Preface

This documentation is both a gentle introduction and a reference guide to the symbolic manipulation program FORM. The tutorial part is meant as a beginners course in FORM, which can be used for self-study. Primary emphasis is on learning the basics of FORM, what can be done with the system, and what is the design philosophy behind it. The didactical guidelines followed in writing this material are: learning by example and learning by practice. Therefore, the tutorial contains many examples and exercises. Have a close look at the examples, try them out with FORM, and change them as you like. Do not look immediately at the answers to the exercises, but first try them yourself. Only through direct confrontation with FORM you can acquire the right skills and experience the applicability and limitations of the system personally. This tutorial does not make the programming guide unnecessary; for details about the program or extended explanations you are referred to this part of the documentation. The reference guide is followed by a collection of more complicated examples that show the power of FORM.

About the FORM version used

In this book, Release 3 of FORM is used. It is publicly available from URL http://www.nikhef.nl/ form (email contact: form@nikhef.nl). the book on CD.

About the Origin of the Book

The tutorial part is partly based on the FORM course that was developed during a visit of André Heck in August 1993 to the Department of Theoretical Physics of the Catholic University of Petrópolis, Brasil, and that was revised in a visit in 1998 to the same institute and in later visits to LNCC (Laboratório Nacional de Computação Científica) in Petrópolis, Brasil. The reference guide is a thorough revision and extension of the original FORM manual ([Vermaseren 91]).

Acknowledgements

Many people have contributed to the turorial. Special thanks go to Renato Doria of the Deparment of Theoretic Physics of the Catholic University of Petrópolis and his students. Through their enthusiasm and eager in learning FORM they provided the right spirit for preparing the course material of the tutorial part.

Despite of all the help the authors got, they are sure that readers of this book will come up with remarks, suggestions, corrections, user programs, and so on. Please send them to one of the authors

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Chapter 1

Introduction

1.1 Getting Started

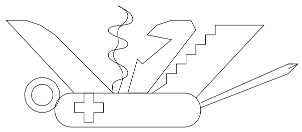
1.1.1 Characteristics of FORM

The origin of computer algebra is high energy physics, where the need for algebra engines to handle big computations was felt earliest. The programs REDUCE and SCHOONSCHIP were born in the sixties. REDUCE transformed gradually into a general purpose system and can be considered as one of the predecessors of the modern general purpose systems such as Maple and *Mathematica*. The design of such general purpose systems make them useful for various areas, but when it comes to really large computations they are too slow, use too big memory, and so on. SCHOONSCHIP was a program completely dedicated towards and optimized for large computations in high energy physics: it was written in assembler language and operated fast. SCHOONSCHIP can be considered as the predecessor of the FORM program.

FORM has been designed along the same philosophy: make a symbolic manipulation program that is useful for "real computations" on "real computers". FORM has been developed over the last fifteen years by Jos Vermaseren at NIKHEF. The first version was released in 1989 and made available by anonymous ftp from ftp.nikhef.nl. The second enhanced version, FORM 2, was commercially released in 1991. The present release FORM 3 is the next major improvement of the software. FORM3 is not only useful for computations in high energy physics (the original application area), but it is also well-suited for large symbolic manipulations of general nature, in cases where other systems give up. Pattern matching in formula manipulation and computing in noncommutative algebras are two other examples of application areas outside the field of high energy physics where FORM has proved to be champion in symbol crunching.

Before starting to learn FORM, let us compare the program with other general purpose systems such as Maple and *Mathematica* so that it is clear where and why FORM is different. The following comparison originates from a short course on FORM [Oldenborgh 95].

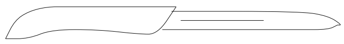
Maple, *Mathematica*, etc.



Swiss army knife

- Much built-in knowledge (integration, solving equations, special functions, etc.)
- Designed for an infinitely large computer
- Big and slow (especially on large problems)
- Very general
- Fancy user interface (typesetting, graphics, sound, drag-and-drop, etc.)
- Tries to do everything

FORM



Chef knife

- Limited mathematical knowledge (calculus with tensors and gamma matrices, ...)
- Designed for real computers
- Small and fast (also on large problems)
- Optimized for certain classes of problems
- Batch program (edit-run cycle)
- Does only what you ask for

1.1.2 Our First Example

In order to do a calculation in FORM, you have to write a program, store it in a file, and call FORM with this file as an argument. Let us look at a simple example.

Suppose you have a file `sample1.frm` with the following contents.

```
Symbols a,b;
Local [(a+b)^2] = (a+b)^2;
Print;
.end
```

When you call FORM by `form sample1`, the following will appear on your terminal screen.

```
    FORM version 3.-(Nov 29 1997). Run at: Tue Jan  6 19:15:59 1998
    Symbols a,b;
    Local [(a+b)^2] = (a+b)^2;
    Print;
    .end

Time =       0.10 sec    Generated terms =          3
       [(a+b)^2]         Terms in output =          3
                         Bytes used      =         52

   [(a+b)^2] =
      2*a*b + a^2 + b^2;
```

Let us have a closer look at the above session. The basic object in FORM is a term. Formulae consist of terms, and terms are separated by addition and subtraction. If a formula contains parentheses these are immediately removed as in our example: the formula `[(a+b)^2]` contains three terms. Note that objects have to be declared in FORM before they can be used. The first line informs the system that `a` and `b` are algebraic symbols. The second line defines `[(a+b)^2]` as a local expression that has to be manipulated.

The `Print` statement is necessary to see the result. The word `.end` marks the end of the program: it is for FORM the signal to execute the last program block and to finish afterwards. FORM shows the original program and the runtime statistics on the screen.

Some general remarks about a regular FORM program:

- Every FORM program must be stored in a file with extension `frm`.
- Every FORM statement consists of a keyword such as `Symbols`, `Local`, `Print`, and `.end`, and some continuation. FORM statements can be continued over several lines, but they must end with a semicolon. You can also have more than one statement in a program line.
- FORM does not distinguish between plural and singular declarations: `Symbols` and `Symbol` are equally valid, and for FORM `Indices` and `Index` are the same.
- Abbreviations for keywords exist: instead of `Symbols` you can just use `S`, and `Local` can be abbreviated as `L`. For clarity of the examples, we have not used the abbreviation possibilities of FORM in this tutorial.
- Items in a FORM statement are separated by commas or spaces. FORM considers these characters most of the time as equivalents.
- FORM is case insensitive with respect to keywords and built-in objects. It is however case sensitive with user defined objects, allowing in this way maximum flexibility.
- In FORM, the simplest kind of names of variables and expressions are sequences of letters and digits, the first of which must be a letter. For example, `alpha`, `a1`, `H2O`, and `firstExercise` are valid names. If you want to use spaces in a name for reasons of readability, or if you want to use special characters such as dots, colons, or slashes, then you must surround the name with square brackets [ ]. Names that contain an underscore are reserved by the FORM system. Good advice: Avoid using names that contain an underscore if they are not absolutely necessary.
- You can log a FORM session by adding the flag `-l` in the starting command: if in our example we had said `form -l sample1`, the effect would have been the creation of the file `sample1.log` with the contents that normally appears on the screen.
- You can interrupt a FORM computation by entering CTRL C, i.e., by pressing the C key while holding down the CONTROL key.

1.1.3 Exercises

1. Does anything happen when you change the `Local [(a+b)^2] = (a+b)^2` statement in the first example by `Local [(a+b)^2] = (a+b)*(a+b)` ?
2. Does anything happen when you change the `Symbols a,b` declaration in the first example by `Functions a,b` ?
3. Consider the following FORM program.

```
Symbol a;
Functions B, C;
Local F1 = (a+B+C)^2;
Local F2 = (a+(B+C))^2;
Print;
.end;
```

   What is the difference in working out expression `F1` and expression `F2`?

4. Check whether the following is a valid FORM program.

```
s,t,u; L,F=
(t+u)
^2;
Print;
.end
```

1.2 Types of Variables

In the second exercise of the last section you have seen that FORM has an easy way to deal with noncommuting objects, viz., through the variable type `Function`. There are more types in FORM: commuting functions, vectors, and tensors, to name a few. In this section we shall discuss some of them.

1.2.1 Functions

Commuting vs. Noncommuting Functions

FORM distinguishes between noncommuting functions, declared by the keyword `Functions`, and commuting functions, declared by the keyword `CFunctions` or `Commuting`. The next example clearly demonstrates the difference.

```
 *
 * Declarations
 *
 Functions f,g;
 CFunctions F,G;
 Symbol x;
 *
 * Specifications, e.g. no runtime statistics
 *
 Off statistics;
 *
 * Definitions
 *
 * local expression with only noncommuting functions
 *
 Local F1 = f(x)*g(x) + g(x)*f(x);
 *
 * Output
 *
 Print;
 *
 * end of module
 *
 .sort

F1 =
   f(x)*g(x) + g(x)*f(x);

 *
 * local expression with only commuting functions
 *
 Local F2 = F(x)*G(x) + G(x)*F(x);
 Print F2;
 *
 * terminate the program
```

```
 *
 .end
```

```
F2 =
   2*F(x)*G(x);
```

We deliberately made the above example more complicated than necessary in order to explain some more FORM features.

- You can write comments by starting these lines with an asterisk $*$.
- The printing of runtime messages can be suppressed by the statement `Off statistics`.

  > *Henceforth, it is assumed in all* FORM *examples that the initialization file* `form.set` *contains the line* `nwritestatistics on`. *This will automatically turn off the printing of runtime messages unless the statement* `On statistics` *is present in a* FORM *program.*

- There is a fixed order of types of statements in a program block:
  1. *Declarations*: starting with keywords `Symbol`, `Function`, ...
  2. *Specifications*: starting with keywords `statistics`, `skip`, `drop`, `hide`, ...
  3. *Definitions*: starting with keywords `Local` or `Global`.
  4. *Executable statements*: starting with keywords `id`, `trace`, `contract`, ...
  5. *Output control*: such as `Print` and `Bracket`.
- The `.sort` statement is a directive to FORM to execute a program block, sort the result (i.e., bring them in standard ordering), and prepare for further processing.

This brings us to the very short description of how FORM operates; we will come back to this issue in section 1.3. FORM consists of a *preprocessor* and a *compiler*. The preprocessor reads from the input stream and prepares input for the compiler. The preprocessor prepares program blocks, also called *modules*, which are translated by the compiler, and immediately executed. A command for the preprocessor is called a *preprocessor instruction*. It always starts with the sharp symbol (#), it does not have to end with a semicolon, and it is executed when it is encountered in the input stream. A module is terminated by a statement that starts with a period. Such a statement is called a *module instruction*. It marks the end of a module, it halts the compiler, and it initiates the execution of the module. Like a preprocessor instruction, a module instructions does not have to end with a semicolon, although it does no harm. The module instructions and their meanings are listed below.

Instruction	*Meaning*
	execute, sort, print, +
`.sort`	continue
`.end`	terminate
`.clear`	restart softly
`.store`	store globals, remove locals, continue

There is one other module instruction, viz., `.global`. It makes the definitions and declarations in the module global so that they cannot be so easily removed again. Most of these directives will be discussed in later chapters.

Finally, in our example we have used an option of `Print`, viz., to print only one expression instead of all expressions known in the module.

Symmetry Properties of Functions

Besides distinguishing commutating and noncommuting functions, FORM can also deal with the following four symmetry properties of functions.

Symmetry Property	*Meaning*
`symmetric`	$F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every permutation σ.
`antisymmetric`	$F(x_1, x_2, \ldots, x_n) = \mathrm{sgn}(\sigma)\, F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every permutation σ, where $\mathrm{sgn}(\sigma)$ denotes the signature of σ.
`cyclic`	$F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every permutation σ in the group generated by the cycle $(1\ 2\ 3\ \ldots\ n)$.
`rcyclic`	$F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every permutation σ in the group generated by the cycle $(1\ 2\ 3\ \ldots\ n)$ and the cycle $(1\ \ n)\,(2\ \ n-1)\,(3\ \ n-2)\,\cdots\,(\lfloor\frac{n}{2}\rfloor\ \ \lfloor\frac{n}{2}\rfloor+1)$.

An example:

```
Symbols x1,x2,x3,x4,x5;
Functions S(symmetric), A(antisymmetric), C(cyclic), R(rcyclic);
Local [S(x2,x3,x4,x1,x5)] = S(x2,x3,x4,x1,x5);
Local [A(x2,x3,x4,x1,x5)] = A(x2,x3,x4,x1,x5);
Local [C(x2,x3,x4,x1,x5)] = C(x2,x3,x4,x1,x5);
Local [R(x2,x3,x4,x1,x5)] = R(x2,x3,x4,x1,x5);
Print;
.end
```

```
[S(x2,x3,x4,x1,x5)] =
   S(x1,x2,x3,x4,x5);

[A(x2,x3,x4,x1,x5)] =
    - A(x1,x2,x3,x4,x5);

[C(x2,x3,x4,x1,x5)] =
   C(x1,x5,x2,x3,x4);

[R(x2,x3,x4,x1,x5)] =
   R(x1,x4,x3,x2,x5);
```

You see that FORM automatically uses the symmetry properties of the functions to bring the arguments into standard order (determined by the order in which objects have been declared). Three remarks:

- Symmetry properties can only be defined for the entire argument field. If you want to have more complicated constructions you should split up the function into more than one function.
- The words describing the symmetry can be abbreviated: the first character already suffices. FORM is also case insensitive with respect to these keywords. So, the above example would have worked equally well (but less readable) when you change `S(symmetric)` into `S(S)`.
- The symmetry properties can also be defined for tensors.

Function Calls

Consider the following FORM session.

```
Symbols x,y;
Commuting f;
```

```
Local F = f(x)+f(x,y)+f(x,,y);
Print;
.end

F =
   f(x) + f(x,y) + f(x,0,y);
```

Notice that calls of the commuting function `f` have different number of arguments. This is a general feature: there is no check on the type of a function and on the number of arguments in a function call until something has to be done with the function. The empty argument in the third call of `f` is replaced by zero. In FORM, empty arguments and arguments that are zero are the same.

Built-in Mathematical Functions

We list the mathematical functions that FORM knows. Like any other built-in object you recognize such a function by its name: it always ends with an underscore. We distinguish between functions that have really been implemented and those whose names have been reserved only. For the latter functions, some safe fixed values and relations may be implemented in future versions of FORM, but do not expect too much of it. Because of potential problems with multivalued functions and with analytic continuations of functions, the number of relations will be limited.

Implemented Function	*Meaning*
`abs_`	absolute value
`bernoulli_`	Bernoulli number
`binom_`	binomial coefficient
`delta_`	delta function
`deltap_`	delta prime function
`fac_`	factorial
`invfac_`	inverse factorial
`max_`, `min_`	maximum and minimum value
`mod_`	modulo arithmetic of integers
`root_`	root function
`sig_`	sign function
`sign_`	signature for integers
`theta_`	theta function
`thetap_`	theta prime function
Reserved Function	*Meaning*
`acos_`, `asin_`, `atan_`, `atan2_`	inverse trigonometric functions
`acosh_`, `asinh_`, `atanh_`	inverse hyperbolic functions
`cos_`, `sin_`, `tan_`	trigonometric functions
`cosh_`, `sinh_`, `tanh_`	hyperbolic functions
`li2_`	dilogarithm
`lin_`	polylogarithm
`ln_`	natural logarithm
`sqrt_`	square root function

Precise definitions of implemented functions are:

- $\texttt{abs\_}(x) = \begin{cases} |x| & \text{if } x \text{ is numerical} \\ \texttt{abs\_}(x) & \text{otherwise} \end{cases}$
- $\texttt{bernoulli\_}(n)$, for some integer n, is defined as the coefficient of t^n in the series expansion of $\dfrac{t}{1-\exp(-t)}$ about zero, i.e., $\dfrac{t}{1-\exp(-t)} = \sum_{n=0}^{\infty} \texttt{bernoulli\_}(n)\, t^n$.

- $\texttt{binom\_}(n,k) = \dfrac{n!}{k!\,(n-k)!}$.
- $\texttt{delta\_}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \text{ is numerical and } x \neq 0 \\ \texttt{delta\_}(x) & \text{otherwise} \end{cases}$

  and

  $\texttt{delta\_}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \text{ and } y \text{ are numerical and } x \neq y \\ \texttt{delta\_}(x,y) & \text{otherwise} \end{cases}$

  So, basically, $\texttt{delta\_}(x,y) = \texttt{delta\_}(x-y)$
- $\texttt{deltap\_}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \\ \texttt{deltap\_}(x) & \text{otherwise} \end{cases}$

  and

  $\texttt{deltap\_}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \text{ and } y \text{ are numerical and } x \neq y \\ \texttt{deltap\_}(x,y) & \text{otherwise} \end{cases}$

  So, basically, $\texttt{deltap\_}(x) = 1 - \texttt{delta\_}(x)$ and $\texttt{deltap\_}(x,y) = 1 - \texttt{delta\_}(x,y)$
- $\texttt{fac\_}(n) = n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.

  $\texttt{invfac\_}(n) = \dfrac{1}{n!}$.
- If all x_i's in $\texttt{max\_}(x_1, x_2, \ldots, x_n)$ are numerical, it evaluates to the maximum value of them. If not, the formula is returned.

  If all x_i's in $\texttt{min\_}(x_1, x_2, \ldots, x_n)$ are numerical, it evaluates to the minimum value of them. If not, the formula is returned.
- $\texttt{mod\_}(n,k) = n \bmod k$ for integers n and k.
- If n is a positive integer, if x is a rational number, and if the nth root of x is also a rational number, say y, then $\texttt{root\_}(n,x)$ is replaced by y. Otherwise, the expression is left untouched. For example, $\texttt{root\_}(2,9) = 3$. This function is mainly intended for internal use. It does not try partial roots such as $\texttt{root\_}(2,8) = 2\,\texttt{root\_}(2,2)$.
- $\texttt{sig\_}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \\ \texttt{sig\_}(x) & \text{otherwise} \end{cases}$
- $\texttt{sign\_}(n) = (-1)^n$ for integer n. The function exists for efficiency reasons: evaluation is much faster than working out the power which it would normally do.
- $\texttt{theta\_}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \\ \texttt{theta\_}(x) & \text{otherwise} \end{cases}$

  and

  $\texttt{theta\_}(x,y) = \begin{cases} 1 & \text{if } x = y \text{ or if } x \text{ and } y \text{ in natural order} \\ 0 & \text{if } x \text{ and } y \text{ not in natural order} \end{cases}$
- $\texttt{thetap\_}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \\ \texttt{theta\_}(x) & \text{otherwise} \end{cases}$

  and

$$\texttt{thetap\_}(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ in natural order and } x \neq y \\ 0 & \text{if } x = y \text{ or if } x \text{ and } y \text{ not in natural order} \end{cases}$$

So, basically, $\texttt{thetap\_}(x) = 1 - \texttt{theta\_}(-x)$ and $\texttt{thetap\_}(x,y) = 1 - \texttt{theta\_}(-x,-y)$

The following FORM session illustrates how some of the built-in functions work.

```
Symbol x;
Local F1 = invfac_(3) + x*fac_(3);
Local F2 = cos_(0) + cos_(x)^2 + sin_(x)^2;
Local F3 = x^3*sign_(3) + x*abs_(-1/2) + sig_(-3) + sig_(x);
Local F4 = binom_(5,2) + sqrt_(4) + x*root_(2,4);
Local F5 = bernoulli_(0) + bernoulli_(1)*x + bernoulli_(2)*x^2;
Local F6 = max_(1/2,2) + min_(1,x);
Local F7 = mod_(7,2);
Print;
.end;
```

```
F1 =
   1/6 + 6*x;

F2 =
   sin_(x)^2 + cos_(x)^2 + cos_(0);

F3 =
    - 1 + 1/2*x - x^3 + sig_(x);

F4 =
   10 + 2*x + sqrt_(4);

F5 =
   1 + 1/2*x + 1/12*x^2;

F6 =
   2 + min_(1,x);

F7 =
   1;
```

1.2.2 Vectors and Indices

Vectors are one of the favorite data types of FORM. They can appear in two ways: with symbolic indices, like in `v(i)`, and with specific integer indices such as `v(1)` and `v(2)`. In the former case, the indices must be declared. The default dimension of the underlying vector space is four, but it can be changed by the `Dimension` statement. An example of the use of vectors and indices:

```
Vectors u,v;
Indices i,j;
Function f;
Local w1 = u(1) + v(i);
Local w2 = u(i) * v(j);
Local w3 = u(i) * u(i);
Local w4 = v(i) * u(i);
Local w5 = f(i,j) * u(i) * v(j);
Print;
.end
```

```
w1 =
   u(1) + v(i);

w2 =
   u(i)*v(j);

w3 =
   u.u;

w4 =
   u.v;

w5 =
   f(u,v);
```

The formulas `w3` and `w4` show that FORM uses the so-called *Einstein summation convention*: indices that occur twice inside the same term are considered to be summed over. So, `v(i)*u(i)` becomes the inner product or dot product of `u` and `v`, which is $\sum_i u_i v_i$. Because (components of) vectors are assumed to be commuting, the order in the dot product is unimportant: FORM will choose one depending on the order of declaration.

Formula `w5` illustrates another convention, called the SCHOONSCHIP *notation*, that FORM uses: when an index is summed over and in one of its occurrences it is the argument of a vector, then this vector is put at the place of the other occurrence. In this notation, $\sum_i f_i v_i$, where v is a vector and f some function, is abbreviated as f_v.

The automatic summation of indices is also called *contraction of indices*. You can overrule the contraction of indices in FORM by specifying a zeroth dimension for the index in the declaration. In this case, explicit summation is still possible by the `sum` statement. This statement is also applicable if indices are arguments of functions or tensors.

```
 Vector u;
 Index i=0;
 * no contraction over index i
 Local P = u(i) * u(i);
 Print;
 .sort

P =
   u(i)*u(i);

 sum i;
 Print;
 .sort

P =
   u.u;

 Function f;
 Local F = f(i);
 sum i,1,3,5;
 Print F;
 .end

F =
   f(1) + f(3) + f(5);
```

Keep in mind that FORM does not distinguish between upper (contravariant) and lower (covariant) indices. We shall see in the next chapter how this concept from tensor calculus can be implemented in FORM.

1.2.3 Tensors

The keyword to declare a tensor in FORM is `Tensor` or `CTensor`. The latter declaration makes clear that FORM assumes (components of) tensors to be commuting. To declare a noncommuting version you must use `NTensor`. Tensors are in FORM special kinds of functions: their arguments can only be indices and vectors of which it is assumed that they have been contracted with an index. The advantage is that the system can manipulate them more efficiently than the general functions.

In the example below, we consider a sum of two products of tensors and explicitly tell FORM that common indices are summed over. In this way, the system will recognize the equal terms in the expression.

```
 Tensors S,T;
 Indices i,j,k,l;
 Local F = S(i,k)*T(k,j) + S(i,l)*T(l,j);
 Print;
 .sort

F =
   S(i,k)*T(k,j) + S(i,l)*T(l,j);

 sum k,l;
 Print;

F =
   2*S(i,N1_?)*T(N1_?,j);
```

The dummy index generated by FORM is denoted by a name that ends with an underscore and a question mark.

FORM has convenience methods to replace tensors by a product of vector components and vice versa. They are called `ToVector` and `ToTensor`, respectively. The commands have two arguments, a tensor and a vector. The order in which these arguments occur is irrelevant. Replacements from vector to tensor occur not only when components of the vector are used, but also when the vector is contracted with other vectors or tensors. An example that shows it all.

```
 Tensor t;
 Vector u,v;
 Indices i,j,k;
 Local F1 = v(i)*v(j)*v(k)*v(1);
 Local F2 = v;
 Local F3 = (u.v)^2 * v.v;
 ToTensor v,t;
 Print;
 .sort

F1 =
   t(1,i,j,k);

F2 =
   v;

F3 =
   t(u,u,N1_?,N1_?);
```

```
 Local F4 = t;
 ToVector t,v;
 Print;
 .end

F1 =
   v(1)*v(i)*v(j)*v(k);

F2 =
   v;

F3 =
   u.v^2*v.v;

F4 =
   1;
```

1.2.4 Exercises

1. Check how FORM handles the summation convention for the following expressions.
   (i) $a_{ij}x_j$
   (ii) $a_{ii}x_j$
   (iii) $a_{ij}x_iy_j$
   (iv) $\delta_{ij}x_ix_j$
2. Demonstrate with FORM the following equalities.
   (i) $a_{ij}x_iy_j = a_{ji}x_jy_i$
   (ii) $(a_{ij} + a_{ji})x_ix_j = 2a_{ij}x_ix_j$
3. Let a be an antisymmetric tensor of rank two. Demonstrate with FORM the following properties.
   (i) $a_{ij}x_ix_j = 0$ for any vector x.
   (ii) the tensor b of rank two defined by the contraction $b_{ij} = a_{ik}a_{kj}$ is symmetric.
4. There are three ways to control the printing of powers of functions:

   ```
   FunPowers nofunpowers;
   FunPowers commutingonly;
   FunPowers allfunpowers;
   ```

   Find out by experimentation what the statements actually do and check also how they affect the printing of powers of tensors.

1.3 Inner Workings of FORM

When writing or studying FORM programs, it is useful to have at least some idea of what FORM internally does. First, you need to know what objects it actually manipulates. The answer is that FORM works with expressions that are sums of terms; each term consisting of a rational coefficient times a product of factors, possibly to some power. The factors can be symbols or more complicated structures such as functions or tensors. The expressions that can be manipulated are called *active expressions*. There can be many active expressions at the same time.

The method of operation of FORM is as follows.

1. The *preprocessor* reads from the input stream and changes the input on a purely textual level into code that can be interpreted by the compiler. The preprocessor has variables, loops, conditional statements, and offers the possibility to include other files or define procedures. Preprocessor instructions always start with the sharp symbol (#).
2. The *compiler* interprets the prepared code and translates it into machine code.
3. The compiler is halted when the preprocessor reads a module instruction, which starts with a period and marks the end of a module.
4. FORM runs the machine code: it applies the operations specified in the module to the active expressions, one by one, and works out brackets. For each expression, FORM applies the requested operations in the order as specified in the module to the first term of the expression (possibly generating new terms) and brings the resulting terms into a standard form. It does the same thing to the second term, and so on until all terms have been processed.
5. FORM sorts the results and calls this the input for the next module.

So, you see that FORM sequentially processes expressions term by term. This mode of operation means that FORM has no operations that use more than one term at the same time. For example, a substitution rule like $a + b \to c$ cannot be expressed as such. You will have to use tricks such as the replacement rule $a \to c - b$. It also means that there is no factorization built into FORM because the whole expression must be taken into account for this mathematical operation. To summarize in one sentence:

All operations in FORM *are local to one term of an active expression.*

1.4 Some FORM Examples

The examples in this section are to acquaint you with FORM and to show some of its built-in facilities.

1.4.1 Summation

FORM provides you of course with tools to compose an expression. For example, to obtain $\sum_{i=0}^{5} \frac{x^i}{i!}$ you can enter the following:

```
Symbols x,i;
Local expr = sum_( i, 0, 5, x^i/fac_(i) );
Print;
.end

expr =
   1 + x + 1/2*x^2 + 1/6*x^3 + 1/24*x^4 + 1/120*x^5;
```

1.4.2 Levi-Civita Tensor

The *Levi-Civita tensor* or *permutation tensor* $\epsilon_{i_1 i_2 \ldots i_n}$ plays an important role in tensor calculus. When the indices range from 1 to n, it is defined as

$$\epsilon_{i_1 i_2 \ldots i_n} = \begin{cases} 0 & \text{if two indices are the same} \\ 1 & \text{if } (i_1, i_2, \ldots, i_n) \text{ is a even permutation of } (1, 2, \ldots, n) \\ -1 & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n) \end{cases}$$

The Levi-Civita tensor is denoted in FORM by e_. The product of a pair of Levi-Civita tensors can be rewritten in terms of Kronecker deltas:

$$\epsilon_{i_1 i_2 \ldots i_n} \, \epsilon_{j_1 j_2 \ldots j_n} = \det \begin{pmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \cdots & \delta_{i_n j_n} \end{pmatrix}$$

The *Kronecker delta* δ_{ij}, denoted in FORM by d_(i,j), is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The Kronecker delta function d_ differs from the delta function delta_. The Kronecker delta has two indices as arguments and serves as a metric tensor (this implies that it is symmetric). The second delta function, delta_ has either one or two arguments, which need not be indices, but can be general expressions. FORM does not symmetrize the delta function delta_.

The contract statement will do the work of writing a product of Levi-Civita tensors in terms of Kronecker deltas. In the example below, the vector space is declared three-dimensional by the Dimension statement (recall that the default dimension in FORM is four).

```
 Dimension 3;
 Indices i,j,k,p,q,r;
 Local f0 = e_(i,j,k) * e_(p,q,r);
 Local f1 = e_(i,j,k) * e_(p,q,k);
 Local f2 = e_(i,j,k) * e_(p,j,k);
 Local f3 = e_(i,j,k) * e_(i,j,k);
 contract;
 Print +s;
 .end

f0 =
    + d_(i,p)*d_(j,q)*d_(k,r)
    - d_(i,p)*d_(j,r)*d_(k,q)
    - d_(i,q)*d_(j,p)*d_(k,r)
    + d_(i,q)*d_(j,r)*d_(k,p)
    + d_(i,r)*d_(j,p)*d_(k,q)
    - d_(i,r)*d_(j,q)*d_(k,p)
   ;

f1 =
    + d_(i,p)*d_(j,q)
    - d_(i,q)*d_(j,p)
   ;

f2 =
    + 2*d_(i,p)
   ;

f3 =
    + 6
   ;
```

The flag +s in the second last command causes FORM to print each term of an expression on a separate line.

1.4.3 Vector Calculus: Outer Product

Now that the Levi-Civita tensor has been introduced, we can also look at the way how to represent or compute outer products (cross products) of 3-dimensional vectors in FORM. We shall concentrate on three well-known formulae:

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})_k &= \epsilon_{ijk}u_iv_j\\ \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} &= \epsilon_{ijk}u_iv_jw_k\\ \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \mathbf{v}(\mathbf{u}\cdot\mathbf{w})-\mathbf{w}(\mathbf{u}\cdot\mathbf{v})\end{aligned}$$

The transcription into FORM is as follows:

```
 Dimension 3;
 Vectors u,v,w;
 Indices i,j,k,l,m,n;
 Local [uxv] = e_(i,j,k) * u(i) * v(j);
 Local [uxv.w] = e_(i,j,k) * u(i) * v(j) * w(k);
 Local [ux(vxw)] = e_(i,j,k) * u(i) * (e_(m,n,j) * v(m) * w(n));
 contract;
 Print;

[uxv] =
   e_(u,v,k);

[uxv.w] =
   e_(u,v,w);

[ux(vxw)] =
   v(k)*u.w - w(k)*u.v;
```

This “coordinate free” FORM description can be made more explicit.

```
 Dimension 3;
 Vectors u,v,w;
 Indices i,j,k,l,m,n;
 Local [uxv](k) = e_(1,2,3) * e_(i,j,k) * u(i) * v(j);
 Local [uxv.w] = e_(1,2,3) * e_(i,j,k) * u(i) * v(j) * w(k);
 Global [ux(vxw)](k) = e_(i,j,k) * u(i) * (e_(m,n,j) * v(m) * w(n));
 contract;
 Bracket w;
 Print [uxv.w];
 .sort

[uxv.w] =
    + w(1) * ( u(2)*v(3) - u(3)*v(2) )

    + w(2) * (  - u(1)*v(3) + u(3)*v(1) )

    + w(3) * ( u(1)*v(2) - u(2)*v(1) );

 AntiBracket u,v;
 Print [uxv];
 .store

[uxv](k) =
    + d_(1,k) * ( u(2)*v(3) - u(3)*v(2) )
```

```
   + d_(2,k) * (  - u(1)*v(3) + u(3)*v(1) )

   + d_(3,k) * ( u(1)*v(2) - u(2)*v(1) );

Local [(ux(vxw)(1)] = [ux(vxw)](1);
Local [(ux(vxw)(2)] = [ux(vxw)](2);
Local [(ux(vxw)(3)] = [ux(vxw)](3);
Print;

[(ux(vxw)(1)] =
   v(1)*u.w - w(1)*u.v;

[(ux(vxw)(2)] =
   v(2)*u.w - w(2)*u.v;

[(ux(vxw)(3)] =
   v(3)*u.w - w(3)*u.v;
```

A few remarks about new concepts used in the above program.

- The expression `[ux(vxw)](k)` is made global so that it survives the `.store` command at the end of the second module and can be used in the last part of the program.
- The `Bracket w` instruction forces the expression `[uxv.w]` to be printed as a polynomial in the components of the vector `w`.
- The `AntiBracket u,v` instruction forces the expression `[uxv]` to be printed in such way that `u` and `v` are put inside the brackets, and that the rest is taken out of the brackets. Thus — nome est omen — the `AntiBracket` statement does just the opposite of the `Bracket` statement.

1.4.4 Linear Algebra: Determinant

The determinant of a square matrix $M = (M_{ij})$ of dimension n is given by

$$\det(M) = \epsilon_{i_1 i_2 \cdots i_n} M_{1i_1} M_{2i_2} \cdots M_{ni_n}.$$

This allows us to compute determinants in FORM in a straightforward way. Below we compute the determinant of the general 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

```
Symbols a,b,c,d;
CFunction M;
Indices i,j;
Local det = e_(1,2) * e_(i,j) * M(1,i) * M(2,j);
contract;
id M(1,1) = a;
id M(1,2) = b;
id M(2,1) = c;
id M(2,2) = d;
Print;
.end

det =
   a*d - b*c;
```

At first sight, it may look superfluous to put in the local expression to the front `e_(1,2)`, which is by definition equal to 1. However, FORM first uses it in the contraction of Levi-Civita tensors, and in this way, the determinant comes out in explicit form.

In the above example we use the most important command in FORM, viz., the identify statement `id`. An identification is a substitution or replacement. Here we do a straightforward replacement of matrix elements by their (symbolic) values. As we shall see in the next chapter, more general patterns are possible in FORM.

1.4.5 Linear Algebra: Gram Determinant

For vectors $v_1, v_2, \ldots, v_n$, the Gram determinant is defined as the determinant of the matrix G with matrix coefficients G_{ij} equal to $v_i \cdot v_j$ (the inner product of vectors v_i and v_j). FORM is the ultimate program for computing such determinants. First we show how to compute them for $n = 2$ and $n = 3$ with the help of the Levi-Civita tensor.

```
Vectors v1,v2,v3;
Local G2 = e_(v1,v2)^2;
Local G3 = e_(v1,v2,v3)^2;
contract;
Print;
.end
```

```
G2 =
   v1.v1*v2.v2 - v1.v2^2;

G3 =
   v1.v1*v2.v2*v3.v3 - v1.v1*v2.v3^2 + 2*v1.v2*v1.v3*v2.v3 - v1.v2^2*v3.v3
    - v1.v3^2*v2.v2;
```

To understand the above example, it suffices to recall the definition of Levi-Civita tensors, the Einstein summation convention, and the SCHOONSCHIP notation. Let us show this for $n = 3$:

$$\begin{vmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_2 \cdot v_2 & v_3 \cdot v_3 \end{vmatrix} = \begin{vmatrix} \delta_{ip} v_1(i) v_1(p) & \delta_{iq} v_1(i) v_2(q) & \delta_{ir} v_1(i) v_3(r) \\ \delta_{jp} v_2(j) v_1(p) & \delta_{jq} v_2(j) v_2(q) & \delta_{jr} v_2(j) v_3(r) \\ \delta_{kp} v_3(k) v_1(p) & \delta_{kq} v_3(k) v_2(q) & \delta_{kr} v_3(k) v_3(r) \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} v_1(i) v_2(j) v_3(k) v_1(p) v_2(q) v_3(r)$$

$$= \epsilon_{ijk} \epsilon_{pqr} v_1(i) v_2(j) v_3(k) v_1(p) v_2(q) v_3(r) = \epsilon_{v_1 v_2 v_3} \epsilon_{v_1 v_2 v_3} .$$

To illustrate that FORM is indeed a very powerful symbol cruncher, let us compute a large Gram determinant of 10 vectors. Actually we only compute the number of terms in the output, because we throw the output away after the program has finished. The computation has been done on a Pentium 166Mhz PC with 16 MB RAM. As you can see, the computation takes less than 9 minutes. If you do the same computation with a general purpose system like Maple or *Mathematica* on this type of computer, your machine is going to crash or the computation takes basically forever. During the FORM computation many runtime statistics appear, but we have omitted most of them in the printout below.

```
AutoDeclare Vector v;
On statistics;
Local G10 = e_(v1,...,v10)^2;
contract;
.end
```

```
Time =         0.60 sec    Generated terms =       6572
                G10      1 Terms left      =       3550
                           Bytes used      =     144066

Time =         0.93 sec    Generated terms =      13043
```

```
                G10       1 Terms left      =       8611
                            Bytes used      =     340904

Time =       1.48 sec       Generated terms =      19562
                G10       1 Terms left      =      13910
                            Bytes used      =     538524

                          :
                          :
                          :

Time =     283.00 sec       Generated terms =    3628800
                G10       1 Terms left      =    3075840
                            Bytes used      =  107001398

Time =     283.60 sec
                G10         Terms active    =    3070880
                            Bytes used      =  106917432

Time =     521.78 sec       Generated terms =    3628800
                G10         Terms in output =    1436714
                            Bytes used      =   50113622
```

Brute force calculation of a 10 by 10 determinant generates $10! = 3,628,800$ terms. In the last message you can read that all 3,628,800 terms have been processed and that FORM has computed the Gram determinant as a sum of 1,436,714 terms.

