Unfortunately this solution is not very elegant. In many cases one can make simplifications. For instance one can write (using equation (139))

 $= \frac{1}{6} C_A^2$ 

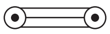
$= I_4(A)$  $+ \frac{2}{3} I_{2,2}(A)$  (156)

which then reduces to

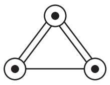
$I_4(A)$  $= \frac{1}{6} C_A^2 \frac{N_A - 8}{N_A + 2}$  (157)

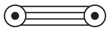
We notice that whenever $I_4(A) = 0$ or $d_\perp^{i_1 i_2 i_3 i_4} = 0$ we must have that either  $= 0$ or $N_A = 8$ (assuming that C_A is never zero). The last is indeed the case for $SU(3)$. Hence a general application of this relation is rather dangerous.

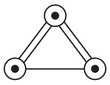
The equations for $I_{4,2}(R)$, $I_{3,3}(R)$ and $I_{2,2,2}(R)$ can be derived by taking the definition of the decomposition in equation (152) and multiplying either by $\delta^{i_1 i_2}\delta^{i_3 i_4}\delta^{i_5 i_6}$ or $d_\perp^{i_1 i_2 i_3 i_4}\delta^{i_5 i_6}$ or $d_\perp^{i_1 i_2 i_3} d_\perp^{i_4 i_5 i_6}$. With the use of the equations (142) and (144) we obtain:

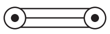
$0 =$  $(\frac{2}{5} I_{3,3}(R))$

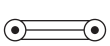
$+ N_A(N_A+2)(\frac{N_A+4}{15} I_{2,2,2}(R) - \frac{1}{3}(C_R - \frac{1}{3}C_A) I_{2,2}(R) - \frac{1}{90} I_{2,2}(A) I_2(R))$ (158)

$0 =$  $(\frac{3}{5} I_{3,3}(R))$

$+$  $(\frac{N_A+8}{15} I_{4,2}(R) - (C_R - \frac{1}{3}C_A) I_4(R) - \frac{1}{30} I_4(A) I_2(R))$ (159)

$0 =$  $(\frac{3}{5} I_{4,2}(R) + \frac{3}{8} C_A I_4(R))$

$+($  $)^2 (\frac{N_A+9}{10 N_A} I_{3,3}(R) - \frac{1}{N_R}(I_3(R))^2)$

$+$  $(\frac{2}{5} I_{2,2,2}(R) + \frac{1}{4} C_A I_{2,2}(R) + \frac{1}{4} C_A^2 I_2(R))$ (160)

Because for each group either some of the objects in these equations are zero, or there are simplifications, it does not seem wise to solve this system in this form. For all groups with the exception of $SU(N)$ we have that $d_\perp^{i_1i_2i_3} = 0$, and hence the system reduces to two equations in $I_{4,2}(R)$ and $I_{2,2,2}(R)$. For $SU(3)$ we have that $d_\perp^{i_1i_2i_3i_4} = 0$ and again we have a simpler system. For the other $SU(N)$ groups we can apply equation (157) improving the solutions somewhat. Because of the singularity (zero divided by zero) of the solutions for all groups but $SU(N), N > 3$, we do not present the general solution here. It serves no purpose because we will not use them. Along the same lines one can derive the equations for $I_{5,2}(R)$, $I_{4,3}(R)$ and $I_{3,2,2}(R)$.

The expressions at rank 6 and 7 become rather complicated due to the fact that the tensors in the right hand side of the equations (152) and (153) are not orthogonal. Hence the various orthogonality relations mix and this gives the complicated result. It should be clear that invariants with a higher rank will give even more complicated relations. The exception is the adjoint representation. For this representation all invariants with an odd number of indices are zero. Hence $I_3(A) = I_5(A) = I_{3,2}(A) = \cdots = 0$.

We can use the above relations for the reduction of some contractions of invariants. We have seen in the previous section that for the adjoint representation more things are possible than for the other representations. However we can rewrite the invariants only to invariants of the adjoint representation when the corresponding d's can indeed be expressed as such. This means that the corresponding $I(A)$ should never be zero. Unfortunately this excludes many contractions of invariants. Hence we do not see many benefits here.

7.1 Orthogonal versus Reference tensors

Here we will compare the computation of a symmetrized trace in two ways, using the orthogonal basis (satisfying (21)) and using characters, with tensors defined for a reference representation. The latter will be referred to as "reference tensors". We consider fourth order traces in $SU(N)$. In this case an explicit expression exists for any representation

$$\mathrm{Str}\ T_R^a T_R^b T_R^c T_R^d = I_4(R) d_\perp^{abcd} + \frac{3}{N_A+2} I_2(R)^2 \left[\frac{N_A}{N_R} - \frac{1}{6}\frac{I_2(A)}{I_2(R)}\right] d_{2,2}^{abcd} \ , \tag{161}$$

where

$$d_{2,2}^{abcd} \equiv \frac{1}{3}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \ . \tag{162}$$

Here "d" without subscript denotes the orthogonal tensor. The reference representation is the vector, and the reference fourth rank tensor is by definition equal to (161) with "R" equal to the reference representation:

$$d_r^{abcd} = d_\perp^{abcd} + \frac{3}{N_A+2}\left(\frac{\frac{2}{3}N^2 - 1}{N}\right) d_{2,2}^{abcd} \tag{163}$$

(Note that $d_{2,2}$ is the same in both bases). This allows us to express (161) in terms of reference rather than orthogonal tensors:

$$\begin{aligned}\mathrm{Str}\ T_R^a T_R^b T_R^c T_R^d &= I_4(R) d_r^{abcd} \\ &+ \frac{3}{N_A+2}\left(I_2(R)^2\left[\frac{N_A}{N_R} - \frac{1}{6}\frac{I_2(A)}{I_2(R)}\right] - \frac{\frac{2}{3}N^2-1}{N}\right) d_{2,2}^{abcd}\end{aligned} \tag{164}$$

Consider now for example the anti-symmetric tensor representation. Its character, expanded to fourth order is

$$\begin{aligned}\mathrm{Ch}_{[2]}(F) &= \frac{1}{2}N(N-1) + \frac{1}{2}(N-2)\mathrm{Tr}\ F^2 + \frac{1}{6}(N-4)\mathrm{Tr}\ F^3 \\ &+ \frac{1}{24}(N-8)\mathrm{Tr}\ F^4 + \frac{1}{8}\mathrm{Tr}\ F^2 \mathrm{Tr}\ F^2 + \dots \ ,\end{aligned} \tag{165}$$

where all traces are over the vector representation. From the fourth order terms we deduce, by differentiating with respect to $F^a, \ldots, F^b$:

$$\text{Tr } T^a_{[2]} T^b_{[2]} T^c_{[2]} T^d_{[2]} = (N-8) d_r^{abcd} + 3 d_{2,2}^{abcd} \tag{166}$$

We may now verify this using (164). Although the coefficient of the second term does not look very encouraging, substituting $I_2(R) = N-2$, $N_A = N^2-1$ and $N_R = \frac{1}{2}N(N-1)$, it does indeed produce the coefficient 3 in (166).

