Two Photon Processes at Very High Energies

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Formulae are presented by which a numerically stable computer program for two photon processes above $\sqrt{s} = 100$ GeV can be constructed. For this purpose a new method for evaluating matrix elements is introduced. It is then shown that two photon processes should be no background to Drell-Yan signals in pp collisions. A comparison is made between two photon physics at e^+e^-, ep and pp collisions.

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Introduction

The first events at the $p\bar{p}$ collider[1] have started a new era in experimental high energy physics. A new generation of machines like the $p\bar{p}$ collider, LEP, HERA, the Fermilab collider, ISABELLE and the SPC will open a whole new range of energies to investigation. At most of these machines 2 photon events will be copiously present even though they may not be the primary target of attention. Whether these events are judged a signal of interest or a potential background it will be necessary to have a computer program with which one can study their theoretical properties and a Monte Carlo event generator to be able to compare theory with experiment.

For e^+e^- collisions much work has been done already[2,3] and also a number of papers[4,5,6] exist concerning $p\bar{(p)}$ collisions. Nevertheless as of yet no good event generator exists for ep or $p\bar{(p)}$ 2 photon reactions while the generator for the reaction $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$[7] suffers occasionally from numerical instabilities. Even though these instabilities occur at the moment only in unobservable corners of phase space, higher centre of mass energies could change this. It is therefore the aim of this paper to remedy this situation by presenting the formulae that are necessary for a numerically stable computer program that can handle 2γ reactions at any energy for any set of incoming particles. Using such a program then, the observable cross sections are calculated for various machines and energy ranges. This is done for the 2γ reactions $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$, $ep \rightarrow e\mu^+\mu^-X$ and $p\bar{(p)} \rightarrow \mu^+\mu^-X$, assuming a muon acceptance that might be called typical for the machines involved. The muons can be considered as a reasonable test case for most of 2γ physics as high $p_\perp$ hadron physics in 2γ collisions can also be treated as the production of 2 pointlike fermions, be it with different charges and masses. Experimental information has become available, recently, that confirms this[8].

It should be noted that in many cases the two photon diagrams are not the only ones that contribute to a specific final state[7]. Usually these extra diagrams contribute only on the level of 1% of the two photon diagrams or in an easily recognizable separate corner of phase space. In the case of small angle tagging experiments their contribution stays limited to less than 10% but in the case of double tagging at large angles these extra diagrams may even become dominant[9].

The calculation of these extra diagrams presents however fewer technical difficulties than the calculation of the multiperipheral two photon diagrams so they will not be considered in this paper.

The outline of the paper is as follows. The kinematics of general two photon reactions is described in section II and the appendices A and B. It is also explained in this section how one should make the extension to specific two photon reactions. Special emphasis is placed on the strategy concerning the numerical integration and the possibility of event generation. Section III shows a new method to formulate matrix elements when gauge cancellations can cause severe numerical problems. The result is a numerically stable formula for the matrix element of the two photon production of a pair of fermions in which the beam particles can have arbitrary structure functions. In section IV it is shown what kind of two photon signals can roughly be expected at the new generation of e^+e^-, ep or pp colliders. The whole is finished with some conclusions in section V.

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II The Kinematics

The main feature of 2γ processes is their multiperipheral structure in which there are two photons in the t-channel. This is responsible for the large cross sections and also for the very strong gauge cancellations in the matrix element. Most of the cross section is found at very small values of t for both photons. This can cause severe problems for a numerical integration program. Additional difficulties arise when one needs cross sections which are integrated over an experimental acceptance. Many mathematically oriented integration methods are excellent for integrating smooth functions but have great difficulties with the discontinuities caused by experimental cuts. The traditional solution for this problem is that used in 1γ production processes, namely to generate events according to the total cross section and only see afterwards which events survive the cuts. Such programs are called event generators and we will call this specific procedure event generation of the first kind. This method is not very practical for 2γ processes or other t-channel dominated reactions like Bhabha scattering since a typical detector only measures a fraction of the total cross section. It would be very time consuming to generate events according to the total cross section and eventually throw most of them away because they do not satisfy the experimental acceptance. A more practical method is to implement cuts during the integration or during the generation of events. There exist two main lines of thought on how this should be done.

In the first method one rewrites the phase space integral in such a way that those cuts which reduce the observable event rates most, like for instance the angle cut on an electron in a tagging experiment, can each be expressed in a single integration variable. The advantage of this approach is that one can remove most of the discontinuities by readjusting the integration boundaries. A first disadvantage is that the peaks of the matrix element now show up as a correlation between several variables so it is harder to integrate over them. It might be necessary therefore to split the phase space up into various

pieces such that each piece contains a single peak. One can then concentrate separately on each piece in order to integrate it accurately. A second disadvantage is that sometimes there are more cuts than variables so it is impossible to remove all the discontinuities. The fact that one may also need a completely different reformulation of the phase space integrals if the nature of the cuts is changed drastically can be considered a third disadvantage. Event generation according to this method we call event generation of the second kind.

The different method which is the one used here relies on a complete reformulation of the basic phase space integrals. The integral is rewritten so that the denominators of the propagators which are responsible for the severe peaks of the matrix element are used directly as integration variables. It is then rather easy to control the near divergences which are caused by the inverse photon propagators t_1 and t_2. The differential cross section behaves roughly like $t_1^{-1}t_2^{-1}$ so by changing the integrals over t_1 and t_2 into integrals over $\ln(-t_1)$ and $\ln(-t_2)$ the integrand is not so badly peaked anymore. The drawback of this method is of course that now nearly all cuts become conditions that are functions of more than one variable. For a good automatic integration routine this is usually not too hard a problem. The program used here is VEGAS [10] and the results show it to be satisfactory. The combination of VEGAS, its extension by Kawabata [11] and this reformulation of phase space yields a good event generator — of the third kind — that can compete with a dedicated generator of the second kind [11]. The great advantage of this method is its universality as one program can deal with any kind of experimental cuts.

