1 Getting started

When computing a crosssection or the decay of a particle the calculation is usually split in three pieces:

1. Overal factors
2. Phase space integrals
3. Matrix element

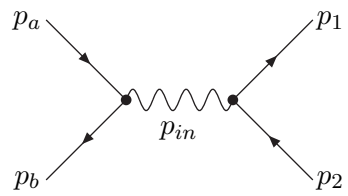
The overal factors concern things like normalization, a flux factor, conversion to the proper units etc. The phase space integrals represent the summation over all possible final states. In rare cases they can be done analytically. We will use the more general approach in which they are done numerically by a Monte Carlo program. Generic Monte Carlo integration routines like the one we use (VEGAS) usually integrate over a hypercube of the proper number of dimensions. In our case we have a Lorenz invariant space which is definitely not a cube and hence the transformation between the two is the object of the second part. It should be done in such a way that the Monte Carlo integration can obtain accurate results. In this step we also incorporate experimental acceptance in the form of kinematical cuts (like the energy of a photon has to be more than a given minimum). The integrant is called the matrix element. It represents the probability density for each final state. One way to obtain it is to write down all Feynman diagrams for the given reaction and apply a set of algebraic operations to them. These operations will be performed with the symbolic manipulation program FORM and the output will be a formula in the language C which can be integrated over.

In perturbative field theory we expand in terms of the Planck constant. It turns out that this is equivalent to an expansion in the number of closed loops in the Feynman diagrams for a given reaction. A diagram without loops we call a tree graph. Calculating loop graphs is a science by itself. Even the one-loop graphs have not been solved in all generality although there are no fundamental problems remaining. Much research goes nowadays into the evaluation of two loop graphs. It is expected that many of those will be needed for an accurate analysis of the LHC data. In the case of more loops only special cases can be solved currently. Yet one can at times obtain very interesting results with these limited means. During the limited time for this course however we will concentrate on tree level diagrams. If you practise well with the homework, you should be capable to calculate tree level reactions for which there are not too many Feynman diagrams. The special techniques for reactions with very many diagrams are not part of this course.

2 Overal factors

Let us start with the first two steps. This means that we want to take a reaction and its matrix element should already be in a form we can put inside the C program that will integrate over it. At a later stage we will look at how we can

obtain such a matrix element. The reaction we will start with is $e^-e^+ \to \mu^-\mu^+$. In the region where only QED is relevant we have a single Feynman diagram:



The notation is as we use it in the program. The matrix element connected to this reaction is

$$\begin{aligned} |\mathcal{M}|^2 &= 8 + 16\frac{t^2}{s^2} + 16\frac{t}{s} + 32\frac{m_e^2 m_\mu^2}{s^2} \\ &\quad -32\frac{m_\mu^2 t}{s^2} + 16\frac{m_\mu^4}{s^2} - 32\frac{m_e^2 t}{s^2} + 16\frac{m_e^4}{s^2} \end{aligned} \tag{1}$$

in which

$$\begin{aligned} s &= (p_a + p_b)^2 \\ t &= (p_a - p_1)^2 \end{aligned}$$

For the moment we will not be concerned with how we obtain this formula. We will concentrate on what to do with it. Once we know what to do with such matrix elements we will have a look at how to generate them.

The complete formula for the crosssection of the above reaction is:

$$\begin{aligned} \sigma &= c^2 e^4 \frac{1}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \frac{1}{4} \int \frac{d^4 p_1}{(2\pi)^4} (2\pi)\delta(p_1^2 - m_1^2)\theta(E_1) \frac{d^4 p_2}{(2\pi)^4}(2\pi) \\ &\quad \delta(p_2^2 - m_2^2)\theta(E_2)(2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - p_2)|\mathcal{M}|^2 \end{aligned} \tag{2}$$

Everything before the integral sign we define as overal constants. They are in this case:

$$\begin{aligned} c &= 19.7327 GeV\ 10^{-15} cm \\ &= 19732.7 GeV\ 10^{-18} cm \\ e^2 &= 4\pi\alpha \\ \lambda(x, y, z) &= x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \end{aligned}$$