This illustrates several points. Trace formulas in terms of orthogonal tensors such as (161) have a simpler form than those in terms of reference tensors, if one tries to write down expressions for arbitrary representations R. However, expressions such as (166) can be written down fairly easily for any representation although not (easily) in closed form. Furthermore they can be extended to arbitrary order in a straightforward way while this rapidly becomes extremely difficult for (161) or (164).

Note that the indices (the coefficients of fundamental tensors) are basis independent (apart from normalizations), whereas the sub-indices (coefficients of combinations of fundamental tensors) are not. In the orthogonal basis it is not hard to see that all sub-indices can in fact be expressed in terms of indices, so that they do not constitute an additional set of variables. In any reference basis the same is then true, since it can be related to an orthogonal basis, but the expression are (even) more complicated, as in (164). For orders larger than six, expression of sub-indices in terms of indices are not available and hard to obtain.

In our method the sub-indices are essentially treated as additional variables, which can be computed for any representation as easily as the indices themselves. This allows the computation of a trace for any representation, which was our goal. The result is a combination of symmetric fundamental tensors with explicit numerical coefficients, as in (166), or an expression involving both indices and sub-indices. Unfortunately it is much harder to present the result in minimal form, with all representation dependence encapsulated in the indices.

A Indices and Trace identities for Exceptional Algebras

In this appendix we summarize our results on traces for exceptional algebras. All the indices for the lowest dimensional representations are given, including all basic representations. Trace identities are given for all traces of order less than the dual Coxeter number g.

For the classical algebras $A \ldots D$ the trace identities were already given in chapter 6, and index formulas for some representations are given in [6].

The indices provide only a small part of the information contained in the full characters, but it is impractical to present the latter in printed form. We do have an efficient procedure to generate the characters of any representation of any simple Lie-algebra to any desired order. This procedure uses a combination of `Kac` [31] (to compute tensor products) and `FORM` [32] (to multiply, add and subtract characters according to these tensor products), and is available via http://norma.nikhef.nl/pub/∼t58. Obviously this then also provides all indices for algebras and representations not listed in this appendix.

A.1 Indices and trace identities for G_2

Trace identity in the representation (7):

$$\text{Tr } F^4 = \frac{1}{4}(\text{Tr } F^2)^2 \tag{167}$$

The indices of the lowest-dimensional representations are shown in table 5.

Rep.	Dimension	$\frac{I_2}{2}$	I_6
(0,1)	7	1	1
(1,0)	14	4	−26
(0,2)	27	9	39
(1,1)	64	32	−208
(0,3)	77	44	494
(2,0)	77	55	−1235
(0,4)	182	156	3666
(1,2)	189	144	−456
(3,0)	273	351	−20709
(2,1)	286	286	−7904
(0,5)	378	450	19500
(1,3)	448	480	2640
(0,6)	714	1122	82212
(2,2)	729	972	−27378
(4,0)	748	1496	−193324
(3,1)	896	1472	−109408

Table 5: *Indices for* G_2*.*

A.2 Indices and trace identities for F_4

Trace identities in the representation (26):

$$\mathrm{Tr}\ F^4 = 3(\frac{1}{6}\mathrm{Tr}\ F^2)^2 \tag{168}$$

$$\mathrm{Tr}\ F^{10} = \frac{9}{4}(\frac{1}{6}\mathrm{Tr}\ F^2)(\mathrm{Tr}\ F^8) - \frac{7}{4}(\frac{1}{6}\mathrm{Tr}\ F^2)^2(\mathrm{Tr}\ F^6) + \frac{21}{16}(\frac{1}{6}\mathrm{Tr}\ F^2)^5 \tag{169}$$

The indices are listed in table 6.

Rep.	Dimension	$\frac{I_2}{6}$	I_6	I_8	I_{12}
(0,0,0,1)	26	1	1	1	1
(1,0,0,0)	52	3	−7	17	−63
(0,0,1,0)	273	21	1	−119	−1959
(0,0,0,2)	324	27	57	153	2073
(1,0,0,1)	1053	108	−132	612	372
(2,0,0,0)	1053	135	−645	2907	−134373
(0,1,0,0)	1274	147	−133	−1309	125811

Table 6: *Indices for* F_4*.*

A.3 Indices and trace identities for E_6

Trace identities in the representation (56):

$$\mathrm{Tr}\ F^4 = 12(\frac{1}{12}\mathrm{Tr}\ F^2)^2 \tag{170}$$

$$\mathrm{Tr}\ F^7 = \frac{7}{2}(\mathrm{Tr}\ F^5)(\frac{1}{12}\mathrm{Tr}\ F^2) \tag{171}$$

$$\mathrm{Tr}\ F^{10} = \frac{9}{2}(\mathrm{Tr}\ F^8)(\frac{1}{12}\mathrm{Tr}\ F^2) - 7(\mathrm{Tr}\ F^6)(\frac{1}{12}\mathrm{Tr}\ F^2)^2 + \frac{7}{40}(\mathrm{Tr}\ F^5)^2 + 42(\frac{1}{12}\mathrm{Tr}\ F^2)^5 \tag{172}$$

$$\mathrm{Tr}\ F^{11} = \frac{11}{36}(\mathrm{Tr}\ F^6)(\mathrm{Tr}\ F^5) + \frac{605}{126}(\mathrm{Tr}\ F^9)(\frac{1}{12}\mathrm{Tr}\ F^2) - \frac{55}{2}(\mathrm{Tr}\ F^5)(\frac{1}{12}\mathrm{Tr}\ F^2)^3 \tag{173}$$

The indices are in table 7.

Rep.	Dimension	$\frac{I_2}{6}$	I_5	I_6	I_8	I_9	I_{12}
(1,0,0,0,0,0)	27	1	1	1	1	1	1
(0,0,0,0,1,0)	27	1	−1	1	1	−1	1
(0,0,0,0,0,1)	78	4	0	−6	18	0	−62
(0,1,0,0,0,0)	351	25	11	−5	−101	−229	−2021
(0,0,0,1,0,0)	351	25	−11	−5	−101	229	−2021
(0,0,0,0,2,0)	351	28	−44	58	154	−284	2074
(2,0,0,0,0,0)	351	28	44	58	154	284	2074
(1,0,0,0,1,0)	650	50	0	60	36	0	116
(0,0,0,0,1,1)	1728	160	−88	−80	664	152	424
(1,0,0,0,0,1)	1728	160	88	−80	664	−152	424
(0,0,0,0,0,2)	2430	270	0	−720	3672	0	−131928
(0,0,1,0,0,0)	2925	300	0	−270	−918	0	122202