The basic kinematics of the two photon reaction can be written as a $2 \rightarrow 3$ process like in figure 1. All two photon physics properties can be found in this system independently of the particle contents of the systems described by the 4 vectors p_3, p_4 or p_5. The treatment of the phase space integrals is according to the following decomposition formula in which we assume that p_3 contains n_3 particles, p_4 contains n_4 particles and p_5 contains n_5 particles:

$$\int \prod_{i=1}^{n_3} \left(\frac{d^3 p_i^{(3)}}{(2\pi)^3 2E_i^{(3)}} \right) \prod_{j=1}^{n_4} \left(\frac{d^3 p_j^{(4)}}{(2\pi)^3 2E_j^{(4)}} \right) \prod_{\ell=1}^{n_5} \left(\frac{d^3 p_\ell^{(5)}}{(2\pi)^3 2E_\ell^{(5)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^{n_3} p_i^{(3)} - \sum_{j=1}^{n_4} p_j^{(4)} - \sum_{\ell=1}^{n_5} p_\ell^{(5)} \right)$$

$$= \int \frac{dp_3^2}{2\pi} \int \frac{dp_4^2}{2\pi} \int \frac{dp_5^2}{2\pi} \frac{1}{(2\pi)^5} \int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \frac{d^3 p_5}{2E_5} \, \delta^{(4)} (p_1 + p_2 - p_3 - p_4 - p_5)$$

$$\times \int \prod_{i=1}^{n_3} \left(\frac{d^3 p_i^{(3)}}{(2\pi)^3 2E_i^{(3)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_3 - \sum_{i=1}^{n_3} p_i^{(3)} \right)$$

$$\times \int \prod_{j=1}^{n_4} \left(\frac{d^3 p_j^{(4)}}{(2\pi)^3 2E_j^{(4)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_4 - \sum_{j=1}^{n_4} p_j^{(4)} \right) \qquad (\text{II},1)$$

$$\times \int \prod_{\ell=1}^{n_5} \left(\frac{d^3 p_\ell^{(5)}}{(2\pi)^3 2E_\ell^{(5)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_5 - \sum_{\ell=1}^{n_5} p_\ell^{(5)} \right)$$

This formula shows that every two photon reaction can be treated kinematically as if it is a 2 to 3 reaction like in figure 1 in which the squares of the masses of the final state systems are also to be integrated over and normal n-body phase space describes the "decay" of these three systems. Normally p_3, p_4 and p_5 contain only one or two particles each so this "decay" will be rather simple. The reformulation of the essential $2 \to 3$ reaction to make it suitable for 2 photon physics is done in appendix A. If the beams carry no polarization the whole system is symmetric under rotations around the beam axis and the integration over such an azimuthal angle is trivial. Consequently the integral $\int d^3 p_3/(2E_3) \, d^3 p_4/(2E_4) \, d^3 p_5/(2E_5)$ $\delta^{(4)}(p_1 + p_2 - p_3 - p_4 - p_5)$ can be reformulated in terms of only 4 relativistic invariants among which one has to take t_1 and t_2. The choice of the other two variables, s_2 and Δ, made it possible to write the formulae of appendix A in such a form that the matrix element can be evaluated in a numerically stable way, eliminating all problems with gauge cancellations.

The expressions for the CM angles and energies in terms of the invariants of appendix A can be found in appendix B. These CM quantities are needed because the experimental cuts are usually expressed in terms of them.

The rest of the kinematics concerns the treatment of the "decay" of the final state systems described by p_3, p_4 and p_5. As an example we will consider here the system described by p_4. If it contains only one particle with mass m_4 its phase space integral is trivial:

$$\int \frac{dp_4^2}{2\pi} \int \prod_{i=1}^{1} \left(\frac{d^3 p_i^{(4)}}{(2\pi)^3 2E_i^{(4)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_4 - \sum_{i=1}^{1} p_i^{(4)} \right) = 1 \qquad (\text{II},2)$$

and particle 4 is on-shell. If there are two particles in the system described by p_4 it is easiest to use angles in the centre of mass of the two particles:

$$\int \prod_{i=1}^{2} \left(\frac{d^3 p_i^{(4)}}{(2\pi)^3 2E_i^{(4)}} \right) (2\pi)^4 \, \delta^{(4)} \left(p_4 - \sum_{i=1}^{2} p_i^{(4)} \right) = \int \frac{d^3 p_6}{2E_6} \frac{d^3 p_7}{2E_7} \frac{\delta^{(4)}(p_4 - p_6 - p_7)}{(2\pi)^2}$$

$$\times \frac{\lambda^{\frac{1}{2}}(p_4^2, m_6^2, m_7^2)}{32\pi^2 \, p_4^2} \int d\Omega^{CM} \qquad (\text{II},3)$$

in which we switched to the notation $p_6 = p_1^{(4)}$ and $p_7 = p_2^{(4)}$. The one problem with this reformulation is that it needs an additional transformation from this centre of mass to the laboratory frame. At first sight this does not seem too difficult as one can just boost this CM frame till system 4 acquires the proper energy and then rotate it into the right direction, indicated by θ_4 and ϕ_4. The angles θ_4 and ϕ_4 can be obtained from the formulae in appendix B. The difficulty arises if the orientation of the CM frame of system 4 is not random. It is for instance best to have the incoming γ's define the z-axis in the CM frame. Many interactions of the two photons will exhibit a forward or a backward peak. With the above orientation of the z-axis, all peaking is confined to the θ variable and any variation with ϕ will be due to spin structure and/or experimental cuts. Having chosen the z-axis this way one needs to rotate over the angles θ_γ and ϕ_γ before boosting and rotating over θ_4 and ϕ_4. The formulae for θ_γ and ϕ_γ can be obtained by inverting this procedure and observing the direction of the γ's after the rotations over $-\phi_4$ and $-\theta_4$ and the boost that brings system 4 to rest.