For us, the declaration and the definition in the above FORM program are interesting, too. The statement

```
AutoDeclare Vector v;
```

has the effect that all undeclared variables starting with the character `v` will be automatically declared as vectors. In other words, `AutoDeclare` makes generic declarations and makes lengthy declarations in many cases unnecessary. In the `AutoDeclare` statement, like in any declaration, you can limit the maximum power of symbols. For example,

```
AutoDeclare Symbol x(:3);
```

makes all undeclared variables starting with the character `x` symbols with maximum power of 3.

The three dots operator `...` is used in the above FORM program to generate a sequence of indices: `i1,...,i10` evaluates to `i1,i2,i3,i4,i5,i6,i7,i8,i9,i10`. This is an example of the following more general rule:

$$\mathit{str}\#1[?]\, O_1\ \ldots\ O_2\, \mathit{str}[?]\#2$$

where *str* is a string that start with an alphabetic character; both strings have to be identical. The string can be empty to generate just a sequence of numbers. #1 and #2 are numbers marking some range of values (so they do not have to be the same). The question marks are optional but you can only have both or none. O_1 and O_2 denote two operators which are in most cases the same and currently only can be comma (,), plus (+), minus (-), times (*), or division (/). There are two exceptions: $O_1 O_2 = -+$ and $O_1 O_2 = +-$. In these cases the operators are used alternatingly: `x1-...+x6` evaluates to `x1-x2+x3-x4+x5-x6`.

Another more general example of using the `...` operator is the following: `<f1(i1)>*...*<f4(i4)>` evaluates to `f1(i1)*f2(i2)*f3(i3)*f4(i4)`. This is an example of the following more general rule:

$$< \mathit{pattern}_1 > O_1\ \ldots\ O_2 < \mathit{pattern}_2 >$$

where the brackets <> delimit the patterns. The patterns are only allowed to differ in numerical parts. The difference in the patterns must all have the same numerical difference in absolute value. For the rest, it works as you expect. For example, `<f4(i2,i6)>*...*<f1(i5,i3)>` evaluates to `f4(i2,i6)*f3(i3,i5)*f2(i4,i4)*f1(i5,i3)`.

1.4.6 Graph Theory: Realizability of Graphs with Prescribed Degrees

There exist a third delta function, denoted by Δ and in FORM by `dd_`, which is totally symmetric and formally equal to a sum of products of Kronecker deltas.

$$
\begin{aligned}
\Delta_{i_1 i_2} &= \delta_{i_1 i_2}\\
\Delta_{i_1 i_2 i_3 i_4} &= \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3}\\
\Delta_{i_1 i_2 i_3 i_4 i_5 i_6} &= \delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + \delta_{i_1 i_2}\delta_{i_3 i_5}\delta_{i_4 i_6} + \delta_{i_1 i_2}\delta_{i_3 i_6}\delta_{i_4 i_5} + \delta_{i_1 i_3}\delta_{i_2 i_4}\delta_{i_5 i_6} + \delta_{i_1 i_3}\delta_{i_2 i_5}\delta_{i_4 i_6} +\\
&\quad \delta_{i_1 i_3}\delta_{i_2 i_6}\delta_{i_4 i_5} + \delta_{i_1 i_4}\delta_{i_2 i_3}\delta_{i_5 i_6} + \delta_{i_1 i_4}\delta_{i_2 i_5}\delta_{i_3 i_6} + \delta_{i_1 i_4}\delta_{i_2 i_6}\delta_{i_3 i_5} + \delta_{i_1 i_5}\delta_{i_2 i_3}\delta_{i_4 i_6} +\\
&\quad \delta_{i_1 i_5}\delta_{i_2 i_4}\delta_{i_3 i_6} + \delta_{i_1 i_5}\delta_{i_2 i_6}\delta_{i_3 i_4} + \delta_{i_1 i_6}\delta_{i_2 i_3}\delta_{i_4 i_5} + \delta_{i_1 i_6}\delta_{i_2 i_4}\delta_{i_3 i_5} + \delta_{i_1 i_6}\delta_{i_2 i_5}\delta_{i_3 i_4}
\end{aligned}
$$

and so on. When the function `dd_` generates its terms, it takes symmetries due to identical arguments into account. Hence, the evaluation of `dd_(v,v,v,v)`, where `v` is a vector, generates directly only one term, viz., `v.v`, with coefficient 3. This coefficient has a combinatorial meaning in graph theory: it equals the number of ways a graph with only one vertex of degree 4 can be realized.

Let us linger upon the application of the function `dd_` in graph theory. First, we associate with vertices v_1, v_2, $v_3, \ldots$ in a graph $G(V,E)$ vectors v_1, v_2, $v_3, \ldots$, and with edges in the graph, say $v_1 v_2$ and $v_1 v_1$, inner products $v_1 \cdot v_2$ and $v_1 \cdot v_1$. The latter inner product represents a loop at vertex v_1, also called a self-loop at vertex v_1. For a vertex v in a graph the degree of v, denoted by $\deg(v)$, is the number of edges that are incident with v. A self-loop is considered as two incident edges. In the vector notation $\deg(v)$ equals the number of inner products that contain vector v. A well-known problem in graph theory is to decide whether a given sequence of nonnegative integers can be realized as the degrees of the vertices of a graph, and if so, how many graphs are possible and in how many ways each graph can be realized. The vector notation for graphs and the FORM function `dd_` are particularly convenient for studying this problem because of automatic contraction in FORM. The following small example gives the idea: for two vectors `v1` and `v2`, the formula

```
dd_(v1,v1,v2,v2) = v1.v1*v2.v2 + 2*v1.v2^2
```

means that there exists one graph with two vertices `v1`, `v2` and each vertex having one self-loop, and that we can construct in two ways a graph with two edges going from one vertex to the other.

Similarly, the expression `dd_(v1,v1,v2,v3)` has to do with the graphs consisting of three nodes, labeled `v1`, `v2` and `v3`, and with prescribed degrees deg(`v1`)=2, deg(`v1`)=deg(`v2`)=1. The formula

```
dd_(v1,v1,v2,v3) = v1.v1*v2.v3 + 2*v1.v2*v1.v3
```

means that there exists one graph with vertex `v1` having a self-loop and with an edge connecting the other two vertices, and that there exists a graph with two edges connecting vertex `v1` with the other two vertices, which can be constructed in two ways.

In general, the expression `dd_`$(v_1, v_2, v_3, \ldots, v_{2n})$ has to do with the graphs consisting of n edges and with prescribed degrees of the vertices. A vertex v occurs d times as argument in the function call of `dd_` if $\deg(v)=d$. Each term in the result of the function call corresponds with a graph with the prescribed degrees. The coefficient of a term tells us in how many ways the graph represented by the term can be constructed. The following FORM session shows that 18 graphs can be made with degree sequence 3,3,3,1. Three of these graphs are loop-free, i.e., have no self-loops, and they are isomorphic. There exists no loop-free graph without a multiple edge.

```
    AutoDeclare Vector v;
    On Statistics;
    Local F = dd_(v1,v1,v1,v2,v2,v2,v3,v3,v3,v4);
    Print +s F;
    .sort

Time =       0.02 sec    Generated terms =         18
                F        Terms in output =         18
```

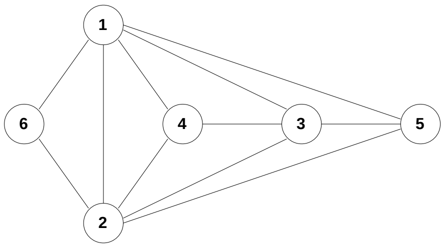
```
                         Bytes used      =        570

   F =
       + 27*v1.v1*v1.v2*v2.v2*v3.v3*v3.v4
       + 54*v1.v1*v1.v2*v2.v3*v2.v4*v3.v3
       + 54*v1.v1*v1.v2*v2.v3^2*v3.v4
       + 54*v1.v1*v1.v3*v2.v2*v2.v3*v3.v4
       + 27*v1.v1*v1.v3*v2.v2*v2.v4*v3.v3
       + 54*v1.v1*v1.v3*v2.v3^2*v2.v4
       + 27*v1.v1*v1.v4*v2.v2*v2.v3*v3.v3
       + 18*v1.v1*v1.v4*v2.v3^3
       + 54*v1.v2*v1.v3*v1.v4*v2.v2*v3.v3
       + 108*v1.v2*v1.v3*v1.v4*v2.v3^2
       + 54*v1.v2*v1.v3^2*v2.v2*v3.v4
       + 108*v1.v2*v1.v3^2*v2.v3*v2.v4
       + 108*v1.v2^2*v1.v3*v2.v3*v3.v4
       + 54*v1.v2^2*v1.v3*v2.v4*v3.v3
       + 54*v1.v2^2*v1.v4*v2.v3*v3.v3
       + 18*v1.v2^3*v3.v3*v3.v4
       + 54*v1.v3^2*v1.v4*v2.v2*v2.v3
       + 18*v1.v3^3*v2.v2*v2.v4
      ;

    * only loop-free graphs
    Off Statistics;
    id v1.v1 = 0;
    id v2.v2 = 0;
    id v3.v3 = 0;
    Print +s F;
    .sort

   F =
       + 108*v1.v2*v1.v3*v1.v4*v2.v3^2
       + 108*v1.v2*v1.v3^2*v2.v3*v2.v4
       + 108*v1.v2^2*v1.v3*v2.v3*v3.v4
      ;

    * no multiple edges
    id v1.v2^2 = 0;
    id v1.v3^2 = 0;
    id v2.v3^2 = 0;
    Print +s F;
    .end

   F = 0;
```

The conditions for “loop-free” and “no multiple edges” can be expressed much shorter in FORM than was done in the above session. As we shall see in the next chapter `id v?.v? = 0` is short notation for saying that every inner product of a vector with itself equals zero. `id u?.v?^2 = 0` is short notation for saying that no inner product with exponent 2 occurs. Hence, the following session proves that there exists only one loop-free graph without multiple edges and with degree sequence 5,5,4,3,3,2.

```
AutoDeclare Vector v;
Local F = dd_(v1,v1,v1,v1,v1,
              v2,v2,v2,v2,v2,
              v3,v3,v3,v3,
              v4,v4,v4,
              v5,v5,v5,
              v6,v6);
id v?.v?=0;     * loop-free
id v1?.v2?^2=0; * no multiple edges
Format 65;
Print F;
.end
```

```
F =
   24883200*v1.v2*v1.v3*v1.v4*v1.v5*v1.v6*v2.v3*v2.v4*v2.v5*
   v2.v6*v3.v4*v3.v5;
```

The third last command `Format 65` is used to control the width of the output: 65 columns at most. The expression corresponds with the following graph:



1.4.7 Exercises

1. Compose in FORM the expression $\sum_{i,j=0}^{3} a_{ij} x^i y^j$.

2. FORM contains a second summation function called `sump_`. It works like the regular function `sum_`, except that the last argument is not the nth element of the sum, but the quotient of the nth element and the $(n-1)$th element. The first element of the sum is normalized to one. So, `sump_(i,0,10,x)` evaluates to the series expansion of $\dfrac{1}{1-x}$ up to order ten.

   Use the function `sump_` to compose the expression $\sum_{i,j=0}^{3} \dfrac{x^i}{i!}\dfrac{y^j}{j!}$, and write it as a polynomial in x.

3. Compose in FORM the expression $\sum_{i=0}^{10} (x+1)^i$, but throw away all powers of degree 4 and higher.

4. Consider the four-dimensional space-time with coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Suppose you have a coordinate transformation $(x_0, x_1, x_2, x_3) = (-ct, x, y, z)$. Show with FORM that $x_\mu x^\mu = -c^2t^2 + x^2 + y^2 + z^2$.

5. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be vectors in $\mathbf{R}^3$. Show with FORM the following equations known in vector analysis.

   [i] $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2$ (identity of Lagrange).

   [ii] $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} - \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$.

   [iii] $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$ (identity of Jacobi).

[iv] $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{C}\cdot\mathbf{D})\mathbf{B} - (\mathbf{B}\times\mathbf{C}\cdot\mathbf{D})\mathbf{A}$.

[v] $(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B}) = 2\mathbf{A}\times\mathbf{B}$.

[vi] $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) + (\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = 0$.

6. Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be polynomials with positive degree in x_n and with coefficients in a field K (e.g, the set of rational numbers). We write

$$\begin{aligned} f &= a_r x_n^r + a_{r-1}x_n^{r-1} + \cdots + a_1 x_n + a_0, \\ g &= b_s x_n^s + b_{s-1}x_n^{s-1} + \cdots + b_1 x_n + b_0, \end{aligned}$$

where $a_r, \ldots, a_0, b_s, \ldots, b_0$ are polynomials in $x_1, x_2, \ldots, x_{n-1}$, and $a_r \neq 0$, $b_s \neq 0$. The *Sylvester matrix* of f and g is the $(r+s)\times(r+s)$ matrix

$$\begin{pmatrix}
a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 \\
0 & a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 \\
0 & \cdots & 0 & 0 & a_r & a_{r-1} & \cdots & a_1 & a_0 \\
b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 & 0 & \cdots & 0 \\
0 & b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\
0 & & 0 & b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 \\
0 & & 0 & 0 & b_s & b_{s-1} & \cdots & b_1 & b_0
\end{pmatrix}$$

where there are s lines constructed with the a_i, and r lines constructed with the b_i. The *resultant* of f and g, denoted by $\mathrm{Res}(f,g)$, or $\mathrm{Res}_{x_n}(f,g)$ if there has to be a variable x_n, is the determinant of the Sylvester matrix. The importance of the resultant lies in the following theorem.

Theorem 1 (Resultant Theorem) *Let $c_1, c_2 \ldots, c_{n-1}$ be number in the algebraic closure of the field K.*
$\mathrm{Res}_{x_n}(f,g)(c_1, c_2, \ldots, c_{n-1}) = 0$ if and only if $f(c_1, c_2, \ldots, c_{n-1}, x_n)$ and $g(c_1, c_2, \ldots, c_{n-1}, x_n)$ have a factor in common or $a_r(c_1, c_2, \ldots, c_{n-1}) = b_s(c_1, c_2, \ldots, c_{n-1}) = 0$.

(i) Use this theorem to find out when a quadratic polynomial in one variable and its derivative have common zeros.

(ii) If f is a univariate polynomial of degree n and with leading coefficient a_n, then the *discriminant* of f is equal to $(-1)^{n(n-1)/2}\mathrm{Res}(f,f')$. Use this property to compute with FORM the discriminant of a third degree univariate polynomial.

7. There is a different way to compute determinants with vectors rather than with commuting functions. For vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$, we have in SCHOONSCHIP notation:

$$\epsilon_{u_1\cdots u_n}\epsilon_{v_1\cdots v_n} = \begin{vmatrix} u_1\cdot v_1 & \cdots & u_1\cdot v_n \\ \vdots & \ddots & \vdots \\ u_n\cdot v_1 & \cdots & u_n\cdot v_n \end{vmatrix}$$

So, if you identify a matrix element M_{ij} with the dot product $u_i\cdot v_j$, then contraction of the above product of two Levi-Civita tensors yields the determinant of the matrix M. The following FORM session show how the determinant example of this chapter can be carried out by this method.

```
AutoDeclare Vectors u,v;
Symbol a,b,c,d;
Local det = e_(u1,u2)*e_(v1,v2);
contract;
```

```
id u1.v1 = a;
id u1.v2 = b;
id u2.v1 = c;
id u2.v2 = d;
Print;
.end

det =
   a*d - b*c;
```

(i) Prove that this method of computing a determinant is correct.

(ii) Experiment a bit to find out whether there is a difference in efficiency between the two method of computing determinants.

8. In the following exercise you can experience the power of the delta function `dd_` in graph theoretical enumeration problems.

   (i) Show that that there exist three loop-free graphs with degree sequence 4,3,2,1. Verify that each graph has multiple edges.

   (ii) Show that that there does not exist a loop-free graph without multiple edges and with degree sequence 7,5,4,3,2,1,1,1.

   (iii) Show that that there exists only one loop-free graph without multiple edges and with degree sequence
   7,4,3,3,2,1,1,1.

   (iv) Show that up to isomorphism there exists only one connected loop-free graph without multiple edges and with degree sequence 3,2,2,1,1,1.

   (v) The terms of the Gram determinant for n vectors are in one-to-one-correspondence with the graphs having degree sequence 2,2,. . .,2 (n numbers). Compare the efficiency of the computation of such Gram determinants via contraction of a square of Levi-Civita tensors with the compuation of graphs with degree sequence 2,2,. . .,2 using `dd_`.

Chapter 2

Pattern Matching

2.1 Substitution

Many operations in a FORM program are in the form of substitutions: replacing one pattern by another one. The `identify` or shortly `id` statement does this in various ways. In the previous chapter we have already seen how it can be used for a straightforward substitution.

In fact, it will only be a one-time substitution as the following example illustrates.

```
 Symbol x;
 Local expr = x + 1/x;
 id x = x+1;
 Print;
 .sort

expr =
   1 + x^-1 + x;

 id x = x+1;
 Print;
 .end

expr =
   2 + x^-1 + x;
```

The replacement rule $x \to x + 1$ does not result in an infinite loop because an `id` statement will never act on its own right hand side. It will only take any natural power of x and replace it by $x + 1$. FORM actually has no other choice than selecting natural powers because pattern matching of rational expressions is hardly available in the system.

Another rather straightforward substitution is the replacement of an integer power of a symbol (exponent 0 is forbidden) or of products of such powers. There will be as many substitutions as possible, e.g., the replacement $x^2 \to y$ transforms x^5 into xy^2. Examples of substitution:

```
 Symbols x,y,z,k;
 Local expr = sum_(k,-2,5,x^k);
 Print;
 .sort

expr =
   1 + x^-2 + x^-1 + x + x^2 + x^3 + x^4 + x^5;

 id x^2 = y;
```

```
 Print;
 .sort

expr =
   1 + x^-2 + x^-1 + x*y + x*y^2 + x + y + y^2;

 id x*y = z;
 Print;
 .sort

expr =
   1 + x^-2 + x^-1 + x + y*z + y + y^2 + z;

 id 1/x = z^2;
 Print;
 .end

expr =
   1 + x + y*z + y + y^2 + z + z^2 + z^4;
```

Expressions can be used in the right-hand side of statement; so, also in the `id` statement. FORM uses the definitions of the expressions as they are present at the start of the module in which the `id` statement is applied. An example:

```
 Symbol x,y;
 Local expr = x*y;
 id x = expr;
 Print;
 .sort

expr =
   x*y^2;

 id x = expr;
 id x = expr;
 Print;
 .end

expr =
   x*y^6;
```

To summarize straightforward substitution: the left-hand side of the replacement rule may be a product of a few factors with exponents, but may not contain a numerical factor, or be a sum of terms. For example, `id 2*x*y=z` and `id x+y=z` are invalid statements. The right-hand side only has to be a valid expression.

Often one wants to apply a substitution rule repeatedly until it causes no further change anymore. This is accomplished by surrounding the command by `repeat` and `endrepeat`. Three examples will do. The first example is a computation of a Fibonacci number. The last two examples come from quantum mechanics: working out the commutation relations of position and momentum operator, and working out a product of Pauli matrices.

Fibonacci Numbers

In the following example we shall use a replacement rule to compute the nineteenth Fibonacci number F_{19}. Recall that the Fibonacci numbers F_n are recursively defined as

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

In other words, every Fibonacci number is the sum of the previous two. The sequence $1, x, x^2, x^3, x^4, \ldots$ also has this property, provided $x^2 = x + 1$.

```
Symbol x;
Local Fibonacci19 = x^18;
repeat;
  id x^2 = x + 1;
endrepeat;
id x = 1;
Print;
.end

Fibonacci19 =
   4181;
```

Working out a Commmutator

We use the commutation relation $[x, p] = \hbar\, i$ between position operator x and momentum operator p repeatedly to work out the commutation relation $[H, x]$ for the Hamiltonian $H = \dfrac{p^2}{2m}$. In the example we shall use the built-in variable i_ for the complex unit $i = \sqrt{-1}$. The mass m and Planck's constant h ($\hbar \stackrel{\text{def}}{=} \frac{h}{2\pi}$) are declared as symbols as they commute with everything else; The Hamiltonian, position, and momentum operators are declared as noncommuting functions.

```
Symbols hbar,m;
Functions x,p,H;
Local [H,x] = H*x - x*H;
id H = p^2/(2*m);
Print;
.sort

[H,x] =
    - 1/2*x*p*p*m^-1 + 1/2*p*p*x*m^-1;

repeat;
  id x*p = p*x + hbar*i_;
endrepeat;
Print;
.end

[H,x] =
    - p*i_*hbar*m^-1;
```

In no time, FORM gives the answer $[H, x] = -\dfrac{\hbar\, i}{m}\, p$.

Pauli Matrices

We consider the algebra generated by σ_1, σ_2, and σ_3 satisfying the relations

$$\sigma_p\, \sigma_q = i\epsilon_{pqr}\sigma_r\,, \quad \text{for } p \neq q, \qquad \{\sigma_p, \sigma_q\} \stackrel{\text{def}}{=} \sigma_p\sigma_q + \sigma_q\sigma_p = 2\delta_{pq},$$

for $p, q, r = 1, 2, 3$. The generators are called Pauli matrices because of the following matrix representation.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We work out the expression $(\sigma_1\sigma_2 + \sigma_1 + \sigma_2 + \sigma_3)^4$. The following FORM program could be improved in various ways, but that is not an issue now.

```
 Function s;
 Index k;
 Dimension 3;
 Local [s(1)*s(2)] = i_*e_(1,2,3)*e_(1,2,k)*s(k);
 Local [s(1)*s(3)] = i_*e_(1,2,3)*e_(1,3,k)*s(k);
 Local [s(2)*s(3)] = i_*e_(1,2,3)*e_(2,3,k)*s(k);
 contract;
 Print;
 .sort

[s(1)*s(2)] =
   s(3)*i_;

[s(1)*s(3)] =
    - s(2)*i_;

[s(2)*s(3)] =
   s(1)*i_;

 Local F = ( s(1)*s(2) + s(1) + s(2) + s(3) )^4;
 repeat;
   id s(2)*s(1) = -s(1)*s(2);
   id s(3)*s(1) = -s(1)*s(3);
   id s(3)*s(2) = -s(2)*s(3);
   id s(1)*s(2) = [s(1)*s(2)];
   id s(1)*s(3) = [s(1)*s(3)];
   id s(2)*s(3) = [s(2)*s(3)];
   id s(1)^2 = 1;
   id s(2)^2 = 1;
   id s(3)^2 = 1;
 endrepeat;
 Print F;
 .end

F =
   8*i_;
```

In the first module, we let FORM compute various products of Pauli matrices. These results are used in the second module to work out formulas.

2.1.1 Exercises

1. The kinetic energy T is given in terms of mass m and momentum p as $T = \dfrac{p^2}{2m}$. Express T in terms of
   (i) the velocity v, which is related to momentum and mass by $p = mv$.
   (ii) acceleration a and time t, which are related to the velocity by $v = at$.
2. Transform in FORM the expression $x^2 + x + \dfrac{1}{x}$ into
   (i) $y^2 + y + \dfrac{1}{y}$
   (ii) $y^2 + y + \dfrac{1}{x}$
   (iii) $x^2 + x + y$

3. How can you replace in FORM $a + b$ by d in the expression $a + b + c$.
4. Transform the expression $(x^2 + 1)^2 + x^2 + 2$ into $y^2 + y + 1$.
5. The basic element of the quaternions are i, j, and k, subject to the rules $ij = k$ (and cyclic permutations of i, j, and k), $ji = -ij$, $ki = -ik$, $kj = -jk$, and $i^2 = j^2 = k^2 = -1$. Write a FORM program that computes the quaternion $(u + 3i - k)^3(-1 + tj)$, where u and t are unknowns. See [Cohen et al 92] for a comparison of a FORM program and a Maple program.
6. Let D denote the differentiation operator $\dfrac{d}{dx}$. So, the following commutation relations hold:
$$[D, x] = 1, \quad [D, \cos x] = -\sin x, \quad [D, \sin x] = \cos x\,.$$
Write a FORM program that uses these relations to transform $D^3 x^3 \sin^2 x \cos x$ into a sum of terms with the D operator to the right, and finally replace D by 0. The remaining terms will form the third derivative of $x^3 \sin^2 x \cos x$. Check your answer by pencil and paper or via a general purpose computer algebra system like Derive, Maple, or *Mathematica*.
7. Consider the vector fields $\{\dfrac{d}{dx},\, x\dfrac{d}{dx},\, x^2\dfrac{d}{dx}\}$.
   (i) Compute the commutation relations between these vector fields and verify that they form a Lie algebra.
   (ii) Determine the center of the Lie algebra, i.e., the set of elements that commute with any other element of the Lie algebra.
   (iii) Show that the vector fields $\{\dfrac{d}{dx},\, x\dfrac{d}{dx},\, x^2\dfrac{d}{dx},\, x^3\dfrac{d}{dx}\}$ do not form a Lie algebra.
8. We consider the Lie algebra of type $\mathcal{SU}_2$ generated by the triple h,e,f, which satisfy the following commutation relations
$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$
The enveloping algebra has Poincaré-Birkhoff-Witt basis given by
$$h^i e^j f^k,$$
where i, j, k are nonnegative integers.
   (i) Use the commutation relation to write $f^3 e^2 h$ in terms of the Poincaré-Birkhoff-Witt basis.
   (ii) What is the commutation relation between h and $he^2 f^3$?

2.2 Pattern Objects

In the previous section we have looked at straightforward substitution via the `id` statement: a symbol, a power of a symbol, and products of powers of symbols were replaced by new expressions. However, very often you do not only want particular objects being replaced, but actually all objects of a certain type. For example, if you are interested in integrating polynomials in one variable x, you do not want to replace x^n for specific values of the natural number n by $\frac{1}{n+1}x^{n+1}$, but instead for all natural numbers. The rest of this chapter will be about such *pattern matching*.

A pattern in FORM, also called a *wildcard*, is an elegant way of representing the syntactical structure of an expression. The atomic pattern objects are denoted by a variable followed by a question mark and represent one single object. For example, if `x` is a symbol, then `x?` will represent any symbol and `x?^2` will represent any square of a symbol. If `p(i)` represents the vector `p` with index `i`, then `p(i?)` represents the vector `p` with any index, `p?(i)` represents any vector with index `i`, and `p?(i?)` represents any vector with any index. And so on.

Because patterns are in general not restricted to single objects they can become rather complicated. For example, `x * y^n? * f(i,g?(i))` is a valid FORM pattern. In case `x`, `y`, `i`, `n` are symbols, and `f`, `g` are functions, the above pattern represents a product of `x`, any integer power of `y`, and a function `f` of which the first argument is equal to `i` and the second one is any function with argument `i`.

One more thing that you have to take into account when dealing with more complicated patterns: when a wildcard such as `x?` appears twice in the left-hand side of an `id` statement both occurrences have to match the same argument. For example, for two vectors `u` and `v`, the pattern `u?.u?` represents any dot product of a vector with itself, and `u?.v?` represents any dot product, including the case of identical vectors in the dot product.

2.2.1 Exercises

Advice: if you find it difficult to answer right now the exercises below, then work through the examples of the next section and come back to these exercises afterwards. Or try things out in FORM to get the idea.

1. Given the symbols `x`, `n`, explain what the following patterns mean.
   - (i) `x`
   - (ii) `1/x`
   - (iii) `x?`
   - (iv) `x^n?`
   - (v) `x?^n`
   - (vi) `x?^n?`
2. Given the vectors `u`, `v` and the indices `i`, `j`, explain what the following patterns mean.
   - (i) `u(i)`
   - (ii) `u(i?)`
   - (iii) `u?(i)`
   - (iv) `u?(i?)`
   - (v) `u(i?)*v(j?)`
   - (vi) `u(i?)*v(i?)`
   - (vii) `u.v?`
   - (viii) `u?.v?`
   - (ix) `v?.v?`
   - (x) `u?.v?^2`

2.3 Patterns in Replacement Rules

Patterns are used in replacement rules. In this section, we shall give examples that show you how to use wildcards in FORM.

2.3.1 Polynomial Substitutions

```
Symbols x,y,z,n;
Local F = x^2 + y^3 + 1;
id x? = z;
Print;
.sort
```

```
F =
```

```
    1 + z^2 + z^3;

  id z^n? = x;
  Print;
  .sort

F =
    3*x;

  Local G = F + y^2 + 1;
  id x?^n? = z;
  Print G;
  .end

G =
    1 + 4*z;
```

Let us have a closer look at the above session. Because x has been declared as a symbol, the first identification reads like "replace every symbol by z." So, the expression $F = x^2 + y^3 + 1$ is replaced by $F = z^2 + z^3 + 1$ and sorted into $1 + z^2 + z^3$. The next identification reads like "replace any integer power of z by x." The result $3x$ becomes less obscure when you realize that 1 can be described as z^0. Finally we compose the expression $G = 3x + y^2 + 1$ and apply the replacement rule "any symbol to any power goes into z". Now, FORM leaves the constant term intact because it expects at least a symbol.

2.3.2 Fibonacci Numbers

In the next two examples we shall use replacement rules to compute the nineteenth Fibonacci number F_{19}. Recall that the Fibonacci numbers F_n are recursively defined as

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

```
  Symbols last, secondlast, dummy;
  Function F;
  On statistics;
  Local Fibonacci19 = F(1,1) * dummy^17;
  repeat;
    id F(last?, secondlast?) * dummy = F(last + secondlast, last);
  endrepeat;
  id F(last?, secondlast?) = last;
  Print;
  .end

Time =       0.04 sec    Generated terms =          1
      Fibonacci19        Terms in output =          1
                         Bytes used      =         10

   Fibonacci19 =
      4181;
```

Three remarks:

- We tagged the expression with `dummy^17` and use a replacement rule that lowers the power of `dummy` by one. The repetition stops when no power of `dummy` is left. This trick of tagging an expression by a dummy variable to control repetition or to apply certain operations from left to right or vice versa in the expression is often used in FORM programs.

- The symbol wildcard, used as a function argument, matches a whole expression (in this case a natural number). You may say that the symbol wildcard as a function argument denotes "any function argument." We shall come back to this issue in the next section.
- There are two symbol wildcards on the left hand-side of `id`-statement, viz. `last?` and `secondlast?`. They both match any function argument in the call of `F`. Then, in the right hand-side of the `id`-statement, `last` and `secondlast` will be replaced by the matched numbers.

A downward recursion would result in a less efficient program with respect to computing time and number of terms generated.

```
Symbols n;
Function F;
On statistics;
Local Fibonacci19 = F(19);
repeat;
  id F(1) = 1;
  id F(2) = 1;
  id F(n?) = F(n-1) + F(n-2);
endrepeat;
Print;
.end
```

```
Time =       6.19 sec    Generated terms =       4181
      Fibonacci19        Terms in output =          1
                         Bytes used      =         10

   Fibonacci19 =
      4181;
```

2.3.3 Differentiation of Polynomials

Differentiation of bivariate polynomials, say computing the derivative $\frac{\partial^2}{\partial x \partial y}\left(x^2y^3 + x^3 + x^4y^4\right)$, can be done as follows in FORM.

```
Symbols x,y,m,n;
Local P = x^2*y^3 + x^3 + x^4*y^4;
Print;
.sort
```

```
   P =
      x^2*y^3 + x^3 + x^4*y^4;
```

```
id  x^m? * y^n? = m*x^(m-1) * n*y^(n-1);
Print;
.end
```

```
   P =
      6*x*y^2 + 16*x^3*y^3;
```

2.3.4 Vector Calculus: Coordinate Transformations

An example from vector calculus: given a basis transformation $y_k = b_{ir}x_r$, a vector T in the y-basis defined as $T_i = g_{ij}a_{jk}y_k$, and the restriction that the matrices a and b are inverses of each other, describe T in the x-basis.

```
Vector x,y;
Tensors g,a,b;
Indices i,j,k;
Local [y(k)] = b(k,x);
Local [T(i)] = g(i,j) * a(j,k) * [y(k)];
id a(i?,k?) * b(k?,j?) = d_(i,j);
Print [T(i)];
.end

[Tx(i)] =
  g(i,x);
```

So, we have proved with FORM that $T_i = g_{ir}x_r$. You may wonder why it works, but in FORM an index matches another index or a vector that would have been an index if the SCHOONSCHIP notation had not been used.

2.3.5 Levi-Civita Tensor

Let us illustrate the rules about repeated wildcards for a user-defined Levi-Civita tensor `eps` and a user-defined Kronecker symbol `delta` with various contractions in 3-dimensional space.

```
Dimension 3;
Tensors eps(antisymmetric), delta(symmetric);
Indices i,j,k,l,m,n;
*
* three repeated indices
*
Local F1 = eps(i,j,k) * eps(i,j,k);
id eps(i?,j?,k?) * eps(i?,j?,k?) = 6;
Print F1;
.sort

F1 =
  6;

*
* two repeated indices; antisymmetry is applied
*
Local F2 = eps(i,j,k) * eps(i,j,l);
id eps(i?,j?,k?) * eps(i?,j?,l?) = delta(k,l);
Print F2;
.sort

F2 =
  delta(k,l);

*
* one repeated index
*
Local F3 = eps(i,j,k) * eps(i,l,m);
id eps(i?,j?,k?) * eps(i?,l?,m?) = delta(j,l) * delta(k,m) -
                                   delta(j,k) * delta(l,m);
Print F3;
.sort
```

```
F3 =
    - delta(j,k)*delta(l,m) + delta(j,l)*delta(k,m);

 *
 * no repeated index; only antisymmetry is applied
 *
 Local F4 = eps(i,j,k) * eps(n,m,l);
 id eps(i?,j?,k?) * eps(l?,m?,n?) = eps(i,j,k) * eps(l,m,n);
 Print F4;
 .end

F4 =
    - eps(i,j,k)*eps(l,m,n);
```

2.3.6 Exercises

1. How can you implement the rules for odd and even functions in FORM.

2. Implement the transformation rule $\exp x \exp y \longrightarrow \exp(x+y)$ and apply it to the expression $(\exp a + \exp b + \exp c)^3$. Simplify the result as far as possible.

3. Implement the simplification $\sin^2 x + \cos^2 x = 1$, and apply it to the expressions $\sin^2 a - 1$, and $\sin^3 a$.

4. Write a FORM program that computes the Laguerre polynomials $L_n(a,x)$. Recall that these polynomials are recursively defined as

$$\begin{aligned} L_0(a,x) &= 1, \\ L_1(a,x) &= 1 + a - x, \\ L_n(a,x) &= \frac{(2n+a-1-x)}{n} L_{n-1}(a,x) - \frac{(n+a-1)}{n} L_{n-2}(a,x), \end{aligned}$$

   for $n > 1$.

5. Write a FORM program that integrates univariate polynomials.

6. Consider the following Lorentz transformation in four-dimensional space-time

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - \frac{vx}{c^2}), \end{aligned}$$

   where

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

   Show with FORM that

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2.$$

7. Consider a space of n dimensions with coordinate functions $\phi^1, \phi^2, \ldots, \phi^n$ and metric tensor given by

$$g_{ij} = \delta_{ij} + \frac{\phi^i \phi^j}{1 - (\phi^k)^2},$$

   where Einstein's summation convention is used so that $\phi_k^2 = \sum_i^n (\phi^i)^2$.

(i) Verify with FORM that the inverse of the metric tensor is given by

$$g^{ij} = \delta_{ij} - \phi^i \phi^j$$

(ii) The Christoffel symbol Γ_{jkl} of the first kind is defined by

$$\Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial \phi^k} + \frac{\partial g_{lk}}{\partial \phi^j} - \frac{\partial g_{kj}}{\partial \phi^l}\right)$$

Compute this symbol with FORM.

(iii) The Christoffel symbol Γ^i_{jk} of the second kind is defined by

$$\Gamma^i_{jk} = g^{il}\Gamma_{jkl} \ ,$$

where we Einstein's summation convention is used again. Compute this symbol with FORM.

2.4 Patterns and Functions

For functions there are basically four types of wildcarding:

1. one or more wildcard parameters of a function;
2. wildcards for functions;
3. a combination of the above two;
4. wildcards for groups of arguments.

2.4.1 Wildcard Parameters

We have already seen examples of such wildcarding in the previous section. But let us look at another one. Suppose that we want to implement in FORM the simplification $x\sqrt{y} = \sqrt{x^2 y}$, and apply it to $a\sqrt{b}$, $\sqrt{2}\sqrt{b}$, and $\sqrt{\sqrt{a}+1} \cdot \sqrt{\sqrt{a}-1}$.

```
 Symbols a,b,x,y;
 CFunction sqrt;
 Local F1 = a * sqrt(b);
 Local F2 = sqrt(2) * sqrt(b);
 Local F3 = sqrt(sqrt(a)+1) * sqrt(sqrt(a)-1);
 id x? * sqrt(y?) = sqrt(x^2*y);
 Print;
 .sort

F1 =
   sqrt(a^2*b);

F2 =
   sqrt(2)*sqrt(b);

F3 =
   sqrt( - 1 + sqrt(a))*sqrt(1 + sqrt(a));

 repeat;
   id sqrt(x?) * sqrt(y?) = sqrt(x*y);
 endrepeat;
 id sqrt(4) = 2;
```

```
 Print F2,F3;
 .sort

F2 =
   sqrt(2*b);

F3 =
   sqrt( - 1 + sqrt(a)^2);

 argument;
   id sqrt(x?) * sqrt(x?) = x;
 endargument;
 Print F3;
 .end

F3 =
   sqrt( - 1 + a);
```

As you see the pattern `x? * sqrt(y?)` only matches the first formula; the factors `sqrt(sqrt(a)+1)` and `sqrt(2)` are no symbols. If you specify a pattern `sqrt(x?) * sqrt(y?)`, then these expressions match. Note that in this case the wildcard may match a whole composite expression. Alas, the identification has not replaced the product of square roots inside the last formula. For efficiency reason, the rule in FORM is that substitutions are *not* executed inside the arguments of functions. Therefore FORM has a special environment that allows manipulation of function arguments. This environment is indicated by the keywords `argument` and `endargument`, as shown in the third module in the above program.