Table 7: *Indices for E_6.*

A.4 Indices and trace identities for E_7

Trace identities in the representation (56):

$$\mathrm{Tr}\ F^4 = 24(\frac{1}{24}\mathrm{Tr}\ F^2)^2 \tag{174}$$

$$\begin{aligned}\mathrm{Tr}\ F^{16} \quad = \quad & -\frac{8567}{5220}(\frac{1}{24}\mathrm{Tr}\ F^2)(\mathrm{Tr}\ F^6)(\mathrm{Tr}\ F^8) \\ & +\frac{2360}{319}(\frac{1}{24}\mathrm{Tr}\ F^2)(\mathrm{Tr}\ F^{14}) \\ & +\frac{61607}{23490}(\frac{1}{24}\mathrm{Tr}\ F^2)^2(\mathrm{Tr}\ F^6)^2 \\ & -\frac{63700}{2871}(\frac{1}{24}\mathrm{Tr}\ F^2)^2(\mathrm{Tr}\ F^{12}) \\ & +\frac{21164}{783}(\frac{1}{24}\mathrm{Tr}\ F^2)^3(\mathrm{Tr}\ F^{10}) \\ & +\frac{7397}{522}(\frac{1}{24}\mathrm{Tr}\ F^2)^4(\mathrm{Tr}\ F^8) \\ & -\frac{72254}{783}(\frac{1}{24}\mathrm{Tr}\ F^2)^5(\mathrm{Tr}\ F^6) \\ & +\frac{222898}{261}(\frac{1}{24}\mathrm{Tr}\ F^2)^8 \\ & +\frac{13}{54}(\mathrm{Tr}\ F^6)(\mathrm{Tr}\ F^{10}) \\ & +\frac{13}{160}(\mathrm{Tr}\ F^8)^2\end{aligned} \tag{175}$$

The indices are in table 8.

Rep.	Dim.	$\frac{I_2}{12}$	I_6	I_8	I_{10}	I_{12}	I_{14}	I_{18}
(0,0,0,0,0,1,0)	56	1	1	1	1	1	29	1229
(1,0,0,0,0,0,0)	133	3	−2	10	−2	−30	542	−111658
(0,0,0,0,0,0,1)	912	30	−10	−82	230	−2082	−39170	96018190
(0,0,0,0,0,2,0)	1463	55	90	174	570	2134	238650	161267970
(0,0,0,0,1,0,0)	1539	54	24	−72	−456	−1992	−235944	−161018664
(1,0,0,0,0,1,0)	6480	270	30	774	−210	534	73350	−102108810
(2,0,0,0,0,0,0)	7371	351	−354	2682	−834	−63438	4748094	−14489069226
(0,1,0,0,0,0,0)	8645	390	−200	40	760	57480	−4368520	14620498520
(0,0,0,0,0,3,0)	24320	1440	3600	10176	50160	292896	59512080	167838228720
(0,0,0,1,0,0,0)	27664	1430	−10	−3442	−7450	63998	32976190	149694252430
(0,0,0,0,0,1,1)	40755	2145	530	−3658	13490	−171138	2436850	−9081228710
(0,0,0,0,1,1,0)	51072	2832	2872	256	−16568	−172464	−46178632	−158703316792
(1,0,0,0,0,0,1)	86184	4995	−3165	963	36195	−366717	−37725705	−137019575865
(1,0,0,0,0,2,0)	150822	9450	8400	41328	59280	410928	30093840	30366263760
(1,0,0,0,1,0,0)	152152	9152	−328	9320	−78088	−197560	−28617992	−27126731432
(3,0,0,0,0,0,0)	238602	17940	−26380	271676	−116620	−13615284	1492228660	−16354668799100
(0,0,0,0,0,0,2)	253935	17820	−9000	−94824	404280	−6024744	−323856360	12685209865560
(0,0,0,0,0,4,0)	293930	24310	88400	329888	2153840	17066368	4880546320	30243257914480
(2,0,0,0,0,1,0)	320112	21762	−9318	155826	−75318	−3178974	303759378	−674257133022
(0,1,0,0,0,1,0)	362880	23760	600	12672	22440	3531792	−239671080	806089955880
(0,0,1,0,0,0,0)	365750	24750	−9000	−63240	79800	2601720	278208600	−12473996293800

Table 8: *Indices for E_7.*

A.5 Indices and trace identities for E_8

Trace identities in the representation (248):

$$\mathrm{Tr}\ F^4 = 36(\frac{1}{60}\mathrm{Tr}\ F^2)^2 \tag{176}$$

$$\mathrm{Tr}\ F^6 = 30(\frac{1}{60}\mathrm{Tr}\ F^2)^3 \tag{177}$$

$$\mathrm{Tr}\ F^{10} = \frac{15}{4}(\mathrm{Tr}\ F^8)(\frac{1}{60}\mathrm{Tr}\ F^2) - \frac{315}{4}(\frac{1}{60}\mathrm{Tr}\ F^2)^5 \tag{178}$$

$$\begin{aligned}\mathrm{Tr}\ F^{16} \quad = \quad & \frac{3003}{64}(\mathrm{Tr}\ F^8)(\frac{1}{60}\mathrm{Tr}\ F^2)^4 \\ & +\frac{143}{1920}(\mathrm{Tr}\ F^8)^2 \\ & -\frac{273}{20}(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2)^2 \\ & +\frac{29}{5}(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^2) \\ & -\frac{147147}{128}(\frac{1}{60}\mathrm{Tr}\ F^2)^8 \end{aligned} \tag{179}$$

$$\begin{aligned}\mathrm{Tr}\ F^{22} \quad = \quad & -\frac{29393}{39360}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2) \\ & +\frac{323}{1920}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{14})\end{aligned}$$