Finally the x-z plane still has to be defined. One can do this by requiring p_1 to lie in the xz plane of the CM frame of system 4. This means that there will be another rotation around the z-axis over the angle ψ. This angle can be found by studying p_1 under the rotations $-\phi_4$ and $-\theta_4$, the boost that brings system 4 at rest, and the rotations $-\phi_\gamma$, $-\theta_\gamma$. Some of the formulae that are obtained this way may not be very stable numerically. In that case it may be better to obtain quantities like $\sin\theta_1^{CM}$ directly from the Levi-Civita tensors of appendix A, the same way this is done for the lab variables in appendix B, and then use the results of the transformations only to determine the sign of $\cos\theta_1^{CM}$.

If the systems 3 and 5 are only representing a general inelastic reaction as in deep inelastic scattering the corresponding structure functions do not depend on the kinematics of the "decay". This means that the "decay" integrals can be done leaving a function of p_3^2 or p_5^2 which is absorbed into the structure functions. In such a case only the dp_3^2 or dp_5^2 integrals are left to represent the systems 3 and 5.

Using the above set of variables all severe peaks that occur in the matrix element of a two photon reaction can be controlled. The $t_1^{-1}t_2^{-1}$ behaviour by changing the integration variables t_1 and t_2 into $\ln(-t_1)$ and $\ln(-t_2)$, the forward backward peaking of for instance the subreaction $\gamma^*\gamma^* \to \mu^+\mu^-$ by changing $\cos\theta^{CM}$ into a more exotic variable and the $1/(p_4^2)^2$ behaviour of $d\sigma/dp_4^2$ by replacing dp_4^2 by $d(1/p_4^2)$. Other changes of variables are shown in appendix A. Usually the change concerning $\cos\theta^{CM}$ is not needed in the presence of experimental cuts as the observation of some of the centre particles is demanded. This means that they should have a reasonable value for their $p_\perp$ which in its turn means that θ^{CM} should not be too near the forward or backward peaks. One should also realize that experimental cuts can change the asymptotic behaviour of the cross section considerably, requiring different changes of variables. Most cuts require for example the replacement of dp_4^2 by $d\ln p_4^2$ instead of $d(1/p_4^2)$.

III The Matrix element

Normally the derivation of a matrix element is a rather straightforward exercise with Feynman rules if one does not need to worry about the practical problem of numerical evaluation. Consider for instance the reaction $e^+e^- \to e^+e^-\mu^+\mu^-$ at PETRA or LEP energies. If the matrix element were to be evaluated in the standard fashion -i.e. expressing it completely in terms of 4-vector products and then substituting their numerical values for each Monte Carlo generated point in phase space - then the cancellations between the various terms would be so bad that even the 60 bit accuracy of a CDC computer would not suffice. It is therefore necessary to use a different form for the matrix element - and thus a different derivation -, which is slightly more tedious.

To get a feeling for how this can be done in a very simple system let us calculate the matrix element of the reaction $e^+e^- \to e^+e^-\pi^0$ under the assumption that the π^0 behaves like a point particle, i.e. without formfactors. The coupling between the photons and the π^0 can then be written as $\frac{1}{2}g\pi^0 F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}g\pi^0 F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$. Adopting the notation of figure 2 the matrix element can be written as:

$$|\mathcal{M}|^2 = q^2 e^4 \left| \frac{\bar{u}(p_3)\gamma_\mu u(p_1)\epsilon^{\mu\nu\rho\sigma}q_{1\rho}q_{2\sigma}\bar{v}(p_2)\gamma_\nu v(p_4)}{q_1^2\,q_2^2} \right|^2 \quad \text{(III.1)}$$

From now on we will use the notation that if an index of a Levi-Civita tensor is contracted with the index of a 4-vector this index of the Levi-Civita tensor is replaced by the 4-vector, so $\epsilon^{\mu\nu q_1 q_2}$ stands for $\epsilon^{\mu\nu\rho\sigma}q_{1\rho}q_{2\sigma}$. The matrix element is now brought into its final form by working out the square, summing over the spins, taking the 2 traces and making the substitutions $p_3 = p_1 - q_1$ and $p_5 = p_2 - q_2$. This gives the expression

$$|\mathcal{M}|^2 = q^2 e^4 \left[-64\,\epsilon^{p_1 q_1 p_2 q_2}\epsilon_{p_1 q_1 p_2 q_2} + 16 q_1^2\,\epsilon^{p_1 q_1 q_2 \mu}\epsilon_{p_1 q_1 q_2 \mu} \right.$$
$$\left. + 16 q_2^2\,\epsilon^{p_1 q_1 q_2 \mu}\epsilon_{p_1 q_1 q_2 \mu} - 4 q_1^2 q_2^2\,\epsilon^{q_1 q_2 \mu\nu}\epsilon_{q_1 q_2 \mu\nu} \right] \Big/ (q_1^2 q_2^2)^2 \quad \text{(III.2)}$$

If the Levi-Civita tensors are contracted into 4-vector dot products this formula becomes numerically unstable, but in its Levi-Civita form all 4 terms are positive so no cancellations occur. It should be noted that the first term $\varepsilon^{p_1 q_1 p_2 q_2} \varepsilon_{p_1 q_1 p_2 q_2}$ is the Gram determinant of the system and the other 3 terms are minors of it, so all four terms have a meaning in the kinematics. When these kinematics are treated according to the method of appendix A all 4 terms in formula (III.2) will be obtained in a numerically stable fashion. The reason for the numerical stability of formula (III.2) is due to the fact that the amplitude of the subreaction $\gamma^* \gamma^* \to \pi^0$, $g\varepsilon^{e_1 q_1 e_2 q_2}$ (e_1 and e_2 are the polarization vectors of the two photons) is a single term which is manifestly gauge invariant. Formula (III.2) is derived by multiplying $\varepsilon^{\mu q_1 \nu q_2} \varepsilon_{\mu' q_1 \nu' q_2}$ with the tensors from the electron and the positron lines, so each term in (III.2) is necessarily gauge independent and there can be no more gauge cancellations.