`argument` environments can be nested. The next example illustrates this.

```
 CFunction f;
 Symbols x,y;
 Local expr = f(x,f(x));
 id x=y;
 Print;
 .sort

expr =
   f(x,f(x));

 argument;
   id x=y;
 endargument;
 Print;
 .sort

expr =
   f(y,f(x));

 argument;
   argument;
     id x?=y^2;
   endargument;
 endargument;
 Print;
 .end

expr =
```

```
   f(y,f(y^2));
```

2.4.2 Wildcards for Functions

There is nothing special about wildcards of functions or about the combination of this and the previous type of wildcarding. We give one example that shows all.

```
    Symbols x,y,z;
    CFunctions f,g,h;
    Local expr1 = f(x) + g(y);
    id f?(x) = h(x);
    Print;
    .sort;

   expr1 =
      g(y) + h(x);

    Local expr2 = f(x) + g(y);
    id f?(x?) = z;
    Print expr2;
    .sort;

   expr2 =
      2*z;

    Local expr3 = f(x+y) + f(x,y);
    id f?(x?) = z;
    Print expr3;
    .end

   expr3 =
      z + f(x,y);
```

We added the last module to illustrate once more that a symbol can match as a function argument a whole expression. But from this example, you also see that it can only match one argument and not more. To achieve this we need a special kind of wildcard that refers to a group of arguments in a function. This is the topic of the next subsection.

2.4.3 Wildcarding for Groups of Parameters

The wildcarding in FORM allows you to refer to a group of parameters. For example, the `id` statement in the example below uses a variable that starts with a question mark to represent any sequence of adjacent arguments in a function call. We call this an *argument sequence wildcard*.

```
    Symbols x,y;
    CFunctions f,g;
    Local F = f(x,x,x) + f;
    id f(?a) = g(0,?a,0,?a,0);
    Print;
    .end

   F =
      g(0,x,x,x,0,x,x,x,0) + g(0,0,0);
```

The variables `?a` in the right hand side of the identify statement refer to the match of the wildcard.

A few remarks:

- Argument sequence wildcards are variables that start with a question mark followed by a nonempty alphanumeric string. These variables do not have to be declared. Their use declares them and after the statement they are forgotten again.
- An argument sequence wildcard can also refer to the sequence of all arguments in a function call.
- An argument sequence wildcard can also refer to an empty sequence.
- You can have more than one argument sequence wildcard. For example, if `f` is a function and `x`, `y` are symbols, then the pattern `f?(x,?a)*f?(y,?b)` matches the product of any function which starts with `x` and the same function that starts with `y`. The pattern `f?(?a, x, ?b)` matches any function that has an argument `x`.
- If there are several options in matching the patterns, there is only one general rule that determines which option will be taken: from left to right, the argument sequence wildcards will refer to lesser numbers of arguments. Symmetry properties may change the way in which the option of matching is chosen or in case of (anti)symmetric tensors even ignore the replacement rule (pattern matching would often be too costly). An example of multiple options:

```
Symbols w,x,y,z;
Indices W,X,Y,Z;
CFunction F;
Tensor S(symmetric), C(cyclic);
Local expr = F(x,y,z) + S(X,Y,Z) + C(X,Y,Z);
id F(?a,w?,?b) = F(w,0,?a,0,?b);
id S(?a,W?,?b) = S(W,0,?a,0,?b);
id C(?a,W?,?b) = C(W,0,?a,0,?b);
Print;
.end

expr =
   F(z,0,x,y,0) + S(X,Y,Z) + C(0,0,Y,Z,X);
```

Let us illustrate the wildcarding with argument fields by another example taken from tensor calculus. We consider the metric tensor g, and the Riemann tensor and Ricci tensor denoted by R. We show how you can implement in FORM the contractions $g_i^j R_{jk} = R_{ik}$ and $g^{ij} R_{ikjl} = R_{kl}$. It will also be an illustration of how one can compute with upper (contravariant) and lower (covariant) indices in FORM.

```
Tensors g,R;
Indices i,j,k,l,m,n,low,up;
Local T1 = g(i,low,j,up) * R(j,low,k,low);
Local T2 = g(i,up,j,up) * R(i,low,k,low,j,low,l,low);
id g(i?,low,j?,up) * R?(?a,j?,low,?b) = R(?a,i,low,?b);
id g(i?, up,j?,up) * R?(?a,i?,low,?b,j?,low,?c) = R(?a,?b,?c);
id g(i?, up,j?,up) * R?(?a,j?,low,?b,i?,low,?c) = R(?a,?b,?c);
Print;
.end

T1 =
   R(i,low,k,low);

T2 =
   R(k,low,l,low);
```

As you see, we simply keep track of the type of the index by putting next to the index in the function call a special index `low` or `up`, and we distinguish the indices by type in the identifications. The third identify statement has only been added for the general case where repeated indices may be interchanged.

Of course, the above implementation of upper and lower indices is somewhat cumbersome. So let us introduce a shorter notation such as `U(i)` and `L(j)` for an upper index `i` and lower index `j`, respectively. The example now looks as follows:

```
Functions g,R,L,U;
Indices i,j,k,l,m,n;
Local T1 = g(L(i),U(j)) * R(L(j),L(k));
Local T2 = g(U(i),U(j)) * R(L(i),L(k),L(j),L(l));
id g(L(i?),U(j?)) * R?(?a,L(j?),?b) = R(?a,L(i),?b);
id g(U(i?),U(j?)) * R?(?a,L(i?),?b,L(j?),?c) = R(?a,?b,?c);
id g(U(i?),U(j?)) * R?(?a,L(j?),?b,L(i?),?c) = R(?a,?b,?c);
Print;
.end
```

```
T1 =
   R(L(i),L(k));

T2 =
   R(L(k),L(l));
```

It works! But it is good to realize that we rely on certain aspects of FORM. Firstly, note that there is ambiguity in the matching of the pattern `R?(?a,L(j?),?b)` with the expression `R(L(j),L(k))`: should the first question mark variable be an empty argument sequence or should the second question mark variable match an empty statement. In the latter case, the first identification in the above example will have no match. Apparently FORM searches until it finds a match. Secondly, note that a nested wildcarding for functions is used. Although FORM allows this, for efficiency reasons, it will in general not try out all possible matchings. Once it has found a match, it will stop looking for further matches. Apparently we were lucky in the wildcarding of our example: FORM selected the correct pattern match for getting the work done.

By denesting you can get more control about the wildcarding of nested functions. Below we show you how to do this in our example.

```
Functions g,R,L,U;
Indices i,j,k,l,m,n,low,up;
Local T1 = g(L(i),U(j)) * R(L(j),L(k));
Local T2 = g(U(i),U(j)) * R(L(i),L(k),L(j),L(l));
*
* denest functions
*
repeat;
  id R?(?a,L(i?),?b) = R(?a,i,low,?b);
  id R?(?a,U(i?),?b) = R(?a,i,up,?b);
endrepeat;
*
* apply rules
*
id g(i?,low,j?,up) * R?(?a,j?,low,?b) = R(?a,i,low,?b);
id g(i?, up,j?,up) * R?(?a,i?,low,?b,j?,low,?c) = R(?a,?b,?c);
id g(i?, up,j?,up) * R?(?a,j?,low,?b,i?,low,?c) = R(?a,?b,?c);
*
* back to original notation
*
repeat;
  id R?(?a,i?,low,?b) = R(?a,L(i),?b);
```

```
  id R?(?a,i?, up,?b) = R(?a,L(i),?b);
endrepeat;
*
* Print the results
*
Print;
.end

T1 =
  R(L(i),L(k));

T2 =
  R(L(k),L(l));
```

In the next section, we shall show another way of handling upper and lower indices which does not lead to two representations of the same mathematical object.

2.4.4 Exercises

1. Implement the simplifications $\ln(xy) \to \ln x + \ln y$ and $\ln(x^n) = n \ln x$ (if n is an integer), and apply them to the expression $\ln(abc)$ and $\ln(ab^3)$.

2. Implement simplification rules for the determinant of matrices such that $\det(M^5)$ simplifies into $\det(M)^5$, and $\det(ABC)$ becomes $(\det A)(\det B)(\det C)$.

3. Show with FORM that if U_i and V_i are the components of covariant vectors $\mathbf{U}$ and $\mathbf{V}$, respectively, then $T_{ij} = U_i V_j - V_i U_j$ are the components of a covariant tensor $\mathbf{T}$ of order 2. Recall that a covariant vector U_i transforms under coordinate changes like $\overline{U}_i = \dfrac{\partial x^k}{\partial \overline{x}^i} U_k$, and that a covariant tensor T_{ij} of order two transforms like $\overline{T}_{ij} = \dfrac{\partial x^k}{\partial \overline{x}^i}\dfrac{\partial x^l}{\partial \overline{x}^j} T_{kl}$.

4. If $n = 2$, write out the triple sum $c_{rst} x_r y_s z_t$ in explicit form using only replacement rules.

5. The `ToVector` command replaces a tensor into a product of vector components. For example, `ToVector t,v` replaces `t(m1,m2,m3)` by `v(m1)*v(m2)*v(m3)`. Use `id` statements to get the same job done.

6. In classical electromagnetic theory, the electromagnetic field tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & \dfrac{E_x}{c} & \dfrac{E_y}{c} & \dfrac{E_z}{c} \\ -\dfrac{E_x}{c} & 0 & -B_z & B_y \\ -\dfrac{E_y}{c} & B_z & 0 & -B_x \\ -\dfrac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

   In other words, $F_{00} = 0$, $F_{0\nu} = \dfrac{E_\nu}{c}$ for $\nu = 1,2,3$, and $F_{ij} = -\epsilon_{ijk} B_k$ for $i,j,k = 1,2,3$, where ϵ denotes the Levi-Civita tensor.

   We shall use as metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ for special relativity the one with sign convention

$g_{0\nu} = \delta_{0\nu}$ and $g_{ij} = -\delta_{ij}$, for $i, j = 1, 2, 3$. Then the full contravariant form $F^{\mu\nu}$ is

$$F^{\mu\nu} = g^{\mu\rho} g^{\mu\rho} F_{\rho\sigma} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

Write the expression $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, which are invariant under Lorentz transformations, in terms of the electric field E and magnetic field B.

2.5 Conditions on Wildcards and Replacements

The pattern matching we have seen thus far involved wildcards that would match any variable of proper type. Furthermore, no restrictions on the replacement rules have been made. In this section, we shall see how to get more control over the wildcards and replacements. The last three subsections will contain "real world" examples of differentiation of functions, tensor calculus with upper and lower indices, and computation with gamma matrices.

2.5.1 Sets and Wildcarding

One of the types of variables in FORM is "set" or "array". An example that explains the data type:

```
Symbol a1,a2;
Set a:a1,a2;
Local F = a[1] + a[2];
Print;
.end
```

```
F =
   a1 + a2;
```

As you see, FORM assumes that sets or arrays start with index 1. Furthermore, sets are homogeneous objects, i.e., elements of sets must be of the same type.

Sets are mostly used in wildcarding.

```
Symbols a,b,c,x;
Local F = a + b + c;
id x?{b,c} = 3;
Print;
.end
```

```
F =
   6 + a;
```

`x?` would mean "any symbol", whereas in our example we restrict the symbols to the set $\{b, c\}$. You can also exclude symbols by `?!`.

```
Symbols a,b,c,x;
Local F = a + b + c;
id x?!{b,c} = 3;
Print;
.end
```

```
F =
   3 + b + c;
```

We called sets also arrays because wildcarding sets behave like these.

```
Symbols a1,a2,b1,b2,x,n;
Function f;
Set a : a1,a2;
Set b : b1,b2;
Local F = a[1] + a[2];
id x?a[n] = b[n] + f(n);
Print;
.end
```

```
F =
   b1 + b2 + f(1) + f(2);
```

In the above example, `x` must belong to the set `a`, and `n` becomes the number of the element in the set to which `x` matches. The same number is used at the right hand side of the identity. However, no arithmetic can be done with `n`.

There exists a shortcut for changing names of set elements.

```
Symbols a1,a2,b1,b2,x,n;
Set a : a1,a2;
Set b : b1,b2;
Local F = a[1] + a[2];
id x?a?b = x;
Print;
.end
```

```
F =
   b1 + b2;
```

Here, the identity statement reads as follows: `x` must belong to the set `a`, and in the right hand side each occurrence of `x` will be replaced by the corresponding element of the set `b`.

The built-in sets in FORM are listed below. As all built-in objects, they end with an underscore.

Set	*Meaning*: set of
`even_`	even integers
`fixed_`	fixed indices
`index_`	all indices
`int_`	integers
`neg_`	negative integers
`neg0_`	nonpositive integers
`number_`	all rational numbers
`odd_`	odd integers
`pos_`	positive integers
`pos0_`	nonnegative integers
`symbol_`	only symbols

Intervals can be specified as so-called *ranged sets*. `{>=-3,<5`, `{<10}`, and `{>=2}` denote $[-3, 5)$, $(-\infty, 10)$, and $[2, \infty)$, respectively.

With the built-in sets and the range sets you can restrict wildcards to some infinite sets as is shown in the next example.

```
 Symbols x,n;
 Local F = sum_( n, -3, 3, x^n );
 Print;
 .sort

F =
   1 + x^-3 + x^-2 + x^-1 + x + x^2 + x^3;

 id x^n?pos_ = 0;
 Print;
 .sort

F =
   1 + x^-3 + x^-2 + x^-1;

 id x^n?!even_ = 0;
 Print;
 .end

F =
   1 + x^-2;
```

Here, we replace first all positive powers of `x` by zero, and hereafter we remove all powers with odd exponent.

2.5.2 Restrictions on Replacements

When you create a couple of identifications, there may be a conflict of what rule to apply first. In FORM, you can influence the applicability of an identification by options. One of the options is `select`. It is followed by the names of one or more sets of symbols, functions, vectors, or indices. No built-in sets are allowed in the `select` statement. The replacement rule will only be applied if after the matching of the left-hand side no elements of the mentioned sets are left anywhere in the term. An example explains it better.

```
 Symbols a,b,c,d;
 Functions f,g;
 Index i;
 Set bcSet: b,c;
 Local F1 = a*b*c;
 id select bcSet a*b = b^2;
 id select bcSet a*b*c = b^2*c^2;
 Print;
 .sort

F1 =
   b^2*c^2;

 Local F2 = f(i,a)*b + f(i,b)*c;
 id select bcSet c?=g(d);
 Print;
 .end

F1 =
   b^2*c^2;

F2 =
   f(i,a)*g(d) + f(i,b)*c;
```

In the above example, the first replacement rule is not applied because after matching `ab` in the product `abc` the element `c` of the set `{b,c}` is left behind. The second replacement rules matches `F1` and will be applied. The third replacement rule is applied to first term of expression `F2` only. After matching a symbol in the second term of `F2` there is still an element of the set $\{b, c\}$ left. The same happens in expression `F1` when one tries to apply the replacement rule.

There are more options to identify statements in FORM. Below we tabulate them and we illustrate the use of some of them.

Option	*Meaning*
`disorder`	substitution in non-commutative algebra
`ifmatch`	if there is a match, jump to a label after substitution
`many`	multiple matches
`multi`	single match with multiplicity
`once`	single match
`only`	exact match
`select`	match with no selected symbols, functions, vectors, or indices left

```
 Symbols a,b,x,y,z;
 Local F0 = (x+y)^4;
 Print F0;
 .sort

F0 =
   4*x*y^3 + 6*x^2*y^2 + 4*x^3*y + x^4 + y^4;

 Local F1 = (x+y)^4;
 id once x = z;
 Print F1;
 .sort

F1 =
   6*x*y^2*z + 4*x^2*y*z + x^3*z + 4*y^3*z + y^4;

 Local F2 = (x+y)^4;
 id x*y = z;
 Print F2;
 .sort

F2 =
   4*x^2*z + x^4 + 4*y^2*z + y^4 + 6*z^2;

 Local F3 = (x+y)^4;
 id only x*y = z;
 Print F3;
 .sort

F3 =
   4*x*y^3 + 6*x^2*y^2 + 4*x^3*y + x^4 + y^4;

 Local F4 = (a+b)^2 * (x+y)^2;
 Print F4;
 .sort

F4 =
```

```
    4*a*b*x*y + 2*a*b*x^2 + 2*a*b*y^2 + 2*a^2*x*y + a^2*x^2 + a^2*y^2 + 2*
    b^2*x*y + b^2*x^2 + b^2*y^2;

  id multi x?*y? = z;
  Print F4;
  .end

 F4 =
    2*a*y*z + 2*b*y*z + 4*x*y*z + 2*x^2*z + 2*y^2*z + 4*z^2;
```

The normal rule for pattern matching in FORM is that the pattern is taken out as many times as possible before inserting the right-hand side (the most important exception is a wildcard power). No second attempt of pattern matching is made. In the first identification in the above example, the option `once` overrules the general behavior of the system: only one single match of a pattern is attempted. By the way, the default is `many`. The next two identifications in the example are replacements $xy \to z$ in the polynomial $(x+y)^4$. The option `only` implies that the match must be exact. Finally, the option `multi` is used: it means a single matching with multiplicity. In the given example, it means that when it is applied to the term $4abxy$, only ab is taken out with multiplicity one. The other product, viz., xy, is left untouched. So you obtain the term $4xyz$. You can check that similar things happen for the other terms.

The `ifmatch` option is mostly used in cases where there is a long list of substitutions, but applying them all at once would make the substitution tree too complicated. Below we only give a simple example to illustrate the syntax and semantics of the statement. We remove from a polynomial in two variables x and y all monomials in y and replace `x` by the symbol `z`.

```
  Symbols x,y,z;
  Local F = x^2*y + y + 1;
  id ifmatch->1 x=z;
    id y=0;
  label 1;
  Print;
  .end

 F =
    1 + y*z^2;
```

The statement `id ifmatch->1, x=z;` will lead to a jump to label 1 if there is a match and after the substitution has been made. Logically the above program is equivalent to

```
  Symbols x,y,z;
  Local F = x^2*y + y + 1;
  if ( match(x) );
    id x = z;
  else;
    id y=0;
  endif;
  Print;
  .end

 F =
    1 + y*z^2;
```

Such a nesting of if-statements becomes rather impractical when many statements are involved. Moreover in this setup the matching has to be done twice, while the `ifmatch` construction involves only a single pattern matching.

We shall describe the option disorder with an example from Dirac algebra in dimension 4. The Dirac gamma matrices γ^0, γ^1, γ^2, and γ^3 have the properties

$$\begin{aligned} \gamma^i\gamma^j + \gamma^j\gamma^i &= 0 \quad \text{if } i \neq j, \\ (\gamma^0)^2 = 1, \quad (\gamma^i)^2 &= -1 \quad \text{for } i = 1,2,3. \end{aligned}$$

For $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the following two equations hold:

$$\begin{aligned} \gamma^5\gamma^k + \gamma^k\gamma^5 &= 0 \quad \text{for all } k, \\ (\gamma^5)^2 &= 1. \end{aligned}$$

It is easy to have FORM confirm these rules.

```
Functions g3,...,g0,g,h;
Local [g5] = i_ * g0 * g1 * g2 * g3;
Local [g5^2] = [g5]^2;
Local [g0*g5+g5*g0] = g0 * [g5] + [g5] * g0;
Local [g1*g5+g5*g1] = g1 * [g5] + [g5] * g1;
Local [g2*g5+g5*g2] = g2 * [g5] + [g5] * g2;
Local [g3*g5+g5*g3] = g3 * [g5] + [g5] * g3;
repeat;
  id g0*g0 = 1;
  id g?*g? = -1;
  id disorder g? * h?= - h * g;
endrepeat;
Print;
.end

[g5] =
   g0*g1*g2*g3*i_;

[g5^2] =
   1;

[g0*g5+g5*g0] = 0;

[g1*g5+g5*g1] = 0;

[g2*g5+g5*g2] = 0;

[g3*g5+g5*g3] = 0;
```

You may have expected replacement rules like:

```
id g2 * g1 = -g1 * g2;
id g3 * g1 = -g1 * g3;
id g3 * g2 = -g2 * g3;
id g4 * g1 = -g1 * g4;
id g4 * g2 = -g2 * g4;
id g4 * g3 = -g3 * g4 ;
```

But this is rather cumbersome. It is much easier to rely on the internal ordering of the functions and to have just one identification like

```
id g? * h? = - h*g
```

But when you repeat such a replacement rule, you would get into an infinite loop. To avoid this, the option disorder has been introduced in FORM. With this option, the identification is applied when the normal ordering of terms in the pattern would change the order of the functions, if they were commuting. In subsection 2.5.5 we shall discuss in detail how to compute with gamma matrices in FORM.

2.5.3 A Realistic Differentiation Example

By now, you should be able to understand the following more realistic example of differentiation of trigonometric functions.

```
Symbols x,y,n;
CFunctions sin,cos,tan,g;
Functions [sin], [cos], [tan], [-sin], [1/cos^2], f, dx;
Set commuting: sin, cos, tan;
Set noncommuting: [sin], [cos], [tan];
Set derivative: [cos], [-sin], [1/cos^2];
*
Local expr = sin(x)*tan(x) + cos(x);
*
id g?commuting?noncommuting(x) = g(x);
Multiply left dx;
repeat;
  id dx*g?noncommuting[n](x) = derivative[n](x) + g(x)*dx;
  id [-sin](x) = - [sin](x);
  id [1/cos^2](x) = 1/[cos](x) * 1/[cos](x);
endrepeat;
id dx = 0;
id f?noncommuting?commuting(x) = f(x);
id 1/f?noncommuting?commuting(x) = 1/f(x);
*
Print;
.end
```

```
expr =
    - sin(x) + cos(x)*tan(x) + 1/(cos(x))/(cos(x))*sin(x);
```

2.5.4 Contravariant and Covariant Indices

Now that we know about sets, we can look at another FORM implementation of contravariant and covariant indices. We consider the same example as in Section 2.4.3. Covariant or lower indices like `a` and `b` are denoted by `La` and `Lb`, where the leading `L` stands for "lower". Similarly, contravariant or upper indices like `a` and `b` are denoted by `Ua` and `Ub`, where the leading `U` stands for "upper". We define two set, viz., `LU` and `UL`, that enable us to check whether an index appears twice but of opposite type. This make it easy to define the replacement rules for raising and lowering indices with the metric tensor as the example below shows. We will work out the Ricci curvature tensor and the trace of the metric tensor.

```
Tensors g,R,h;
AutoDeclare Indices U,L;
Indices i,j,k,l,m;
Set UL: Ui, Li, Uj, Lj, Uk, Lk, Ul, Ll;
Set LU: Li, Ui, Lj, Uj, Lk, Uk, Ll, Ul;
Symbol n;
*
Local T1 = g(Li, Uj) * R(Lj, Lk);
Local T2 = g(Ui, Uj) * R(Li, Lk, Lj, Ll);
Local T3 = g(Ui, Lj) * R(Li, Uj);
Local T4 = g(Ui, Lj) * g(Uj, Li);
*
repeat;
  id g(?a, i?UL[n], ?b) * h?(?c, j?LU[n], ?d) = h(?a, ?b, ?c, ?d);
  id h?(?a, i?UL[n], ?b, j?LU[n], ?c) = h(?a, ?b, ?c);
```

```
 endrepeat;
 *
 Print;
 .end

T1 =
   R(Li,Lk);

T2 =
   R(Lk,Ll);

T3 =
   R;

T4 =
   g;
```

No indices in the answers means implicitly that contraction of all indices took place.

2.5.5 Dirac Algebra

First, we shall implement calculus with N-dimensional Dirac gamma matrices and pay extra attention to the case $N = 4$. Later in this subsection we shall look at the class of gamma matrices available in FORM and at how to use them to compute traces.

Theory

Henceforth, we shall use the Bjorken & Drell metric with indices running over time-space from 0 to $N-1$, and with metric tensor $g_{\mu\nu}$ defined by $g_{00} = 1$, $g_{0i} = 0$, for $i = 1, \ldots, N-1$, and $g_{ij} = -\delta_{ij}$, for $i, j = 1, \ldots, N-1$. The inverse matrix $g^{\mu\nu}$ is equal to the metric tensor $g_{\mu\nu}$, but will be used as well. Einstein's summation convention simplifies $g_{\mu\lambda}g^{\lambda\nu}$ to the Kronecker delta symbol δ_μ^ν, but this can also be denoted by g_μ^ν.

The Dirac gamma matrices $\gamma^0, \gamma^1, \ldots, \gamma^{N-1}$ are $N \times N$-matrices satisfying the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,,$$

for $\mu, \nu = 0, \ldots, N-1$. In other words,

$$\begin{aligned} \gamma^i\gamma^j + \gamma^j\gamma^i &= 0 \quad \text{if } i \neq j \\ (\gamma^0)^2 = 1, \quad (\gamma^i)^2 &= -1 \quad \text{for } i = 1, \ldots, N-1 \end{aligned}$$

In case $N = 4$, they can be introduced as a 4×4 generalization of the Pauli-matrices:

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the Pauli-matrices are the usual 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac gamma matrices with their anti-commutation relations form the *Dirac algebra* or *Clifford algebra*.

In case $N = 4$, we define the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,.$$

It is easy to check (and we did it before with FORM) that $(\gamma^5)^2 = 1$ and that γ^5 anti-commutes with all γ^μ. Now define the operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5)\,.$$

These are projection operators: $P_+^2 = P_+, P_-^2 = P_-, P_+P_- = 0$. With these projection operators the underlying vector space splits into a sum of two γ^5 eigenspaces corresponding with eigenvalues ± 1.

The metric tensor can be used to raise or lower indices of tensors: for example, $\gamma_\mu = g_{\mu\nu}\,\gamma^\nu$. In our metric, this leads to $\gamma_0 = \gamma^0, \gamma_i = -\gamma^i$, for $i = 1, \ldots, N-1$. It is easy to check that the following anti-commutation rules hold:

$$\{\gamma^\mu, \gamma_\nu\} = 2g^\mu_\nu = 2\delta^\mu_\nu, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

for $\mu, \nu = 0, 1, \ldots, N-1$. The first relation gives us $\gamma^\mu\,\gamma_\mu = N\,\mathbf{I}_N$, where $\mathbf{I}_N$ denotes the identity matrix.

For any vector p we define

$$\not{p} \stackrel{\text{def}}{=} g_{\mu\nu}p^\mu\gamma^\nu = p_\nu\gamma^\nu = p^\mu\gamma_\mu.$$

In our metric, an N-vector $p = (p^0, p^1, \ldots, p^{N-1})$ gives $\not{p} = p^0\gamma^0 - p^1\gamma^1 - p^2\gamma^2 - \ldots - p^{N-1}\gamma^{N-1}$. The following anti-commutation relations hold:

$$\{\not{p}, \not{q}\} = 2p\cdot q\,\mathbf{I}_N, \quad \{\gamma^\mu, \not{p}\} = 2p^\mu, \quad \{\gamma_\mu, \not{p}\} = 2p_\mu,$$

for $\mu, \nu = 0, 1, \ldots, N-1$ and N-vectors p, q. The first relation gives us $\not{p}^2 = p\cdot p\,\mathbf{I}_N$.

Gamma Matrix Calculus

In Dirac algebra, an important job is to use the anti-commutation relations and other properties to simplify a product of gamma matrices. We shall show how to prove with FORM some useful reduction rules. They are N-dimensional variations of the so-called *Chisholm identity.*

Chisholm Identity

In dimension 4:

$$\gamma^\mu\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma_\mu = -2\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}, \quad \textit{for odd } n\,,$$

$$\gamma^\mu\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma_\mu = 2\gamma^{\mu_n}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{n-1}} + 2\gamma^{\mu_{n-1}}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_n}, \quad \textit{for even } n\,.$$

Special cases are $n = 0, 2$:

$$\gamma^\mu\gamma_\mu = 4\,\mathbf{I}_4, \quad \gamma^\mu\gamma^{\mu_1}\gamma^{\mu_2}\gamma_\mu = 4g^{\mu_1\mu_2}\,\mathbf{I}_4.$$

In the FORM program below, we shall prove the following identities for dimension N:

$$\begin{array}{rcl}
\gamma^m\gamma_m & = & N\,\mathbf{I}_N \\
\gamma^m\gamma^{m_1}\gamma_m & = & (2-N)\gamma^{m_1} \\
\gamma^m\gamma^{m_1}\gamma^{m_2}\gamma_m & = & 4g^{m_1m_2}\,\mathbf{I}_N + (N-4)\gamma^{m_1}\gamma^{m_2} \\
\gamma^m\gamma^{m_1}\gamma^{m_2}\gamma^{m_3}\gamma_m & = & -2\gamma^{m_3}\gamma^{m_2}\gamma^{m_1} - (N-4)\gamma^{m_1}\gamma^{m_2}\gamma^{m_3} \\
\gamma^m\gamma^{m_1}\gamma^{m_2}\gamma^{m_3}\gamma^{m_4}\gamma_m & = & 2\gamma^{m_4}\gamma^{m_1}\gamma^{m_2}\gamma^{m_3} + 2\gamma^{m_3}\gamma^{m_2}\gamma^{m_3}\gamma^{m_4} + (N-4)\gamma^{m_1}\gamma^{m_2}\gamma^{m_3}\gamma^{m_4}
\end{array}$$

The Chisholm identity mentioned above follows easily by taking N equal to four. In our program, the first character of an index classifies it as an upper or lower index: we denote the gamma matrices γ^{m_1} and γ_{m_1} by `g(U1))` and `g(L1)`, respectively. The metric tensor is denoted by `eta`. The simplification is based on applying repetitively the anti-commutation rules of gamma matrices. The order of defining the indices is important here, because it determines in what direction the anti-commutation rules are going to be applied (the option `disorder` plays its role!). Sets are used to match upper and lower indices. Of course we have only one pair of matching upper and lower indices present in the local expressions that we investigate, and therefore the sets could have been kept small, only involving the indices in this pair, but we wanted the program to be more general. This way we can easily extend the FORM program to prove for example Chisholm-like identities with lower indices or mixtures of lower and upper indices.

```
 Functions g;
 CFunction eta;
 Indices Um, Lm, U1, ..., U4, L1, ..., L4;
 Set U: Um, U1, ..., U4;
 Set L: Lm, L1, ..., L4;
 Set UL: Um, Lm, <U1, L1>, ..., <U4, L4>;
 Set LU: Lm, Um, <L1, U1>, ..., <L4, U4>;
 Index i,j;
 Symbol k, N;
 *
 Local F1 = g(Um) * g(U1) * g(Lm);
 Local F2 = g(Um) * g(U1) * g(U2) * g(Lm);
 Local F3 = g(Um) * g(U1) * g(U2) * g(U3) * g(Lm)
      + 2 * g(U3) * g(U2) * g(U1);
 Local F4 = g(Um) * g(U1) * g(U2) * g(U3) * g(U4) * g(Lm)
      - 2 * g(U4) * g(U1) * g(U2) * g(U3)
      - 2 * g(U3) * g(U2) * g(U1) * g(U4);
 *
 * bring g(Lm) to the left to cancel it with g(Um)
 * and use rewrite rule for metric tensor eta
 *
 repeat;
   id g(Um) * g(Lm) = N;
   id g(i?) * g(Lm)= - g(Lm) * g(i) + 2*eta(i,Lm);
   id g(i?U[k]) * eta(?a, j?L[k]) = g(?a);
 endrepeat;
 *
 * bring product of gamma matrices in standard order
 *
 repeat;
   id disorder g(i?U) * g(j?U)= - g(j) * g(i) + 2*eta(i,j);
 endrepeat;
 *
 AntiBracket N;
 Print;
 .sort

F1 =
    + g(U1) * ( 2 - N );

F2 =
    + eta(U2,U1) * ( 4 )

    + g(U1)*g(U2) * (  - 4 + N );

F3 =
    + g(U1)*g(U2)*g(U3) * ( 4 - N );

F4 =
    + g(U1)*g(U2)*g(U3)*g(U4) * (  - 4 + N );

 *
 * specialize to the case N=4
 *
```

```
id N = 4;
AntiBracket eta;
Print;
.end

F1 =
   + g(U1) * (  - 2 );

F2 =
   + 4*eta(U2,U1);

F3 = 0;

F4 = 0;
```

We can improve the above program to make it applicable in more cases (e.g., also for expressions containing gamma matrices with lower indices, or for contractions of gamma matrices with vectors), and to make the notation more similar to the representation used inside FORM for gamma matrices. In the program below, a product of gamma matrices is written in "contracted form":

```
g(m1, m2, ..., mn) = g(m1) * g(m2) * ... * g(mn)
```

where each index `mi` is either `Ui` or `Li`, and where `g(Ui)` $= \gamma^i$, `g(Li)` $= \gamma_i$, for `i`$=1, 2, \ldots, N$. The advantage of this notation is that it allows us to enter a product of gamma matrices in a convenient way and that it displays the results of computations in a clear way. It also makes it simple to include vector contractions such as $\not{p}$ by using a vector object instead of an index object as argument of the function `g`. So, `g(p)`, where `p` is a vector, is short notation for `g(Um) * p(Lm)` and `g(Lm) * p(Um)`, where the index pair (`Um`, `Lm`) of course can be any pair of matching upper and lower indices.

```
Symbol N;
Dimension N;
Function g;
CFunction eta;
Vector p,q;
Indices Um, Lm, U1, ..., U5, L1, ..., L5;
Set U: Um, U1, ..., U5;
Set L: Lm, L1, ..., L5;
Set UL: Um, Lm, <U1, L1>, ..., <U5, L5>;
Set LU: Lm, Um, <L1, U1>, ..., <L5, U5>;
Index i,j,m,n;
Symbol k;
*
Local F1 = g(Lm, p, Um);
Local F2 = g(Um, L1, U2, Lm);
Local F3 = g(Um, p, q, U3, Lm) + 2 * g(U3, q, p);
Local F4 = g(Um, L1, U2, L3, U4, Lm)
     - 2 * g(U4, L1, U2, L3)
     - 2 * g(L3, U2, L1, U4);
Local F5 = g(Lm, U1, p, U3, q, U5, Um)
     - 2 * g(U5, U1, p, U3, q)
     + 2 * g(q, U1, p, U3, U5)
     + 2 * g(U3, p, U1, q, U5);*
*
* change notation to product of gamma matrices
*
repeat;
```

```
   id g(?a, i?, j?, ?b) = g(?a, i) * g(j, ?b);
endrepeat;
*
* bring low index to the left in the hope
* to meet a corresponding upper index
* bring vector arguments to the left
*
repeat;
  repeat;
    id g(i?U[k]) * g(j?L[k]) = d_(m,m);
  endrepeat;
  id g(i?U) * g(j?L) = - g(j) * g(i) + 2*eta(i, j);
  id g(i?UL[k]) * eta(?a, j?LU[k], ?b) = g(?a, ?b);
  id g(i?UL) * g(p?) = - g(p) * g(i) + 2*p(i);
  id p?(i?UL[k]) * eta(m?, j?LU[k]) = p(m);
  id eta(?a, i?UL[k], ?b) * eta(?c, j?LU[k], ?d)
     = eta(?a, ?b, ?c, ?d);
endrepeat;
*
* bring low index to the right in the hope
* to meet a corresponding upper index
*
repeat;
  repeat;
    id g(i?L[k]) * g(j?U[k]) = d_(m,m);
  endrepeat;
  id g(i?L) * g(j?U) = - g(j) * g(i) + 2*eta(i, j);
  id g(i?UL[k]) * eta(?a, j?LU[k], ?b) = g(?a, ?b);
  id p?(i?UL[k]) * eta(j?LU[k], m?) = p(m);
  id eta(?a, i?UL[k], ?b) * eta(?c, j?LU[k], ?d)
     = eta(?a, ?b, ?c, ?d);
endrepeat;
*
* bring product of gamma matrices with respect to
* index arguments in standard order
*
repeat;
  id disorder g(i?UL) * g(j?UL) = - g(j) * g(i) + 2*eta(i,j);
  id g(i?UL[k]) * eta(?a, j?LU[k], ?b) = g(?a, ?b);
endrepeat;
*
* contract vector components with gamma matrices
* bring all vector arguments to the left
*
repeat;
  id g(i?UL[k]) * p?(j?LU[k]) = g(p);
  id g(i?UL) * g(p?) = - g(p) * g(i) + 2 * p(i);
  id p?(i?UL[k]) * eta(m?, j?LU[k]) = p(m);
  id eta(?a, i?UL[k], ?b) * eta(?c, j?LU[k], ?d)
     = eta(?a, ?b, ?c, ?d);
endrepeat;
*
* bring product of gamma matrices with respect to
* vector arguments in standard order
```

```
 *
 repeat;
   id disorder g(p?) * g(q?) = - g(q) * g(p) + 2*p.q;
 endrepeat;
 *
 * symmetrize the metric tensor and
 * go back to short notation
 *
 symmetrize eta;
 repeat;
    id g(i?,?a) * g(j?,?b) = g(i, ?a, j, ?b);
 endrepeat;
 *
 AntiBracket N;
 Print;
 .sort

F1 =
    + g(p) * ( 2 - N );

F2 =
    + eta(U2,L1) * (  - 4 + 2*N )

    + g(U2,L1) * ( 4 - N );

F3 =
    + g(p,q,U3) * ( 4 - N );

F4 =
    + eta(U2,L1)*eta(U4,L3) * (  - 16 + 4*N )

    + g(U2,U4,L1,L3) * ( 4 - N )

    + g(U2,L1)*eta(U4,L3) * ( 8 - 2*N )

    + g(U2,L3)*eta(U4,L1) * (  - 8 + 2*N )

    + g(U4,L3)*eta(U2,L1) * ( 8 - 2*N );

F5 =
    + g(p,q,U1,U3,U5) * (  - 4 + N )

    + g(p,U1,U5)*q(U3) * (  - 8 + 2*N )

    + g(p,U3,U5)*q(U1) * ( 8 - 2*N )

    + g(q,U3,U5)*p(U1) * (  - 8 + 2*N )

    + g(U5)*p(U1)*q(U3) * ( 16 - 4*N );

 *
 * specialize to the case N=4
 *
 id N = 4;
```

```
AntiBracket eta;
Print;
.end

F1 =
   + g(p) * (  - 2 );

F2 =
   + 4*eta(U2,L1);

F3 = 0;

F4 = 0;

F5 = 0;
```

From the examples you see that the Chisholm identity actually holds for all indices, whether they are upper or lower indices or vectors.