$$
\begin{aligned}
&+\frac{3193461271}{1679360}(\mathrm{Tr}\ F^8)(\frac{1}{60}\mathrm{Tr}\ F^2)^7 \\
&+\frac{361151791}{50380800}(\mathrm{Tr}\ F^8)^2(\frac{1}{60}\mathrm{Tr}\ F^2)^3 \\
&-\frac{298544701}{524800}(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2)^5 \\
&+\frac{3438981}{16400}(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^2)^4 \\
&-\frac{163457}{6560}(\mathrm{Tr}\ F^{18})(\frac{1}{60}\mathrm{Tr}\ F^2)^2 \\
&+\frac{623}{82}(\mathrm{Tr}\ F^{20})(\frac{1}{60}\mathrm{Tr}\ F^2) \\
&-\frac{164475086139}{3358720}(\frac{1}{60}\mathrm{Tr}\ F^2)^{11} \qquad (180)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\ F^{26} \;=\; &\frac{1888509563603}{44213686272}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2)^3 \\
&-\frac{13299145169}{1347978240}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^2)^2 \\
&+\frac{22517}{145152}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{18}) \\
&-\frac{400077914058102709}{7074189803520}(\mathrm{Tr}\ F^8)(\frac{1}{60}\mathrm{Tr}\ F^2)^9 \\
&-\frac{10415984073858649}{35370949017600}(\mathrm{Tr}\ F^8)^2(\frac{1}{60}\mathrm{Tr}\ F^2)^5 \\
&-\frac{15835512121}{221838704640}(\mathrm{Tr}\ F^8)^3(\frac{1}{60}\mathrm{Tr}\ F^2) \\
&+\frac{5083}{42336}(\mathrm{Tr}\ F^{12})(\mathrm{Tr}\ F^{14}) \\
&+\frac{96960058189033}{5554483200}(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2)^7 \\
&-\frac{3481855}{6318648}(\mathrm{Tr}\ F^{12})^2(\frac{1}{60}\mathrm{Tr}\ F^2) \\
&-\frac{28373592046607}{4386278400}(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^2)^6 \\
&+\frac{1446721465417}{1973825280}(\mathrm{Tr}\ F^{18})(\frac{1}{60}\mathrm{Tr}\ F^2)^4 \\
&-\frac{1354577341}{7441008}(\mathrm{Tr}\ F^{20})(\frac{1}{60}\mathrm{Tr}\ F^2)^3 \\
&+\frac{929825}{105868}(\mathrm{Tr}\ F^{24})(\frac{1}{60}\mathrm{Tr}\ F^2) \\
&+\frac{506251846536678653}{336866181120}(\frac{1}{60}\mathrm{Tr}\ F^2)^{13} \qquad (181)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\ F^{28} \;=\; &\frac{666245292706591}{2412453120000}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^2)^4 \\
&-\frac{575488458230809}{9649812480000}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^2)^3 \\
&+\frac{572769851}{1766476800}(\mathrm{Tr}\ F^8)(\mathrm{Tr}\ F^{18})(\frac{1}{60}\mathrm{Tr}\ F^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1090453}{6133600}(\mathrm{Tr}\ F^{8})(\mathrm{Tr}\ F^{20}) \\
&-\frac{20281446070347775183}{56144363520000}(\mathrm{Tr}\ F^{8})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{10} \\
&-\frac{19127927}{2770944000}(\mathrm{Tr}\ F^{8})^{2}(\mathrm{Tr}\ F^{12}) \\
&-\frac{1076029152338172071}{561443635200000}(\mathrm{Tr}\ F^{8})^{2}(\frac{1}{60}\mathrm{Tr}\ F^{2})^{6} \\
&-\frac{13886644775887}{36092805120000}(\mathrm{Tr}\ F^{8})^{3}(\frac{1}{60}\mathrm{Tr}\ F^{2})^{2} \\
&+\frac{3432221}{8131200}(\mathrm{Tr}\ F^{12})(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^{2}) \\
&+\frac{108447101840059177}{969830400000}(\mathrm{Tr}\ F^{12})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{8} \\
&-\frac{7079594327}{1941730560}(\mathrm{Tr}\ F^{12})^{2}(\frac{1}{60}\mathrm{Tr}\ F^{2})^{2} \\
&-\frac{23843919029848581}{574393600000}(\mathrm{Tr}\ F^{14})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{7} \\
&+\frac{157757}{1936000}(\mathrm{Tr}\ F^{14})^{2} \\
&+\frac{1607603265138373}{344636160000}(\mathrm{Tr}\ F^{18})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{5} \\
&-\frac{61788727534443}{54567392000}(\mathrm{Tr}\ F^{20})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{4} \\
&+\frac{286680771}{6654560}(\mathrm{Tr}\ F^{24})(\frac{1}{60}\mathrm{Tr}\ F^{2})^{2} \\
&+\frac{565720659186764144207}{58817904640000}(\frac{1}{60}\mathrm{Tr}\ F^{2})^{14} \qquad (182)
\end{aligned}
$$

.

B The computer program for the reduction into invariants

We have implemented the reduction algorithms into a computer program[6] in the language of FORM [32]. This language is particularly suited for these types of problems. Because of all the problems with reduction identities when the number of vertices becomes large, we have restricted the program to the case of no more than 16 vertices. If the user needs to run the program with more vertices, it can be extended by analogy, but many new reduction identities would have to be derived. Alternatively one could decide to not reduce a number of contractions with f's in them and leave them for later evaluation. The user should be warned however that some diagrams with 16 vertices may need quite some computer time and resources for their evaluation.

The program consists of three parts. The first part reduces all traces of matrices which do not belong to the adjoint representation. Much attention is given to a potential contraction of indices. The special cases have been written out in one highly nested loop to take the maximum benefit of these contractions. This saves much work when we have to use the algorithm of equation (30) for the remaining trace. It is quite useful to rewrite each invariant immediately with the notation of equation (100). This removes invariants which have more than one line contracted with the same

[6]This program can be obtained from "http://norma.nikhef.nl/~t68/FORMapplications/Color"

Rep.	Dimension	$\frac{I_2}{30}$	I_8	I_{12}	I_{14}
(1,0,0,0,0,0,0,0)	248	1	1	1	1
(0,0,0,0,0,0,1,0)	3875	25	−17	223	−521
(2,0,0,0,0,0,0,0)	27000	225	393	2073	8961
(0,1,0,0,0,0,0,0)	30380	245	119	−1801	−7945
(0,0,0,0,0,0,0,1)	147250	1425	−801	−3921	90423
(1,0,0,0,0,0,1,0)	779247	8379	357	64677	−207291
(3,0,0,0,0,0,0,0)	1763125	22750	64330	653050	3872050
(0,0,1,0,0,0,0,0)	2450240	29640	576	−300624	−407160
(1,1,0,0,0,0,0,0)	4096000	51200	59264	−176896	−1416448
(0,0,0,0,0,0,2,0)	4881384	65610	−68202	1623078	−5978610
(0,0,0,0,0,1,0,0)	6696000	88200	−64176	344544	2464392
(1,0,0,0,0,0,0,1)	26411008	372736	12544	−928256	20640256
(2,0,0,0,0,0,1,0)	70680000	1083000	991440	15398400	1956600
(0,1,0,0,0,0,1,0)	76271625	1148175	−64071	732969	−67564287
(4,0,0,0,0,0,0,0)	79143000	1404150	6100842	97389402	723747954
(0,0,0,1,0,0,0,0)	146325270	2360085	−942669	−20062029	81822195
(0,2,0,0,0,0,0,0)	203205000	3441375	3576615	−53721225	−402564225
(2,1,0,0,0,0,0,0)	281545875	4843800	10500696	50453496	233862408
(0,0,0,0,0,0,1,1)	301694976	5068800	−4540800	36284160	244435200
(1,0,1,0,0,0,0,0)	344452500	5740875	3591945	−51773175	−133939575
(1,0,0,0,0,0,2,0)	820260000	14773500	−7295820	368355300	−1650963300
(1,0,0,0,0,1,0,0)	1094951000	19426550	−3552406	74388314	342570098
(2,0,0,0,0,0,0,1)	2172667860	40883535	36197469	182867949	3494811285
(0,1,0,0,0,0,0,1)	2275896000	42214200	−2179296	−439704336	1139298552
(0,0,0,0,0,0,3,0)	2903770000	60885500	−95237740	3610174100	−17484769900
(3,0,0,0,0,0,1,0)	3929713760	79228100	167887580	2727186380	10466026340
(0,0,0,0,0,0,0,2)	4076399250	83281275	−77203203	−459950403	17875089309
(0,0,1,0,0,0,1,0)	4825673125	93400125	−36251565	−180072525	−4026565725
(0,0,0,0,1,0,0,0)	6899079264	139094340	−107301348	−484327668	13082745060

Table 9: *Indices for E_8, part one.*

f in a very natural way without any extra pattern matching. For this type of rewriting `FORM` even has a special statement (ToVector).