The procedure to obtain a corresponding result for the reaction $e^+ e^- \to e^+ e^- \mu^+ \mu^-$ should consequently follow a similar path. First the amplitude for the subreaction $\gamma^* \gamma^* \to \mu^+ \mu^-$ has to be written in a manifestly gauge invariant way in which each individual term is gauge independent. Then the matrix element for this subreaction is evaluated and rewritten in a suitable form, after which the contraction with the electron and the positron gives the full matrix element. As the electron and the positron are put in last, it becomes also easy to insert structure functions and obtain the matrix elements of the reactions $p\bar{p} \to p\bar{p}\mu^+\mu^-$, $ep \to ep\mu^+\mu^-$ and their corresponding inelastic versions.

The first step of the above procedure is achieved via the identity:

$$\bar{u}(p_6)\left\{ \not{e}_1 \frac{\not{k}_1+m}{k_1^2-m^2} \not{e}_2 + \not{e}_2 \frac{\not{k}_2+m}{k_2^2-m^2} \not{e}_1 \right\} v(p_7)$$

$$= -\varepsilon^{q_1 e_1 \alpha \mu} \varepsilon_{q_2 e_2 \mu \rho} \, \bar{u}(p_6) \frac{\gamma^\rho \not{k}_1 \gamma_\alpha + \gamma_\alpha \not{k}_2 \gamma^\rho}{(k_1^2-m^2)(k_2^2-m^2)} v(p_7) \qquad (III.3)$$

using the notation of figure 3. To derive the above formula the identities $\bar{u}(p_6)(\not{p}_6 - m) = 0$ and $(\not{p}_7 + m)v(p_7) = 0$ had to be used several times,

together with some γ matrix algebra and the Schouten identity for Levi-Civita tensors. It should be noted that the right-hand side of (III.3) has a common denominator suggesting similarities between the method used here and the derivation of "super"formulae of ref. [12].

The matrix element for the subreaction $\gamma^* \gamma^* \to \mu^+ \mu^-$ is now brought into the form

$$\varepsilon^{q_1 e_1 \alpha \rho} \varepsilon^{q_2 e_2 \gamma \delta} \varepsilon^{q_1 e_1 \alpha' \rho'} \varepsilon^{q_2 e_2 \gamma' \delta'} F_{\alpha\rho\gamma\delta\,\alpha'\rho'\gamma'\delta'} \qquad (III.4)$$

Obtaining the precise form of the tensor F involves a spin sum and the taking of a trace. In this form it is rather difficult to evaluate (III.4) for a given point in phase space. An example of a term that is far from trivial is $\varepsilon^{q_1 e_1 Q\alpha} \varepsilon_{q_2 e_2 \alpha\beta} \varepsilon^{q_2 e_2 \beta\gamma} \varepsilon_{q_1 e_1 \gamma Q}$ with $Q \equiv p_6 - p_7$. To write such forms compactly, let us introduce a new notation inspired by the fact that $\varepsilon^{q_1 e_1 \alpha\beta}$ is an antisymmetric tensor with two indices. In this notation the above "difficult" term becomes $Q \bullet 1 \bullet 2 \bullet 2 \bullet 1 \bullet Q$. Terms like this one are to be reduced to sums and products of scalar quantities that are contractions between two Levi-Civita tensors only. The necessary reduction formulae can be found in appendix C. Once these reduction formulae have been applied the matrix element of the subreaction takes a rather simple form. Using also the notation $Tr[i \cdot j] = \varepsilon^{q_i e_i \alpha\beta} \varepsilon_{q_j e_j \beta\alpha}$ this matrix element is given by the formula:

$$|\mathcal{M}|^2 = \frac{1}{8}\left(8((q_1.q_2)^2 - (Q.q_1)^2)(\frac{1}{4}(Tr[1 \cdot 2])^2 - \varepsilon^{q_1 e_1 q_2 e_2} \varepsilon_{q_1 e_1 q_2 e_2})\right.$$
$$-2(Q.q_1)^2 Tr[1 \cdot 1] Tr[2 \cdot 2] - 8\left\{Q \bullet 1 \bullet 2 \bullet Q - \frac{1}{2}Tr[1 \cdot 2](Q^2 + q_1.q_2)\right\}^2$$
$$+ Tr[1 \cdot 1] (\varepsilon^{q_2 e_2 \alpha\beta} \varepsilon_{q_1 Q\alpha\beta})^2 + Tr[2 \cdot 2] (\varepsilon^{q_1 e_1 \alpha\beta} \varepsilon_{q_2 Q\alpha\beta})^2\right\} \qquad (III.5)$$
$$\times ((q_1.q_2)^2 - (Q.q_1)^2)^{-2}$$