Let us prove with FORM another rule in Dirac algebra. First, some notation for antisymmetrized products of gamma matrices:

$$\Gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) ,$$

$$\Gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} = \frac{1}{6}\left(\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu\right) ,$$

and so on. Similar antisymmetrizations can be done for lower indices and for mixed indices.

Theorem 2 (Theorem) *In dimension 4, using Bjorken-Drell metric:*

$$\Gamma_{\mu\nu\rho} = i\epsilon_{\mu\nu\rho\sigma}\gamma^5\gamma^\sigma .$$

In the following FORM program, we rewrite every term explicitly in terms of gamma matrices with upper indices so that we do not have to distinguish between upper and lower indices all the time. The Bjorken-Drell metric is diagonal. So, γ_μ and γ^μ are related simply by $\gamma_\mu = g_{\mu\mu}\gamma^\mu$ without contraction of repeated indices. The formula under consideration is also worked out so that Einstein's summation convention is not needed anymore.

```
Function g, G;
CFunction eta, eps, del;
Index a, b, c, d, k, m, n, r;
*
* make the left-hand side of the equality
*
Local [Gabc] = 1/6 * e_(1,2,3) * e_(a,b,c) * g(a) * g(b) * g(c);
contract;
id g(1) = g(a) * eta(a,a);
id g(2) = g(b) * eta(b,b);
id g(3) = g(c) * eta(c,c);
Print +s [Gabc];
.sort

[Gabc] =
   + 1/6*g(a)*g(b)*g(c)*eta(a,a)*eta(b,b)*eta(c,c)
   - 1/6*g(a)*g(c)*g(b)*eta(a,a)*eta(b,b)*eta(c,c)
   - 1/6*g(b)*g(a)*g(c)*eta(a,a)*eta(b,b)*eta(c,c)
   + 1/6*g(b)*g(c)*g(a)*eta(a,a)*eta(b,b)*eta(c,c)
```

```
    + 1/6*g(c)*g(a)*g(b)*eta(a,a)*eta(b,b)*eta(c,c)
    - 1/6*g(c)*g(b)*g(a)*eta(a,a)*eta(b,b)*eta(c,c)
   ;


 *
 * define the right-hand side of the equality
 * implicitly assume Einstein's summation convention
 *
 Local [g5] =  i_ * g(0) * g(1) * g(2) * g(3);
 Local F2 =    i_ *  eps(a,b,c,d) * [g5] * g(d);
 sum d,0,1,2,3;
 Print +s F2;
 .sort

F2 =
    - g(0)*g(1)*g(2)*g(3)*g(0)*eps(a,b,c,0)
    - g(0)*g(1)*g(2)*g(3)*g(1)*eps(a,b,c,1)
    - g(0)*g(1)*g(2)*g(3)*g(2)*eps(a,b,c,2)
    - g(0)*g(1)*g(2)*g(3)*g(3)*eps(a,b,c,3)
   ;

 *
 * compute the difference of the left- and right-hand side
 *
 Local F = F2 - [Gabc];
 repeat;
   id g(a?) * g(a?) = eta(a,a);
   id disorder g(a?) * g(b?)= - g(b) * g(a) + 2*eta(a,b);
 endrepeat;
 .sort
 *
 * work out the contraction of repeated indices and
 * show that for all combinations of indices the result equals zero
 *
 Symbols x, y, z;
 Local R = sum_(a, 0, 3, sum_(b, 0, 3, (sum_(c, 0, 3, F*x^a*y^b*z^c))));
 Bracket x, y, z;
 .sort
 repeat;
   id g(a?) * g(a?) = eta(a,a);
   id disorder g(a?) * g(b?)= - g(b) * g(a) + 2*eta(a,b);
 endrepeat;
 id eps(?a) = e_(?a);
 id eta(a?,a?) = -1 + 2*d_(0,a);
 id eta(a?,b?) = d_(a,b);
 id e_(0,1,2,3) = 1;
 Print R;
 .end

R = 0;
```

Gamma Matrices in FORM

Let us now look at the calculus of gamma matrices that is available in FORM. A product of gamma matrices is denoted by `g_(`i`, mu, nu, ... )`. The index i distinguishes between different spin lines. This extra label is necessary because in high energy physics gamma matrices are associated with fermion lines in a Feynman diagram, and if more than one such line occurs, then a different set of gamma matrices (operating in independent spin spaces) is required to represent each line. Gamma matrices associated with different spin lines commute; gamma matrices from the same spin line are normally collected into one `g_` function, but this is not obligatory on the input side. For example, you can input $\not{p}\gamma^\mu\not{p}\gamma^\nu$ as `g_(1,p)*g_(1,mu)*g_(1,p)*g_(1,nu)` and FORM will automatically display it as `g_(1,p,mu,p,nu)`. In FORM, almost all you can do with these expressions is taking the trace of a string of gamma matrices. Taking the trace of a spin line with index i is accomplished by the commands `trace4,`i and with `tracen,`i. In the first case, FORM uses algorithms that are applicable in four dimensions only. For example, it uses the Chisholm identity. See the reference guide for details about the algorithms used. The second command does not assume dimension 4 and it cannot handle properties of γ^5 (denoted by `g5_(`i`)` or `g_(`i`, ..., 5_, ... )`). FORM shortens $1+\gamma^5$ and $1-\gamma^5$ to `g6_(`i`)` (or `g_(`i`, ..., 6_, ... )`) and `g7_(`i`)` (or `g_(`i`, ..., 7_, ... )`), respectively. The identity matrix is denoted by `gi_(`i`)`. It is possible to alter the value of the trace of the identity matrix: its default value is 4, but by using the statement `unittrace` *value* it can be changed into *value*. In the following table we summarize the notations for gamma matrices and the conventions that are used in FORM. Here, the spin line is always denoted by the character i.

FORM *Notation*	*Meaning*
`gi_(i)`	identity
`g_(i,m)`	γ^m
`g5_(i)`	γ^5
`g_(i,5_)`	γ^5
`g6_(i)`	$\gamma^6 \stackrel{\text{def}}{=} 1+\gamma^5$
`g_(i,6_)`	γ^6
`g7_(i)`	$\gamma^7 \stackrel{\text{def}}{=} 1-\gamma^5$
`g_(i,7_)`	γ^7
`g_(i,r,s)`	`g_(i,r)*g_(i,s)`
`g_(i,m,...,r,s)`	`g_(i,m,...r)*g_(i,s)`

It is important to know that FORM uses the Pauli metric instead of the Bjorken & Drell metric. This means that space-time indices usually run from 1 to 4, that the identity matrix represents the metric, that no distinction between upper and lower indices is required, and that the gamma matrices in FORM fulfill the anti-commutation relation

```
{ g_(j1,mu), g_(j1,nu) } = 2 * d_(mu,nu)
```

and the commutation relation

```
[ g_(j1,mu), g_(j2,nu) ] = 0   if j1≠j2
```

The relation between the gamma matrices in Pauli metric and Bjorken & Drell metric is

$$\texttt{g\_(...,1)} = \gamma^0, \qquad \texttt{g\_(...,k)} = i\gamma^{k-1},$$

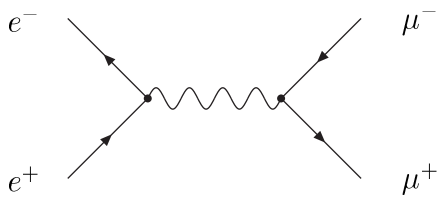
for $k = 2,3,4$ and complex unit i. In Pauli metric, the Levi-Civita tensors that are generated by trace routines are imaginary.

Example from Particle Physics: the Process $e^+e^- \longrightarrow \mu^+\mu^-$

We shall compute the amplitude squared summed over spins for the process $e^+e^- \longrightarrow \mu^+\mu^-$ at tree level, with massless electrons and muons, due to the electromagnetic interaction

$$\mathcal{L}_I = -eA_\mu j^\mu, \quad j^\alpha = {:}\overline{\psi}_e\gamma^\alpha\psi_e{:} + {:}\overline{\psi}_\mu\gamma^\alpha\psi_\mu{:}\,.$$

We choose momenta as follows $e^+(k_1)e^-(k_2) \longrightarrow \mu^+(p_1)\mu^-(p_2)$. There is just one Feynman diagram, namely



Ignoring the labels for the spin lines, the amplitude $\mathcal{M}$ for this process is given by

$$\mathcal{M} = i\frac{e^2}{(k_1+k_2)^2}\overline{v}(k_1)\gamma^\rho u(k_2)\overline{u}(p_2)\gamma_\mu v(p_1)\,.$$

So, the amplitude squared summed over spins, using for massless particles $\sum\limits_{spins} u(p)\overline{u}(p) = \not{p}$ and $\sum\limits_{spins} v(p)\overline{v}(p) = \not{p}$, equals

$$\sum_{spins} |\mathcal{M}|^2 = \frac{e^4}{k^4}\,\mathrm{tr}(\not{k}_1\gamma^\rho\ \not{k}_2\gamma^\sigma)\ \mathrm{tr}(\not{p}_1\gamma_\rho\ \not{p}_2\gamma_\sigma) = 8e^4\frac{(t^2+u^2)}{s^2}\,,$$

where we use the Mandelstam variables

$$k = k_1+k_2 = p_1+p_2, \quad s = k^2, \quad t = (k_1-p_1)^2 = (k_2-p_2)^2, \quad u = (k_1-p_2)^2 = (k_2-p_1)^2\,.$$

For further simplification of this result we refer to the same example in [Schellekens 97]. The following FORM session finds the above answer.

```
Vectors k1, k2, p1, p2;
Symbols s, t, u, e;
Indices mu, nu, rho, sigma;
*
Local   M2 =
*       electron line
          e^2 * g_(1, k1, rho, k2, sigma) *
*       photon propagator
          d_(rho,mu) * d_(sigma,nu) / s^2 *
*       muon spin line
          e^2 * g_(2, p1, mu, p2, nu)
        ;
Trace4,1;
Trace4,2;
Bracket e,s;
Print;
.sort

 M2 =
     + s^-2*e^4 * ( 32*k1.p1*k2.p2 + 32*k1.p2*k2.p1 );

*
id k1.k2 = s/2;
```

```
id p1.p2 = s/2;
id k1.p1 = -t/2;
id k2.p2 = -t/2;
id k1.p2 = -u/2;
id k2.p1 = -u/2;
Bracket e,s;
Print;
.end
```

```
M2 =
    + s^-2*e^4 * ( 8*t^2 + 8*u^2 );
```

Another Example from Particle Physics: Decay of Heavy Leptons

The following is an example from high energy physics that illustrates the efficiency of the trace algorithms that have been implemented in FORM. We are looking at the reaction $e^+e^- \longrightarrow \tau^+\tau^- \longrightarrow u\overline{d}\nu_\tau\overline{u}d\overline{\nu}_\tau$. This is a 2 to 6 reaction, but it has some features that make it easier than one might expect.

```
Vectors p1,...,p8,Q,q1,q2;
Indices m1,m2,m3,n1,n2,n3;
Symbol emass,tmass,mass4,mass5,mass7,mass8;
On Statistics;
Local   F =
*
*   The incoming e+ e- pair. momenta p2 and p1
*
    (g_(1,p2)-emass)*g_(1,m1)
    *(g_(1,p1)+emass)*g_(1,n1)
*
*   The tau line. tau- is q1, tau+ is q2.
*
    *g_(2,p3)*g_(2,m2)*g7_(2)
    *(g_(2,q1)+tmass)*g_(2,m1)
    *(-g_(2,q2)+tmass)*g_(2,m3)*g7_(2)*g_(2,p6)
    *g_(2,n3)*g7_(2)*(-g_(2,q2)+tmass)*g_(2,n1)
    *(g_(2,q1)+tmass)*g_(2,n2)*g7_(2)
*
*   The u d-bar pair. p4 is u, p5 is d-bar.
*
    *(g_(3,p4)+mass4)*g_(3,m2)*g7_(3)
    *(g_(3,p5)-mass5)*g_(3,n2)*g7_(3)
*
*   The d u-bar pair. p7 is d, p8 is u-bar.
*
    *(g_(4,p7)+mass7)*g_(4,m3)*g7_(4)
    *(g_(4,p8)-mass8)*g_(4,n3)*g7_(4)
    ;
trace4,1;
trace4,2;
trace4,3;
trace4,4;
contract;
print +s;
.end
```

```
Time =        0.49 sec    Generated terms =         164
                 F        Terms in output =          27
                          Bytes used      =        1354

   F =
       - 524288*p1.p2*p3.p4*p5.p7*p6.p8*q1.q2*tmass^2
       + 524288*p1.p2*p3.p4*p5.q1*p6.p8*p7.q2*tmass^2
       + 524288*p1.p2*p3.p4*p5.q2*p6.p8*p7.q1*tmass^2
       + 262144*p1.p5*p2.p7*p3.p4*p6.p8*q1.q1*q2.q2
       + 524288*p1.p5*p2.p7*p3.p4*p6.p8*q1.q2*tmass^2
       + 262144*p1.p5*p2.p7*p3.p4*p6.p8*tmass^4
       - 524288*p1.p5*p2.q2*p3.p4*p6.p8*p7.q1*tmass^2
       - 524288*p1.p5*p2.q2*p3.p4*p6.p8*p7.q2*q1.q1
       + 262144*p1.p7*p2.p5*p3.p4*p6.p8*q1.q1*q2.q2
       + 524288*p1.p7*p2.p5*p3.p4*p6.p8*q1.q2*tmass^2
       + 262144*p1.p7*p2.p5*p3.p4*p6.p8*tmass^4
       - 524288*p1.p7*p2.q1*p3.p4*p5.q1*p6.p8*q2.q2
       - 524288*p1.p7*p2.q1*p3.p4*p5.q2*p6.p8*tmass^2
       - 524288*p1.q1*p2.p7*p3.p4*p5.q1*p6.p8*q2.q2
       - 524288*p1.q1*p2.p7*p3.p4*p5.q2*p6.p8*tmass^2
       + 524288*p1.q1*p2.q2*p3.p4*p5.p7*p6.p8*tmass^2
       + 1048576*p1.q1*p2.q2*p3.p4*p5.q1*p6.p8*p7.q2
       - 524288*p1.q2*p2.p5*p3.p4*p6.p8*p7.q1*tmass^2
       - 524288*p1.q2*p2.p5*p3.p4*p6.p8*p7.q2*q1.q1
       + 524288*p1.q2*p2.q1*p3.p4*p5.p7*p6.p8*tmass^2
       + 1048576*p1.q2*p2.q1*p3.p4*p5.q1*p6.p8*p7.q2
       + 262144*p3.p4*p5.p7*p6.p8*q1.q1*q2.q2*emass^2
       + 262144*p3.p4*p5.p7*p6.p8*emass^2*tmass^4
       - 524288*p3.p4*p5.q1*p6.p8*p7.q1*q2.q2*emass^2
       + 1048576*p3.p4*p5.q1*p6.p8*p7.q2*q1.q2*emass^2
       + 1048576*p3.p4*p5.q1*p6.p8*p7.q2*emass^2*tmass^2
       - 524288*p3.p4*p5.q2*p6.p8*p7.q2*q1.q1*emass^2
      ;
```

2.5.6 Exercises

1. Implement the rule $J(-n,z) = (-1)^n J(n,z)$, if n is a natural number. Apply your rule for $n=3$, $n=4$, and general n.

2. For an invertible matrix M holds the equation
$$\frac{dM^{-1}}{dt} = -M^{-1}\left(\frac{dM}{dt}\right)M^{-1}.$$
Write a FORM program that computes the derivative of M^{-3}.

3. Write a FORM program that computes the derivative of $x^4 \ln^2 x$.

4. Write a FORM program that computes the integrals $\int x^4 \cos x\, dx$ and $\int x^4 \sin x\, dx$.

5. Let $\mathbf{T} = \left(T^{ij}_{klm}\right)$ denote a tensor of order and type indicated by the indices.
Prove with FORM that $\mathbf{S} = (T_k) = \left(T^{ij}_{kij}\right)$ is a covariant vector.

6. From the contravariant tensor $\mathbf{S} = (S^{ij})$ and the covariant tensor $\mathbf{T} = (T_{kl})$, both of order two, form the inner product $\mathbf{U} = (U^i_l) = (S^{ij}T_{jl})$. Show with FORM that $\mathbf{U}$ is a mixed tensor of order two.

7. Carry out in FORM the following trace calculation, published in [Veltman 89]: Compute
$$\mathrm{trace}(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{10}}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{10}})$$
and replace the dimension d by $d-4$. The answer should be equal to
$$-31023169536+38971179008d-21328977920d^2+6679521280d^3-1320732160d^4+ \\ 171464832d^5-14710080d^6+816960d^7-27840d^8+520d^9-4d^{10}\,.$$

8. Repeat the following calculation in high energy physics, which is also described in the REDUCE manual: the computation of the Compton scattering cross-section as given in Bjorken and Drell Eqs. (7.72) through (7.74). Requested is the trace of
$$\frac{\alpha^2}{2}\left(\frac{k'}{k}\right)^2\left(\frac{\not p_f+m}{2m}\right)\left(\frac{\not p_f+m}{2m}\right)\left(\frac{\not e'\ \not e\ \not k_i}{2k\cdot p_i}+\frac{\not e\ \not e'\ \not k_f}{2k'\cdot p_i}\right)\left(\frac{\not p_i+m}{2m}\right)\left(\frac{\not k_i\ \not e\ \not e'}{2k\cdot p_i}+\frac{\not k_f\ \not e'\ \not e}{2k'\cdot p_i}\right)$$
where k_i and k_f are the four-momenta of incoming and outgoing photons, with polarization vectors e and e' and laboratory energies k and k', respectively, and where p_i and p_f are incident and final electron four-momenta. It is necessary to put the particles "on the mass shell" in the calculation:
$$k_i^2=0,\quad k_f^2=0,\quad p_i^2=m^2,\quad p_f^2=m^2\,.$$
For the polarization vectors hold
$$p_i\cdot e=0,\quad p_i\cdot e'=0,\quad k_i\cdot e=0,\quad k_f\cdot e'=0,\quad p_f\cdot e=-k_f\cdot e,\quad p_f\cdot e'=k_i\cdot e',\quad e^2=-1,\quad e'^2=-1\,.$$
Furthermore,
$$p_i\cdot p_f=m^2+k_i\cdot k_f,\quad p_i\cdot k_i=m\,k,\quad p_i\cdot k_f=m\,k',\quad p_f\cdot k_i=m\,k',\quad p_f\cdot k_f=m\,k,\quad k_i\cdot k_f=m(k-k')\,.$$
With these relations you should readily get the following Compton scattering cross-section:
$$\frac{\alpha^2}{2m^2}\left(\frac{k'}{k}\right)^2\left(\frac{k'}{2k}+\frac{k}{2k'}+2(e\cdot e')^2-1\right)$$

9. Consider again the annihilation of an electron pair and creation of a muon pair, but now without the assumption of massless particles. Show that the amplitude squared summed over spins is given in the Mandelstam variables by
$$\sum_{spins}|\mathcal{M}|^2=8e^4\frac{(t^2+u^2+4s(m^2+M^2)-2(m^2+M^2)^2)}{s^2}\,,$$
where m and M are the electron and muon mass, respectively.

2.6 Limitations in Wildcarding

2.6.1 Coefficients and Wildcards

We started the subsection about wildcard parameters with the simplification $x\sqrt{y}=\sqrt{x^2y}$. The implemented replacement rule worked well for the formula $a\sqrt{b}$, but look what is the result when applied to $2\sqrt{a}$.

```
Symbol a,x,y;
CFunction sqrt;
Local F = 2 * sqrt(a);
id x? * sqrt(y?) = sqrt(x^2*y);
Print;
.end
```

```
F =
   2*sqrt(a);
```

The coefficient is not recognized in the wildcarding. In FORM, you cannot write an identification like `2*sqrt(y?)`. The best workaround is to put the coefficient inside a dummy function via the `PolyFun` option to a FORM directive. Then you can write the replacement which involves coefficients. In our example it could look like

```
 Symbol a,x,y;
 CFunctions sqrt,dummy;
 Local F = 2 * sqrt(a);
 Print;
 .sort (PolyFun = dummy);

F =
   sqrt(a)*dummy(2);

 id dummy(x?) * sqrt(y?) = sqrt(x^2*y);
 Print;
 .end

F =
   sqrt(4*a);
```

The semicolon at the end of the `PolyFun` option is obligatory. This statement is an example of a module option statement. There can be more than one module option statement. The module option statement(s) are the last statement(s) before the FORM directives at the end of the module. They are local settings that overwrite more general settings. They hold only for the current module.

Another working style is to use a `PolyFun` declaration. Like any other declaration it will remain valid during the session and does not have to be put in all `.sort` instructions. The declaration `PolyFun;` switches the PolyFun option off. So, we can also use the following FORM program:

```
 Symbol a,x,y;
 CFunctions sqrt,dummy;
 Polyfun dummy;
 Local F = 2 * sqrt(a);
 Print;
 .sort
 Polyfun;
 id dummy(x?) * sqrt(y?) = sqrt(x^2*y);
 Print;
 .end
```

Another way of getting hold of the coefficients is by use of the `collect` statement. With this statement you can put data which are between brackets (either by the `Bracket` or `AntiBracket` statement) inside a regular function. Our example would look like

```
 Symbol a,x,y;
 CFunctions sqrt,dummy;
 Local F = 2 * sqrt(a);
 AntiBracket a;
 Print;
 .sort

F =
    + sqrt(a) * ( 2 );

 collect dummy;
```

```
 Print;
 .sort

F =
   sqrt(a)*dummy(2);


 id dummy(x?) * sqrt(y?) = sqrt(x^2*y);
 Print;
 .end

F =
   sqrt(4*a);
```

2.6.2 Sums of Wildcards

Another type of pattern matching which is currently not allowed in FORM is `f(x?+y?)`, and variations thereof. For example, additivity of a function `f` cannot be specified as follows:

```
 Symbols x,y,z;
 CFunction f;
 Local expr = f( x + y + z );
 id f( x? + y? + z? ) = f(x) + f(y) + f(z);
 Print;
 .end

expr =
   f(x + y + z);
```

The reason for not allowing this kind of wildcarding in function arguments is efficiency: the pattern `f(x1?+x2?+...+xn?)` has in general $n!$ possible assignments for the wildcards. Efficiency is also the reason for not implementing pattern matching for composite denominators and for powers with non-integer exponents.

However, the above problem of additivity of a function can be implemented by the following trick. In fact, we implement more: linearity of a function.

```
 Vectors x,y,z;
 Index i;
 CFunction f;
 Local expr = f( 2*x + y + z );
 id f(i?) = f(i);
 Print;
 .end

expr =
   2*f(x) + f(y) + f(z);
```

The explanation is as follows: the wildcard in the function is an index. If FORM gets a vectorlike argument in the function, it assumes that it is there because of contraction of indices. In other words, it assumes that the function is linear in this argument.

2.6.3 Exercises

1. Implement the trigonometric identities

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y,\\ \cos(x+y) &= \cos x\cos y + \sin x\sin y,\end{aligned}$$

and apply them to $\sin(a+b)$ and $\sin(a+b+c)$.

2. Implement the rule $\sin(2x) \to 2\sin x \cos x$, and apply it to $\sin(2a)$, $\sin(3a)$, and $\sin(4a)$.

Chapter 3

Procedural Programming

3.1 Superstructure of the Preprocessor

FORM provides the usual facilities of procedural programming languages: control flow statements such as repetition with "do"- and "while"-loops and choice control structures of type "if-then-else" and "switch" are present, and procedures can be defined and called as one pleases. There is however an important difference with many languages: FORM consists of two parts, viz., the preprocessor and the compiler. At both levels, tools of procedural programming are provided. In this section we shall describe the functionality of the preprocessor. In the second section, the programming facilities at compiler level will be discussed.

The preprocessor reads from the input stream and prepares input for the compiler. The preprocessor prepares program blocks, also called *modules*, which are translated by the compiler, and immediately executed. As such, you can consider the preprocessor as an autonomous unit that facilitates the efficient and easy writing of FORM programs. A command for the preprocessor is called a *preprocessor instruction*. It always starts with the sharp symbol (#), it does not have to end with a semicolon, and it is executed when it is encountered in the input stream. Preprocessor instructions are there to make programming in FORM easier. For example, the preprocessor has its own structures for control flow and it allows you to write procedures. To make preparation of code for the compiler easier, the preprocessor is equipped with variables that can be defined or redefined by the user or by other preprocessor actions. The preprocessor variables can be recognized by the quotes that are around them when they are referred to. In this section we shall concentrate on the control flow preprocessor instructions and the preprocessor variables; the preprocessor instructions that have to do with I/O will be discussed in a later chapter.

3.1.1 Choice

Most Common Choice: #if ... #else ... #endif

The conditional statement in the preprocessor provides a way to include code selectively. Formally, it has the following syntax.

> `#if` *condition*
>
> $statseq_1$
>
> [`#else`
>
> $statseq_2$]
>
> `#endif`

where *condition* is a logical variable or a composite logical expression, $statseq_1$ and $statseq_2$ are expressions, statements, or sequences of statements separated by semicolons, and [ ] denotes an optional part. Omitting the optional part is equivalent to saying "continue". The `#endif` instruction marks the end of the choice. The *condition* is evaluated: if it is true, i.e., if it has a non-zero value, $statseq_1$ is read by the preprocessor and then reading continues after the `#endif` instruction. If *condition* is false, i.e., if it is zero, and if there is an `else` part, $statseq_2$ is read instead. Hence it is allowed to use the following code snippet.

```
#if 'i'
        statements
#endif
```

instead of

```
#if 'i' != 0
        statements
#endif
```

provided that the so-called preprocessor variable `'i'` has a value that can be interpreted as a number. If there is just a string, and not a number, the condition is false.

A composite condition is composed of variables and numbers via relational operators (less than, equal to, etc.) and logical operators (and, or). The operator are tabulated below. Relational operators are in FORM mostly the same as in the C programming language and are either numerical comparisons or string comparisons with respect to the lexicographic ordering of strings. The same holds for logical operators, with the exception that the negation (!) does not exist in FORM.

Operator	*Meaning*
= or ==	equal to
!=	not equal to
>	greater than
<	less than
>=	greater than or equal to
<=	less than or equal to
&&	and
||	or

A typical `#if` statement is

```
#if {'i'%2} == 0
  1
#else
  -1
#endif
```

where `'i'` is a preprocessor variable, which is defined elsewhere in the FORM program, and where we have used the percentage symbol % for modular arithmetic ($x\%y$ is the remainder when x is divided by y). For an even integer the result is 1; odd numbers give -1. The curly brackets in the `if` part are required for this behavior. Otherwise, the preprocessor does not carry out the remainder calculation and it will compare two very different strings, viz., `i`%2 and 0. This behavior will become more clear after preprocessor variables and their calculus have been discussed.

Nested #if Statements and the #elseif Instruction

Nested `#if` statements are allowed, and FORM provides the shortcut `#elseif`.

```
#if 'x' < 0
  -1
#else
  #if 'x' = 0
    0
  #else
```

```
    1
  #endif
#endif
```

is equivalent to

```
#if ‘x’ < 0
  -1
#elseif ‘x’ = 0
   0
#else
   1
#endif
```

The following sequence of `#if`, `#elseif`, `#else`, and `#endif` instructions is the most general way of writing a multi-way decision.

`#if` *condition*$_1$

  statseq$_1$

`#elseif` *condition*$_2$

  statseq$_2$

`#elseif` *condition*$_3$

  statseq$_3$

....

`#elseif` *condition*$_n$

  statseq$_n$

`#else`

  statseq$_{n+1}$

`#endif`

The conditions are evaluated in order; if any condition is true, the sequence of statements associated with it is read by the preprocessor, and this terminates the whole chain so that reading continues after the `#endif` instruction. The last `#else` part handles the "none of the above" or default case where none of the other conditions is satisfied. If there is no explicit reading required for the default, the `#else` part can be omitted, or it can be used for error checking an "impossible" condition.

Multi-way Decision: #switch

The `#switch` instruction, together with `#break`, `#default`, and `#endswitch`, provides a multi-way decision, which tests whether a string matches one of a number of constant string values, and which branches accordingly. In other words, these instructions allow the user to conveniently make code for a number of cases that are distinguished by the string value of a preprocessor variable.

`#switch` *string*

`#case` *string*$_1$

  statseq$_1$

`#break`

`#case` *string*$_2$

  statseq$_2$

`#break`

`#case` *string*$_3$

$statseq_3$

`#break`

....

`#case` $string_n$

$statseq_n$

`#break`

[`#default`

$statseq_{n+1}$

`#break`]

`#endswitch`

Each case is labeled by a string. FORM looks for the first case of which the string matches the string value of the preprocessor variable in the `#switch` instruction and, if found, starts at that case. The case labeled `#default` is read if none of the other cases are satisfied. The `#default` case can be omitted: then, if none of the cases match, reading just continues after the `#endswitch` instruction, i.e., no action at all takes place. The `#break` instruction causes an immediate exit from the switch. It prevents that after the code of one case is read, processing falls through to the next case.

Simple Examples of Choice

Let us have the conditional statements of the previous sections in an executable FORM program.

```
#define two "2"
#define three "3"
#define four "4"
#define quatro "4"
Symbols x2,...,x5;
Local F`two' =
#if {`two'%2} == 0
   1
#else
  -1
#endif
;
Local F`three' =
#if {`three'%2} == 0
   1
#else
  -1
#endif
;
Local F`four' =
#switch {`four'/2}
#case {`quatro'/2}
  + x2
#case `three'
  + x3
#break
#case 2
  + x4
#default
  + x5
#endswitch
```

```
 ;
 Print;
 .end

F2 =
   1;

F3 =
    - 1;

F4 =
   x2 + x3;
```

With the `#define` instruction you can give a string value to a preprocessor variable. Of course, the string values in our example have a numerical interpretation, and the preprocessor knows this. It concatenates regular string characters and preprocessor variables: the names `F2`, `F3`, and `F4` of the local expressions in the above session are constructed in this way. Furthermore, you can do arithmetic operations with preprocessor variables that have a numerical interpretation: in our example, we compute the remainder and quotient of division by two.

Be warned that the preprocessor does not always interpret string values numerically or in the same way as you would do. For example, rational numbers are rounded to integers (in the direction to zero) before any further processing, in case curly brackets are present. The following FORM session illustrates this.

```
 #define minustwothird "-2/3"
 Local expr1a = `minustwothird';
 Local expr1b =
 #if `minustwothird' > -3/2
    1
 #elseif `minustwothird' == 0
    0
 #else
   -1
 #endif
 ;
 Local expr1c =
 #if {`minustwothird'} > -3/2
    1
 #elseif {`minustwothird'} == 0
    0
 #else
   -1
 #endif
 ;
 Local expr1d = {`minustwothird'};
 Print;
 .end

expr1a =
    - 2/3;

expr1b =
    - 1;

expr1c =
   1;
```

```
expr1d = 0;
```

Again, the curly brackets get the integer calculator of the preprocessor activated while reading the input stream.

3.1.2 Repetition

In general, three types of repetitions can be distinguished

- *unconditional repetition*, in which a predetermined set of actions are carried out,
- *conditional repetition*, in which actions are repeated while or until some condition is fulfilled,
- *mixed repetition*, in which above types of repetition are both present.

The last two types of repetition are not available in the preprocessor; they can only be simulated. Unconditional repetition can be used in a straightforward way.

Unconditional Repetition

• Counted Do-Loop

The preprocessor provides two types of unconditional do-loops. The first one is similar to the one found in most programming languages, and is referred to as a *counted do-loop*.

> `#do` *counter* = *start*, *finish* [, *stepsize*]
>
> *statseq*
>
> `#enddo`

where *counter* is a preprocessor variable used for counting with initial value *start*. The counter is incremented at the end of each step in the repetition by the value of *stepsize* until it passes the value of *finish*. Then the repetition stops and FORM continues reading after the `#enddo` instruction. *statseq* represents any valid FORM expression, statement, or sequence of statements separated by semicolons, which are read at each step in the repetition. The *stepsize* is optional: if omitted, the default value 1 will be assumed. You cannot alter the *finish* and *stepsize* from within the loop; it is however allowed to redefine the preprocessor variable *counter*. Furthermore, the preprocessor variable used for counting has no value or no meaning outside the loop.

In the following four examples, we compute the sum of the first ten natural numbers with the preprocessor. The first session uses a simple `#do` loop to compose an expression that is by definition equal to the requested sum.

```
 Local S =
 #do i = 1, 10
    + `i'
 #enddo
 ;
 Print;
 .end

S =
   55;
```

A preprocessor variable can be used to make the program more general.

```
#define MAX "10"
Local S =
#do i = 1, `MAX'
   + `i'
#enddo
;
Print;
.end
```

```
S =
   55;
```

The above two sessions were just there for explaining the simple counted do-loop. In a real FORM program, it is of course easier to use the following code, which is also more readable.

```
#define MAX "10"
Local S = 1 + ... + `MAX';
Print;
.end
```

```
S =
   55;
```

Counted do-loops can be nested.

```
Local S =
#do i = 1, 5
   #do j = 0, 1
      + 2*`i' - `j'
   #enddo
#enddo
;
Print;
.end
```

```
S =
   55;
```

In the above three do-loop programs we have used the `#do` to compose a local expression. This is why we always have a semicolon after the `#enddo` instruction on a separate line in the above programs. Of course the loop control structure can also be used with "complete" FORM statements.

```
Local S0 = 0;
#do i = 1, 10
    Local S`i' = S{`i'-1} + `i';
#enddo
Print S10;
.end
```

```
S10 =
   55;
```

The start, finish, and stepsize of a counted do-loop do not need to be integers: rationals will also do because the preprocessor's arithmetic will always give an integer result. The following rather weird example illustrates that FORM does its best to make sense of the instructions.

```
AutoDeclare Symbol x;
Local S =
```

```
#do i = -2, 9/2, 3/2
   + x‘i’
#enddo
;
Print;
.end

S =
    - 3 + 2*x + x0 + x1 + x2 + x3 + x4;
```

First of all the finish and stepsize of the loop are calculated via integer division as 4 and 1, respectively. The concatenation of strings "x-1" and "x-2" for `i` = -1 and `i` = -2 lead to the summands `2*x-3` in the end result.

- **Listed Do-Loop**

The second variation on the `#do` loop is the *listed do-loop*.

> `#do` *name* `= {` *sequence* `}`
>
> *statseq*
>
> `#enddo`

where *sequence* consists of strings separated by commas or vertical bars, and *statseq* is an expression, statement, or sequences of statements separated by semicolons. The separators of the strings can even be mixed in one loop instruction. The vertical bar is available as separator for reason of backward compatibility; the preferred separator in FORM is the comma. The following example shows all details to know.

```
Indices i,j,k;
Function T;
Local expr =
#do p = {1,i|2\,i|j,(k,k)}

  + T(‘p’)
#enddo
;
Print;
.end

expr =
  T(k + T(k)) + T(i) + T(j) + T(1) + T(2,i);
```

We "escape" a comma in `2\,i` so that it is read as one parameter with string value `"2,i"` and so that it gives the last term in the above expression. The first term in the output is a bit strange: it shows that FORM indeed picks the terms between the round brackets in one step of the loop and processes them recursively, one by one, from left to right.