The second part eliminates the loops of f's. Here we do not have to worry about contracted indices inside the trace, because `FORM` looks each time for the smallest loop to make its next reduction. Hence this part of the program is much simpler. Because this reduction is much faster than the general reduction the first routine calls this reduction routine each time after it has removed contracted indices and generated more f's. Very often such a removal generates a loop of f's and if this is removed it may introduce new contracted indices. The net result can be a significant increase in speed. In some cases this is not faster however. Hence there is the option not to hunt for f-loops until all other representations have been rewritten.

The third part of the program contains the reduction identities. Here the program tries to eliminate contractions with f's that are obviously zero and to rewrite the contractions for which it can construct meaningful identities. This is a rather peculiar piece of programming. The derivation of the reduction identities is by no means a fixed algorithm. The equation (96) can be applied to one of the d_R and one of the f's, but it is not always clear which pair will give good results and which will make things worse. In general it seems to be a good strategy to try to increase the number of contractions between the invariants (number of common indices). In that case there will be more contractions between the f's and hence more chance of loops that contain only f's.

I_{18}	I_{20}	I_{24}	I_{30}
1	41	199	61
−281	720023	−8538743	107370139
131601	20785953	1677921087	32641770621
−130825	−21485681	−1669283839	−32749110565
−3057657	1091333799	69614416281	−13332825829797
3122949	−891843603	−70053431037	13392095601009
158684770	53898275690	19220216027590	4181215457807530
96664440	42322995216	18320504001024	4178331237939240
−129144448	−47849709056	−18734041431424	−4186512285307648
−131658210	336853672758	−90053043268518	53479289344535190
32750232	−373796000256	71666802925296	−57656847733698648
−887500544	601951099904	−72720785548544	50171949070921216
1422281640	−496637707680	79406160046320	−50034441438444840
368688753	−283297282191	−5188200203889	7356020195237373
52000832994	23915855965002	18712064549246118	18617881056766529514
10867212915	−105084506709	−9407941453913691	−16544807547516665625
−32456806065	−12291541050705	−4580281352771055	−1096019298618319485
−14097200472	−10058442950664	−14025344636105496	−17530899917498537112
−22110000000	4164533921280	9773219242968960	16439802976484140800
13626625545	10661778671985	13958084762321535	17581170727669309605
−10809258420	79696508512740	−32082728508187860	−3180131072643324180
19892530658	−97084816917686	17486065830031526	−14251422467900989462
−107677764315	118967521198869	−19927893621180789	12855700625814459585
−134688633528	112636179273744	−29916313535701344	−3637801450738060968
−675793438060	2261894204588980	−1168369134581957620	7589921677844796014660
721988940500	90443981815180	94451334234961220	3296282167094693780
−1346442094371	689963459361477	507354515738178603	−7162275330089795125911
518531279235	147692804386155	73753893160393605	17639532662269149015
104563589460	−617925299730228	−535510167643211772	7128890471280203011860

Table 10: *Indices for E_8, part two.*

This is not always possible in a direct way, and sometimes we have to just try equation (96) in the hope that in the next pass the improvement will follow. The selection of the invariant and the f that take part in this game has to be done carefully and the code consists of two pieces that make a slightly different choice. By running a loop that contains the first choice twice and the second choice once, we could reduce nearly everything up to 16 vertices. Of course we had to define the object of formula (112) as a separate entity. Similarly we had to define three such objects at level 16. The program also defines the topologies at level 16 that we did not present in the text.

C Some examples

We have run a number of color traces with the program. Here we present the results with some timing information. All runs were done on a PentiumPro 200 processor running NextStep. First we look at traces of the type

$$Tr[T_R^{i_1}\cdots T_R^{i_n}T_R^{i_1}\cdots T_R^{i_n}]$$

as such traces represent some type of maximal complexity. Here we show the results to $n=7$ in table 11.

Actually the program can go to $n=8$. For this it took about 1520 sec. We do not give the answer here. Similarly we can calculate this in the adjoint representation only. This is of course much

n	time	result
2	0.23 s	$N_A I_2(R)(C_R - C_A/2)$
3	0.23 s	$N_A I_2(R)(C_R - C_A)(C_R - C_A/2)$
4	0.25 s	$d_R^{abcd} d_A^{abcd} + N_A I_2(R)(C_R^3 - 3\ C_R^2 C_A + 11/4\ C_R C_A^2 - 19/24\ C_A^3)$
5	0.95 s	$d_R^{abcd} d_A^{abcd}(5C_R - 6C_A) + 1/3 I_2(R)\ d_A^{abcd} d_A^{abcd}$ $+N_A I_2(R)(C_R^4 - 5\ C_R^3 C_A + 35/4\ C_R^2 C_A^2 - 155/24\ C_R C_A^3 + 125/72\ C_A^4)$
6	2.59 s	$-8 d_R^{abcdef} d_A^{abcdef} + 6 d_R^{abcd} d_A^{abef} d_A^{cdef} + I_2(R) d_A^{abef} d_A^{cdef}(2\ C_R - 199/60\ C_A)$ $+d_R^{abef} d_A^{cdef}(15\ C_R^2 - 87/2\ C_R C_A + 179/6\ C_A^2) + N_A I_2(R)(C_R^5$ $-15/2\ C_R^4 C_A + 85/4\ C_R^3 C_A^2 - 115/4\ C_R^2 C_A^3 + 905/48\ C_R C_A^4\ - 1405/288\ C_A^5)$
7	34.9 s	$+112/3\ d_R^{abcdef} d_A^{abcg} d_A^{defg} - 328/9\ d_A^{abcdef} d_R^{abcg} d_A^{defg}$ $+d_R^{abcdef} d_A^{abcdef}(-56\ C_R + 296/3\ C_A)$ $+d_R^{abcd} d_A^{abef} d_A^{cdef}(42\ C_R - 749/10\ C_A) + 67/15\ I_2(R) d_A^{abcd} d_A^{abef} d_A^{cdef}$ $+d_R^{abcd} d_A^{abcd}(35\ C_R^3 - 357/2 C_R^2 C_A + 868/3\ C_R C_A^2 - 2695/18\ C_A^3)$ $+I_2(R) d_A^{abcd} d_A^{abcd}(7\ C_R^2 - 1603/60 C_R C_A + 497/20\ C_A^2)$ $+N_A I_2(R)(+C_R^6 - 21/2\ C_R^5 C_A + 175/4\ C_R^4 C_A^2 - 280/3\ C_R^3 C_A^3$ $+5215/48\ C_R^2 C_A^4 - 19075/288\ C_R C_A^5 + 43357/2592\ C_A^6)$