Even though the terms in this formula are not all positive there are no problems with the resulting cancellations. As all the gauge cancellations are hidden inside the Levi-Civita tensors whatever cancellation is left is due to spin structure and therefore not serious. To illustrate this somewhat we calculate the amplitude for the reaction $\gamma^* \gamma^* \to \pi^+ \pi^-$, assuming again that the pions are point particles. Even though there is an extra seagull diagram the same notation can be used as in the $\gamma^* \gamma^* \to \mu^+ \mu^-$ reaction and the amplitude becomes

becomes

$$\mathcal{M} = \frac{-2i\left(Q\cdot 1\cdot 2\cdot Q - \frac{1}{2}Tr[1\cdot 2]\left(Q^2 + q_1 q_2\right)\right)}{\left(q_1\cdot q_2\right)^2 - \left(Q\cdot q_1\right)^2} \qquad (III.6)$$

using $k_1^2 - m^2 = -q_1\cdot q_2 - Q\cdot q_1$ and $k_2^2 - m^2 = -q_1\cdot q_2 + Q\cdot q_1$. One can see now that one of the terms of (III.5) is -2 times the matrix element for the production of two pointlike charged scalar particles. At high energies the cross section of the reaction $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$ is almost an order of magnitude smaller than the cross section of the reaction $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$[13] even if one allows for the fact that the lighter mass of the muon is to be compensated for by a factor m_π^2/m_μ^2 in the cross section, so cancellations involving this term are not severe.

The addition of the electrons (or protons) is now rather simple. The electron gives:

$$Tr\left[(\not{p}_3 + m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] = 2\left(q_1^2 g_{\mu\nu} - q_{1\mu}q_{1\nu}\right) + 2\left(2p_{1\mu} - q_{1\mu}\right)\left(2p_{1\nu} - q_{1\nu}\right) \quad (III.7)$$

using $p_3 = p_1 - q_1$. Since $p_1\cdot q_1 = \frac{1}{2}q_1^2$ this can be rewritten as:

$$2q_1^2\left(g_{\mu\nu} - \frac{q_{1\mu}q_{1\nu}}{q_1^2}\right) + 8\left(p_{1\mu} - \frac{p_1\cdot q_1}{q_1^2}q_{1\mu}\right)\left(p_{1\nu} - \frac{p_1\cdot q_1}{q_1^2}q_{1\nu}\right) \qquad (III.8)$$

This formula is symmetric in μ and ν and therefore the contraction with (III.5) gives no problems as to which e_1 should be replaced by a μ and which one by a ν. For protons we take

$$2q_1^2\left(g_{\mu\nu} - \frac{q_{1\mu}q_{1\nu}}{q_1^2}\right)W_1 + 8\left(p_{1\mu} - \frac{p_1\cdot q_1}{q_1^2}q_{1\mu}\right)\left(p_{1\nu} - \frac{p_1\cdot q_1}{q_1^2}q_{1\nu}\right)W_2 \qquad (III.9)$$

This way the normalization is such that for point particles W_1 and W_2 are both equal to one. Even though these definitions differ from the customary ones by factors $-2q_1^2$ and 8 respectively we believe that this unique way of defining W_1 and W_2 is the better one.

The full matrix element for the two photon production of a pair of spin $\frac{1}{2}$ point particles with mass m can now be written as:

$$|\mathcal{M}|^2 = \frac{W_1^{(1)}W_1^{(2)}M_{11} + W_1^{(1)}W_2^{(2)}M_{12} + W_2^{(1)}W_1^{(2)}M_{21} + W_2^{(1)}W_2^{(2)}M_{22}}{\left(q_1^2 q_2^2 \left(\left(q_1\cdot q_2\right)^2 - \left(Q\cdot q_1\right)^2\right)\right)^2} \qquad (III.10)$$

where the upper index of the structure functions refers to photon number 1 and photon number 2 respectively. The evaluation of M_{12} and M_{21} still needs some reduction formulae. Those are also present in Appendix C. After this the final expression for the M_{ij} becomes:

$$M_{11} = 64 \, q_1^2 q_2^2 \left\{ \left((q_1 \cdot q_2)^2 - (Q \cdot q_1)^2 \right) \left((Q \cdot q_1)^2 - q_1^2 q_2^2 + (q_1 \cdot q_2)^2 \right.\right.$$
$$\left.\left. - Q^2 (q_1^2 + q_2^2 + 2m^2) \right) - 2 (Q \cdot q_1)^2 (q_1^2 + 2m^2)(q_2^2 + 2m^2) \right\}$$

$$M_{12} = 128 \, q_1^2 \left\{ \left((q_1 \cdot q_2)^2 - (Q \cdot q_1)^2 \right) \left(-\epsilon^{p_2 q_1 q_1 \mu} \epsilon_{p_2 q_2 q_1 \mu} - \epsilon^{p_2 q_1 Q \mu} \epsilon_{p_2 q_2 Q \mu} \right) \right.$$
$$\left. - (q_1^2 + 2m^2) \left((Q \cdot q_1)^2 \epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} + \tfrac{1}{2} \left(\epsilon^{p_1 q_1 \mu \nu} \epsilon_{q_2 Q \mu \nu} \right)^2 \right) \right\}$$

$$M_{21} = 128 \, q_2^2 \left\{ \left((q_1 \cdot q_2)^2 - (Q \cdot q_1)^2 \right) \left(-\epsilon^{p_1 q_2 q_1 \mu} \epsilon_{p_1 q_1 q_1 \mu} - \epsilon^{p_1 q_2 Q \mu} \epsilon_{p_1 q_1 Q \mu} \right) \right.$$
$$\left. - (q_2^2 + 2m^2) \left((Q \cdot q_1)^2 \epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} + \tfrac{1}{2} \left(\epsilon^{p_1 q_1 \mu \nu} \epsilon_{q_2 Q \mu \nu} \right)^2 \right) \right\}$$