- **Realistic Examples of Unconditional Do-Loops**

We end this subsection about unconditional repetition with three, more realistic examples of unconditional `#do` loops. In the first example we are going to take the series expansion of $\ln(1+x)$ about $x=0$ up to 50 terms and substitute in it the series expansion of e^x-1 about $x=0$. The expansions we need are:

$$\ln(1+x) = \sum_{i=1}^{N} (-1)^{i+1} x^i/i + \mathcal{O}(x^{N+1})$$

and

$$e^x - 1 = \sum_{i=1}^{N} \frac{1}{i!} x^i + \mathcal{O}(x^{N+1}) = x(1 + \frac{x}{2}(1 + \frac{x}{3}(1 + \frac{x}{4}(1 + \cdots)))) \,.$$

```
* check that exp(ln(1+x))-1 = x up to order 50 in series expansions
#define N "50"
On Statistics;
Symbol i, x(:‘N’), y(:‘N’);
*   define ln(1+x)
Local X =  - sum_(i, 1, ‘N’, sign_(i)/i*x^i);
*   tag x by y
id  x = x*y;
*   so that we can use the telescope formula of exp(x)-1.
*   in this example, the expansion will be slow.
#do i=2,’N’+1
    id  y = 1 + x*y/‘i’;
#enddo
* print the result
Print;
.end
```

```
Time =    1068.25 sec    Generated terms =    1295970
                X        Terms in output =          1
                         Bytes used      =         18

   X =
      x;
```

The program gives the expected answer x, but it uses quite some time to do so because it generates more than a million terms, even though powers with degree higher than 50 are discarded during the computation. The program becomes about 125 times faster when this build up is suppressed by sorting the terms after each substitution. This is done by adding a `.sort` instruction inside the loop. Then, only about 15000 terms are generated during the computation.

```
* check that exp(ln(1+x))-1 = x up to order 50 in series expansions
#define N "50"
On Statistics;
Symbol i, x(:‘N’), y(:‘N’);
*   define series expansion of ln(1+x)
Local X =  - sum_(i, 1, ‘N’, sign_(i)/i*x^i);
*   tag x by y
id  x = x*y;
*   so that we can use the telescope formula of exp(x)-1.
*   in this example, sorting takes place at each step of the expansion.
#do i=2,’N’+1
    id  y = 1 + x*y/‘i’;
    .sort: step ‘i’;

Time =       1.12 sec    Generated terms =        675
                X        Terms in output =        675
                 step 2  Bytes used      =       9638
#enddo

Time =       3.79 sec    Generated terms =       4247
                X        Terms in output =        433
                 step 3  Bytes used      =       7182

Time =       4.97 sec    Generated terms =       2206
                X        Terms in output =        348
```

```
                step 4  Bytes used      =       7030
  .
  .
  .

Time =       8.42 sec    Generated terms =          6
               X         Terms in output =          3
               step 50  Bytes used      =        158

Time =       8.43 sec    Generated terms =          3
               X         Terms in output =          1
               step 51  Bytes used      =         18
    * print the result
    Print;
    .end

Time =       8.43 sec    Generated terms =          1
               X         Terms in output =          1
                         Bytes used      =         18

   X =
      x;
```

We have added a commentary to the `.sort` instruction so that the statistics shows which step of the do-loop is involved. The commentary to the module instruction is initiated by a colon, it is terminated by a semicolon, and the characters in between form the message printed.

The second example is about simplifying an expression in a 6-dimensional Clifford algebra generated by elements $e_1, e_2, \ldots, e_6$ with relations

$$e_i^2 = 1, \quad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j \in \{1, \ldots 6\}.$$

In this case, you do not want to write down the relations one by one; it is much easier to use a `#do` loop for this purpose.

```
    #define DIM "6";
    Symbols i;
    Functions e;
    Local  expr = sum_(i, 1, 'DIM', e(i)) ^ 3;
    repeat;
      #do i = 1, 'DIM'
         #do j = 'i'+1, 'DIM'
             id e('j') * e('i') = -e('i') * e('j');
         #enddo
         id e('i') * e('i') = 1;
      #enddo
    endrepeat;
    Print;
    .end

   expr =
      6*e(1) + 6*e(2) + 6*e(3) + 6*e(4) + 6*e(5) + 6*e(6);
```

The third example is about computing Fibonacci numbers F_n, which are recursively defined by

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

They can be efficiently generated with preprocessor instructions.

```
#define MAX "7"
Local F1 = 1;
Local F2 = 1;
#do n = 3, `MAX'
   .sort
   Drop F{`n'-2};
   Skip F{`n'-1};
   Local F`n' = F{`n'-1} + F{`n'-2};
   Print;
#enddo

  F3 =
     2;


  F4 =
     3;


  F5 =
     5;


  F6 =
     8;

 .end

  F7 =
     13;
```

There are more reasons for showing this example. It makes once more clear that curly brackets, instead of round brackets, are used to set precedence in the preprocessor. But we also want to point out the `Drop` and `Skip` instruction. When the `Drop` statement is used, the expression can be used in the current module, but after the next `.sort` or `.store` instruction the expression does not exist anymore. The `Skip` instruction inactivates the expression only for the range of the current module. The skipped expression may be used in the right hand side of an identification, but no operation is performed on the expression in the current module. In our example, this means that we forget about Fibonacci numbers when they are not needed anymore in the recursive computation. Furthermore, when it comes to printing of results, we forget about the Fibonacci number computed in the previous step; we only show the result of the newly computed number. The printing is inside the loop; therefore the `.end` instruction appears in the above output much later than in the source file, where it is just after the `#enddo` instruction.

Conditional Repetition

The following example, in which the Fibonacci number F_{19} is computed, shows how a conditional repetition like a post-checked do-loop can be constructed. Other conditional repetitions can be treated similarly.

```
Symbol x;
Local F19 = x^18;
#do i = 1, 1
   id x^2 = x+1;
   if (count(x,1)>1) redefine i "0";
   .sort
```

```
#enddo
id x = 1;
Print;
.end

F19 =
   4181;
```

As long as there exist powers of x of degree higher than 1 — the `count` function is for power counting — we reset the preprocessor variable to the value 0 inside the loop. In effect, the do-loop becomes a post-checked do-loop.

A few remarks about the above FORM code are still necessary. The condition, which determines whether the loop should be terminated or not, is checked during program execution, via a regular `if` statement. The redefinition of the loop variable `i` is not carried out by the preprocessor instruction `#redefine`, but instead by a regular command called `redefine`, because it is part of a choice control structure at compiler level and not at preprocessor level. This command should be before the last `.sort` inside the loop, because the `#do` instruction is part of the preprocessor. This implies that the value of the loop variable `i` is considered before the module is executed. This means that if the `redefine` command would be after the last `.sort` inside the loop, two things would go wrong: First, the loop would be terminated before the `redefine` command would ever make a chance of being executed. Secondly, the statement would be compiled in the expectation that there exists a variable `i`, but then the loop would be terminated. Afterwards, when the statement is being executed, the `redefine` statement would refer to a variable that does not exist anymore.

The above construction can also be used to simulate a kind of multi-module repeat; as we shall see in the next section, the `repeat` loop may only contain statements from within a single module. The code will look as follows:

```
#do i = 1, 1
    some statements
    if ( match(pattern) ) redefine i "0";
.sort
#enddo
```

As long as there are terms with the pattern — the `match` function will select them — the statements will be repeated. Note that if there are more expressions, they will all be repeated, even those expressions that do not contain the pattern any longer. The above code emulates a do–until loop that can contain one or more `.sort` instructions. In subsection 3.1.5 we shall use this kind of control structure in processing word problems in Coxeter groups.

3.1.3 Preprocessor Variables

In order to help in the preparation of code for the compiler, the preprocessor is equipped with variables that can be defined or redefined by the user or by other preprocessor actions. We have already seen examples of preprocessor variables in the choice and repetition examples of the previous subsections. Preprocessor variables have regular names that are composed of strings of alphanumeric characters of which the first must be alphabetic. Recall that FORM is case-sensitive with respect to variables. In the table below we list all preprocessor instructions that deal with definition and removal of preprocessor variable.

Instruction	*Meaning*
`#define` *var* "*string*"	initialize the preprocessor variable *var* with the *string* value
`#redefine` *var* "*string*"	(re)set the preprocessor variable *var* to the *string* value
`#undefine` *var*	remove the preprocessor variable *var*

The preprocessor variables can be recognized in a FORM program and be distinguished from regular variables by the quotes that are around them when they are referred to. This convention makes it possible to concatenate regular strings of character and preprocessor variables to form larger strings of characters. The following example, which involves four preprocessor variables, viz., `'i'`, `'j'`, `'MAX'`, and `'max'`, illustrates the concatenation of strings.

```
#define MAX "2"
#define max "2"
AutoDeclare Symbol x;
#do i = 1, 'MAX'
   #do j = 1, 'max'
      Local F'i''j' = x^'i''j';
   #enddo
#enddo
Print;
.end

F11 =
   x^11;

F12 =
   x^12;

F21 =
   x^21;

F22 =
   x^22;
```

The left and right quotes can be nested. Hence `'max'i''` will result in the preprocessor variable `i` to be substituted first. If it happens to be the string `"1"`, the result after the first substitution is `max1` and then FORM first looks up its string value. This explains the following session.

```
#define max1 "2"
#define max2 "3"
AutoDeclare Symbol x;
#do i = 1, 2
   #do j = 1, 'max'i''
      Local F'i''j' = x^'i''j';
   #enddo
#enddo
Print;
.end

F11 =
   x^11;

F12 =
   x^12;

F21 =
   x^21;

F22 =
   x^22;

F23 =
   x^23;
```

When the preprocessor encounters a left curly bracket it will read till the matching right curly bracket and test whether the characters, after substitution of the preprocessor variables, can be interpreted numerically. If so, the preprocessor calculator will compute the numeric result and the original string is replaced by a textual representation of the number computed. If no numerical interpretation of an expression between curly brackets is possible, the whole string, including the curly brackets, will be passed on to the later stages of the FORM program. An example is the FORM program of the previous section, in which Fibonacci numbers were computed: the statement

```
Local F`n' = F{`n'-1} + F{`n'-2};
```

evaluates to

```
Local F6 = F5 + F4;
```

if the preprocessor variable `` `n' `` has the value 6.

A valid numerical expression can contain digits and the characters `+`, `-`, `*`, `/`, `^`, `%`, `!`, `&`, |, `(`, `)`, `{`, and `}`. Very important is that the comma `,` is <u>not</u> a legal character for the preprocessor calculator. This explains the difference between

```
if f(x?!{0}) = 1/x;
```

which the preprocessor changes into the syntacticly incorrect statement

```
if f(x?!0) = 1/x;
```

and the statement

```
if f(x?!{0,0}) = 1/x;
```

which the preprocessor leaves untouched, because of the presence of the comma, so that the identify statement is carried out for nonzero `x`. Parentheses `( )` and curly brackets `{ }` are used for setting priority rules in preprocessor calculations. The other characters listed above occur in arithmetic operators. All arithmetic is done over integers in a finite range: from -2^{31} to $2^{31}-1$ on 32 bit platforms, and from -2^{63} to $2^{63}-1$ on 64 bit systems. The table below lists all operators that the preprocessor has at its disposal.

Operator	*Meaning*
`+`	addition
`-`	subtraction
`*`	multiplication
`/`	division
`^`	exponentiation
`%`	remainder after division
`!`	factorial
`&`	bitwise AND (as in C)
`^%`	a postfix ^{2}log
`^/`	a postfix integer square root

3.1.4 Procedure

One of the most important features of the preprocessor is the use of procedures. A FORM procedure has the following syntax

`#procedure` *name* [(*parameterseq*)]

statements

`#endprocedure`

where the optional (*parameterseq*) is a sequence of names consisting of regular alphanumerical characters; the parameters are separated by commas. The parameters are preprocessor variables and hence they must be referred to between left and right quotes. The procedure is activated by the `#call` instruction, which has the following syntax

`#call` *name* [(*argumentseq*)]

where the *argumentseq* is a sequence of FORM expressions separated by commas or vertical bars (as in the listed do-loop). If the definition of a procedure has no parameters, the procedure call should not contain arguments either.

Let us look at an example: the Fibonacci numbers again.

```
#procedure fibonacci(F,n)
*
* Procedure to compute Fibonacci numbers
* Input: F: the function that represents the number
* It should have one argument, viz. n, which is
* for internal use and must be declared as a symbol
* before the procedure call.
*
  repeat;
    id ‘F’(1) = 1;
    id ‘F’(2) = 1;
    id ‘F’(‘n’?) = ‘F’(‘n’-1) + ‘F’(‘n’-2);
  endrepeat;
#endprocedure

Symbol n;
CFunction F;
On Statistics;
Local F19 = F(19);
#call fibonacci(F,n)
Print;
.end
```

```
Time =       6.71 sec    Generated terms =       4181
              F19        Terms in output =          1
                         Bytes used      =         10

   F19 =
      4181;
```

Although the procedure looks like a subroutine, it works in fact more like a macro. When the procedure is called, the preprocessor checks whether the procedure is defined, and if so, whether the number of arguments matches the number of parameters in the procedure definition. When everything seems to be okay, the preprocessor substitutes the program block. In our example, the program block consists of a repetition of identifications that compute the requested Fibonacci number by downward recursion (as in Section 2.3.2).

The very deep recursion in the above procedure is rather costly. One way to solve this problem is to tabulate the ten lowest Fibonacci numbers. Then the reduction stops when a table element is reached. This brings us to the creation of tables in FORM. A table is a special function in FORM: it is a kind of array filled with data. A table must be declared first with the `Table` statement and it must be filled before any executable statement. It declares the table automatically as a commuting function. This could have been done more explicitly via the `CTable` instruction. A noncommuting table is declared with the `NTable` instruction. The array can be multidimensional, be of special type such as “sparse”, and have wildcards as formal parameters. The creation of a table of Fibonacci numbers and its use goes as follows.

```
#define MAXTAB "10"
Table Ftbl(1:‘MAXTAB’);
Fill Ftbl(1) = 1;
Fill Ftbl(2) = 1;
#do i = 3, ‘MAXTAB’
   Fill Ftbl(‘i’) = Ftbl({‘i’-1}) + Ftbl({‘i’-2});
#enddo

#procedure fibonacci(F,n)
*
* Procedure to compute Fibonacci numbers
* Input: F: the function that represents the number
* It should have one argument, viz. n, which is
* for internal use and must be declared as a symbol
* before the procedure call.
*
  id ‘F’(?x) = Ftbl(?x);    * replace F by Ftbl
  repeat;
    id Ftbl(‘n’?) = Ftbl(‘n’-1) + Ftbl(‘n’-2);
  endrepeat;
#endprocedure

Symbol n;
CFunction F;
On Statistics;
Local F19 = F(19);
#call fibonacci(F,n)
Print;
.end
```

```
Time =       0.89 sec    Generated terms =       4181
              F19        Terms in output =          1
                         Bytes used      =         10

   F19 =
      4181;
```

The curly brackets in the statement

```
Fill Ftbl(‘i’) = Ftbl({‘i’-1}) + Ftbl({‘i’-2});
```

are essential for efficient use of the table. Of course, the first part of the above program could have been used on its own to compute the nineteenth Fibonacci number.

```
#define MAXTAB "19"
Table Ftbl(1:‘MAXTAB’);
Fill Ftbl(1) = 1;
Fill Ftbl(2) = 1;
#do i = 3, ‘MAXTAB’
   Fill Ftbl(‘i’) = Ftbl({‘i’-1}) + Ftbl({‘i’-2});
#enddo
On Statistics;
Local F19 = Ftbl(19);
Print;
.end
```

```
Time =       0.61 sec    Generated terms =       4181
              F19        Terms in output =          1
                         Bytes used      =         10

   F19 =
      4181;
```

However, this is still not the most efficient FORM program to compute the Fibonacci number. Nevertheless, it is a clear and easy implementation. The version below, which we have explained before, is the fastest (CPU-time 6 times faster) and the least memory consuming, but it is a little bit more complicated.

```
#define MAX "19"
On Statistics;
Local F1 = 1;
Local F2 = 1;
#do n = 3, `MAX'
   .sort
   Drop F{`n'-2};
   Skip F{`n'-1};
   Local F`n' = F{`n'-1} + F{`n'-2};
#enddo
Print;
.end
```

The `.sort`, `Drop`, and `Skip` are essential for the efficiency. They make that not 4181 terms are generated but only $2 \times 18 = 36$ terms, or better to say, that an algorithm of running time $O(2^n)$ is replaced by an algorithm of running time $O(2n)$.

As a second example of the use of procedures in FORM, we rewrite our example of differentiation of trigonometric functions from Section 2.5.3.

```
#procedure diff(x,dx)
  id g?commuting?noncommuting(`x') = g(`x');
  Multiply left `dx';
  repeat;
    id `dx'*g?noncommuting[n](`x') = derivative[n](`x') + g(`x')*`dx';
    id [-sin](`x') = - [sin](`x');
    id [1/cos^2](`x') = 1/[cos](`x') * 1/[cos](`x');
  endrepeat;
  id `dx' = 0;
  id f?noncommuting?commuting(`x') = f(`x');
  id 1/f?noncommuting?commuting(`x') = 1/f(`x');
#endprocedure
*
* The following statements could be put in a standard include file
*
Symbols x,n;
CFunctions sin,cos,tan,g;
Functions [sin], [cos], [tan], [-sin], [1/cos^2], f, dx;
Set commuting: sin, cos, tan;
Set noncommuting: [sin], [cos], [tan];
Set derivative: [cos], [-sin], [1/cos^2];
FunPowers allfunpowers;
*
* And the rest of the program is as follows
*
```

```
 Local expr = sin(x)*tan(x) + cos(x);
 #call diff(x,dx)
 Print;
 .end

expr =
    - sin(x) + cos(x)*tan(x) + 1/(cos(x))^2*sin(x);
```

The comments are even more important here as they give a clue to how to work more easily with FORM. In our example, we can put the differentiation procedure in a file called `diff.prc`, without the left and right quotes around the parameters. The accompanying declarations can be stored in a file called `diff.h` (the declarations of `x`, and `dx` are kept outside). The two files look as follows.

diff.prc

```
#procedure diff(x,dx)
  id g?commuting?noncommuting(x) = g(x);
  Multiply left dx;
  repeat;
    id dx*g?noncommuting[n](x) = derivative[n](x) + g(x)*dx;
    id [-sin](x) = - [sin](x);
    id [1/cos^2](x) = 1/[cos](x) * 1/[cos](x);
  endrepeat;
  id dx = 0;
  id f?noncommuting?commuting(x) = f(x);
  id 1/f?noncommuting?commuting(x) = 1/f(x);
#endprocedure
```

diff.h

```
#-
Symbols n;
CFunctions sin,cos,tan,g;
Functions [sin], [cos], [tan], [-sin], [1/cos^2], f, dx;
Set commuting: sin, cos, tan;
Set noncommuting: [sin], [cos], [tan];
Set derivative: [cos], [-sin], [1/cos^2];
FunPowers allfunpowers;
#+
```

Now, the program can just look like

```
#include diff.h
Symbol x;
Function dx;
Local expr = sin(x)*tan(x) + cos(x);
#call diff(x,dx)
Print;
.end
```

How does it run under FORM? When the preprocessor encounters the `#include diff.h`, it inserts the contents of the file into the input. The `#-` instruction means that the listing of the input, when running the program, will be turned off until further notice. With the `#+` instruction, logging of input is resumed again. We added these instructions in the file `diff.h` because normally it is unnecessary to see the declarations needed for the procedure definition. When the preprocessor encounters the procedure call, it will first look whether there is a definition in the program given. If not, it will look for the definition in a file whose name is the name of the procedure extended with `.prc`. This file is searched for in the current directory or in the directories listed in the environment variable FORMPATH or indicated by the `-p` option in the command that starts the system. The proof of the pudding is in the eating: let us run the FORM program.

```
#include diff.h
#-
Symbol x;
Function dx;
Local expr = sin(x)*tan(x) + cos(x);
#call diff(x,dx)
Print;
.end

   expr =
      - sin(x) + cos(x)*tan(x) + 1/(cos(x))^2*sin(x);
```

We end with a few remarks about procedures in FORM.

- When a procedure is called, the system checks whether the number of arguments matches the number of parameters in the procedure call.
- The number of arguments is available by the `nargs_` function.
- The parameters inside the procedures are local preprocessor variables; in this way, name conflicts are avoided.
- Recursive procedure definitions are not allowed.

3.1.5 Processing Word Problems in Coxeter Groups

What we have learned so far in this chapter can be applied in a realistic FORM application: processing word problems in Coxeter groups.

Following [du Cloux 99], let us first give a brief introduction to the topic. A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(st)^{m_{s,t}} = 1\,, \quad \text{for } s, t \in S\,,$$

where 1 denotes the identity element in the group W and the $m_{s,t}$ are natural numbers or ∞ satisfying the following conditions:

$$m_{s,s} = 1, \quad m_{s,t} \geq 2 \quad \text{for } s \neq t\,,$$

and $m_{s,t} = \infty$ means that no relation occurs for the pair s, t. Formally, W is the quotient F/N, where F is the free group on the set S and N is the smallest normal subgroup of F that contains all elements $(st)^{m_{s,t}}$. We allow ourselves to write $s \in W$ for the image of $s \in S$ and we refer to W as the *Coxeter group* generated by S. The information contained in the numbers $m_{s,t}$ can be given in the form of the so-called *Coxeter graph* for the group, with vertex set S, and an edge between s and t if and only if $m_{s,t} > 2$, labeled by $m_{s,t}$ if $m_{s,t} > 3$. In most cases, the generators are numbered s_1, s_2, etc., and the vertices of the Coxeter graph are numbered accordingly.

For example, the symmetric group S_{n+1} of all permutations of a sequence of $n+1$ objects, with the following n adjacent transpositions as generators $s_1 = (1\,2)$, $s_2 = (2\,3), \ldots s_n = (n\,n+1)$ forms a Coxeter group with Coxeter graph

$$\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\overset{2}{\circ}\!\!-\!\!\!-\!\!\!-\cdots\cdots-\!\!\!-\!\!\!-\overset{n}{\circ}$$

where we have labeled the vertices by the corresponding indices of the generators. The Coxeter group is said to be of type A_n. The finite Coxeter groups, also referred to as the *finite reflection groups*, have been classified. For details see [Humphreys 90].

For a Coxeter system (W, S), the generators $s \in S$ have order 2 in W. Hence, each element $w \neq 1$ in W can be written in the form $w = s_1 s_2 \cdots s_r$ for some s_i (not necessarily distinct) in S. If r is as small as possible, we call it the *length* of w, written $l(w)$, and call any expression of w as a product of r elements of S a *reduced expression*. The Poincaré series of W is the formal power series $W(t)$ in indeterminate t defined by $W(t) = \sum_{w \in W} t^{l(w)}$.

Let J be an arbitrary subset of S and denote by W_J the subgroup generated by J. Furthermore, define the set $W^J = \{\, w \in W \mid l(sw) > l(w) \text{ for } s \in J \,\}$. Then, each left coset $W_J w$ contains a unique element w^J of minimal length, $l(w_J w^J) = l(w^J) + l(w_J)$ for all $w_J \in W_J$, and $W^J = \{\, w \in W \mid w \text{ is unique element } w^J \text{ in coset } W_J w \,\}$. It follows that each $w \in W$ can be uniquely written as product $w_J w^J$ with $w^J \in W^J$ and $w_J \in W_J$ such that $l(w) = l(w^J) + l(w_J)$.

We choose once and for all an increasing sequence of subsets $J_0 = \emptyset \subset J_1 \subset \cdots \subset J_n = S$, with $\#J_j = j$ for $j = 0, 1, \ldots, n$ and define W_j as the subgroup of W generated by J_j. We also set $X_j = {W_j}^{J_{j-1}}$, for $j = 1, \ldots, n$. Then, the canonical multiplication map $(x_1, \ldots, x_n) \to x_1 \cdots x_n$ from $X_1 \times \cdots \times X_n$ to W is bijective. The decomposition $w = x_1 \cdots x_n$ of a group element w given by the previous bijection is called the *special decomposition* of w. We also have chosen now a linear ordering on S: the first element is the unique element of J_1, the second one is the unique element of J_2 not in J_1, and so on. This allows us to define a well-ordering on the free monoid S^* on the set S by ordering words first by length, then lexicographically from the left in each given length. We call this the *ShortLex ordering*.

For each $w \in W$ we denote by $\mathrm{NF}(w)$ the unique smallest reduced expression for w in the ShortLex ordering. $\mathrm{NF}(w)$ is called the *normal form* of w with respect to the ShortLex ordering. In principal, the normal form of w can be recursively found as follows: The normal form of the identity element is the empty string. Let $w \neq 1$, then $\mathrm{NF}(w) = s_1 \,\mathrm{NF}(s_1 w)$, where s_1 is the smallest generator $s \in S$ such that $l(sw) < l(w)$. We say here "in principle" because we have no effective way described yet to determine which $s \in S$ satisfy the length condition $l(sw) < l(w)$.

From $\mathrm{NF}(w) = s_1 \cdots s_p$ one can recursively read off the special decomposition $w = x_1 \cdots x_n$ in the following way. Define q to be the smallest integer ≥ 0 such that $s_{q+1} = s_n$, $q = p$ if there is no such integer. Then $x_n = s_{q+1} \cdots s_p$, $x_n = 1$ if $q = p$, and $s_1 \cdots s_q \in W_{n-1}$. Roughly stated, the term x_n can be read off from $\mathrm{NF}(w)$ as the last "slice" taken from the first appearance of the generator s_n (empty if there is no such appearance). Continue with $s_1 \cdots s_q \in W_{n-1}$ in a similar way to compute x_{n-1}, and so on.

A *rewrite system* in the free monoid S^* on the set of generators S is a set of ordered pairs $\mathcal{R} = \{\, (x, y) \mid x, y \in S^* \,\}$, with $x > y$ in the ShortLex ordering for each $(x, y) \in \mathcal{R}$; we shall write $x \to y$ instead of (x, y). If a word $z \in S^*$ contains the left-hand side of a rule $x \to y$ as an interval, i.e., if $z = uxv$ for some $u, v \in S^*$, we say that the rule $x \to y$ applies to z and that uyv is the reduction of z corresponding to $x \to y$. If none of the rules in $\mathcal{R}$ applies to z, we say that z is *$\mathcal{R}$-reduced*. Because reductions are strictly decreasing in the ShortLex ordering, one can obtain from a given word an $\mathcal{R}$-reduced word in a finite number of steps, i.e., all sequences of rewrites terminate. However, several distinct $\mathcal{R}$-reduced words might be obtained from a given word. A rewrite system is said to be *confluent* or *complete* if each word possesses a unique $\mathcal{R}$-reduction, which is also referred to as the *$\mathcal{R}$-normal form* of the word. The rewrite system is *reduced* if the following conditions hold:

- for each $x \to y$ in $\mathcal{R}$, y is $\mathcal{R}$-reduced.
- for each $x \to y$ in $\mathcal{R}$, x is reduced with respect to all "lower" rules.

With a complete rewrite system, the word problem is easily solved: two words are equivalent if and only they have the same normal form. The so-called Knuth-Bendix completion procedure constructs from any finite set $\mathcal{R}$ of rewrite rules a (possibly infinite) complete and reduced set $\mathcal{R}'$ of rules. For finite Coxeter groups, the starting rewrite system $\mathcal{R}$ could be the following set of rules:

$$s^2 \to 1 \text{ for } s \in S \text{ and } \alpha_{s,t} u_{s,t} \to t \alpha_{s,t} \text{ for } s > t \text{ in the ShortLex ordering.}$$

Here, $\alpha_{s,t}$ denotes the element $stst\ldots$ obtained by multiplying $m_{s,t} - 1$ generators alternatingly equal to s and t, for $m_{s,t} < \infty$, and $u_{s,t} = t$ if $m_{s,t}$ is even, $u_{s,t} = s$ otherwise. A result of le Chenadec [le Chenadec 86] states that when no two generators in S commute, i.e., if $m_{s,t} \geq 3$ for all $s \neq t$, then the completion procedure creates the following rewrite rules other than the rules $s^2 \to 1$ for s in S:

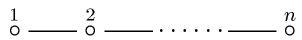
$$\alpha_{s_{i_1}, s_{j_1}} \ldots \alpha_{s_{i_p}, s_{j_p}} u_{s_{i_p}, s_{j_p}} \to s_{j_1} \alpha_{s_{i_1}, s_{j_1}} \ldots \alpha_{s_{i_p}, s_{j_p}} ,$$

(a "concatenation" of the defining relations) where $i_1, \ldots, i_p$ and $j_1, \ldots, j_p$ satisfy

$$\begin{aligned} i_1 &> j_1 & & \\ i_k &< j_k & & \text{for } 1 < k \leq p \\ u_{i_k, j_k} &= s_{j_{k+1}} & & \text{for } 1 \leq k < p \end{aligned}$$

In [le Chenadec 86, du Cloux 98] complete rewrite systems are given for many types of Coxeter groups. Below we list complete rewrite systems for the Coxeter groups of type A_n and B_n.

Type $A_n, n \geq 1$



Completion gives the following $n^2 - n + 1$ rules:

$$
\begin{array}{rcll}
s_i^2 & \to & 1 & 1 \le i \le n \\
s_i s_j & \to & s_j s_i & 1 \le j \le i-2,\ 3 \le i \le n \\
s_i s_{i-1} \ldots s_{i-j} s_i & \to & s_{i-1} s_i s_{i-1} \ldots s_{i-j} & 1 \le j < i \le n
\end{array}
$$

In case $n = 3$ we get the following complete rewrite system.

$$
\begin{array}{rcl}
s_1^2 & \to & 1 \\
s_2^2 & \to & 1 \\
s_3^2 & \to & 1 \\
s_3 s_1 & \to & s_1 s_3 \\
s_2 s_1 s_2 & \to & s_1 s_2 s_1 \\
s_3 s_2 s_3 & \to & s_2 s_3 s_2 \\
s_3 s_2 s_1 s_3 & \to & s_2 s_3 s_2 s_1
\end{array}
$$

Note that the last rewrite rule is clear from the relations between s_1, s_2, and s_3, but it cannot be derived from earlier rules; you need the relation $s_3 s_1 = s_1 s_3$ in the right-to-left direction. This rule is obtained by the Knuth-Bendix completion procedure.

Type $B_n, n \ge 2$ $\quad \overset{1}{\circ} \text{---} \overset{2}{\circ} \text{---} \cdots\cdots \text{---} \overset{n-1}{\circ} \underset{4}{\text{---}} \overset{n}{\circ}$

Completion gives the following $n^2 - n + 1$ rules:

$$
\begin{array}{rcll}
s_i^2 & \to & 1 & 1 \le i \le n \\
s_i s_j & \to & s_j s_i & 1 \le j \le i-2,\ 3 \le i \le n \\
s_i s_{i-1} \ldots s_{i-j} s_i & \to & s_{i-1} s_i s_{i-1} \ldots s_{i-j} & 1 \le j < i < n \\
(s_n s_{n-1} \ldots s_{n-j})^2 & \to & s_{n-1} s_n s_{n-1} \ldots s_{n-j} s_n s_{n-1} \ldots s_{n-j+1} & 1 \le j < n
\end{array}
$$

The FORM session below shows the result of computing all elements of the group S_4 via the above rewrite system for the Coxeter group of type A_3 and the result of computing the Poincaré polynomial. The calculation is set up in such a way that it works for any Coxeter group of type A_n. Further explanation follows immediately after the sample session.

```
#define n "3"
* define a procedure to generate reduction rules
#procedure reduce()
    id s?^2 = 1;
    #do i = 3,‘n’
      #do j = 1,{‘i’-2}
        id s‘i’ * s‘j’ = s‘j’ * s‘i’;
      #enddo
    #enddo
    #do i = 2,‘n’
      #do j = 1,{‘i’-1}
        id s‘i’ * ... * s‘j’ * s‘i’ =
           s{‘i’-1} * s‘i’ * ... * s‘j’;
      #enddo
    #enddo
#endprocedure
*
AutoDeclare Functions s;
CFunctions dummy;
Symbol x, t;
* initialize: start with trivial element
Local expr = 1;
Local lastElements = 1;
```

```
* create recursively new elements of the group
#do dummyindex = 1,1
  .sort
* generate new words with word length raised by one
  Drop lastElements;
  Skip expr;
  Local elements = lastElements * t * (s1+...+s`n');
* generate reduction rules and apply them on the newly created elements
  repeat;
    #call reduce
  endrepeat;
* remove newly created elements that have too small word length
  if (count(<s1,1>,...,<s`n',1>) < count(t,1)) discard;
  .sort (polyfun=dummy);
* make coefficients equal to 1
  Skip expr;
  id dummy(x?) = 1;
  .sort
* terminate loop if no new elements are added anymore
  #if (termsin(elements)!=0)
     Local lastElements = elements;
     Local expr = expr + elements;
     #redefine dummyindex "0"
  #endif
  .sort
#enddo
* list all group elements; words of length l are tagged by the power t^l
On statistics;
Drop elements;
Bracket t;
Print +s;
.sort

Time =       0.05 sec    Generated terms =         24
             expr        Terms in output =         24
                         Bytes used      =        658

   expr =

       + t * (
          + s1
          + s2
          + s3
          )

       + t^2 * (
          + s1*s2
          + s1*s3
          + s2*s1
          + s2*s3
          + s3*s2
          )

       + t^3 * (
```

```
      + s1*s2*s1
      + s1*s2*s3
      + s1*s3*s2
      + s2*s1*s3
      + s2*s3*s2
      + s3*s2*s1
      )

   + t^4 * (
      + s1*s2*s1*s3
      + s1*s2*s3*s2
      + s1*s3*s2*s1
      + s2*s1*s3*s2
      + s2*s3*s2*s1
      )

   + t^5 * (
      + s1*s2*s1*s3*s2
      + s1*s2*s3*s2*s1
      + s2*s1*s3*s2*s1
      )

   + t^6 * (
      + s1*s2*s1*s3*s2*s1
      )

   + 1
     ;

* compute the Poincare polynomial
Off statistics;
id s? = 1;
Print;

expr =
   1 + 3*t + 5*t^2 + 6*t^3 + 5*t^4 + 3*t^5 + t^6;
```

The first part of the FORM program is a procedure, called `reduce`, that generates the complete and reduced rewrite system for the Coxeter group of type A_n. In the next part of the FORM program, all elements of the group are computed recursively. In fact, we compute the element $\sum_{w\in W} t^{l(w)}w$ in the group algebra $\mathbf{Q}[t]\,W$ over the ring of polynomials in the indeterminate t. We start with the identity element and multiply it with $t\,(s_1 + s_2 + \ldots + s_n)$. At this point, no reduction takes place. Hereafter, we multiply each term that was newly created with $t\,(s_1 + s_2 + \ldots + s_n)$ and use the rewrite rules to verify if a new group element has been found or not. If the reduction process leads to an element of shorter length the term is discarded (via the `discard` command). The verification is most easily done by comparing the sum of exponents of the generators in the word under investigation with the exponent of the power of t. The `count` procedure is used for power counting in the statement `if (count(<s1,1>,...,<s'n',1>) < count(t,1));` . This process of multiplying newly created elements with $t\,(s_1 + s_2 + \ldots + s_n)$ and verifying whether new elements are created or not is repeated. At each step in the repetition we set the coefficient of a newly found element w equal to $t^{l(w)}$, i.e., we replace any numerical coefficient with 1. The algorithm terminates when no new elements are found anymore. This is most easily checked with the `termsin` procedure, which counts the number of terms in a given FORM expression.. The last step of computing the Poincaré polynomial is easy to do: just replace all generators with 1. Our result for $n = 3$ is in perfect agreement with the following general formula for the Poincaré polynomial $W(t)$ of the Coxeter group of type A_n.

$$W(t) = \prod_{i=1}^{n}(t^i + t^{i-1} + \ldots + t + 1) = \prod_{i=1}^{n} \frac{t^{i+1}-1}{t-1}$$

To get an idea of the performance of FORM, we list the timings for $n = 3, \ldots, 8$ and relate the CPU-time with the maximum word length, denoted $l_{\max}$.

n	CPU-time (sec)	$\#W$	$l_{\max}$
3	0.05	24	6
4	0.28	120	10
5	3.04	720	15
6	40.58	5040	21
7	633.32	40320	28
8	11069.00	362880	36

The data suggest the following recursive formula for the CPU-time (or at least for the upper bound of the CPU-time):

$$\text{CPU-time}(n) = \text{CPU-time}(n-1) \times l_{\max}(n-1).$$

3.1.6 Exercises

1. Let the Fibonacci polynomial $F_n(x)$ be given by $F_1(x) = 1$, $F_2(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n > 2$. Write a program that computes the Fibonacci polynomial. Can your program compute $F_{50}(x)$?

2. How would you generate in FORM the expression $\displaystyle\sum_{i=0}^{25}(-1)^i a_i$ and change it with identifications into $\displaystyle\sum_{i=0}^{25}(-1)^i b_i$?

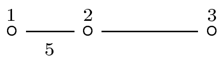
3. Write a FORM procedure that allows you to work out contracted expressions like dot products of vectors and FORM expressions of type $a(p,q)$, where a is a matrix, and p, q are vectors.