Table 11: *Results for traces of the type* $Tr[T_R^{i_1} \cdots T_R^{i_n} T_R^{i_1} \cdots T_R^{i_n}]$.

faster because the program selects automatically the smallest loops. These results can be found in table 12. For $n = 8$ the program took 1.5 sec We notice that here the computer time does not

n	time	result
2	0.15 s	$\frac{1}{2} N_A C_A^2$
3	0.20 s	0
4	0.23 s	$d_A^{abcd} d_A^{abcd} - \frac{1}{24} N_A C_A^4$
5	0.78 s	$\frac{2}{3} C_A d_A^{abcd} d_A^{abcd} - \frac{1}{36} N_A C_A^5$
6	0.81 s	$d_A^{abcd} d_A^{abef} d_A^{cdef} + \frac{1}{4} C_A^2 d_A^{abcd} d_A^{abcd} - \frac{13}{864} N_A C_A^6$
7	0.89 s	$-\frac{8}{9} d_A^{abcdef} d_A^{abcg} d_A^{defg} + \frac{53}{30} C_A\ d_A^{abcd} d_A^{abef} d_A^{cdef}$ $-\frac{5}{648} N_A C_A^7$

Table 12: Like the previous table but now in the adjoint representation.

increase very much with the number of crossing lines (the number of vertices and hence the weight is $2n$). There is actually more 'compilation time' than 'execution' time. The jump in time going from $n = 4$ to $n = 5$ represents the use of the reduction algorithms to eliminate f's. In that case the program needs considerably more compilation time.

The fact that diagrams with only vertices in the adjoint representation are easier to evaluate than the diagrams with vertices in the other representations is exactly the opposite of what happens with the Cvitanovic algorithms [1]. For them each f is converted to one or more terms with one or more matrices in the fundamental representation. This can lead to an avalanche of terms at the intermediate stages, because no advantage is taken from the potentially simpler structures. In the case of the traces in the fundamental representation the Cvitanovic algorithms are much faster. These algorithms do not worry about symmetrizations and are directly applicable to such traces.

As an example of high complexity for purely adjoint diagrams we take the one topology of girth 6 with 14 vertices. It is also called the Coxeter graph. In this the smallest loop has 6 vertices. The result is rather short:

$$G_6(n=14) \;=\; \frac{16}{9} d_A^{abcdef} d_A^{abcg} d_A^{defg} - \frac{8}{15} C_A\, d_A^{abcd} d_A^{abef} d_A^{cdef} + \frac{1}{648} N_A C_A^7 \qquad (183)$$

This took 1.6 sec.

D Explicit expressions

Here we present some expressions for a number of invariants. These are mostly invariants for representations that can be used as reference representations. The expressions are given in terms of the normalization factor η defined in (27).

In all cases the tensors (referred to as $d_n(R)$) are defined as

$$d_R^{a_1 \ldots a_n} = \mathrm{Str}\ T_R^{a_1} \ldots T_R^{a_n}\ . \qquad (184)$$

In particular no traces are subtracted and no overall factors are included. For $SO(N)$ we deviate from the preferred index normalization of table (4), since otherwise we would have to deal with a few low-N $SO(N)$ cases separately.

Results for the fundamental (vector) representation V of SU(N):

$$\begin{aligned}
C_V &= \frac{a}{N}(N^2-1) && (185)\\
C_A &= 2aN && (186)\\
d_{33}(VV) &= \frac{a^3}{2N}(N^2-1)(N^2-4) && (187)\\
d_{44}(VV) &= \frac{a^4}{6N^2}(N^2-1)(N^4-6N^2+18) && (188)\\
d_{55}(VV) &= \frac{a^5}{24N^3}(N^2-1)(N^2-2)(N^4+24) && (189)\\
d_{433}(VVV) &= \frac{a^5}{6N^2}(N^2-1)(N^2-4)(N^2-6) && (190)\\
d_{66}(VV) &= \frac{a^6}{120N^4}(N^2-1)(N^8+6N^6-60N^4+600) && (191)\\
d_{633}(VVV) &= \frac{a^6}{480N^3}(N^2-1)^2(N^2-4)^2 && (192)\\
d_{543}(VVV) &= \frac{a^6}{288N^3}(N^2-1)(N^2-4)(N^4-6N^2+18) && (193)\\
d_{444}(VVV) &= \frac{a^6}{27N^3}(N^2-1)(N^6-9N^4+81N^2-189) && (194)\\
d_{3333}(VVVV) &= \frac{a^6}{8N^2}(N^2-1)(N^2-4)(N^2-12) && (195)
\end{aligned}$$

with $a = \frac{1}{2}\eta$ (see equation(27)). The choice $\eta = 1; a = \frac{1}{2}$ corresponds to the most commonly used normalization. Then Tr $T_V^a T_V^b = \frac{1}{2}\delta^{ab}$. In SU(N) the vector representation is always equal to the reference representation.

For the vector representation of SO(N) we have:

$$\begin{aligned}
C_V &= \frac{a}{2}(N-1) && (196)\\
C_A &= a(N-2) && (197)
\end{aligned}$$

$$
\begin{aligned}
N_A &= \frac{1}{2}N(N-1) && (198)\\
d_{44}(VV) &= \frac{a^4}{24}N_A(N^2-N+4) && (199)\\
&= \frac{a^4}{12}N_A(N_A+2) && (200)\\
d_{66}(VV) &= \frac{a^6}{1920}N_A(N^4-2N^3+33N^2-32N+52) && (201)\\
&= \frac{a^6}{480}N_A(N_A^2+16N_A+13) && (202)\\
d_{444}(VVV) &= \frac{a^6}{432}N_A(2N^3-3N^2+33N-16) && (203)\\
& && (204)
\end{aligned}
$$

with $a = \eta$ (see equation(27)). In this case $a = \eta = 2$ is the most frequently used convention. Then $\mathrm{Tr}\ T_V^a T_V^b = 2\delta^{ab}$.

Note that for SO(N), $N \leq 6$ the vector representation is *not* the reference representation. The formulas for the reference representation for those groups can be read off from the appropriate SU(N) or Sp(N) results. The tensors used in the foregoing formulas are normalized so that $I_4(V) = I_6(V) = 1$ in (95). As explained above, this differs from the index normalization chosen in table (4) for $SO(7)$ and $SO(8)$. For these groups our convention is to make I_4 twice as large, and hence d_4 twice as small.