$$\text{(III.11)}$$

$$M_{22} = 512 \left\{ \left((q_1 \cdot q_2)^2 - (Q \cdot q_1)^2 \right) \left(\left(\tfrac{1}{2} \epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} \right)^2 - \epsilon^{p_1 q_1 p_2 q_2} \epsilon_{p_1 q_1 p_2 q_2} \right) \right.$$
$$- \left(\epsilon^{p_1 q_1 Q \mu} \epsilon_{p_2 q_2 Q \mu} - \tfrac{1}{2} \epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} (Q^2 + q_1 \cdot q_2) \right)^2$$
$$- \tfrac{1}{8} \epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_1 q_1 \mu \nu} \left(\epsilon^{p_2 q_2 \varrho \sigma} \epsilon_{q_1 Q \varrho \sigma} \right)^2$$
$$- \tfrac{1}{8} \epsilon^{p_2 q_2 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} \left(\epsilon^{p_1 q_1 \varrho \sigma} \epsilon_{q_2 Q \varrho \sigma} \right)^2$$
$$\left. - \tfrac{1}{4} (Q \cdot q_1)^2 \left(\epsilon^{p_1 q_1 \mu \nu} \epsilon_{p_2 q_2 \mu \nu} \right) \left(\epsilon^{p_2 q_2 \varrho \sigma} \epsilon_{p_2 q_2 \varrho \sigma} \right) \right\}$$

The Levi-Civita tensors contracted with the 4-vector Q have to be evaluated carefully as they are not contained in the kinematics of the reaction discussed in the appendices. It is however not too complicated to do so as the only relevant cancellations come from the interplay between p_1 and q_1 or between

p_2 and q_2. To evaluate for example $\epsilon^{p_1 q_1 Q \, \mu} \epsilon_{p_1 q_1 Q \, \mu}$ one could calculate, analytically, the components of the axial 4-vector $a^\mu = \epsilon^{p_1 q_1 Q \, \mu}$ in the lab frame and then compute $a^\mu a_\mu$. There are some cancellations between $(a^0)^2$ and $(a^3)^2$ and only after cancelling these terms analytically should one substitute numerical values. It is also possible to do this in the CM frame of the two colliding photons.

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IV Results

With the aid of the previous two chapters and the appendices it is possible to construct a computer program for two photon reactions that produce two fermions at the centre. Whenever a beam particle is a proton it is necessary to specify its structure functions. Most of the cross section will be at small values for the invariant mass of the hadron system so it is necessary to use structure functions that are good in the resonance region. In principle it is possible to use the data from ep and μp scattering to get good structure functions in the whole Q^2 and invariant mass range that is relevant for the two photon processes under consideration. This would go far beyond the scope of this paper. To get a rough impression of the inelastic results we decided to use the structure functions from the paper by Suri and Yennie[14] which are a reasonable fit to the measured structure functions in the resonance region. For the elastic case the usual dipole form factors were used.

As a first check the total cross section of the reaction $e^+e^- \to e^+e^-\mu^+\mu^-$ was calculated at a CM energy of 30 GeV because this number is known to be between 119 nb and 120 nb from previous programs[7]. The current result is 119.555 $\pm$ 0.019 nb. This accuracy which took about 5 million Monte Carlo points is of course absolutely useless from a physics point of view. The integrand is however a good test case for numerical integration programs. For the rest we consider total cross sections irrelevant for most cases. The cross sections that are relevant are total observable cross sections within a given acceptance.

Let us therefore consider the 2γ production of a muon pair in e^+e^-, ep and pp collisions with the condition that the muon pair should be observable. The detector should have a geometry that is typical for LEP, HERA or ISABELLE. A muon was defined observable when its angle with respect to the beam axis satisfies the condition $|\cos\Theta| < 0.95$. Additionally the muon has to have a perpendicular momentum of at least 1 GeV/c when $|\cos\Theta| < 0.75$ or a longitudinal momentum of at least 1 GeV/c when $0.75 < |\cos\Theta| < 0.95$. This last condition comes from the fact that the muon shield has two endcaps,

thus forming a big cylinder with two holes for the beam. The most significant difference between this "standard" detector and UA1 at the $p\bar{p}$ collider is the momentum that the muon needs to traverse the shield. In the case of the UA1 experiment at least 3 GeV/c is needed in the direction that is perpendicular to the shield. The observable cross sections for the "standard" detector can be found in table 1.

It is possible to completely eliminate 2γ processes as a background to Drell-Yan events if one considers the very steep t dependence of the cross section which indicates that elastic protons will almost never be visible and the inelastic systems also go forward. The separation criterion is that Drell-Yan events are accompanied by hadrons along both beam directions - the remnants of fragmentation -while in the case of 2γ processes it is a very great exception to see hadrons outside both beam pipes simultaneously. Of the 67.5 pb in the elastic-elastic channel for pp collisions about 0.4 pb has one proton come out at an angle larger than 3.7 mrad (this corresponds to $Q^2 = 1$ GeV^2). The cross section for both of them to have $Q^2 > 1$ GeV^2 is much less than 10^{-38} cm^2. Of the 66.8 pb in the elastic-inelastic case only 3.10^{-37} cm^2 has the elastic proton come out at more than 3.7 mrad. This leaves at best the inelastic-inelastic events as a potential background for Drell-Yan processes and most of those will have hadronic systems with a small invariant mass in the resonance region. When such a resonance decays its decay products will have a relatively small $p_\perp$ and a lot of energy - the energy of the resonance is about 270 GeV - so again only a small percentage will be visible. The total cross section of two photon events accompanied by hadrons outside both beam pipes with an angle of at least 10 mrad with respect to the beam directions is estimated to be less than 1 pb. A better estimate needs a Monte Carlo of the inelastic hadronic system. More knowledge about the inelastic hadronic system will also lead to a better selection criterion reducing the possible background even further. This method of separating the 2γ events from the Drell-Yan events is better than the use of a cut in the value of the perpendicular momentum Q_T of the dimuon pair[6]. There will be a number of Drell-Yan events at small Q_T and the experimental resolution will give some 2γ events at larger Q_T. Therefore the experimental resolution will make such a cut less efficient.