4. Write a FORM procedure that given an integer n and a list of three variables returns the sum of all monomials that have n as total degree. E.g., `#call(monomialsum(2,x,y,z)` should return $x^2 + xy + y^2 + xz + yz + z^2$. Generalize your program to any number of variables in the sense that `#call(monomialsum(`n, m, x`)` creates the sum of all monomials in m unknowns $x1, x2, \ldots, xm$ of total degree n

5. Write a FORM procedure that can integrate multivariate polynomials.

6. Write a FORM procedure that can integrate integrals of types $\int x^n \cos x\, dx$ and $\int x^n \cos x\, dx$.

7. The Coxeter group of type H_3 has Coxeter graph



and the following complete rewrite system

$$\begin{array}{rcl}
s_i^2 & \to & 1 \quad \text{for } i = 1,2,3 \\
s_3 s_1 & \to & s_1 s_3 \\
s_2 s_1 s_2 s_1 s_2 & \to & s_1 s_2 s_1 s_2 s_1 \\
s_3 s_2 s_3 & \to & s_2 s_3 s_2 \\
s_3 s_2 s_1 s_2 s_3 s_2 & \to & s_2 s_3 s_2 s_1 s_2 s_3 \\
(s_3 s_2 s_1 s_2 s_1)^2 & \to & s_2 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2
\end{array}$$

Write a FORM program to compute all elements of the Coxeter group and the Poincaré polynomial.

8. Look up the complete rewrite system for the Coxeter group of type B_n.

   (i) Write a FORM program that can be used to compute all elements of the Coxeter group of type B_n and the Poincaré polynomial for $n = 2, \ldots, 5$. By the way, the Poincaré polynomial $W(t)$ is for a Coxeter group of type B_n given by

   $$W(t) = \prod_{i=1}^{n}(t^i + t^{i-1} + \ldots + t + 1) = \prod_{i=1}^{n} \frac{t^{2i} - 1}{t - 1}$$

   Make sure that your result is in agreement with this formula.

   (ii) Conjecture a general formula for the maximum word length in the Coxeter group of type B_n and try to find a reduced expression for the longest element.

   (iii) Determine the order of the Coxeter element $s_1 s_2 \ldots s_n$ for $n = 2, \ldots, 5$. Can you guess a general formula for the order of the Coxeter element?

3.2 Control Structures at Compiler Level

In this section we shall discuss three control structures, known at the compiler level: choice, repetition, and "go to".

3.2.1 Choice

The preprocessor conditional statement only selects statements for execution; it does not restrict the set of expressions to be manipulated. For this purpose, FORM has a "real" `if` statement with the following syntax.

> `if` (*condition*);
>
> *statseq*
>
> [`elseif` (*condition*);
>
> *statseq*]*
>
> [`else`;
>
> *statseq*]
>
> `endif`;

where *condition* is a conditional statement, *statseq* is a sequence of statements separated by semi-colons, [] denotes an optional part, and * denotes a part which can be repeated zero or more times. Omitting an optional part is equivalent to saying "continue". If all optional parts are omitted and only one statement is executed when the condition is met, then

> `if` (*condition*);
>
> *statement*;
>
> `endif`;

can be further abbreviated to

> `if` (*condition*) *statement*;

An example:

```
Symbols a,b,c;
Local F = a*c + b*c;
if (match(a)) id c=a;
endif;
Print;
```

```
.end
```

```
F =
   a^2 + b*c;
```

In the above example, the `match` function selects only those terms that have an `a` inside. Only for these terms, the substitution rule is applied.

One more example:

```
Symbols a,b,c;
Local F = a*c + b*c;
if (match(a)=0) id c=a;
Print;
.end
```

```
F =
   a*b + a*c;
```

Now, only those expressions are selected that do not contain an `a`. The way it works is that `match(`*expression*`)` returns the number of times the *pattern* matches the term. The pattern is given in the same format as the left-hand side of a substitution. So it may contain keywords like `once`, `many`, `select`, etc., and it may contain wildcards.

Three other "questions about terms" can be made in FORM:

- `count(`*object*, *weight*, ...`)` counts the number of times the *object* occurs with given *weight* in the current term.
- `coefficient` return the numerical coefficient of the current term.
- `findLoop` returns the value 1 if a certain loop exists in the current term, and 0 otherwise. A loop is a cyclic contraction of summable indices in (anti)symmetric functions or tensors. Examples of loops of size 3 are
  `f(i1,i2)*f(i2,i3)*f(i3,i1)` and `f(i4,i1,i2)*f(i5,i2,i3)*f(i6,i3,i1)*f(i4,i7,i8)`.

Let us go into details.

The `count` function is for power counting. In the example below, we use it to get rid of all terms of a polynomial of too high degree.

```
Symbols x;
Local F = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6;
if (count(x,1)>3) discard;
endif;
Print;
.end
```

```
F =
   1 + x + x^2 + x^3;
```

The second argument of `count` is the weight of the first object. You can have weights different from the usual number 1. The next example illustrates that when `x` has weight 2, the term `x^2` has a power counted as 4 and therefore can be discarded.

```
Symbols x;
Local F = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6;
if (count(x,2)>3) discard;
Print;
.end
```

```
F =
   1 + x;
```

One can also test whether the exponent of a monomial is a multiple of a given number. For example, the statement

```
if ( count(x,1) = multipleof(3) )
```

tests whether the exponent of a monomial in x is a multiple of 3.

```
Symbols x;
Local F = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6;
if (count(x,1)=multipleof(3)) discard;
Print;
.end

F =
   x + x^2 + x^4 + x^5;
```

coefficient, which is also abbreviated as coeff, returns the numeric coefficient of the current term. It can for instance be used to discard terms with too small coefficients.

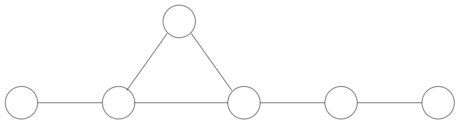
```
Symbols a,b,i,j;
Local F = sum_(i,0,4,a^i/fac_(i)) * sum_(j,0,4,b^j/fac_(j));
if (coefficient < 1/10) discard;
Bracket b;
Print;
.end

F =
    + b * ( 1 + a + 1/2*a^2 + 1/6*a^3 )

    + b^2 * ( 1/2 + 1/2*a + 1/4*a^2 )

    + b^3 * ( 1/6 + 1/6*a )

    + 1 + a + 1/2*a^2 + 1/6*a^3;
```

In the first chapter, in subsection 1.4.6, we have used the symmetric dd_ function to find graphs with prescribed degrees. For FORM to be more useful in graph theory, it must provide ways to detect and manipulate cycles or loops in graphs, i.e., paths in graphs with common begin- and endpoint. The FORM functions replaceLoop and findLoop serve this purpose in the context of (anti)symmetric functions or tensors with summable indices. We give a simple example to illustrate the syntax and use of findLoop: we determine all connected, loop-free graphs with degree sequence 1,1,2,2,3,3 and without multiple edges or loops of size 3.

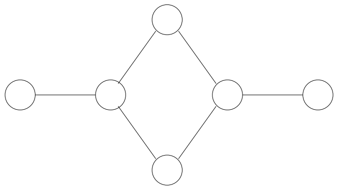
```
AutoDeclare Vector v;
AutoDeclare Index i;
CTensor f(symmetric);
Local G = dd_(v1,v2,v3,v3,v4,v4,v5,v5,v5,v6,v6,v6);
id v?.v?=0;     * loop-free
id v1?.v2?^2=0; * no multiple edges
id v1.v2=0;     * connected
Format 65;
Print;
.sort
```

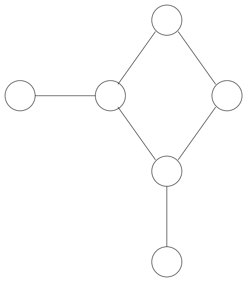
```
G =
   144*v1.v3*v2.v5*v3.v6*v4.v5*v4.v6*v5.v6 + 144*v1.v3*v2.v6*
   v3.v5*v4.v5*v4.v6*v5.v6 + 144*v1.v4*v2.v5*v3.v5*v3.v6*
   v4.v6*v5.v6 + 144*v1.v4*v2.v6*v3.v5*v3.v6*v4.v5*v5.v6 +
   144*v1.v5*v2.v3*v3.v6*v4.v5*v4.v6*v5.v6 + 144*v1.v5*v2.v4*
   v3.v5*v3.v6*v4.v6*v5.v6 + 144*v1.v5*v2.v5*v3.v4*v3.v6*
   v4.v6*v5.v6 + 144*v1.v5*v2.v6*v3.v4*v3.v5*v4.v6*v5.v6 +
   144*v1.v5*v2.v6*v3.v4*v3.v6*v4.v5*v5.v6 + 144*v1.v5*v2.v6*
   v3.v5*v3.v6*v4.v5*v4.v6 + 144*v1.v6*v2.v3*v3.v5*v4.v5*
   v4.v6*v5.v6 + 144*v1.v6*v2.v4*v3.v5*v3.v6*v4.v5*v5.v6 +
   144*v1.v6*v2.v5*v3.v4*v3.v5*v4.v6*v5.v6 + 144*v1.v6*v2.v5*
   v3.v4*v3.v6*v4.v5*v5.v6 + 144*v1.v6*v2.v5*v3.v5*v3.v6*
   v4.v5*v4.v6 + 144*v1.v6*v2.v6*v3.v4*v3.v5*v4.v5*v5.v6;

 repeat;
   #do l=1,6
     #do k=1,‘l’
        id v‘k’.v‘l’ = f(i‘k’,i‘l’);
     #enddo
   #enddo
 endrepeat;
 if (findLoop(f,arguments=2,loopsize=3)) discard;
 repeat;
   #do l=1,6
     #do k=1,‘l’
        id f(i‘k’,i‘l’) = v‘k’.v‘l’;
     #enddo
   #enddo
 endrepeat;
 Print;
 .end

G =
   144*v1.v5*v2.v6*v3.v4*v3.v5*v4.v6*v5.v6 + 144*v1.v5*v2.v6*
   v3.v4*v3.v6*v4.v5*v5.v6 + 144*v1.v5*v2.v6*v3.v5*v3.v6*
   v4.v5*v4.v6 + 144*v1.v6*v2.v5*v3.v4*v3.v5*v4.v6*v5.v6 +
   144*v1.v6*v2.v5*v3.v4*v3.v6*v4.v5*v5.v6 + 144*v1.v6*v2.v5*
   v3.v5*v3.v6*v4.v5*v4.v6;
```

From the expression G in the two stages of the computation it is clear that graphs of type



have been removed and that we are left with graphs of type





In the second stage of the above computation, we first rewrite the dot products of vectors in terms of function calls of the symmetric functions `f` with the indices corresponding with the vectors. In the conditional statement, if a term is encountered that is a loop of size 3 in the function calls of `f` with 2 arguments, then this term is discarded. Hereafter we rewrite the product of function calls as dot products of vectors. These terms represent the graphs that are left.

We have used one of the allowed formats of the `findLoop` statement:

> `findLoop` *function*, `arguments=`*number*, `loopsize=`*number*

where *function* is a symmetric or antisymmetric function or tensor. The `arguments` part says that only occurrences of the function with the specified number of arguments are considered. The `loopsize` part says that only loops of certain size are considered; in our example, only loops of size 3. Other options are `loopsize<`*number*, in which case all loops of size less than the specified number are considered, and `loopsize=all`, in which case all loops are considered. So, we could have left out the `id v1?.v2?^2=0;` and instead we could have used the condition `loopsize<4` to obtain the same result. You can also specify via the option `include=`*index* which summable index must be included in the loop. The order of the option is irrelevant. The `replaceLoop` statement has one more mandatory argument, viz., `outfun=`*name*, in which the name of the (cyclesymmetric) function is given that is used to collect the remaining indices. For example,

```
replaceLoop f,arguments=3,loopsize=3,outfun=ff;
```

replaces

```
f(i4,i1,i2)*f(i5,i2,i3)*f(i6,i3,i1)*f(i4,i7,i8)
```

by

```
f(i4,i5,i6)*f(i4,i7,i8),
```

but the statement

```
replaceLoop f,arguments=3,loopsize=3,outfun=ff,include=i9;
```

leaves the term as it is.

Let us continue with the discussion about the conditional statement at compiler level. As we have seen in the previous section, relational operators are in FORM mostly the same as in the C programming language. The same holds for logical operators with the exception that the negation (! in C) does not exist in FORM.

Operator	*Meaning*
= or ==	equal to
!=	not equal to
>	greater than
<	less than
>=	greater than or equal to
<=	less than or equal to
&&	and
||	or

Furthermore, the conditional statement must be completely within a single module; one cannot have a `.sort` statement inside an `if`-statement. The same holds for other control sturctures at compiler level such as the repetition constructs `repeat` and `while`, which will be discussed in the next subsection.

3.2.2 Repetition

Repetitions can be constructed in several ways. You have already seen the `repeat` statement in several examples. Its syntax is

```
repeat;
      statseq
endrepeat;
```

where *statseq* is a statement or a sequence of statements separated by semicolons. These statements will be executed until they no longer have any effect. FORM also provides a pre-checked repetition, commonly called a `while`-loop, which has the following syntax.

```
while (condition);
      statseq
endwhile;
```

where *condition* is a conditional statement, and *statseq* is a statement or a sequence of statements separated by semicolons. As long as the condition is satisfied, these statements will be executed. An example:

```
Symbols x,y,z;
Local F = x^3 * y^5;
while ( match(x*y^2) );
  id x*y^2 = z;
endwhile;
print;
.end

F =
   x*y*z^2;
```

So, the replacement rule $xy^2 \longrightarrow z$ is applied as long as there is a subexpression xy^2 present.

3.2.3 GO TO

Another way to construct loops is by labels and goto-statements. Below is an example of a post-checked repetition.

```
Symbols x;
Local F = 1;
Label 1;
Multiply x;
if (count(x,1)<10);
   goto 1;
endif;
print;
.end;

 F =
    x^10;
```

The prototype of this post-checked repetition is as follows:

```
label 1;
statseq
if (condition);
      goto 1;
endif;
```

The prototype of a pre-checked repetition is the following.

```
label 1;
if (condition);
      statseq
      goto 1;
endif;
```

Labels are restricted to integers from 0 to 20, but they may be reused in other modules.

3.2.4 Exercises

1. Explain the result of the following program

```
Symbols x,y;
Local F = 1 + y^2*x^4 + y^3*x^5 + y^4*x^6 + y^5*x^7;
if (count(y,-1,x,2)>7);
   discard;
endif;
Print;
.end
```

2. Compute the sum $\sum_{i=-5}^{5} \frac{1}{i^2} x^i$ and then throw away all terms with positive exponent and those with coefficient smaller than $\frac{1}{10}$.

3. Picard's method for generating an approximate solution of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

is to iterate the formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi))\, d\xi$$

starting from $y_0(x) = y_0$.

(i) Write a FORM program that for a given polynomial f in x and y and a given number N computes the approximate solution $p(x)$ that agrees with the exact solution $y(x)$ up to and including degree N, i.e., $y(x) - p(x) = \mathcal{O}(x^{N+1})$. Avoid expression swell by performing computations only up to and including degree N and let the iteration only stop when two successive approximations are equal.

(ii) Compute $y_5(x)$ for the initial-value problem

$$y' = xy^2, \quad y(0) = 1.$$

(iii) Check if your program can also compute y_{20}.

(iv) Compare the result in part (iii) with the series expansion of the exact solution $y(x) = \dfrac{1}{1-x^2}$.

Chapter 4

Answers to Problems of Chapter 1

4.1 Getting Started

1. Does anything happen when you change the `Local [(a+b)^2] = (a+b)^2` statement in the first example by `Local [(a+b)^2] = (a+b)*(a+b)` ?

   The proof is in eating the pudding.

```
    FORM version 3.-(Nov 29 1997). Run at: Tue Jan  6 19:32:18 1998
    Symbols a,b;
    Local [(a+b)^2] = (a+b)^2;
    Print;
    .end
Time =       0.10 sec    Generated terms =          3
                F        Terms in output =          3
                         Bytes used      =         52
    [(a+b)^2] =
       2*a*b + a^2 + b^2;
```

   FORM recognizes the equal terms in the product and treats them as powers so that the binomial formula of Newton can be applied instead of working out the brackets one by one. This is one of the rare cases in which FORM touches the input. In general, FORM never tries to interpret the right hand side of a statement until it needs it and even then it is inserted in its proper place before any interpretation takes place.

2. Does anything happen when you change the `Symbols a,b` declaration in the first example by `Functions a,b` ?

   The FORM session:

```
    FORM version 3.-(Nov 29 1997). Run at: Tue Jan 6 19:42:14 1998
    Functions a,b;
    Local F = (a+b)^2;
    Print;
    .end
Time =       0.10 sec    Generated terms =          4
                F        Terms in output =          4
                         Bytes used      =         74

   F =
      a*a + a*b + b*a + b*b;
```

Here, a and b are used as functions (without arguments). A general property of functions in FORM is that they do not commute. Hence, the binomial formula of Newton cannot be applied. The brackets must be worked out by brute force into a sum of 4 terms.

3. Consider the following FORM program.

```
Symbol a;
Functions b, c;
Local F1 = (a+b+c)^2;
Local F2 = (a+(b+c))^2;
Print;
.end;
```

What is the difference in working out expression F1 and expression F2?

The FORM session:

```
    FORM version 3.-(Nov 29 1997). Run at: Wed Jan  7 13:04:18 1998
    Symbol a;
    Functions B, C;
    Local F1 = (a+B+C)^3;
    Local F2 = (a+(B+C))^3;
    Print;
    .end;
Time =       0.16 sec    Generated terms =         27
               F1        Terms in output =         15
                         Bytes used      =        286

Time =       0.27 sec    Generated terms =         15
               F2        Terms in output =         15
                         Bytes used      =        286

   F1 =
      a^3 + 3*B*a^2 + 3*B*B*a + B*B*B + B*B*C + 3*B*C*a + B*C*B + B*C*C + 3*C*
      a^2 + 3*C*B*a + C*B*B + C*B*C + 3*C*C*a + C*C*B + C*C*C;

   F2 =
      a^3 + 3*B*a^2 + 3*B*B*a + B*B*B + B*B*C + 3*B*C*a + B*C*B + B*C*C + 3*C*
      a^2 + 3*C*B*a + C*B*B + C*B*C + 3*C*C*a + C*C*B + C*C*C;
```

Note that the expansion of expression F1 needs more intermediate terms. The reason is the following: both expressions contain one symbol and two functions, in other words one commuting and two noncommuting objects. The rule that is used by FORM is that binomial expansion is applicable when at most one noncommuting object is involved. Therefore, expression F1 must be expanded by brute force, which generates some extra intermediate terms. In expression F2, the noncommuting functions are grouped into a single noncommuting object. Hence, the third power can be expanded first by the binomial formula of Newton into a sum of four terms. Hereafter the powers of B+C are worked out. You may wonder why FORM does not automatically place brackets, but this would violate the principle of not touching the input if not necessary.

4. Check whether the following is a valid FORM program.

```
s,t,u; * these are the symbols to be used.
L,F =
(t+u)
^2;    * this is the local expression to be manipulated.
Print; .end
```

The following FORM session shows that there are no difficulties.

```
    FORM version 3.-(Nov 29 1997). Run at: Wed Jan  7 13:00:59 1998
    s,t,u; * these are the symbols to be used.
    L,F =
    (t+u)
    ^2;    * this is the local expression to be manipulated.
    Print; .end

Time =       0.01 sec    Generated terms =          3
                F        Terms in output =          3
                         Bytes used      =         52

   F =
      2*t*u + t^2 + u^2;
```

4.2 Types of Variables

1. Check how FORM handles the summation convention for the following expressions.

   [i] $a_{ij}x_j$

   [ii] $a_{ii}x_j$

   [iii] $a_{ij}x_iy_j$

   [iv] $\delta_{ij}x_ix_j$

   In the FORM session below you see examples of the Einstein summation convention and of the SCHOONSCHIP notation.

```
    Vectors x,y;
    Tensor a;
    Indices i,j;
    Local F1 = a(i,j)*x(j);
    Local F2 = a(i,i)*x(j);
    Local F3 = a(i,j)*x(i)*y(j);
    Local F4 = d_(i,j)*x(i)*x(j);
    Print;
    .end

   F1 =
      a(i,x);

   F2 =
      a(i,i)*x(j);

   F3 =
      a(x,y);

   F4 =
      x.x;
```

2. Demonstrate with FORM the following equalities.

   [i] $a_{ij}x_iy_j = a_{ji}x_jy_i$

   [ii] $(a_{ij} + a_{ji})x_ix_j = 2a_{ij}x_ix_j$

We show that the difference of left- and right-hand side of the equations are equal to zero. Automatic contraction will do the work.

```
 Vectors x,y;
 Tensor a;
 Indices i,j;
 Local F1 = a(i,j)*x(i)*y(j) - a(j,i)*x(j)*y(i);
 Local F2 = (a(i,j) + a(j,i))*x(i)*x(j) - 2*a(i,j)*x(i)*x(j);
 Print;
 .end

F1 = 0;

F2 = 0;
```

3. Let a be an antisymmetric tensor of rank two. Demonstrate with FORM the following two properties.
   [i] $a_{ij}x_i x_j = 0$ for any vector x.
   [ii] the tensor b of rank two defined by the contraction $b_{ij} = a_{ik}a_{kj}$ is symmetric.

The first property is immediately clear in FORM when you declare the tensor a to be antisymmetric. The second property is verified by showing that $b_{ij} - b_{ji} = 0$.

```
 Tensor A(antisymmetric);
 Vectors x;
 Indices i,j,k;
 Local F1 = A(i,j)*x(i)*x(j);
 Local F2 = A(i,k)*A(k,j) - A(j,k)*A(k,i);
 Print;
 .end

F1 = 0;

F2 = 0;
```

4. There are three ways to control the printing of powers of functions:

```
FunPowers nofunpowers;
FunPowers commutingonly;
FunPowers allfunpowers;
```

Find out by experimentation what the statements actually do and check also how they affect the printing of powers of tensors.

First a FORM session that shows everything:

```
 Function f;
 Commuting g;
 Tensor t;
 Local expr = f*f*g*g*t*t;
 Print;
 .sort;

expr =
   f*f*g^2*t*t;
```

```
 FunPowers nofunpowers;
 Print;
 .sort

expr =
   f*f*g*g*t*t;

 FunPowers commutingonly;
 Print;
 .sort

expr =
   f*f*g^2*t*t;

 FunPowers allfunpowers;
 Print;
 .end

expr =
   f^2*g^2*t*t;
```

Conclusions:

- Tensors are never printed with powers.
- With `nofunpowers`, products of same functions are never printed with powers.
- With `commutingonly`, products of same commuting functions are printed with powers and products of same noncommuting functions are not. This is the default behavior of FORM.
- With `allfunpowers`, all products of same functions, regardless of there type, are printed with powers.

4.3 Some FORM Examples

1. Compose in FORM the expression $\sum_{i,j=0}^{3} a_{ij}x^i y^j$.

```
 Symbols x,y,i,j;
 CFunction a;
 Local F=sum_(i,0,2,x^i*sum_(j,0,2,a(i,j)*y^j));
 Print;
 .end

F =
   a(0,0) + a(0,1)*y + a(0,2)*y^2 + a(1,0)*x + a(1,1)*x*y + a(1,2)*x*y^2 +
   a(2,0)*x^2 + a(2,1)*x^2*y + a(2,2)*x^2*y^2;
```

2. FORM contains a second summation function called `sump_`. It works like the regular function `sum_`, except that the last argument is not the nth element of the sum, but the quotient of the nth element and the $(n-1)$th element. The first element of the sum is normalized to one. So, `sump_(i,0,10,x)` evaluates to the series expansion of $\frac{1}{1-x}$ up to order ten.

   Use the function `sump_` to compose the expression $\sum_{i,j=0}^{3} \frac{x^i}{i!}\frac{y^j}{j!}$, and write it as a polynomial in x.

```
Symbols x,y,i,j;
Local F=sump_(i,0,3,x/i)*sump_(j,0,3,y/j);
Bracket x;
Print;
.end
```

```
F =
    + x * ( 1 + y + 1/2*y^2 + 1/6*y^3 )

    + x^2 * ( 1/2 + 1/2*y + 1/4*y^2 + 1/12*y^3 )

    + x^3 * ( 1/6 + 1/6*y + 1/12*y^2 + 1/36*y^3 )

    + 1 + y + 1/2*y^2 + 1/6*y^3;
```

3. Compose in FORM the expression $\sum_{i=0}^{10}(x+1)^i$, but throw away all powers of degree 4 and higher.

   In the declaration of a symbol you may already restrict the powers that will appear in the result.

```
Symbol i, x(:3);
Local F = sum_(i,0,10,(x+1)^i);
Print;
.end
```

```
F =
   11 + 55*x + 165*x^2 + 330*x^3;
```

4. Consider the four-dimensional space-time with coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Suppose you have a coordinate transformation $(x_0, x_1, x_2, x_3) = (-ct, x, y, z)$. Show with FORM that $x_\mu x^\mu = -c^2t^2 + x^2 + y^2 + z^2$.

```
Vectors p,P;
Indices mu=0;
Symbols c,t,x,y,z;
Local F = p(mu)*P(mu);
sum mu,0,1,2,3;
id p(0) = c*t;
id p(1) = x;
id p(2) = y;
id p(3) = z;
id P(0) = -c*t;
id P(1) = x;
id P(2) = y;
id P(3) = z;
Print;
.end
```

```
F =
    - c^2*t^2 + x^2 + y^2 + z^2;
```

5. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be vectors in $\mathbf{R}^3$. Show with FORM the following equations known in vector analysis.

   [i] $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})^2$ (identity of Lagrange).

[ii] $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} - \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$.

[iii] $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$ (identity of Jacobi).

[iv] $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{C} \cdot \mathbf{D})\mathbf{B} - (\mathbf{B} \times \mathbf{C} \cdot \mathbf{D})\mathbf{A}$.

[v] $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B}) = 2\mathbf{A} \times \mathbf{B}$.

[vi] $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) + (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{A} \times \mathbf{D}) + (\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{B} \times \mathbf{D}) = 0$.

```
Dimension 3;
Vectors A,B,C,D;
Indices i,j,k,l,m,n,p,q;
Local F1 = e_(i,j,k)*A(i)*B(j) * e_(m,n,k)*A(m)*B(n);
Local F2 = e_(i,j,k) * (e_(m,n,i)*A(m)*B(n)) * C(j) -
           e_(i,j,k) * A(i) * (e_(m,n,j)*B(m)*C(n));
Local F3 = e_(i,j,k) * A(i) * (e_(m,n,j)*B(m)*C(n)) +
           e_(i,j,k) * B(i) * (e_(m,n,j)*C(m)*A(n)) +
           e_(i,j,k) * C(i) * (e_(m,n,j)*A(m)*B(n));
Local F4 = e_(i,j,k) * (e_(m,n,i)*A(m)*B(n)) * (e_(p,q,j)*C(p)*D(q));
Local F5 = e_(i,j,k)*(A(i)-B(i))*(A(j)+B(j));
Local F6 = e_(i,j,k)*A(i)*B(j) * e_(m,n,k)*C(m)*D(n) +
           e_(i,j,k)*B(i)*C(j) * e_(m,n,k)*A(m)*D(n) +
           e_(i,j,k)*C(i)*A(j) * e_(m,n,k)*B(m)*D(n);
contract;
Print;
.end
```

```
F1 =
   A.A*B.B - A.B^2;

F2 =
    - A(k)*B.C + C(k)*A.B;

F3 = 0;

F4 =
   e_(A,C,D)*B(k) - e_(B,C,D)*A(k);

F5 =
   2*e_(A,B,k);

F6 = 0;
```

6. Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be polynomials with positive degree in x_n and with coefficients in a field K (e.g, the set of rational numbers). We write

$$\begin{aligned} f &= a_r x_n^r + a_{r-1} x_n^{r-1} + \cdots + a_1 x_n + a_0, \\ g &= b_s x_n^s + b_{s-1} x_n^{s-1} + \cdots + b_1 x_n + b_0, \end{aligned}$$

where $a_r, \ldots, a_0, b_s, \ldots, b_0$ are polynomials in $x_1, x_2, \ldots, x_{n-1}$, and $a_r \neq 0$, $b_s \neq 0$. The *Sylvester matrix* of f and g is the $(r+s) \times (r+s)$ matrix

$$\begin{pmatrix}
a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 \\
0 & a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_r & a_{r-1} & \cdots & a_1 & a_0 & 0 \\
0 & \cdots & 0 & 0 & a_r & a_{r-1} & \cdots & a_1 & a_0 \\
b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 & 0 & \cdots & 0 \\
0 & b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\
0 & & 0 & b_s & b_{s-1} & \cdots & b_1 & b_0 & 0 \\
0 & & 0 & 0 & b_s & b_{s-1} & \cdots & b_1 & b_0
\end{pmatrix}$$

where there are s lines constructed with the a_i, and r lines constructed with the b_i. The *resultant* of f and g, denoted by $\mathrm{Res}(f,g)$, or $\mathrm{Res}_{x_n}(f,g)$ if there has to be a variable x_n, is the determinant of the Sylvester matrix. The importance of the resultant lies in the following theorem.

Theorem 3 (Resultant Theorem) *Let $c_1, c_2 \ldots, c_{n-1}$ be number in the algebraic closure of the field K.*
$\mathrm{Res}_{x_n}(f,g)(c_1, c_2, \ldots, c_{n-1}) = 0$ if and only if $f(c_1, c_2, \ldots, c_{n-1}, x_n)$ and $g(c_1, c_2, \ldots, c_{n-1}, x_n)$ have a factor in common or $a_r(c_1, c_2, \ldots, c_{n-1}) = b_s(c_1, c_2, \ldots, c_{n-1}) = 0$.

(i) Use this theorem to find out when a quadratic polynomial in one variable and its derivative have common zeros.

(ii) If f is a univariate polynomial of degree n and with leading coefficient a_n, then the *discriminant* of f is equal to $(-1)^{n(n-1)/2}\mathrm{Res}(f,f')$. Use this property to compute with FORM the discriminant of a third degree univariate polynomial.

(i) Consider the polynomial $f = ax^2 + bx + c$. To find common zeros of the polynomial f and its derivate f', we need to work out the condition $\mathrm{Res}(f,f') = 0$. Note that in this case

$$\mathrm{Res}(f,f') = \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix}.$$

The FORM session looks as follows:

```
Symbols a,b,c,x;
CFunction M;
Indices i1,i2,i3;
Local R = e_(1,2,3)*e_(i1,i2,i3)*M(1,i1)*M(2,i2)*M(3,i3);
contract;
id M(1,1) = a;
id M(1,2) = b;
id M(1,3) = c;
id M(2,1) = 2*a;
id M(2,2) = b;
id M(2,3) = 0;
id M(3,1) = 0;
id M(3,2) = 2*a;
id M(3,3) = b;
Print;
.end
```

```
R =
   - a*b^2 + 4*a^2*c;
```

We get $\mathrm{Res}(f, f') = -a(b^2 - 4ac)$. Assuming that $a \neq 0$, we conclude that the discriminant of f must be equal to zero.

(ii) Consider the polynomial $f = ax^3 + bx^2 + cx + d$. To find discriminant of f, we use the formula $\mathrm{Res}(f, f') = -a\,\mathrm{discriminant}(f)$. Note that in this case

$$\mathrm{Res}(f, f') = \begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ 3a & 2b & c & 0 & 0 \\ 0 & 3a & 2b & c & 0 \\ 0 & 0 & 3a & 2b & c \end{vmatrix}.$$

The FORM session to compute the discriminant of f looks as follows:

```
Symbols a,b,c,d,x;
CFunction M;
AutoDeclare Indices i;
Local D = - e_(1,2,3,4,5)*e_(i1,i2,i3,i4,i5)*
          M(1,i1)*M(2,i2)*M(3,i3)*M(4,i4)*M(5,i5) / a;
contract;
id M(1,1) = a;
id M(1,2) = b;
id M(1,3) = c;
id M(1,4) = d;
id M(1,5) = 0;
id M(2,1) = 0;
id M(2,2) = a;
id M(2,3) = b;
id M(2,4) = c;
id M(2,5) = d;
id M(3,1) = 3*a;
id M(3,2) = 2*b;
id M(3,3) = c;
id M(3,4) = 0;
id M(3,5) = 0;
id M(4,1) = 0;
id M(4,2) = 3*a;
id M(4,3) = 2*b;
id M(4,4) = c;
id M(4,5) = 0;
id M(5,1) = 0;
id M(5,2) = 0;
id M(5,3) = 3*a;
id M(5,4) = 2*b;
id M(5,5) = c;
Print;
.end
```

```
D =
   18*a*b*c*d - 4*a*c^3 - 27*a^2*d^2 + b^2*c^2 - 4*b^3*d;
```

7. There is a different way to compute determinants: with vectors rather than with commuting functions. For vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$, we have in SCHOONSCHIP notation:

$$\epsilon_{u_1 \cdots u_n} \epsilon_{v_1 \cdots v_n} = \begin{vmatrix} u_1 \cdot v_1 & \cdots & u_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ u_n \cdot v_1 & \cdots & u_n \cdot v_n \end{vmatrix}$$

So, if you identify a matrix element M_{ij} with the dot product $u_i \cdot v_j$, then contraction of the above product of two Levi-Civita tensors yields the determinant of the matrix M. The following FORM session show how the determinant example in this chapter can be carried out by this method.

```
AutoDeclare Vectors u,v;
Symbol a,b,c,d;
Local det = e_(u1,u2)*e_(v1,v2);
contract;
id u1.v1 = a;
id u1.v2 = b;
id u2.v1 = c;
id u2.v2 = d;
Print;
.end

det =
   a*d - b*c;
```

(i) Prove that this method of computing a determinant is correct.

(ii) Experiment a bit to find out whether there is a difference in efficiency between the two method of computing determinants.

(i)

$$\begin{vmatrix} u_1 \cdot v_1 & \cdots & u_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ u_n \cdot v_1 & \cdots & u_n \cdot v_n \end{vmatrix} = \begin{vmatrix} \delta_{i_1 j_1} u_1(i_1) v_1(j_1) & \cdots & \delta_{i_1 j_n} u_1(i_1) v_n(j_n) \\ \vdots & \ddots & \vdots \\ \delta_{i_n j_1} u_n(i_n) v_1(j_1) & \cdots & \delta_{i_n j_n} u_n(i_n) v_n(j_n) \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{i_1 j_1} & \cdots & \delta_{i_1 j_n} \\ \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \cdots & \delta_{i_n j_n} \end{vmatrix} u_1(i_1) \cdots u_n(i_n) v_1(j_1) \cdots v_n(j_n)$$

$$= \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \, u_1(i_1) \cdots u_n(i_n) v_1(j_1) \cdots v_n(j_n)$$

$$= \epsilon_{u_1 \cdots u_n} \epsilon_{v_1 \cdots v_n} \, .$$

(ii)

Below you see the printouts of the computation of a general 10×10 matrix with both methods under default settings of FORM and using a Pentium 166Mhz PC with 16 MB RAM. Most statistics messages have been left out.

```
*
* Method with functions
*
AutoDeclare Indices i;
CFunction M;
On statistics;
Local detF = e_(1,...,10)*
             e_(i1,...,i10)*
             M(1,i1)*M(2,i2)*M(3,i3)*M(4,i4)*M(5,i5)*
             M(6,i6)*M(7,i7)*M(8,i8)*M(9,i9)*M(10,i10);
contract;
.end;

:
```

```
    :
    :

Time =    1208.26 sec    Generated terms =    3628800
             detF        Terms in output =    3628800
                         Bytes used      =  130839812

    *
    * Method with vectors
    *
    AutoDeclare Vectors u,v;
    On statistics;
    Local detV = e_(u1,...,u10)*
                 e_(v1,...,v10);
    contract;
    .end

    :
    :
    :

Time =     410.66 sec    Generated terms =    3628800
             detV        Terms in output =    3628800
                         Bytes used      =   88215006
```

As you see the method with commuting functions takes about 20 minutes to generate all $10! = 3{,}628{,}800$ terms in the determinant, whereas the method with vectors only takes 7 minutes. Also the memory usage of the second method is less because less sorting is needed.

8. In the following exercise you can experience the power of the delta function dd_ in graph theoretical enumeration problems.
   (i) Show that that there exist three loop-free graphs with degree sequence 4,3,2,1. Verify that each graph has multiple edges.
   (ii) Show that that there does not exist a loop-free graph without multiple edges and with degree sequence 7,5,4,3,2,1,1,1.
   (iii) Show that that there exists only one loop-free graph without multiple edges and with degree sequence
   7,4,3,3,2,1,1,1.
   (iv) Show that up to isomorphism there exists only one connected loop-free graph without multiple edges and with degree sequence 3,2,2,1,1,1.
   (v) The terms of the Gram determinant for n vectors are in one-to-one-correspondence with the graphs having degree sequence 2,2,. . .,2 (n numbers). Compare the efficiency of the computation of such Gram determinants via contraction of a square of Levi-Civita tensors with the compuation of graphs with degree sequence 2,2,. . .,2 using dd_.