For the vector representation of Sp(N) we have:

$$
\begin{aligned}
C_V &= \frac{a}{2}(N+1) && (205)\\
C_A &= a(N+2) && (206)\\
N_A &= \frac{1}{2}N(N+1) && (207)\\
d_{44}(VV) &= \frac{a^4}{24}N_A(N^2+N+4) && (208)\\
&= \frac{a^4}{12}N_A(N_A+2) && (209)\\
d_{66}(VV) &= \frac{a^6}{1920}N_A(N^4+2N^3+33N^2+32N+52) && (210)\\
&= \frac{a^6}{480}N_A(N_A^2+16N_A+13) && (211)\\
d_{444}(VVV) &= \frac{a^6}{432}N_A(2N^3+3N^2+33N+16) && (212)\\
& && (213)
\end{aligned}
$$

with $a = \frac{1}{2}\eta$ (see equation(27)). In this case the vector coincides with the reference representation.

For all groups for which $I_4(A) = 0$ we can derive a number of invariants with relatively simple methods. This is of particular interest for the exceptional algebras, which have $I_4(R) = 0$ for *any* representation. We will present the following formulas for tensors defined in the adjoint representation, which is not the reference representation (except for E_8). The reason for doing this is that it allows us to write a single set of relations for all algebras. It is straightforward to re-express these results in terms of the reference representation. To do so one needs the relation between adjoint tensors and reference tensors, which follows directly from the characters of both representations; the latter can be computed by means of the methods used in Appendix A. The adjoint representation is unsuitable for the odd traces of E_6, which are discussed separately below.

To do the computations, first we notice that

$$d_A^{i_1 i_2 i_3 i_4} = I_{22}(A)(\delta^{i_1 i_2}\delta^{i_3 i_4} + \delta^{i_1 i_3}\delta^{i_2 i_4} + \delta^{i_1 i_4}\delta^{i_2 i_3})/3 \tag{214}$$

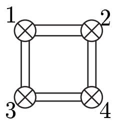
with $I_{22}(A) = \frac{5}{2}C_A^2/(N_A+2)$. Using this and equation (118) we can determine $d_{66}(AA)$. Next we can go even further by using a technique similar to the one used to derive equation(118): We run the program for the product of two traces with 8 vertices. First we run it for a representation R in one trace and the adjoint representation in the other. After the run we put R equal to A. This gives an expression that includes $d_{88}(AA)$ and $d_{844}(AAA)$ and objects that contain combinations of d_4 and d_6. We can run the same traces with the program, but starting with both of them in the adjoint representation. In that case we obtain an expression that does not contain $d_{88}(AA)$. This gives us the required equation. Now we substitute d_4 and we need an equation for $d_8^{jji_1\cdots i_6}$ which is also easy to obtain with the program:

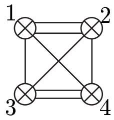
$$d_A^{jji_1\cdots i_6} = \frac{10}{21}d_A^{i_1\cdots i_6} + \frac{1}{6}(d_A^{ji_1i_2i_3}d_A^{ji_4i_5i_6} + \cdots)/10 \tag{215}$$

in which we have to take the 10 symmetric combinations over the indices in the last term. For d_6 we have a similar equation which is given by equation (142). In total we obtain:

$$\begin{aligned}
d_{44}(AA) &= \frac{25\ C_A^4}{12(N_A+2)} && (216)\\
d_{66}(AA) &= \frac{C_A^6 N_A}{(N_A+2)^2}(\frac{797}{288} + \frac{8}{27}N_A - \frac{1}{864}N_A^2) && (217)\\
d_{444}(AAA) &= \frac{C_A^6 N_A}{(N_A+2)^2}(\frac{125}{27} + \frac{125}{216}N_A) && (218)\\
d_{644}(AAA) &= \frac{175}{48}\frac{C_A^7 N_A}{(N_A+2)^2} && (219)\\
d_{88}(AA) &= \frac{C_A^8 N_A}{(N_A+2)^3}(\frac{3425}{1008} + \frac{111025}{145152}N_A + \frac{125}{6804}N_A^2 + \frac{25}{435456}N_A^3) && (220)\\
d_{844}(AAA) &= \frac{C_A^8 N_A}{(N_A+2)^3}(\frac{125}{24} + \frac{625}{288}N_A) && (221)\\
d_{664}(AAA) &= \frac{C_A^8 N_A}{(N_A+2)^3}(\frac{5455}{864} + \frac{3485}{2592}N_A - \frac{5}{2592}N_A^2) && (222)\\
d_{4444a}(AAAA) &= \frac{C_A^8 N_A}{(N_A+2)^3}(\frac{3125}{324} + \frac{625}{216}N_A + \frac{625}{1296}N_A^2) && (223)\\
d_{4444b}(AAAA) &= \frac{C_A^8 N_A}{(N_A+2)^3}(\frac{6875}{648} + \frac{3125}{1296}N_A) && (224)
\end{aligned}$$

The last two topologies are defined as

$d_{4444a}(p_1,p_2,p_3,p_4) =$  (225)

$d_{4444b}(p_1,p_2,p_3,p_4) =$  (226)

group	G_2	F_4	E_6	E_7	E_8
N_A	14	52	78	133	248
C_A	$4\ \eta$	$9\ \eta$	$24\ \eta$	$18\ \eta$	$30\ \eta$

Table 13: *Values of N_A and C_A for the exceptional groups.*

The appropriate values to be substituted for the various groups are given in table 13.

For E_6 we also have to consider the invariants with 5 indices. These we can obtain with the reference representation r. For invariants that involve this representation we have:

$$\begin{aligned}
C_r &= \frac{52}{3}\ \eta && (227)\\
N_r &= 27 && (228)\\
d_{44}(rr) &= 18720\ \eta^4 && (229)\\
d_{55}(rr) &= 291200\ \eta^5 && (230)\\
d_{66}(rr) &= 1023152\ \eta^6 && (231)\\
d_{444}(rrr) &= 536640\ \eta^6 && (232)\\
d_{77}(rr) &= \frac{11211200}{3}\ \eta^7 && (233)\\
d_{644}(rrr) &= \frac{20616544}{5}\ \eta^7 && (234)\\
d_{554}(rrr) &= 582400\ \eta^7 && (235)\\
d_{88}(rr) &= \frac{466346192}{27}\ \eta^8 && (236)
\end{aligned}$$

A number of these quantities can be obtained in various ways and serve as a check of our programs.

E Chiral representations of $SO(2m)$

The algebra D_m has an index of order m that vanishes for the vector representation, and is non-zero only for chiral representations. The latter are characterized by having unequal values for the last two Dynkin labels, $a_{m-1} \neq a_m$. The simplest representations of this type are the spinors, $S = (0, \ldots, 0, 1, 0)$ and $S' = (0, \ldots, 0, 1)$. If m is odd S' is the complex conjugate of S. Note that for odd m the extra index (henceforth referred to as the "chiral index") has odd order, unlike all other $SO(N)$ indices, whereas for m even there are two distinct indices of even order, namely the chiral index and one of the regular indices.