Of course the above numbers are very sensitive to the specific expression that

is used for the structure functions. In due time it will be necessary to have a reliable set of structure functions that is specifically fitted to the data for the use in two photon physics. A comparison between the results here and those of ref. [4] which uses the parametrization of ref. [15] shows the current calculation to give a significantly larger inelastic cross section.

It should also be noted that 2γ events were discovered at the ISR[16] via the observation of an excess of dimuon events with no accompanying hadrons. Finally the total observable cross section for dimuon events coming from two photon collisions is only about 7 pb if the muons need at least 3 GeV to traverse the shield. This shows that two photon physics should cause absolutely no problems for the $p\bar{p}$ collider experiments.

Dimuon production by two photons can also give a rough idea of the hadron signal that can be expected in 2γ collisions. For estimating the number of 2 jet events in two photon collisions one needs the value of $R_{\gamma\gamma}$ and $d\sigma/dM_{\mu\mu}$. This $d\sigma/dM_{\mu\mu}$ is given in figure 4, while the value of $R_{\gamma\gamma}$ is 34/27 in the region where 4 flavours are relevant and 35/27 if five flavours contribute. From figure 4 it is clear that HERA will need a significantly better luminosity than LEP in order to be better at two photon physics in the mass range of 10-20 GeV/c^2 for the 2γ system. For ISABELLE to compete with LEP it will be necessary to realize a luminosity that is an order of magnitude better than that of LEP.

The rather strange looking dip in the dimuon mass distribution at 2 GeV/c^2 is due to the cuts. When the $p_\perp$ cut is relevant ($|\cos\theta_\mu| < 0.75$) there are almost no events below 2 GeV/c^2, but when the cut is on $p_\parallel$ there is a completely separate signal that peaks at a smaller value of the dimuon mass. If a muon has $\cos\theta = 0.95$ and $p_\parallel = 1$ GeV/c its $p_\perp$ is slightly larger than 300 MeV/c so one would expect this second peak to be above 600 MeV/c^2. The figure shows it slightly above 1 GeV/c^2 in the elastic channels while it is partially washed out in the inelastic channels.

The total cross section with two muons and one electron observed is another signal of interest. For this it was assumed that an electron or positron is observable if it has at least a 20 mrad angle with respect to the beam direction. The cross sections are presented in table 2. It should be mentioned

that these signals will actually be larger due to the other diagrams that contribute to the reactions $e^+e^- \to e^+e^-\mu^+\mu^-$ or $ep \to e\mu^+\mu^-X$. In the case of double tagging there are some cuts that even make them the dominant diagrams, but, as mentioned in the introduction, we will not consider them in this paper.

The most remarkable feature of table 2 is the fact that the event rate at HERA for tagging experiments will not be better than at LEP unless the luminosity at HERA is two times higher. The main disadvantage of HERA is due to the fact that at LEP one can tag on both sides while the HERA advantage of the smaller electron energy (this means a smaller Q^2 for equal angles) is not quite enough to make up for it.

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V   Conclusions

A working program has been constructed that can calculate two photon processes for the accelerators of the near future. At equal luminosities LEP is the better two photon machine but if the luminosities at HERA and ISABELLE will be significantly higher this situation might be reversed. A careful study of the hadron system in the inelastic channels will eliminate two photon processes completely as a background to Drell-Yan signals. Very simple criteria like calling an event a Drell-Yan event if there are hadrons at an angle of more than 10 mrad near both beam pipes should already reduce the two photon signals in pp collisions at $\sqrt{s} = 540$ GeV by two orders of magnitude.

To conclude, the author would like to thank J. Smith of SUNY Stony Brook and W. van Neerven for stimulating discussions.

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Appendix A

The nucleus of the kinematics as used for the two photon calculations in this paper consists of a reformulation of the two to three body phase space such that the integration variables are relativistic invariants[17]. One of these invariants is then replaced by a slightly more exotic variable

$\Delta \equiv p_1 \cdot p_2 \; q_1 \cdot q_2 - p_1 \cdot q_2 \; p_2 \cdot q_1$ according to the notation of figure 1. This gives then for the phase space integral:

$$\int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \frac{d^3 p_5}{2E_5} \; \delta^{(4)}(p_1 + p_2 - p_3 - p_4 - p_5) =$$

$$= \frac{\pi}{16 \lambda^{1/2}(s, m_1^2, m_2^2)} \int \frac{ds_1 \, ds_2 \, dt_1 \, dt_2}{\sqrt{-\Delta_4(p_1, p_2, p_3, p_4)}} \tag{A1}$$

$$= \frac{\pi}{4 \lambda^{1/2}(s, m_1^2, m_2^2)} \int \frac{d\Delta \, ds_2 \, dt_1 \, dt_2}{(s_2 - t_1 - m_2^2)\sqrt{-\Delta_4(p_1, p_2, p_3, p_4)}} \tag{A2}$$

with Δ_4 being the Gram determinant of the system. This Gram determinant can be written as:

$$\Delta_4 = \epsilon^{p_1 p_2 p_3 p_4} \epsilon_{p_1 p_2 p_3 p_4}$$

$$= \epsilon^{p_1 q_1 p_2 q_2} \epsilon_{p_1 q_1 p_2 q_2} \tag{A3}$$

When Δ_4 is written in terms of Δ, s_2, t_1, t_2 it will yield the integration boundaries once the condition $-\Delta_4 > 0$ is imposed. These boundaries are derived below for three different orders of integration:

(i) $\int d\Delta \, ds_2 \, dt_2 \, dt_1$

(ii) $\int d\Delta \, dt_2 \, ds_2 \, dt_1$

(iii) $\int d\Delta \, dt_2 \, dt_1 \, ds_2$

The first of these three orders of integration requires most care while the other two can almost be considered as special cases of it. Therefore the main effort will concern the derivation of the first one while the modifications that are needed for the others will be mentioned at the appropriate places. It should be mentioned that many of the formulae presented here can also be found in ref. [18], however we repeat them for completeness. The treatment here differs from ref. [7] in the use of the variable Δ and the emphasis on numerical stability of the formulae.