The conversion factor c^2 converts GeV^{-2} into microbarn or $10^{-30}\mathrm{cm}^2$ when we use the first value. With the second value it converts into picobarn. Usually we put the overal coupling constants in front as well. For the value of α we have to be careful. At zero energy it is about 1/137.0368, but if we use larger energies we have to take the running of the coupling constants into account. The formula we use is

$$\frac{\alpha_0}{\alpha} = 1 - \frac{\alpha_0}{3\pi} \log\left(\frac{s}{m_e^2 e^{5/3}}\right) \tag{3}$$

The why and how of this formula should be explained in a more formal course. Around the mass of the Z (91.19 GeV) the value of α has already increased to roughly 1/128.5. The lambda function comes from the so-called flux factor which has to do with the relative speed of the colliding objects. When the masses involved are negligible the complete function just becomes s^2.

It should be noted that sometimes there is a discussion in the literature as to which value should be substituted for the quantity s (also called the scale). This is mainly the case when there are subreactions that correspond to only one type of interaction and that have a significantly smaller value for their total invariant. Such problems would get resolved if we could calculate to an infinite number of loops. Here we ignore that problem. We just want you to be aware of it.

Usually we put also an extra factor in front of the integral. When we calculate the matrix element we tradionally sum over the spins of the external particles. The incoming particles however have a definite spin and hence we should actually average over this. This means that for fermions and massless vector particles we have to divide by two. Because we have two incoming fermions it means that we have to divide by 4. Other people may put this factor with the matrix element. One should remember that factors like this are easily forgotten.

If we would have other quantum numbers like color, we would have the same problem: to obtain the matrix element we sum over the colors, but for the incoming particles we actually have to average over the colors. This means that for incoming quarks we have to divide by three (apart from the factor two for the spin) and for incoming gluons we have to divide by 8 (apart from the factor two for its spin).

Another factor that goes here is what is called the symmetry factor. If there are n identical particles in the final state one needs a factor $1/n!$ at this spot.

Important advise: before you present numbers always check that you have the overal constants correct. You would not be the first one to get the difficult part of the calculation right and then miss a simple overal factor. Always think of what your opinion would be if you see someone else do that.

3 Some kinematics

The more interesting part is the phase space integral. As with all difficult integrals we will have to transform it to something that we know how to do. The phase space integral represents the summation over all possible final states. This means the integration over all 4-momenta, coupled with the physical conditions that the particles are on-shell and that their energy is positive. First we rewrite

$$\int \frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2 - m_i^2)\theta(E_i) \quad = \quad \int \frac{1}{(2\pi)^3}\frac{d^3p_i}{2E_i} \tag{4}$$

The next rewriting for a two particles phase space is very important:

$$\int \frac{1}{(2\pi)^3}\frac{d^3p_1}{2E_1}\frac{1}{(2\pi)^3}\frac{d^3p_2}{2E_2}(2\pi)^4\delta^{(4)}(P - p_1 - p_2) \quad =$$

$$\frac{1}{16\pi^2}\frac{p^{CM}}{E^{CM}}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \qquad (5)$$

in which ϕ and θ are to be taken in the Center of Mass frame (CM). E^{CM} is the total energy in the CM frame and p^{CM} is the size of the three momentum of each of the outgoing particles. Hence we write in short

$$\int \frac{1}{(2\pi)^3}\frac{d^3p_1}{2E_1}\frac{1}{(2\pi)^3}\frac{d^3p_2}{2E_2}(2\pi)^4\delta^{(4)}(P-p_1-p_2) \quad = \quad \frac{1}{16\pi^2}\frac{p^{CM}}{E^{CM}}\int d\Omega^{CM} \qquad (6)$$

Homework: Suppose we are in the CM frame and the square of the total energy is called s. The masses of the two outgoing particles are m_1 and m_2. Derive the formulas for their energies and their three-momenta.

Homework: derive this equation.