(i) In the session below the three graphs with requested properties are calculated.

```
AutoDeclare Vectors v;
Local F = dd_(v1,v1,v1,v1,
              v2,v2,v2,
              v3,v3,
              v4);
id v?.v?=0;        * loop-free
```

```
 Print +s F;
 .end

F =
    + 144*v1.v2^2*v1.v3*v1.v4*v2.v3
    + 72*v1.v2^2*v1.v3^2*v2.v4
    + 48*v1.v2^3*v1.v3*v3.v4
   ;
```

(ii) The following session proves that no graph with requested properties exists.

```
 AutoDeclare Vectors v;
 Local F = dd_(v1,v1,v1,v1,v1,v1,v1,
              v2,v2,v2,v2,v2,
              v3,v3,v3,v3,
              v4,v4,v4,
              v5,v5,
              v6,v7,v8);
 id v?.v?=0;        * loop-free
 id v1?.v2?^2 = 0;  * no multiple edges
 Print F;
 .end

F = 0;
```

(iii) In the session below the one graph with requested properties are calculated.

```
 AutoDeclare Vectors v;
 Local F = dd_(v1,v1,v1,v1,v1,v1,v1,
              v2,v2,v2,v2,
              v3,v3,v3,
              v4,v4,v4,
              v5,v5,
              v6,v7,v8);
 id v?.v?=0;        * loop-free
 id v1?.v2?^2 = 0;  * no multiple edges
 format 65;
 Print F;
 .end

F =
   8709120*v1.v2*v1.v3*v1.v4*v1.v5*v1.v6*v1.v7*v1.v8*v2.v3*
   v2.v4*v2.v5*v3.v4;
```

(iv) The session below differs from the ones we already met in this exercises that identify statements are added that are necessary to guarantee connectedness of the graph. They turn out to be sufficient as well.

```
 AutoDeclare Vectors u,v,w;
 Local F = dd_(u,u,u,v,v,w,w,v1,v2,v3);
 id v?.v?=0;     * loop-free
 id u?.v?^2 = 0; * no multiple edges
 id v1.v2 = 0;   * connected
 id v1.v3 = 0;
```

```
 id v2.v3 = 0;
 Print +s F;
 .end

F =
    + 24*u.v*u.w*u.v1*v.v2*w.v3
    + 24*u.v*u.w*u.v1*v.v3*w.v2
    + 24*u.v*u.w*u.v2*v.v1*w.v3
    + 24*u.v*u.w*u.v2*v.v3*w.v1
    + 24*u.v*u.w*u.v3*v.v1*w.v2
    + 24*u.v*u.w*u.v3*v.v2*w.v1
    + 24*u.v*u.v1*u.v2*v.w*w.v3
    + 24*u.v*u.v1*u.v3*v.w*w.v2
    + 24*u.v*u.v2*u.v3*v.w*w.v1
    + 24*u.w*u.v1*u.v2*v.w*v.v3
    + 24*u.w*u.v1*u.v3*v.w*v.v2
    + 24*u.w*u.v2*u.v3*v.w*v.v1
   ;
```

It is easy to see that up to isomorphism there is only one graph. The above graphs are all isomorphic to
`u.v*u.w*u.v1*v.v2*w.v3` via a permutation of `v1`, `v2`, `v3`, in combination with a permutation of `v` and `w`.

(v) Comparison of the following FORM program would show that the use of the `dd_` function is more efficient. By the way, only the value of the preprocessor variable `'n'` has to be changed in the program listed below.

```
#define n "10";
Vectors v1,...,v`n';
Local G`n' = e_(v1,...,v`n')^2;
contract;
.end
```

versus

```
#define n "10";
Autodeclare Vector v;
Local F = dd_(<v1,v1>,...,<v`n',v`n'>);
.end
```

Chapter 5

Answers to Problems of Chapter 2

5.1 Substitution

1. The kinetic energy T is given in terms of mass m and momentum p as $T = \frac{p^2}{2m}$. Express T in terms of

   (i) the velocity v, which is related to momentum and mass by $p = mv$.

   (ii) acceleration a and time t, which are related to the velocity by $v = at$.

   Straightforward substitution does the work.

```
Symbols m,a,p,t,v;
Local T = p^2/(2*m);
id p = m*v;
Print;
.sort

T =
   1/2*m*v^2;

id v = a*t;
Print;
.end

T =
   1/2*m*a^2*t^2;
```

2. Transform in FORM the expression $x^2 + x + \frac{1}{x}$ into

   (i) $y^2 + y + \frac{1}{y}$

   (ii) $y^2 + y + \frac{1}{x}$

   (iii) $x^2 + x + y$

   In this exercise you only have to take into account differences between patterns of type `x` and `1/x`, where `x` is some symbol.

```
Symbols x,y;
Local F1 = x^2 + x + 1/x;
id x = y;
```

```
id 1/x = 1/y;
Print F1;
.sort

F1 =
   y^-1 + y + y^2;

Local F2 = x^2 + x + 1/x;
id x = y;
Print F2;
.sort

F2 =
   x^-1 + y + y^2;

Local F3 = x^2 + x + 1/x;
id 1/x = y;
Print F3;
.end

F3 =
   x + x^2 + y;
```

3. How can you replace in FORM $a + b$ by d in the expression $a + b + c$.

   The trick is to replace `a` by `d-b` instead of trying to replace `a+b` by `d`.

```
Symbols a,b,c,d;
Local F = a + b + c;
id a = d - b;
Print;
.end

F =
   c + d;
```

4. Transform the expression $(x^2 + 1)^2 + x^2 + 2$ into $y^2 + y + 1$.

   Writing the substitution as $x^2 = y - 1$ will replace all higher powers of `x`.

```
Symbols x,y;
Local F = (x^2+1)^2 + x^2 + 2;
id x^2 = y - 1;
Print;
.end

F =
   1 + y + y^2;
```

5. The basic element of the quaternions are i, j, and k, subject to the rules $ij = k$ (and cyclic permutations of i, j, and k), $ji = -ij$, $ki = -ik$, $kj = -jk$, and $i^2 = j^2 = k^2 = -1$. Write a FORM program that computes the quaternion $(u + 3i - k)^3(-1 + tj)$, where u and t are unknowns. See [Cohen et al 92] for a comparison of a FORM program and a Maple program.

   In one `repeat`-loop we shall work out products into canonical form.

```
Functions i,j,k;
Symbols t,u;
Local F = (u+3*i-k)^3*(-1+t*j);
repeat;
  id i*j = k;
  id j*k = i;
  id k*i = j;
  id j*i = -k;
  id i*k = -j;
  id k*j = -i;
  id i*i = -1;
  id j*j = -1;
  id k*k = -1;
endrepeat;
Bracket i,j,k;
Print;
.end

F =
    + i * ( 30 + 3*t*u^2 - 10*t - 9*u^2 )

    + j * (  - 30*t*u + t*u^3 )

    + k * (  - 10 + 9*t*u^2 - 30*t + 3*u^2 )

    + 30*u - u^3;
```

The Bracket statement has been added to increase readability.

6. Let D denote the differentiation operator $\frac{d}{dx}$. So, the following commutation relations hold:

$$[D, x] = 1, \quad [D, \cos x] = -\sin x, \quad [D, \sin x] = \cos x\,.$$

Write a FORM program that uses these relations to transform $D^3 x^3 \sin^2 x \cos x$ into a sum of terms with the D operator to the right, and finally replace D by 0. The remaining terms will form the third derivative of $x^3 \sin^2 x \cos x$. Check your answer by pencil and paper or via a general purpose computer algebra system like Derive, Maple, or *Mathematica*.

The trick of augmenting a formula with some extra factor, carrying out operations defined in terms of this extra factor, and finally removing this extra factor, can be applied in many cases. Computing with differential operators to calculate a derivative is just one of the examples.

```
Functions D,x,[sin(x)],[cos(x)];
FunPowers allfunpowers;
Local F = D^3*x^3*[sin(x)]^2*[cos(x)];
Print;
.sort

F =
   D^3*x^3*[sin(x)]^2*[cos(x)];

repeat;
  id D*x = x*D + 1;
  id D*[sin(x)] = [sin(x)]*D + [cos(x)];
  id D*[cos(x)] = [cos(x)]*D - [sin(x)];
```

```
endrepeat;
id D = 0;
*
* write all terms in the order of
* power of x, power of sin(x), power of cos(x).
*
repeat;
  id [sin(x)]*x = x*[sin(x)];
  id [cos(x)]*x = x*[cos(x)];
  id [cos(x)]*[sin(x)] = [sin(x)]*[cos(x)];
endrepeat;
Print;
.end
```

```
F =
   7*x^3*[sin(x)]^3 - 20*x^3*[sin(x)]*[cos(x)]^2 - 63*x^2*[sin(x)]^2*
   [cos(x)] + 18*x^2*[cos(x)]^3 - 18*x*[sin(x)]^3 + 36*x*[sin(x)]*[cos(x)]^
   2 + 6*[sin(x)]^2*[cos(x)];
```

7. Consider the vector fields $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$.
   (i) Compute the commutation relations between these vector fields and verify that they form a Lie algebra.
   (ii) Determine the center of the Lie algebra, i.e., the set of elements that commute with any other element of the Lie algebra.
   (iii) Show that the vector fields $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}, x^3\frac{d}{dx}\}$ do not form a Lie algebra.

In the FORM program below, `D` denotes the differential operator $\frac{d}{dx}$. All we need to do is repeatedly use the commutation relation $[\frac{d}{dx}, x] = 1$ to move the differential operator the right-hand side of expressions.

```
Functions x,D;
Symbols a,b,c;
FunPowers allfunpowers;
*
* (i) The requested commutation relations
*
Local [D,x*D] = D*x*D - x*D*D;
Local [D,x^2*D] = D*x^2*D - x^2*D*D;
Local [x*D,x^2*D] = x*D*x^2*D - x^2*D*x*D;
repeat;
  id D*x = x*D + 1;
endrepeat;
Print;
.sort
```

```
[D,x*D] =
   D;

[D,x^2*D] =
   2*x*D;
```

```
[x*D,x^2*D] =
   x^2*D;

 *
 * (ii) The center of the Lie algebra
 *
 Local Z = a*D + b*x*D + c*x^2*D;
 Local [Z,D] = Z*D - D*Z;
 Local [Z,x*D] = Z*x*D - x*D*Z;
 Local [Z,x^2*D] = Z*x^2*D - x^2*D*Z;
 repeat;
   id D*x = x*D + 1;
 endrepeat;
 Print [Z,D], [Z,x*D],[Z,x^2*D];
 .sort

[Z,D] =
    - 2*x*D*c - D*b;

[Z,x*D] =
    - x^2*D*c + D*a;

[Z,x^2*D] =
   x^2*D*b + 2*x*D*a;

 *
 * (iii) Proof that [x^2*D, x^3*D] = x^4*D
 *
 Local [x^2*D,x^3*D] = x^2*D*x^3*D - x^3*D*x^2*D;
 repeat;
   id D*x = x*D + 1;
 endrepeat;
 Print [x^2*D,x^3*D];
 .end

[x^2*D,x^3*D] =
   x^4*D;
```

You see that the center of the Lie algebra is equal to 0 because the system of equations that a, b, and c must satisfy has clearly only the trivial solution set. The last part of the computations shows that the given set of four vector fields in part (iii) is not closed under computing commutators and therefore cannot be a basis of a Lie algebra.

8. We consider the Lie algebra of type $\mathcal{SU}_2$ generated by the triple h,e,f, which satisfy the following commutation relations

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h.$$

The enveloping algebra has Poincaré-Birkhoff-Witt basis given by

$$h^i e^j f^k,$$

where i,j,k are nonnegative integers.

(i) Use the commutation relation to write f^3e^2h in terms of the Poincaré-Birkhoff-Witt basis.

(ii) What is the commutation relation between h and he^2f^3?

Because noncommutative algebra is the default choice in FORM, the system is very convenient for this kind of computations.

```
Functions h,e,f;
FunPowers allfunpowers;
Local F1 = f^3*e^2*h;
Local F2 = h * h*e^2*f^3 - h*e^2*f^3 * h;
repeat;
  id e*h = h*e - 2*e;
  id f*h = h*f + 2*f;
  id f*e = e*f - h;
endrepeat;
Print;
.end
```

```
F1 =
   6*h^3*f - 6*h^2*e*f^2 + 30*h^2*f + h*e^2*f^3 - 18*h*e*f^2 + 48*h*f + 2*e
   ^2*f^3 - 12*e*f^2 + 24*f;

F2 =
    - 2*h*e^2*f^3;
```

5.2 Pattern Objects

1. Given the symbols `x`, `n`, explain what the following patterns mean.

   (i) `x`

   (ii) `1/x`

   (iii) `x?`

   (iv) `x^n?`

   (v) `x?^n`

   (vi) `x?^n?`

Pattern	*Meaning*
`x`	The symbol x
`1/x`	x^{-1}
`x?`	Any symbol
`x^n?`	Any nonnegative power of x
`x?^n`	Any symbol x raised to a fixed integer power n
`x?^n?`	Any symbol to any power

2. Given the vectors `u`, `v` and the indices `i`, `j`, explain what the following patterns mean.

   (i) `u(i)`

   (ii) `u(i?)`

   (iii) `u?(i)`

   (iv) `u?(i?)`

   (v) `u(i?)*v(j?)`

   (vi) `u(i?)*v(i?)`

   (vii) `u.v?`

(viii) `u?.v?`

(ix) `v?.v?`

(x) `u?.v?^2`

Pattern	*Meaning*
`u(i)`	The vector u with index i
`u(i?)`	The vector u with any index
`u?(i)`	Any vector with index i
`u?(i?)`	Any vector with any index
`u(i?)*v(j?)`	Product of vectors u and v with any indices
`u(i?)*v(i?)`	Product of vectors u and v with equal indices
`u.v?`	Dot product of vector u with any vector
`u?.v?`	Any dot product
`v?.v?`	Dot product of any vector with itself
`u?.v?^2`	Square of any dot product

5.3 Patterns in Replacement Rules

1. How can you implement the rules for odd and even functions in FORM.

```
Symbol x,y;
Functions odd, even;
Local ODD = odd(-x);
Local EVEN = even(-x);
Local F = odd(-x-y) + even(-x-y) + odd(x-y) + odd(-x+y);
id odd(-x?) = - odd(x);
id even(-x?) = even(x);
Print;
.end

ODD =
    - odd(x);

EVEN =
   even(x);

F =
   odd( - x - y) + odd( - x + y) + odd(x - y) + even( - x - y);
```

We added the expression `F` to illustrate how rudimentary our current implementation is.

2. Implement the transformation rule $\exp x \exp y \longrightarrow \exp(x+y)$ and apply it to the expression $(\exp a + \exp b + \exp c)^3$. Simplify the result as far as possible.

A `repeat`-loop is necessary to simplify results as far as possible.

```
CFunction exp;
Symbols x,y,a,b,c;
Local F = (exp(a) + exp(b) + exp(c))^3;
id exp(x?) * exp(y?) = exp(x+y);
Print +s;
.sort;
```

```
F =
    + exp(a)*exp(2*a)
    + 3*exp(b)*exp(a + b)
    + 3*exp(b)*exp(2*a)
    + exp(b)*exp(2*b)
    + 6*exp(c)*exp(a + b)
    + 3*exp(c)*exp(a + c)
    + 3*exp(c)*exp(2*a)
    + 3*exp(c)*exp(b + c)
    + 3*exp(c)*exp(2*b)
    + exp(c)*exp(2*c)
   ;
```

```
 repeat;
   id exp(x?) * exp(y?) = exp(x+y);
 endrepeat;
 Print +s;
 .end
```

```
F =
    + 6*exp(a + b + c)
    + 3*exp(a + 2*b)
    + 3*exp(a + 2*c)
    + 3*exp(2*a + b)
    + 3*exp(2*a + c)
    + exp(3*a)
    + 3*exp(b + 2*c)
    + 3*exp(2*b + c)
    + exp(3*b)
    + exp(3*c)
   ;
```

3. Implement the simplification $\sin^2 x + \cos^2 x = 1$, and apply it to the expressions $\sin^2 a - 1$, and $\sin^3 a$.

   All you need to do is to write the substitution rule in the form of $\sin^2 x = 1 - \cos^2 x$.

```
 CFunctions sin, cos;
 Symbols a,x;
 Local F1 = sin(a)^2 - 1;
 Local F2 = sin(a)^3;
 id sin(x?)^2 = 1 - cos(x)^2;
 Bracket sin;
 Print;
 .end
```

```
F1 =
    - cos(a)^2;

F2 =
    + sin(a) * ( 1 - cos(a)^2 );
```

4. Write a FORM program that computes the Laguerre polynomials $L_n(a, x)$. Recall that these polynomials are recursively defined as

$$L_0(a, x) \quad = \quad 1,$$

$$
\begin{aligned}
L_1(a,x) &= 1+a-x,\\
L_n(a,x) &= \frac{(2n+a-1-x)}{n}L_{n-1}(a,x) - \frac{(n+a-1)}{n}L_{n-2}(a,x),
\end{aligned}
$$

for $n > 1$.

We shall give two solutions to the problem: one simple, but inefficient program, and another more complicated program that works much more efficient.

```
*
* easy, but inefficient program
*
Symbols a,x,n;
CFunction L;
On statistics;
Local Laguerre8 = L(8,a,x);
repeat;
  id L(1,a,x) = 1;
  id L(2,a,x) = 1+a-x;
  id L(n?,a,x) = (2*n+a-1-x)/n*L(n-1,a,x) - (n+a-1)/n*L(n-2,a,x);
endrepeat;
Print;
.end
```

```
Time =      11.83 sec    Generated terms =      20000
       Laguerre8       1 Terms left      =         35
                         Bytes used      =        492

Time =      19.02 sec    Generated terms =      32181
       Laguerre8       1 Terms left      =         70
                         Bytes used      =        984

Time =      19.03 sec    Generated terms =      32181
       Laguerre8         Terms in output =         36
                         Bytes used      =        508

   Laguerre8 =
      1 - 1389/140*a*x + 76529/10080*a*x^2 - 1507/672*a*x^3 +
      295/1008*a*x^4 - 17/1008*a*x^5 + 1/2880*a*x^6 + 481/140*a
       - 381/80*a^2*x + 50177/20160*a^2*x^2 - 9883/20160*a^2*x^3
       + 157/4032*a^2*x^4 - 1/960*a^2*x^5 + 349/144*a^2 - 5953/
      5040*a^3*x + 8129/20160*a^3*x^2 - 1/21*a^3*x^3 + 1/576*a^3
      *x^4 + 329/360*a^3 - 467/2880*a^4*x + 131/4032*a^4*x^2 - 1/
      576*a^4*x^3 + 115/576*a^4 - 59/5040*a^5*x + 1/960*a^5*x^2
       + 73/2880*a^5 - 1/2880*a^6*x + 1/576*a^6 + 1/20160*a^7 -
      7*x + 2441/280*x^2 - 3187/840*x^3 + 247/336*x^4 - 347/5040
      *x^5 + 61/20160*x^6 - 1/20160*x^7;
```

```
*
* less obvious, but efficient program
*
Symbols a, x, n, last, secondlast, dummy;
CFunction L;
On statistics;
Local Laguerre8 = L(2,1+a-x,1)*dummy^6;
```

```
repeat;
  id L( n?, last?, secondlast?) * dummy =
     L( n+1, (2*n+a+1-x)/(n+1)*last - (n+a)/(n+1)*secondlast, last );
endrepeat;
id L( n?, last?, secondlast? ) = last;
.end
```

```
Time =       0.29 sec    Generated terms =         36
       Laguerre8         Terms in output =         36
                         Bytes used      =        508
```

5. Write a FORM program that integrates univariate polynomials.

```
Symbols x,dx,n;
*
* Next could be any polynomial in x
*
Local P = 1+x+x^3;
*
* Tag the expression and apply differentiation rule
*
multiply dx;
id dx*x^n? = 1/(n+1)*x^(n+1);
Print;
.end
```

```
P =
   x + 1/2*x^2 + 1/4*x^4;
```

6. Consider the following Lorentz transformation in four-dimensional space-time

$$
\begin{aligned}
x' &= \gamma(x - vt), \\
y' &= y, \\
z' &= z, \\
t' &= \gamma(t - \frac{vx}{c^2}),
\end{aligned}
$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Show with FORM that

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2.$$

Below, we use uppercase characters `X`, `Y`, `Z`, and `T` to denote the new coordinates x', y', z', t'.

```
Symbols t, x, y, z, v, c, gamma, [c^2-v^2];
Local X = gamma*(x-v*t);
Local Y = y;
Local Z = z;
Local T = c*gamma*(t-v*x/c^2);
Local S = X^2 + Y^2 + Z^2 - T^2;
id gamma^2 = c^2/[c^2-v^2];
repeat;
```

```
  id c^2/[c^2-v^2] = 1 + v^2/[c^2-v^2];
endrepeat;
Print S;
.end

S =
   - t^2*c^2 + x^2 + y^2 + z^2;
```

7. Consider a space of n dimensions with coordinate functions $\phi^1, \phi^2, \ldots, \phi^n$ and metric tensor given by

$$g_{ij} = \delta_{ij} + \frac{\phi^i \phi^j}{1-(\phi^k)^2},$$

where Einstein's summation convention is used so that $\phi_k^2 = \sum_i^n (\phi^i)^2$.

(i) Verify with FORM that the inverse of the metric tensor is given by

$$g^{ij} = \delta_{ij} - \phi^i \phi^j$$

(ii) The Christoffel symbol Γ_{jkl} of the first kind is defined by

$$\Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial \phi^k} + \frac{\partial g_{lk}}{\partial \phi^j} - \frac{\partial g_{kj}}{\partial \phi^l}\right)$$

Compute this symbol with FORM.

(iii) The Christoffel symbol Γ^i_{jk} of the second kind is defined by

$$\Gamma^i_{jk} = g^{il}\Gamma_{jkl},$$

where we Einstein's summation convention is used again. Compute this symbol with FORM.

FORM is not good in working with rational expressions: you have to write the denominator as one function, say `[1-phi(n)^2]`, and then design appropriate rewrite rules for getting the job done. Another point in the program below is that for computing derivates we need to work with noncommuting objects. This is why we distinguish between commuting and noncommuting functions. Uppercase characters are used to denote commuting functions; corresponding names in lowercase refer to the corresponding noncommuting functions.

```
Functions g, ginv, del, phi, [1-phi(n)^2];
CFunctions PHI, [1-PHI(n)^2];
FunPowers allfunpowers;
Indices i,j,k,l,n;
*
* (i) metric tensor and its inverse
*
Local [g(ik)] = d_(i,k) + PHI(i)*PHI(k)/[1-PHI(n)^2];
Local [ginv(kj)] = d_(k,j) - PHI(k)*PHI(j);
Local I = [g(ik)] * [ginv(kj)];
id PHI(i?)^2 = 1 - [1-PHI(n)^2];
id 1/PHI? * PHI? = 1;
Print I;
.sort

I =
  d_(i,j);
```

```
*
* (ii) Christoffel symbols of the 1st kind
*
Local C1 = 1/2*( del(k)*g(j,l) + del(j)*g(l,k) - del(l)*g(j,k));
*
* definition of metric tensor
*
id g(i?,j?) = d_(i,j) + phi(i)*phi(j)/[1-phi(n)^2];
*
* rules for differentiation
*
repeat;
  id del(i?)*phi(j?) = d_(i,j)+ phi(j)*del(i);
endrepeat;
id del(i?)/[1-phi(n)^2] = 2*phi(i)/[1-phi(n)^2]^2;
id del(i?) = 0;
Print +s C1;
.sort;

C1 =
    - phi(j)*phi(k)*phi(l)*([1-phi(n)^2])^(-2)
    + phi(j)*phi(l)*phi(k)*([1-phi(n)^2])^(-2)
    + phi(l)/([1-phi(n)^2])*d_(j,k)
    + phi(l)*phi(k)*phi(j)*([1-phi(n)^2])^(-2)
   ;

*
* normalization:
*   bring over one denominator = (1-phi(n)^2)^2
*   first we simplify the numerator
*
id 1/[1-phi(n)^2]^2 = 1;
id 1/[1-phi(n)^2] = 1-phi(n)^2;
*
* move to commuting functions
*
id phi(i?) = PHI(i);
*
* further simplification of numerator
*
id PHI(i?)^2 = 1 - [1-PHI(n)^2];
*
* The requested formula
*
multiply 1/[1-PHI(n)^2]^2;
id PHI?^(-2) * PHI? = 1/PHI;
Print +s C1;
.sort

C1 =
    + ([1-PHI(n)^2])^(-2)*PHI(j)*PHI(k)*PHI(l)
    + 1/([1-PHI(n)^2])*PHI(l)*d_(j,k)
   ;
```

```
*
* (iii) Christoffel symbols of 2nd kind
*
Local C2 = ginv(i,l) * C1;
id ginv(i?,j?) = d_(i,j) - PHI(i)*PHI(j);
Print C2;
.sort

C2 =
  ([1-PHI(n)^2])^(-2)*PHI(i)*PHI(j)*PHI(k) - ([1-PHI(n)^2])^(-2)*PHI(i)*
  PHI(j)*PHI(k)*PHI(l)^2 + 1/([1-PHI(n)^2])*PHI(i)*d_(j,k) - 1/(
  [1-PHI(n)^2])*PHI(i)*PHI(l)^2*d_(j,k);

*
* normalization:
*   bring over one denominator = (1-PHI(n)^2)^2
*   first we simplify the numerator
*
id 1/[1-PHI(n)^2]^2 = 1;
id 1/[1-PHI(n)^2] = 1-PHI(n)^2;
*
* further simplification of numerator
*
id PHI(i?)^2 = 1 - [1-PHI(n)^2];
Print +s C2;
.sort

C2 =
   + PHI(i)*PHI(j)*PHI(k)*[1-PHI(n)^2]
   + PHI(i)*[1-PHI(n)^2]^2*d_(j,k)
  ;

*
* bringing back the denominator
*
id [1-PHI(n)^2]*[1-PHI(n)^2] = 1;
id [1-PHI(n)^2] = 1/[1-PHI(n)^2];
AntiBracket PHI;
Print +s C2;
.end

C2 =

   + d_(j,k) * (
      + PHI(i)
      )

   + 1/([1-PHI(n)^2]) * (
      + PHI(i)*PHI(j)*PHI(k)
      );
```

So, we have

$$\Gamma_{jkl} = \frac{\delta_{jk}\phi^l}{1-(\phi^n)^2} + \frac{\phi^j\phi^k\phi^l}{(1-(\phi^n)^2)^2}$$

and

$$\Gamma^i_{jk} = \delta_{jk}\phi^i + \frac{\phi^i\phi^j\phi^k}{1-(\phi^n)^2}$$

5.4 Patterns and Functions

1. Implement the simplifications $\ln(xy) \to \ln x + \ln y$ and $\ln(x^n) = n \ln x$ (if n is an integer), and apply them to the expression $\ln(abc)$ and $\ln(ab^3)$.

   We give two solutions to the problem. The first one is a very direct, but somewhat elaborate way of getting the job done.

```
    Symbols x,y,z,a,b,c,dummy,n;
    CFunction ln;
    Local F1 = ln(a*b);
    Local F2 = ln(a*b*c);
    id ln(x?*y?) = ln(x) + ln(y);
    Print;
    .sort

   F1 =
      ln(a) + ln(b);

   F2 =
      ln(a*b*c);

    argument;
      id a = dummy/b;
    endargument;
    id ln(x?*y?) = ln(x) + ln(y);
    argument;
      id dummy = a*b;
    endargument;
    id ln(x?*y?) = ln(x) + ln(y);
    Print F2;
    .sort

   F2 =
      ln(a) + ln(b) + ln(c);

    Local F3 = ln(a*b^3);
    id ln(x?)=x;
    id x?*y?^n? = x+n*y;
    id x? = ln(x);
    Print F3;

   F3 =
      ln(a) + 3*ln(b);
```

   The second solution is more delicate, but works in many more circumstances.

```
    Symbols x,y,a,b,c,n;
    Function ln;
    Local F1 = ln(a*b);
    Local F2 = ln(a*b*c);
```

```
Local F3 = ln(a*b^3);
argument ln;
  multiply left ln;
  repeat;
    id ln*x? = ln(x) + ln;
    id ln(x?)*y? = ln(x);
  endrepeat;
  id ln=0;
endargument;
id ln(x?) = x;
Print;

F1 =
  ln(a) + ln(b);

F2 =
  ln(a) + ln(b) + ln(c);

F3 =
  ln(a) + 3*ln(b);
```

2. Implement simplification rules for the determinant of matrices such that $\det(M^5)$ simplifies into $\det(M)^5$, and $\det(ABC)$ becomes $(\det A)(\det B)(\det C)$.

```
Symbol M,A,B,C,n;
CFunction det;
FunPowers allfunpowers;
Local F5 = det(M^5);
Local F3 = det(A*B*C);
id det(M?)=M;
repeat;
  id M?=det(M);
endrepeat;
Print;
.end

F5 =
  det(M)^5;

F3 =
  det(A)*det(B)*det(C);
```

Alternatively:

```
Symbol M,A,B,C,n;
CFunction det;
FunPowers allfunpowers;
Local F5 = det(M^5);
Local F3 = det(A*B*C);
id det(M?)=M;
repeat;
  id M?=det(M);
endrepeat;
Print;
```

```
.end

F5 =
   det(M)^5;

F3 =
   det(A)*det(B)*det(C);
```

3. Show with FORM that if U_i and V_i are the components of covariant vectors **U** and **V**, respectively, then $T_{ij} = U_iV_j - V_iU_j$ are the components of a covariant tensor **T** of order 2. Recall that a covariant vector U_i transforms under coordinate changes like $\overline{U}_i = \frac{\partial x^k}{\partial \overline{x}^i} U_k$, and that a covariant tensor T_{ij} of order two transforms like $\overline{T}_{ij} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j} T_{kl}$.

   In the program below `t(dx,i,up,dxbar,j,low)` denotes $\frac{\partial x^i}{\partial \overline{x}^j}$.

```
Tensors [T_kl],U,V,t;
Indices i,j,k,l,low,up,dx,dxbar;
Local [Tbar_ij] = U(i,low)*V(j,low) - V(i,low)*U(j,low);
id U?(i,low) = t(dx,k,up,dxbar,i,low) * U(k,low);
id U?(j,low) = t(dx,l,up,dxbar,j,low) * U(l,low);
id U(k,low) * V(l,low) = [T_kl] + V(k,low)*U(l,low);
Print;
.end

[Tbar_ij] =
   [T_kl]*t(dx,k,up,dxbar,i,low)*t(dx,l,up,dxbar,j,low);
```

4. If $n = 2$, write out the triple sum $c_{rst}x_ry_sz_t$ in explicit form using only replacement rules.

```
Tensor a,dummy;
Vectors x,y,z;
Indices r,s,t;
Local expr = a(r,s,t)*x(r)*y(s)*z(t);
multiply dummy();
repeat;
  id a(?p,z?)*dummy(?q)=a(?p)*dummy(1,?q)*z(1)+a(?p)*dummy(2,?q)*z(2);
endrepeat;
id a()*dummy(?p)=a(?p);
Format 65;
Print;
.end

expr =
   a(1,1,1)*x(1)*y(1)*z(1) + a(1,1,2)*x(1)*y(1)*z(2) + a(1,2,
   1)*x(1)*y(2)*z(1) + a(1,2,2)*x(1)*y(2)*z(2) + a(2,1,1)*
   x(2)*y(1)*z(1) + a(2,1,2)*x(2)*y(1)*z(2) + a(2,2,1)*x(2)*
   y(2)*z(1) + a(2,2,2)*x(2)*y(2)*z(2);
```

   Of course, in real computations you would use the `sum_` statement.

5. The `ToVector` command replaces a tensor into a product of vector components. For example, `ToVector t,v` replaces `t(m1,m2,m3)` by `v(m1)*v(m2)*v(m3)`. Use `id` statements to get the same job done.

The following program shows that the `ToVector` command is present in FORM for convenience and some efficiency only.

```
Tensor t;
Vector v;
Indices m1,m2,m3;
Local F = t(m1,m2,m3);
repeat;
  id t(m1?,?m) = v(m1)*t(?m);
endrepeat;
id t() = 1;
Print;
.end

F =
   v(m1)*v(m2)*v(m3);
```

6. In classical electromagnetic theory, the electromagnetic field tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & \dfrac{E_x}{c} & \dfrac{E_y}{c} & \dfrac{E_z}{c} \\ -\dfrac{E_x}{c} & 0 & -B_z & B_y \\ -\dfrac{E_y}{c} & B_z & 0 & -B_x \\ -\dfrac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

In other words, $F_{00} = 0$, $F_{0\nu} = \dfrac{E_\nu}{c}$ for $\nu = 1, 2, 3$, and $F_{ij} = -\epsilon_{ijk} B_k$ for $i, j, k = 1, 2, 3$, where ϵ denotes the Levi-Civita tensor.

We shall use as metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ for special relativity the one with sign convention $g_{0\nu} = \delta_{0\nu}$ and $g_{ij} = -\delta_{ij}$, for $i, j = 1, 2, 3$. Then the full contravariant form $F^{\mu\nu}$ is

$$F^{\mu\nu} = g^{\mu\rho} g^{\mu\rho} F_{\rho\sigma} = \begin{pmatrix} 0 & -\dfrac{E_x}{c} & -\dfrac{E_y}{c} & -\dfrac{E_z}{c} \\ \dfrac{E_x}{c} & 0 & -B_z & B_y \\ \dfrac{E_y}{c} & B_z & 0 & -B_x \\ \dfrac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

Write the expression $F^{\mu\nu} F_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$, which are invariant under Lorentz transformations, in terms of the electric field E and magnetic field B.

We work out the expression by expansion of the indices μ, ν, ρ, and σ into $0, i$, $0, j$, $0, k$, and $0, l$, respectively. The indices i, j, k, l run from 1 to 3. Hereafter we use the definitions of the field tensors.

```
Dimension 3;
CFunctions F(antisymmetric), Ft, L, U, eps(antisymmetric);
Vector E,B;
Symbol c;
Indices mu, nu, rho, sigma, i, j, k, l;
Local expr1 = F(U(mu),U(nu)) * F(L(mu),L(nu));
Local expr2 = eps(L(mu),L(nu),L(rho),L(sigma)) *
```

```
                F(U(mu),U(nu)) * F(U(rho),U(sigma));
 *
 * expand mu,nu,rho,sigma into 0,i and 0,j, and 0,k, and 0,l, respectively
 *
 sum mu 0,1;
 sum nu 0,2;
 sum rho 0,3;
 sum sigma 0,4;
 argument;
   id L?(1) = L(i);
   id L?(2) = L(j);
   id L?(3) = L(k);
   id L?(4) = L(l);
 endargument;
 repeat;
   id eps(?a,L(i?),?b) = eps(?a,i,?b);
 endrepeat;
 id eps(0,?a) = e_(?a);
 id eps(?a) = 0;
 Format 65;
 Print;
 .sort

expr1 =
   F(L(i),L(0))*F(U(i),U(0)) + F(L(j),L(i))*F(U(j),U(i)) + F(
   L(j),L(0))*F(U(j),U(0));

expr2 =
   F(U(i),U(0))*F(U(l),U(k))*e_(i,k,l) + F(U(j),U(i))*F(U(k),
   U(0))*e_(i,j,k) + F(U(j),U(i))*F(U(l),U(0))*e_(i,j,l) + F(
   U(j),U(0))*F(U(l),U(k))*e_(j,k,l);

 *
 * Apply definitions
 *
 id F(U(0),U(j?)) = - E(j)/c;
 id F(U(i?),U(j?)) = - e_(i,j,B);
 id F(L(0),L(j?)) =  E(j)/c;
 id F(L(i?),L(j?)) = - e_(i,j,B);
 contract;
 Print;
 .end

expr1 =
    - 2*E.E*c^-2 + 2*B.B;

expr2 =
   8*E.B*c^-1;
```

So, the invariants can be written as $2(B^2 - \dfrac{E^2}{c^2})$ and $\dfrac{8}{c}E \cdot B$.

5.5 Conditions on Wildcards and Replacements

1. Implement the rule $J(-n,z) = (-1)^n J(n,z)$, if n is a natural number. Apply your rule for $n = 3$, $n = 4$, and general n.

```
Symbols z,n;
Function J;
Local  [J(-3,z)] = J(-3,z);
Local  [J(-4,z)] = J(-4,z);
Local  [J(-n,z)] = J(-n,z);
Local  [J( 4,z)] = J( 4,z);
id J(n?neg_, ?z) = (-1)^(-n) * J(-n, ?z);
Print;
.end
```

```
[J(-3,z)] =
    - J(3,z);

[J(-4,z)] =
   J(4,z);

[J(-n,z)] =
   J( - n,z);

[J(4,z)] =
   J(4,z);
```

2. For an invertible matrix M holds the equation

$$\frac{dM^{-1}}{dt} = -M^{-1}\left(\frac{dM}{dt}\right)M^{-1}.$$

Write a FORM program that computes the derivative of M^{-3}.

```
Functions M, [dM/dt], dt;
FunPowers allfunpowers;
Local F = 1/M * 1/M * 1/M;
multiply left dt;
repeat;
  id dt*1/M = -1/M*[dM/dt]*1/M + 1/M*dt;
endrepeat;
id dt=0;
Print;
.end
```

```
F =
    - 1/(M)^3*[dM/dt]/(M) - 1/(M)^2*[dM/dt]/(M)^2 - 1/(M)*[dM/dt]/(M)^3;
```

3. Write a FORM program that computes the derivative of $x^4 \ln^2 x$.

```
Symbols x,y,n;
CFunctions log,g;
Functions [log],[1/x],f,dx;
Set commuting:log;
```

```
Set noncommuting:[log];
Set derivative:[1/x];
*
Local expr = x^4*log(x)^2;
id g?commuting?noncommuting(x) = g(x);
Multiply left dx;
repeat;
  id dx*g?noncommuting[n](x) = derivative[n](x)+g(x)*dx;
endrepeat;
id dx*x^n? = n*x^(n-1);
id [1/x](x) = 1/x;
id f?noncommuting?commuting(x) = f(x);
*
Print;
.end

expr =
   2*log(x)*x^3 + 4*log(x)^2*x^3;
```

4. Write a FORM program that computes the integrals $\int x^4 \cos x\, dx$ and $\int x^4 \sin x\, dx$.

   We compute the integrals by recursion. The `select` option in the `id` statements makes that these rules are only applied if after matching of the left-hand side of an expression no elements of the set `fromx` are left. So, the first to rules are only applied if no power of `x` comes in front of the trigonometric function.

```
Symbols x, y, [sin(x)], [cos(x)], n, dx;
Set fromx: x, [sin(x)], [cos(x)];
*
Local exprcos = x^4*[cos(x)];
Local exprsin = x^4*[sin(x)];
Multiply dx;
repeat;
  id select fromx  dx*[sin(x)] = - [cos(x)];
  id select fromx  dx*[cos(x)] = [sin(x)];
  id select fromx dx*x^n?*[sin(x)] =
      - x^n*[cos(x)] + dx*n*x^(n-1)*[cos(x)];
  id select fromx dx*x^n?*[cos(x)] =
      x^n*[sin(x)] - dx*n*x^(n-1)*[sin(x)];
endrepeat;
*
Format 60;
Print;
.end

exprcos =
    - 24*x*[cos(x)] - 12*x^2*[sin(x)] + 4*x^3*[cos(x)] + x^4*[sin(x)]
    + 24*[sin(x)];

exprsin =
    - 24*x*[sin(x)] + 12*x^2*[cos(x)] + 4*x^3*[sin(x)] - x^4*[cos(x)]
    - 24*[cos(x)];
```

5. Let $\mathbf{T} = \left(T^{ij}_{klm}\right)$ denote a tensor of order and type indicated by the indices.

Prove with FORM that $\mathbf{S} = (T_k) = \left(T^{ij}_{kij}\right)$ is a covariant vector.