Since the chiral index vanishes for the vector representation, whereas all indices are non-zero for the spinor, it might be argued that the latter is perhaps a more suitable choice for the reference representation. However, the vector representation has other advantages, perhaps most importantly that the trace-identities for traces of order larger than $2m$ are simpler. The main drawback of this choice is that it requires a separate discussion of chiral traces, which we give here.

Our conventions for $SO(2m)$ are as follows. The generators in the vector representation are

$$M^{\mu\nu}_{ij} = i\sqrt{\eta/2}(\delta^{\mu}_{i}\delta^{\nu}_{j} - \delta^{\nu}_{i}\delta^{\mu}_{j}) \tag{237}$$

This is a complete set of generators for $\mu < \nu$. The normalization factor $\sqrt{\eta/2}$ is introduced in (27). Clearly for $SO(N)$ the most attractive choice is $\eta = 2$.

The generators of the (chiral plus anti-chiral) spinor representation are then

$$\Sigma^{\mu\nu} = \frac{i}{4}\sqrt{\eta/2}\,[\gamma^\mu, \gamma^\nu] \ , \tag{238}$$

with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The dimension of this Clifford algebra is 2^m.

For the chiral γ-matrix (often referred to as γ_5) we choose

$$\gamma_c = (i)^m \gamma^1 \dots \gamma^{2m} \tag{239}$$

The phase is chosen so that γ_c is hermitean (assuming the other γ-matrices are chosen hermitean as well). Hermiticity fixes the overall factor up to an m-dependent sign, for which we have made a conventional choice.

The chiral spinor generators are

$$\Sigma^{\mu\nu}_{\pm} = \frac{1}{2}\Sigma^{\mu\nu}(1 \pm \gamma_c) \tag{240}$$

The symmetrized trace of m such generators is

$$\mathrm{Str}\Sigma^{\mu_1\nu_1}_{\pm} \dots \Sigma^{\mu_m\nu_m}_{\pm} = \frac{1}{2}\mathrm{Str}\Sigma^{\mu_1\nu_1} \dots \Sigma^{\mu_m\nu_m} \pm \frac{1}{2}\mathrm{Str}\Sigma^{\mu_1\nu_1} \dots \Sigma^{\mu_m\nu_m}\gamma_c \tag{241}$$

The first term yields (for m even) an ordinary d-tensor, the second one yields the $\tilde{d}$ tensor. By general arguments,

$$\begin{aligned}\mathrm{Str}\Sigma^{\mu_1\nu_1}_{\pm} \dots \Sigma^{\mu_n\nu_n}_{\pm} &= I_n(S_\pm) d^{[\mu_1\nu_1],\dots,[\mu_n\nu_n]} + \text{lower order} \\ &\quad \pm \tilde{I}_n(S_\pm)\tilde{d}^{[\mu_1\nu_1],\dots,[\mu_n\nu_n]} \ ,\end{aligned} \tag{242}$$

where $[\mu_1\nu_1]$ denotes an adjoint index. By convention

$$\tilde{I}_n(S_\pm) = \pm 1 \tag{243}$$

then

$$\tilde{d}^{[\mu_1\nu_1],\dots,[\mu_m\nu_m]} = \frac{1}{2}\mathrm{Str}\Sigma^{\mu_1\nu_1} \dots \Sigma^{\mu_m\nu_m}\gamma_c = \frac{1}{2}(\eta/2)^{m/2}\epsilon^{\mu_1,\dots,\nu_m} \ , \tag{244}$$

with the definition $\epsilon^{1,\dots,2m} = 1$. Note that the interpretation requires *pairs* of indices to be identified with an adjoint index, and that the ϵ tensor is indeed fully symmetric under permutation of pairs of indices.

This gives us an explicit expression for the extra tensor, and computing invariants that involve this tensor is now straightforward. One can either do that by computing all other tensors also in the spinor representation. Then computing the invariants amounts to simple γ-algebra. Since expressions for the spinor characters in terms of vector traces are explicitly known (see (65) and (66)), all tensors in the spinor representation can be re-expanded in terms of reference tensors. Alternatively it is also very easy to work directly with the reference tensors, which, using (237) can be expressed completely in terms of kronecker δ's with vector indices. These indices are to be taken in pairs and identified with adjoint indices, and can then be pairwise contracted with the ϵ tensor.

For the normalization of the tensor $\tilde{d}$ of $SO(2m)$ we get

$$d_{mm}(CC) = \frac{1}{4}2^{-m}(2m)!$$

The factor 2^{-m} compensates the double counting of index pairs $\mu\nu$ as compared to adjoint indices. The argument "CC" indicates that two chiral tensors $\tilde{d}$ are used. Furthermore we have

$$d_{mm}(CV) = d_{mm}(CS) = 0 \ .$$

Here "V" is the fundamental representation, as before, and "S" refers to the non-chiral tensor computed in the spinor representation,

$$d_S^{[\mu_1\nu_1],\ldots,[\mu_m\nu_m]} = \frac{1}{2}\mathrm{Str}\Sigma^{\mu_1\nu_1}\ldots\Sigma^{\mu_m\nu_m}$$

This tensor is related to the one in the reference representation V.

To order 12, the maximal order we used for the other $SO(N)$ tensors, the only other quantity of interest is d_{444}, which has chiral tensor contributions only for $SO(8)$. Here is a complete set of results for that group.

$$\begin{aligned}
d_{44}(VV) &= 70\eta^4 && (245)\\
d_{44}(SS) &= \tfrac{245}{8}\eta^4 && (246)\\
d_{44}(CC) &= \tfrac{315}{8}\eta^4 && (247)\\
d_{66}(VV) &= \tfrac{581}{8}\eta^6 && (248)\\
d_{444}(VVV) &= 70\eta^6 && (249)\\
d_{444}(SSS) &= \tfrac{665}{32}\eta^6 && (250)\\
d_{444}(SSC) = d_{444}(VVC) &= 0 && (251)\\
d_{444}(SCC) &= \tfrac{525}{32}\eta^6 && (252)\\
d_{444}(VCC) &= \tfrac{-105}{8}\eta^6 && (253)\\
d_{444}(CCC) &= 0 && (254)
\end{aligned}$$

The fundamental quantities here are the ones involving $d_4(V), d_6(V)$ and $d_4(C)$. The tensor $d_4(S)$ is related to $d_4(V)$:

$$d_S^{abcd} = -\frac{1}{2}d_V^{abcd} + \frac{3}{2}d_{2,2}^{abcd}$$

The results involving $d_4(S)$ are given here because one may use them to check the triality relations

$$d_{44}(VV) = d_{44}(SS) + 2d_{444}(SC) + d_{44}(CC)$$

and

$$d_{444}(VVV) = d_{444}(SSS) + 3d_{444}(SSC) + 3d_{444}(SCC) + d_{444}(CCC)$$

which are due to the fact that the tensors $d_4(V)$ and the combinations $d_4(S) \pm d_4(C)$ are given by symmetrized traces of the triality-related representations (8_v), (8_s) and (8_c).

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