Homework: One can also rewrite this integral as an integral over the variable t, as defined in equation 2, and ϕ^{CM}. Please try to do this. Can you determine the limits of the t-integral? Can you see some potential problems when $s >> m_\mu^2, m_e^2$? *Hint: assume that p_a is along the z-axis.*

One nice feature of reactions without polarization is that they have usually an axial symmetry. This means that the matrix element doesn't depend on ϕ^{CM}. As a consequence the integral over this ϕ variable is trivial and we are left with:

$$\int \frac{1}{(2\pi)^3}\frac{d^3p_1}{2E_1}\frac{1}{(2\pi)^3}\frac{d^3p_2}{2E_2}(2\pi)^4\delta^{(4)}(P-p_1-p_2) \quad = \quad \frac{1}{8\pi}\frac{p^{CM}}{E^{CM}}\int_{-1}^{1} d\cos\theta^{CM} \qquad (7)$$

In the case of the reaction $e^-e^+ \rightarrow \mu^-\mu^+$ that we looked at before the matrix element is in terms of angles in the CM frame:

$$|\mathcal{M}|^2 \quad = \quad 4(1+\cos^2\theta) + 16\sin^2\theta(\frac{m_e^2}{s}+\frac{m_\mu^2}{s}) + 64\frac{m_e^2 m_\mu^2}{s^2}\cos^2\theta \qquad (8)$$

and we see indeed that there is no dependence on ϕ.

Homework: Try to derive this formula, starting at formula 1. The angle θ is the angle between the incoming electron p_a and the outgoing negative muon (p_1) in the CM frame.

Homework: Try to show what kind of formula we get if we were to do this in a frame in which the electron would be at rest and the positron would come in with very much energy and hit the electron. You don't have to do the full derivation of the formula. Just show what is needed to derive it and what kind of objects will be in the answer. Or in other words: just show how messy the answer becomes without approximations.

In the case of three particles things become a bit more complicated. In the CM-frame one can transform the integral into a five fold integral over two energies and three angles. In the case that we have the decay of an unpolarized

particle into three particles, there is a complete rotational symmetry and the angles can be integrated over. In that case we have

$$\int \frac{1}{(2\pi)^3}\frac{d^3p_1}{2E_1}\frac{1}{(2\pi)^3}\frac{d^3p_2}{2E_2}\frac{1}{(2\pi)^3}\frac{d^3p_3}{2E_3}(2\pi)^4\delta^{(4)}(P-p_1-p_2-p_3) \quad = \quad \frac{1}{32\pi^3}\int dE_1\ dE_2 \qquad (9)$$

in which the boundaries of the integrals are determined by the traditional kinematical restrictions. This is the famous Dalitz integral. Note that in these variables there are no extra energy dependent factors. We say that the phase space is completely flat.

Homework: Try to find out what a Dalitz plot is (use either your field theory book/particles physics book or Google). What is it used for?

Just for trying (if you want to understand things really well): How would you derive the above formula? Can you determine the boundaries on the integrals? *Hint:* Start with the integral

$$(\prod_{i=1}^{3}\int \frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2-m_i^2)\theta(E_i))(2\pi)^4\delta^{(4)}(P-p_1-p_2-p_3) \qquad (10)$$

and insert

$$1 \quad = \quad \int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^{(4)}(q-p_2-p_3)\theta(E_q)\int \frac{dW^2}{2\pi}(2\pi)\delta(q^2-W^2) \qquad (11)$$

in which W is the combined mass of the fictitious combination of the particles 2 and 3. This decomposes the system into two two-body decays $P \to p_1 + q$ and $q \to p_2 + p_3$, and an integral over the mass squared of the intermediate state. This last integral can be translated to an integral over E_1. Most angle integrals are trivial and the $\cos\theta$ integral in the CM frame of the $2-3$ system can be translated into the E_2 integral with a bit of work.

In the case of the two body phase space we have usually only an integral over the single variable $\cos\theta$. When we compute total crosssections this integral can usually be done analytically, leading to relatively simple results. For the Dalitz integral things are already different. The boundaries of the innermost integral contain square roots and only in special cases the second integral can be done analytically in terms of simple functions. When the extra angles are involved because we study either the decay of polarized particles or the collision of two particles, things definitely become too complicated and we either resort to severe simplifications or to numerical techniques.