The tensor $\mathbf{T} = \left(T^{ij}_{klm}\right)$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ is related to the tensor $\overline{\mathbf{T}} = \left(\overline{T}^{ij}_{klm}\right)$ in a coordinate system $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$ by the transformation equation

$$\left(\overline{T}^{ij}{}_{klm}\right) = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^k}{\partial \overline{x}^r}\frac{\partial x^l}{\partial \overline{x}^s}\frac{\partial x^m}{\partial \overline{x}^t}$$

In the following FORM program we shall denote $\dfrac{\partial \overline{x}^a}{\partial x^b}$ and $\dfrac{\partial x^a}{\partial \overline{x}^b}$ by `xbar(a,up,od,b,low)` and `x(a,up,odbar,b,low)`, respectively.

```
Tensors [T^ij_ij], T, x, xbar;
Indices i, j, k, l, m, p, q, r, s, t, low, up, od, odbar;
Indices k1,...,k5;
set orig: i, j, k;
Local [Tbar^ij_ij](k) = T(i,up,j,up,k,low,i,low,j,low);
*
* Create the transformed expression
*
id T(?a, i?orig, low, ?b) = x(t, up, odbar, i, low) * T(?a, t, low, ?b);
id T(?a, i?orig, low, ?b) = x(s, up, odbar, i, low) * T(?a, s, low, ?b);
id T(?a, i?orig, low, ?b) = x(r, up, odbar, i, low) * T(?a, r, low, ?b);
id T(?a, i?orig,  up, ?b) = xbar(i, up, od, q, low) * T(?a, q,  up, ?b);
id T(?a, i?orig,  up, ?b) = xbar(i, up, od, p, low) * T(?a, p,  up, ?b);
*
Print;
.sort

[Tbar^ij_ij](k) =
   T(p,up,q,up,r,low,s,low,t,low)*x(r,up,odbar,k,low)*x(s,up,odbar,i,low)*
   x(t,up,odbar,j,low)*xbar(i,up,od,p,low)*xbar(j,up,od,q,low);

*
* Apply transformation rules
*
repeat;
  id x(i?, up, odbar, j?, low) * xbar(j?, up, od, k?, low) = d_(i,k);
endrepeat;
id T(i?, up, j?, up, k?, low, i?, low, j?, low) = [T^ij_ij](k);
*
Print;
.end

[Tbar^ij_ij](k) =
   [T^ij_ij](r)*x(r,up,odbar,k,low);
```

You see that the quantity transforms like a covariant vector.

6. From the contravariant tensor $\mathbf{S} = (S^{ij})$ and the covariant tensor $\mathbf{T} = (T_{kl})$, both of order two, form the inner product $\mathbf{U} = (U^i_l) = (S^{ij}T_{jl})$. Show with FORM that $\mathbf{U}$ is a mixed tensor of order two.

We use the same notation as in the previous exercise.

```
Tensors U, S, T, x, xbar;
```

```
Indices i, j, k, l, m, p, q, r, s, low, up, od, odbar;
set orig: i, j, l;
Local Ubar(i,up,l,low) = S(i,up,j,up) * T(j,low,l,low);
*
* Create the transformed expression
*
id T(?a, i?orig, low, ?b) = x(s, up, odbar, i, low) * T(?a, s, low, ?b);
id T(?a, i?orig, low, ?b) = x(r, up, odbar, i, low) * T(?a, r, low, ?b);
id S(?a, i?orig, up, ?b) = xbar(i, up, od, q, low) * S(?a, q, up, ?b);
id S(?a, i?orig, up, ?b) = xbar(i, up, od, p, low) * S(?a, p, up, ?b);
*
* Apply transformation rules
*
repeat;
  id x(i?, up, odbar, j?, low) * xbar(j?, up, od, k?, low) = d_(i,k);
endrepeat;
id S(i?, up, j?, up) * T(j?, low, l?, low) = U(i, up, l, low);
Print;
.end
```

```
Ubar(i,up,l,low) =
   U(p,up,s,low)*x(s,up,odbar,l,low)*xbar(i,up,od,p,low);
```

You see that the quantity transforms like a mixed tensor of order two.

7. Carry out in FORM the following trace calculation published in [Veltman 89]: Compute

$$\mathrm{trace}(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{10}}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{10}})$$

and replace the dimension d by $d-4$. The answer should be equal to

$$-31023169536 + 38971179008d - 21328977920d^2 + 6679521280d^3 - 1320732160d^4 + 171464832d^5 - 14710080d^6 + 816960d^7 - 27840d^8 + 520d^9 - 4d^{10}\,.$$

```
Symbol d;
Dimension d;
Indices m1, ..., m10;
On statistics;
Local F1 = g_(1,m1,m2,...,m10,m1,m2,...,m10);
tracen,1;
.sort
```

```
Time =       3.51 sec    Generated terms =      20000
               F1      1 Terms left      =          9
                         Bytes used      =        138

Time =       6.97 sec    Generated terms =      40000
               F1      1 Terms left      =         18
                         Bytes used      =        276

Time =      10.44 sec    Generated terms =      60000
               F1      1 Terms left      =         28
                         Bytes used      =        430

Time =      12.59 sec    Generated terms =      72379
```

```
               F1      1 Terms left      =        37
                         Bytes used      =       568

Time =       12.59 sec   Generated terms =     72379
               F1        Terms in output =        10
                         Bytes used      =       154
        id d = d-4;
        Print;
        .end

Time =       12.62 sec   Generated terms =        65
               F1        Terms in output =        11
                         Bytes used      =       182

   F1 =
       - 31023169536 + 38971179008*d - 21328977920*d^2 + 6679521280*d^3 -
      1320732160*d^4 + 171464832*d^5 - 14710080*d^6 + 816960*d^7 - 27840*d^8
       + 520*d^9 - 4*d^10;
```

8. Repeat the following calculation in high energy physics, which is also described in the REDUCE manual: the computation of the Compton scattering cross-section as given in Bjorken and Drell Eqs. (7.72) through (7.74). Requested is the trace of

$$\frac{\alpha^2}{2}\left(\frac{k'}{k}\right)^2\left(\frac{\not{p}_f+m}{2m}\right)\left(\frac{\not{p}_f+m}{2m}\right)\left(\frac{\not{e}'\;\not{e}\;\not{k}_i}{2k\cdot p_i}+\frac{\not{e}\;\not{e}'\;\not{k}_f}{2k'\cdot p_i}\right)\left(\frac{\not{p}_i+m}{2m}\right)\left(\frac{\not{k}_i\;\not{e}\;\not{e}'}{2k\cdot p_i}+\frac{\not{k}_f\;\not{e}'\;\not{e}}{2k'\cdot p_i}\right)$$

where k_i and k_f are the four-momenta of incoming and outgoing photons, with polarization vectors e and e' and laboratory energies k and k', respectively, and where p_i and p_f are incident and final electron four-momenta. It is necessary to put the particles "on the mass shell" in the calculation:

$$k_i^2 = 0, \quad k_f^2 = 0, \quad p_i^2 = m^2, \quad p_f^2 = m^2\,.$$

For the polarization vectors hold

$$p_i\cdot e = 0, \quad p_i\cdot e' = 0, \quad k_i\cdot e = 0, \quad k_f\cdot e' = 0, \quad p_f\cdot e = -k_f\cdot e, \quad p_f\cdot e' = k_i\cdot e', \quad e^2 = -1, \quad e'^2 = -1\,.$$

Furthermore,

$$p_i\cdot p_f = m^2+k_i\cdot k_f, \quad p_i\cdot k_i = m\,k, \quad p_i\cdot k_f = m\,k', \quad p_f\cdot k_i = m\,k', \quad p_f\cdot k_f = m\,k, \quad k_i\cdot k_f = m(k-k')\,.$$

With these relations you should readily get the following Compton scattering cross-section:

$$\frac{\alpha^2}{2m^2}\left(\frac{k'}{k}\right)^2\left(\frac{k'}{2k}+\frac{k}{2k'}+2(e\cdot e')^2-1\right)$$

```
V pi, pf, ki, kf, e, ep;
S k, kp, [alpha^2/(2*m^2) * (kp/k)^2], m;
Local C = [alpha^2/(2*m^2) * (kp/k)^2] / 16
        * (g_(1,pf) + m)
        * (g_(1,ep,e,ki)/ki.pi + g_(1,e,ep,kf)/kf.pi)
        * (g_(1,pi) + m)
        * (g_(1,ki,e,ep)/ki.pi + g_(1,kf,ep,e)/kf.pi)
        ;
trace4,1;
.sort
*
```

```
repeat;
  id ki.ki = 0;
  id kf.kf = 0;
  id pi.pi = m^2;
  id pf.pf = m^2;
  id pi.e  = 0;
  id pi.ep = 0;
  id pi.pf = m^2 + ki.kf;
  id pi.ki = m*k;
  id 1/(pi.ki)=1/(m*k);
  id pi.kf = m*kp;
  id 1/(pi.kf)=1/(m*kp);
  id pf.e  = -kf.e;
  id pf.ep = ki.ep;
  id pf.ki =  m*kp;
  id pf.kf = m*k;
  id ki.e  = 0;
  id ki.kf = m*(k-kp);
  id kf.ep = 0;
  id  e.e  = -1;
  id ep.ep = -1;
endrepeat;
*
Bracket [alpha^2/(2*m^2) * (kp/k)^2];
Format 60;
Print;
.end

C =
   + [alpha^2/(2*m^2)*(kp/k)^2] * (  - 1 + 1/2*k^-1*kp
      + 1/2*k*kp^-1 + 2*e.ep^2 );
```

5.6 Limitations in Wildcarding

1. Implement the trigonometric identities

$$\begin{aligned}\sin(x+y) &= \sin x\cos y+\cos x\sin y,\\ \cos(x+y) &= \cos x\cos y+\sin x\sin y,\end{aligned}$$

and apply them to $\sin(a+b)$ and $\sin(a+b+c)$.

```
Indices mu,nu;
Vectors a,b;
CFunction sin,cos;
Local expr1 = sin(a+b);
id sin(nu?)=sin(nu)*cos(nu);
id sin?(?x)*cos?(?x)=0;
Print;
.sort

expr1 =
  sin(a)*cos(b) + sin(b)*cos(a);

Local expr2 = cos(a+b);
```

```
id cos(nu?)=1/2*cos(nu)*cos(nu)+1/2*sin(nu)*sin(nu);
id sin?(?x)*sin?(?x)=0;
id cos?(?x)*cos?(?x)=0;
Print expr2;
.end
```

```
expr2 =
   sin(a)*sin(b) + cos(a)*cos(b);
```

2. Implement the rule $\sin(2x) \to 2\sin x\cos x$, and apply it to $\sin(2a)$, $\sin(3a)$, and $\sin(4a)$.

```
CFunctions sin,cos,f;
Symbols a,b,x;
Local expr1 = sin(2*a);
Local expr2 = sin(3*a);
Local expr3 = sin(4*a);
repeat;
  id sin(2*x?)=2*sin(x)*cos(x);
  id sin(4*x?)=2*sin(2*x)*cos(2*x);
endrepeat;
Print;
.end
```

```
expr1 =
   2*sin(a)*cos(a);

expr2 =
   sin(3*a);

expr3 =
   4*sin(a)*cos(a)*cos(2*a);
```

Chapter 6

Answers to Problem of Chapter 3

6.1 Introduction

1. Let the Fibonacci polynomial $F_n(x)$ be given by $F_1(x) = 1$, $F_2(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n > 2$. Write a program that computes the Fibonacci polynomial. Can your program compute $F_{50}(x)$?

```
#procedure Fibonacci(F,n,x)
  repeat;
    id `F'(1,`x'?) = 1;
    id `F'(2,`x'?) = `x';
    id `F'(`n'?,`x'?)=`x'*`F'(`n'-1,`x') + `F'(`n'-2,`x');
  endrepeat;
#endprocedure

Symbol x,y,n;
CFunction F;
Local F3 = F(3,y);
Local F20 = F(20,y);
#call Fibonacci(F,n,y)
Print;
.end
```

```
F3 =
   1 + y^2;

F20 =
   10*y + 165*y^3 + 792*y^5 + 1716*y^7 + 2002*y^9 + 1365*y^11 + 560*y^13 +
   136*y^15 + 18*y^17 + y^19;
```

The above program computes Fibonacci polynomials, but it would take a long time to compute F_{50}. The next FORM program performs better.

```
#define MAX "50"
Symbols x;
Local F1 = 1;
Local F2 = x;
#do i = 3, `MAX'
  .sort
  drop F{`i'-2};
  Local F`i'= x*F{`i'-1} + F{`i'-2};
```

```
#enddo
print F‘MAX’;
.end

F50 =
   25*x + 2600*x^3 + 80730*x^5 + 1184040*x^7 + 10015005*x^9 + 54627300*x^11
    + 206253075*x^13 + 565722720*x^15 + 1166803110*x^17 + 1855967520*x^19
    + 2319959400*x^21 + 2310789600*x^23 + 1852482996*x^25 + 1203322288*x^27
    + 635745396*x^29 + 273438880*x^31 + 95548245*x^33 + 26978328*x^35 +
   6096454*x^37 + 1086008*x^39 + 148995*x^41 + 15180*x^43 + 1081*x^45 + 48*
   x^47 + x^49;
```

2. How would you generate in FORM the expression $\sum_{i=0}^{25}(-1)^i a_i$ and change it with identifications into $\sum_{i=0}^{25}(-1)^i b_i$?

```
#define MAX "25"
AutoDeclare Symbol a, b;
Local expr = a0 - ... + a‘MAX’;
Print;
.sort

expr =
   a0 - a1 + a2 - a3 + a4 - a5 + a6 - a7 + a8 - a9 + a10 - a11 + a12 - a13
    + a14 - a15 + a16 - a17 + a18 - a19 + a20 - a21 + a22 - a23 + a24 - a25
   ;

#do i = 0, ‘MAX’
   id a‘i’ = b‘i’;
#enddo
Print;
.end

expr =
   b0 - b1 + b2 - b3 + b4 - b5 + b6 - b7 + b8 - b9 + b10 - b11 + b12 - b13
    + b14 - b15 + b16 - b17 + b18 - b19 + b20 - b21 + b22 - b23 + b24 - b25
   ;
```

3. Write a FORM procedure that allows you to work out contracted expressions like dot products of vectors and FORM expressions of type $a(p, q)$, where a is a matrix, and p, q are vectors.

```
Set mat: ;
Set vec: ;

#procedure workout(mat,vec,dim)
  id p?‘vec’.q?‘vec’ = <p(1)*q(1)> + ... + <p(‘dim’)*q(‘dim’)>;
  id t?‘mat’(?a,p?‘vec’,?b) =
     <p(1)*t(?a,1,?b)> + ... + <p(‘dim’)*t(?a,‘dim’,?b)>;
#endprocedure

Vectors p,q,u,v,w;
Indices i,j,k;
Tensor t,a,b;
```

```
 Local F1 = u.v;
 Local F2 = u.v+u.w;
 Local F3 = a(i,j)*v(j);
 Local F4 = a(i,j)*u(i);
 Local F5 = a(i,j)*u(i)*v(j);
 Local F6 = b(i,j,k)*u(i)*v(j)*w(k);
 Print;
 .sort

F1 =
   u.v;

F2 =
   u.v + u.w;

F3 =
   a(i,v);

F4 =
   a(u,j);

F5 =
   a(u,v);

F6 =
   b(u,v,w);

 Set matrices:  a,b;
 Set vectors: u,v,w;
 repeat;
   #call workout(matrices, vectors, 2)
 endrepeat;
 Print;
 .end

F1 =
   u(1)*v(1) + u(2)*v(2);

F2 =
   u(1)*v(1) + u(1)*w(1) + u(2)*v(2) + u(2)*w(2);

F3 =
   a(i,1)*v(1) + a(i,2)*v(2);

F4 =
   a(1,j)*u(1) + a(2,j)*u(2);

F5 =
   a(1,1)*u(1)*v(1) + a(1,2)*u(1)*v(2) + a(2,1)*u(2)*v(1) + a(2,2)*u(2)*
   v(2);

F6 =
   b(1,1,1)*u(1)*v(1)*w(1) + b(1,1,2)*u(1)*v(1)*w(2) + b(1,2,1)*u(1)*v(2)*
   w(1) + b(1,2,2)*u(1)*v(2)*w(2) + b(2,1,1)*u(2)*v(1)*w(1) + b(2,1,2)*u(2)
```

```
*v(1)*w(2) + b(2,2,1)*u(2)*v(2)*w(1) + b(2,2,2)*u(2)*v(2)*w(2);
```

First we introduce two auxiliary sets, viz., `mat` and `vec`, to specify that the first two arguments of the procedure `workout` are sets. In the main program we specify the sets of matrices and vectors that we are actually going to work out.

4. Write a FORM procedure that given an integer n and a list of three variables returns the sum of all monomials that have n as total degree. E.g., `#call(monomialsum(2,x,y,z)` should return $x^2 + xy + y^2 + xz + yz + z^2$. Generalize your program to any number of variables in the sense that `#call(monomialsum(`n, m, x`)` creates the sum of all monomials in m unknowns $x1, x2, \ldots, xm$ of total degree n

```
#procedure monomialsum3(n,X,Y,Z)
  id dummy =
  #do i = 0, 'n'
     #do j = 0, {'n'-'i'}
    + 'X'^'i' * 'Y'^'j' * 'Z'^{'n'-'i'-'j'}
     #enddo
  #enddo
  ;
#endprocedure

Symbols x,y,z,dummy;
Local expr = dummy;
#call monomialsum3(2,x,y,z)
Print;
.end

expr =
   x*y + x*z + x^2 + y*z + y^2 + z^2;
```

and generally,

```
#define m "4"
Symbols X1,...,X'm';
Set indets: X1,...,X'm';
Set s: ;

#procedure monomialsum(n,s)
  #do i = 1, 'm'
     id dummy = dummy * (
       #do j = 0, 'n'
          + 's'['i']^'j'
       #enddo
     );
  #enddo
  id dummy = 1;
  repeat;
    id x?'s'?indets = x;
  endrepeat;
  if ( count(<X1,1>,...,<X'm',1>) != 'n' ) discard;
  repeat;
    id x?indets?'s' = x;
  endrepeat;
#endprocedure
```

```
Symbols w,x,y,z,dummy;
Set unknowns: w,x,y,z;
Local expr = dummy;
#call monomialsum(2,unknowns)
Print;
.sort

expr =
   w*x + w*y + w*z + w^2 + x*y + x*z + x^2 + y*z + y^2 + z^2;


AutoDeclare Symbol u;
Set UNKNOWNS: u1,...,u`m';
Local expr = dummy;
#call monomialsum(3,UNKNOWNS)
Print;
.end

expr =
   u1*u2*u3 + u1*u2*u4 + u1*u2^2 + u1*u3*u4 + u1*u3^2 + u1*u4^2 + u1^2*u2
    + u1^2*u3 + u1^2*u4 + u1^3 + u2*u3*u4 + u2*u3^2 + u2*u4^2 + u2^2*u3 +
   u2^2*u4 + u2^3 + u3*u4^2 + u3^2*u4 + u3^3 + u4^3;
```

The general program screams for explanation. First of all, we have defined a procedure with two arguments:

- The first argument `n` is used for the total degree in which we are interested.
- The second argument stands for the set of unknowns. We use a set because it allows us easy use of various names for the unknowns: we can have the indeterminates w, x, y, z as easily as the unknowns u_1, u_2, u_3, u_4.

Actually we introduce another set of unknowns, called `indets`, for use inside the procedure only. The reason for this is that the statement

```
if ( count(<`s'[1],1>,...,<`s'[`m'],1>) != `n' ) discard;,
```

which one may expect in the above FORM program, does not parse well. The error message that the argument is not a symbol, function, vector or dot product would appear. With the auxiliary set `indets`, we first replace the variables in the given argument set by the variables `X1`, `X2`, etc., then we discard all terms of wrong total degree via the command

```
if ( count(<X1,1>,...,<X`m',1>) != `n' ) discard;,
```

and finally we backsubstitute the original unknowns.

5. Write a FORM procedure that can integrate multivariate polynomials.

```
#procedure int(u,du)
  multiply `du';
  id `du'*`u'^n? = `u'^(n+1)/(n+1);
#endprocedure;

Symbols x,y,z,n,dx,dy,dz;
Local F = x^2*y^3 + x*z^2;
```

```
#call int(x|dx) * integration with respect to x
#call int(y|dy) * integration with respect to y
#call int(z|dz) * integration with respect to z
Print;
.end;
```

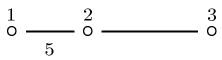
```
F =
   1/6*x^2*y*z^3 + 1/12*x^3*y^4*z;
```

6. Write a FORM procedure that can integrate integrals of types $\int x^n \cos x \, dx$ and $\int x^n \cos x \, dx$.

```
#procedure int(x,dx)
  #call tosymbols;
  multiply 'dx';
  repeat;
    id select fromx dx*[sin(x)] = - [cos(x)];
    id select fromx dx*[cos(x)] = [sin(x)];
    id select fromx dx*x^n?*[sin(x)] =
                    - x^n*[cos(x)] + dx*n*x^(n-1)*[cos(x)];
    id select fromx dx*x^n?*[cos(x)] =
                    x^n*[sin(x)] - dx*n*x^(n-1)*[sin(x)];
  endrepeat;
  #call tofunctions;
#endprocedure;

#procedure tosymbols()
  id sin(x) = [sin(x)];
  id cos(x) = [cos(x)];
#endprocedure

#procedure tofunctions()
  id [sin(x)] = sin(x);
  id [cos(x)] = cos(x);
#endprocedure


Symbols x,[sin(x)],[cos(x)],n,dx;
CFunctions sin,cos;
Set fromx:x,[sin(x)],[cos(x)];
Local exprcos = x^6*cos(x);
Local exprsin = x^7*sin(x);
#call int(x|dx);
Print;
.end
```

```
exprcos =
    - 720*sin(x) + 360*sin(x)*x^2 - 30*sin(x)*x^4 + sin(x)*x^6 + 720*cos(x)
   *x - 120*cos(x)*x^3 + 6*cos(x)*x^5;

exprsin =
    - 5040*sin(x) + 2520*sin(x)*x^2 - 210*sin(x)*x^4 + 7*sin(x)*x^6 + 5040*
   cos(x)*x - 840*cos(x)*x^3 + 42*cos(x)*x^5 - cos(x)*x^7;
```

7. The Coxeter group of type H_3 has Coxeter graph



and the following complete rewrite system

$$
\begin{array}{rcl}
s_i^2 & \to & 1 \quad \text{for } i = 1, 2, 3 \\
s_3 s_1 & \to & s_1 s_3 \\
s_2 s_1 s_2 s_1 s_2 & \to & s_1 s_2 s_1 s_2 s_1 \\
s_3 s_2 s_3 & \to & s_2 s_3 s_2 \\
s_3 s_2 s_1 s_2 s_3 s_2 & \to & s_2 s_3 s_2 s_1 s_2 s_3 \\
(s_3 s_2 s_1 s_2 s_1)^2 & \to & s_2 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2
\end{array}
$$

Write a FORM program to compute all elements of the Coxeter group and the Poincaré polynomial.

```
* define a procedure to generate reduction rules
#procedure reduce()
    id s?^2 = 1;
    id s3*s1 = s1*s3;
    id s2*s1*s2*s1*s2 = s1*s2*s1*s2*s1;
    id s3*s2*s3 = s2*s3*s2;
    id s3*s2*s1*s3 = s1*s3*s2*s1;
    id s3*s2*s1*s2*s3*s2 = s2*s3*s2*s1*s2*s3;
    id (s3*s2*s1*s2*s1)^2 = s2*s3*s2*s1*s2*s1*s3*s2*s1*s2;
#endprocedure
*
AutoDeclare Functions s;
CFunctions dummy;
Symbol x, t;
* initialize: start with trivial element
Local expr = 1;
Local lastElements = 1;
* create recursively new elements of the group
#do dummyindex = 1,1
  .sort
* generate new words with word length raised by one
  Drop lastElements;
  Skip expr;
  Local elements = lastElements * t * (s1+s2+s3);
* generate reduction rules and apply them on the newly created elements
  repeat;
    #call reduce
  endrepeat;
* remove newly created elements that have too small word length
  if (count(s1,1,s2,1,s3,1) < count(t,1));
    discard;
  endif;
  .sort (polyfun=dummy);
* make coefficients equal to 1
  Skip expr;
  id dummy(x?) = 1;
  .sort
* terminate loop if no new elements are added anymore
  #if (termsin(elements)!=0)
     Local lastElements = elements;
     Local expr = expr + elements;
     redefine dummyindex "0";
  #endif
  .sort
```

```
#enddo
* list all group elements; words of length l are tagged by the power t^l
On statistics;
Drop elements;
Bracket t;
Print +s;
.sort

Time =       0.92 sec    Generated terms =        120
             expr        Terms in output =        120
                         Bytes used      =       5380

   expr =

       + t * (
          + s1
          + s2
          + s3
          )

       + t^2 * (
          + s1*s2
          + s1*s3
          + s2*s1
          + s2*s3
          + s3*s2
          )

       + ....

       + t^14 * (
          + s1*s2*s1*s2*s1*s3*s2*s1*s2*s1*s3*s2*s1*s2
          + s1*s2*s1*s2*s3*s2*s1*s2*s1*s3*s2*s1*s2*s3
          + s2*s1*s2*s1*s3*s2*s1*s2*s1*s3*s2*s1*s2*s3
          )

       + t^15 * (
          + s1*s2*s1*s2*s1*s3*s2*s1*s2*s1*s3*s2*s1*s2*s3
          )

       + 1
         ;

* compute the Poincare polynomial
Off statistics;
id s? = 1;
Format 65;
Print;
.end

   expr =
      1 + 3*t + 5*t^2 + 7*t^3 + 9*t^4 + 11*t^5 + 12*t^6 + 12*t^7
       + 12*t^8 + 12*t^9 + 11*t^10 + 9*t^11 + 7*t^12 + 5*t^13 +
      3*t^14 + t^15;
```

The answer mimics the code of the FORM example about the Coxeter group of type A_n in this section, except that there no number n is involved and that only a small system of rewrite rules is needed. By the way, we have not listed all 120 group elements in the above printout; we manually replaced most elements by a sequence of dots.

8. Look up the complete rewrite system for the Coxeter group of type B_n.

   (i) Write a FORM program that can be used to compute all elements of the Coxeter group of type B_n and the Poincaré polynomial for $n = 2, \ldots, 5$. By the way, the Poincaré polynomial $W(t)$ is for a Coxeter group of type B_n given by

   $$W(t) = \prod_{i=1}^{n}(t^i + t^{i-1} + \ldots + t + 1) = \prod_{i=1}^{n}\frac{t^{2i}-1}{t-1}$$

   Make sure that your result is in agreement with this formula.

   (ii) Conjecture a general formula for the maximum word length in the Coxeter group of type B_n and try to find a reduced expression for the longest element.

   (iii) Determine the order of the Coxeter element $s_1 s_2 \ldots s_n$ for $n = 2, \ldots, 5$. Can you guess a general formula for the order of the Coxeter element?

Below, we only show the results for $n = 3$, but the program works for general n.

```
#define n "3"
* define a procedure to generate reduction rules
#procedure reduce()
    id s?^2 = 1;
    #do i = 3,‘n’
      #do j = 1,{‘i’-2}
        id s‘i’ * s‘j’ = s‘j’ * s‘i’;
      #enddo
    #enddo
    #do i = 2,{‘n’-1}
      #do j = 1,{‘i’-1}
        id s‘i’ * ... * s‘j’ * s‘i’ =
           s{‘i’-1} * s‘i’ * ... * s‘j’;
      #enddo
    #enddo
    #do i = 1,{‘n’-1}
        id (s‘n’ * ... * s‘i’)^2 =
           s{‘n’-1} * (s‘n’ * ... * s‘i’) * (s‘n’ * ... * s{‘i’+1});
    #enddo
#endprocedure
*
AutoDeclare Functions s;
CFunctions dummy;
Symbol x, t;
* initialize: start with trivial element
Local expr = 1;
Local lastElements = 1;
* create recursively new elements of the group
#do dummyindex = 1,1
  .sort
* generate new words with word length raised by one
  Drop lastElements;
  Skip expr;
```

```
  Local elements = lastElements * t * (s1+...+s`n');
* generate reduction rules and apply them on the newly created elements
  repeat;
    #call reduce
  endrepeat;
* remove newly created elements that have too small word length
  if (count(<s1,1>,...,<s`n',1>) < count(t,1));
    discard;
  endif;
  .sort (polyfun=dummy);
* make coefficients equal to 1
  Skip expr;
  id dummy(x?) = 1;
  .sort
* terminate loop if no new elements are added anymore
  #if (termsin(elements)!=0)
     Local lastElements = elements;
     Local expr = expr + elements;
     redefine dummyindex "0";
  #endif
  .sort
#enddo
* list all group elements; words of length l are tagged by the power t^l
On statistics;
Drop elements;
Bracket t;
Print +s;
.sort

Time =       0.37 sec    Generated terms =         48
             expr        Terms in output =         48
                         Bytes used      =       1588

   expr =

       + t * (
          + s1
          + s2
          + s3
          )

       + t^2 * (
          + s1*s2
          + s1*s3
          + s2*s1
          + s2*s3
          + s3*s2
          )

       + t^3 * (
          + s1*s2*s1
          + s1*s2*s3
          + s1*s3*s2
          + s2*s1*s3
```

```
   + s2*s3*s2
   + s3*s2*s1
   + s3*s2*s3
   )

+ t^4 * (
   + s1*s2*s1*s3
   + s1*s2*s3*s2
   + s1*s3*s2*s1
   + s1*s3*s2*s3
   + s2*s1*s3*s2
   + s2*s3*s2*s1
   + s2*s3*s2*s3
   + s3*s2*s1*s3
   )

+ t^5 * (
   + s1*s2*s1*s3*s2
   + s1*s2*s3*s2*s1
   + s1*s2*s3*s2*s3
   + s1*s3*s2*s1*s3
   + s2*s1*s3*s2*s1
   + s2*s1*s3*s2*s3
   + s2*s3*s2*s1*s3
   + s3*s2*s1*s3*s2
   )

+ t^6 * (
   + s1*s2*s1*s3*s2*s1
   + s1*s2*s1*s3*s2*s3
   + s1*s2*s3*s2*s1*s3
   + s1*s3*s2*s1*s3*s2
   + s2*s1*s3*s2*s1*s3
   + s2*s3*s2*s1*s3*s2
   + s3*s2*s1*s3*s2*s3
   )

+ t^7 * (
   + s1*s2*s1*s3*s2*s1*s3
   + s1*s2*s3*s2*s1*s3*s2
   + s1*s3*s2*s1*s3*s2*s3
   + s2*s1*s3*s2*s1*s3*s2
   + s2*s3*s2*s1*s3*s2*s3
   )

+ t^8 * (
   + s1*s2*s1*s3*s2*s1*s3*s2
   + s1*s2*s3*s2*s1*s3*s2*s3
   + s2*s1*s3*s2*s1*s3*s2*s3
   )

+ t^9 * (
   + s1*s2*s1*s3*s2*s1*s3*s2*s3
   )
```

```
    + 1
      ;

 * compute the Poincare polynomial
 Off statistics;
 Drop expr;
 id s? = 1;
 Format 65;
 Print;
 .sort
 *determine the length of the Coxeter element s1 * s2 * ... * sn
 Off statistics;
 Local coxeterElement = s1 * ... * s`n';
 Local order = 1;
 #do dummyindex = 1,1
   .sort
   Local order = order+1;
   Local coxeterElement = coxeterElement * s1 * ... * s`n';
   repeat;
     #call reduce
   endrepeat;
   if (count(<s1,1>,...,<s`n',1>) > 0);
      redefine dummyindex "0";
   endif;
   .sort
 #enddo
 Print order;
 .sort

order =
   6;

 .end
```

The results of the computations for $n = 2, 3, 4, 5$ suggest that the maximum word length in the group of type B_n is equal to n^2 and that the order of the Coxeter element $s_1 s_2 \dots s_n$ equals $2n$.

6.2 Control Structures

1. Explain the result of the following program

```
Symbols x,y;
Local F = 1 + y^2*x^4 + y^3*x^5 + y^4*x^6 + y^5*x^7;
if (count(y,-1,x,2) > 7);
   discard;
endif;
Print;
.end

Symbols x,y;
Local F = 1 + y^2*x^4 + y^3*x^5 + y^4*x^6 + y^5*x^7;
if (count(y,-1,x,2) > 7);
   discard;
```

```
endif;
Print;
.end

F =
   1 + x^4*y^2 + x^5*y^3;
```

The variables y and x have weights -1 and 2 respectively. So, the term $y^3 * x^5$ has actually degree $3 \times (-1) + 5 \times 2 = 7$ and will not be discarded.

2. Compute the sum $\sum_{i=-5}^{5} \frac{1}{i^2} x^i$ and then throw away all terms with positive exponent and those with coefficient smaller than $\frac{1}{10}$.

```
Symbols i,x;
Local SUM=sum_(i, 1, 5, 1/i^2*(x^i+x^-i));
Print;
.sort

SUM =
   1/25*x^-5 + 1/16*x^-4 + 1/9*x^-3 + 1/4*x^-2 + x^-1 + x + 1/4*x^2 + 1/9*
   x^3 + 1/16*x^4 + 1/25*x^5;

if ((count(x,1)>0) || (coefficient<1/10)) discard;
Print;
.end

SUM =
   1/9*x^-3 + 1/4*x^-2 + x^-1;
```

3. Picard's method for generating an approximate solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

is to iterate the formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi))\, d\xi$$

starting from $y_0(x) = y_0$.

(i) Write a FORM program that for a given polynomial f in x and y and a given number N computes the approximate solution $p(x)$ that agrees with the exact solution $y(x)$ up to and including degree N, i.e., $y(x) - p(x) = \mathcal{O}(x^{N+1})$. Avoid expression swell by performing computations only up to and including degree N and let the iteration only stop when two successive approximations are equal.

(ii) Compute $y_5(x)$ for the initial-value problem

$$y' = xy^2, \quad y(0) = 1.$$

(iii) Check if your program can also compute y_{20}.

(iv) Compare the result in part (iii) with the series expansion of the exact solution $y(x) = \dfrac{1}{1-x^2}$.

It suffices to show the FORM session of part (iii).

```
#define N "20"
#define f "2*x*y^2"                  * the function f in ODE

#procedure int(u,du)                 * integration routine
  multiply 'du';
  id 'du'*'u'^k? = 'u'^(k+1)/(k+1);
#endprocedure;

Symbols x(:'N'), y, dx, k;
Local X0 = 0;
Local Y0 = 1;
Local previous = 1;

#do dummyindex = 1,1
   .sort
   Skip Y0, X0, previous;
   Local approx = 'f';
   id y = previous;
   #call int(x,dx);                  * compute indefinite integral
   .sort;
   Skip approx, previous;
   Local c = approx;                 * compute constant
   id x = X0;
   .sort
   Drop c;
   Local approx = Y0 + approx - c; * compute next approximation
   .sort;
   Local difference = approx - previous;
   .sort;                            * compare with previous approximation
   #if (termsin(difference)!=0)
         Local previous = approx;
         redefine dummyindex "0";
   #endif
   .sort
#enddo
.sort
Print previous;
.end;
```

```
previous =
   1 + x^2 + x^4 + x^6 + x^8 + x^10 + x^12 + x^14 + x^16 + x^18 + x^20;
```

This is in perfect agreement with the series approximation of the exact solution $y(x) = \dfrac{1}{1-x^2}$. You can easily verify that the program works fine for other intial value problems as well.

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