For a given value σ of s_2 the minimum and maximum of t_1 are:

$$t_1^{\max} = m_1^2 + m_3^2 - 2\left(E_1^{CM} E_3^{CM} + p_1^{CM} p_3^{CM}\right)$$

$$= m_1^2 + m_3^2 - \left\{(s + m_1^2 - m_2^2)(s + m_3^2 - \sigma) \right.$$

$$\left. + \lambda^{1/2}(s, m_1^2, m_2^2)\, \lambda^{1/2}(s, m_3^2, \sigma)\right\}/_{2s} \tag{A4}$$

$$t_1^{\min} \, t_1^{\max} = (m_3^2 - m_1^2)(\sigma - m_2^2) + \left\{(m_3^2 - m_1^2) - (\sigma - m_2^2)\right\}$$

$$\times \left\{ m_2^2(m_3^2 - m_1^2) - m_1^2(\sigma - m_2^2)\right\}/_s \tag{A5}$$

where max and min refer to absolute values. We will use the same notation for the boundaries on t_2. In case (iii) σ is a value of s_2 between its minimum

$(m_4 + m_5)^2$ and its maximum $(\sqrt{s} - m_3)^2$ while in the other cases one substitutes for σ only the minimum value $(m_4 + m_5)^2$. For numerical reasons it is advisable to keep $\delta_1 = m_3^2 - m_1^2$ as one unit, since if $m_1 = m_3$ δ_1 becomes exactly zero and the numerical accuracy of an expression like $m_3^2 - m_1^2 - \sigma + m_2^2$ will not be affected, even if m_1 and m_3 are individually much larger than m_2 or σ.

In numerical integration one can now begin by choosing a value for t_1 between $t_1^{\max}$ and $t_1^{\min}$. Then the boundaries for s_2 can be found by inverting (A4) substituting t_1 for $t_1^{\max}$ and s_2 for σ. This yields:

$$s_2^{\pm} = \left\{ s(t_1 - \delta_1) + \delta_2(t_1 - m_1^2 - m_3^2) + 2m_1^2 m_3^2 \right.$$

$$\left. \pm \lambda^{1/2}(s, m_1^2, m_2^2)\, \lambda^{1/2}(t_1, m_1^2, m_3^2)\right\}/_{2m_1^2} \tag{A6}$$

$$s_2^+ s_2^- = \left\{ s(t_1(s + t_1 - m_1^2 - m_2^2 - m_3^2) - m_2^2 \delta_1) \right.$$

$$\left. + m_3^2(\delta_2 \delta_3 + m_1^2 m_2^2)\right\}/_{m_1^2} \tag{A7}$$

with $\delta_2 = m_1^2 - m_2^2$ and $\delta_3 = t_1 - m_2^2$. In the case that $m_1 = 0$ there is only one root (see also (A10)):

$$s_2^+ = \frac{s(t_1(s + t_1 - m_2^2 - m_3^2) - m_2^2 m_3^2) + m_2^2 m_3^2(m_2^2 + m_3^2 - t_1)}{(s - m_2^2)(t_1 - m_3^2)} \tag{A8}$$

The real boundaries of s_2 are now (for order (ii)):

$$\max\left((m_4 + m_5)^2, s_2^-\right) < s_2 < s_2^+ \tag{A9}$$

Equation (A7) is again used to provide a numerically stable evaluation of one limit $\left(\text{like in (A5)}\right)$. The stable root of (A6) is usually s_2^- which is obvious in the case that m_1 is small and t_1 negative. In the limit that $m_1 \to 0$ ($t_1 < 0$) s_2^- goes to $-\infty$ and s_2^+ becomes the difference of two very big terms, so it is best to calculate s_2^+ from (A7). Even when the order of integration is given by (i) or (iii) the values of $s_2^\pm$ are needed because they facilitate a stable evaluation of the quantity D_1 which is defined by:

$$D_1 = \tfrac{1}{4} m_1^2 (s_2^+ - s_2)(s_2^- - s_2) \qquad (m_1 \neq 0)$$

$$D_1 = -\tfrac{1}{4}(s - m_1^2)(t_1 - m_3^2)(s_2 - s_2^+) \qquad (m_1 = 0)$$

(A10)

D_1 can also be expressed in terms of Levi-Civita tensors as

$$D_1 = \epsilon^{p_1 p_2 q_1 \mu} \; \epsilon_{p_1 p_2 q_1 \mu}$$

(A11)

In this form it is easy to recognize that D_1 is a minor of the Gram determinant, and we can exploit this fact during the evaluation of the matrix element (see section III).

For a given value σ' of s_2 the extrema of t_2 can now also be found:

$$t_2^{max} = m_1^2 + m_3^2 - \tfrac{1}{2}(\sigma' - t_1 + m_1^2)(\sigma' - m_4^2 + m_5^2)$$

$$+ \lambda^{1/2}(\sigma', t_1, m_1^2)\, \lambda^{1/2}(\sigma', m_4^2, m_5^2)\Big\}\big/_{2\sigma'}$$

(A12)

$$t_2^{min} t_2^{max} = \delta_4 \delta_5 + \frac{(\delta_4 - \delta_5)(\delta_4 t_1 - \delta_5 m_1^2)}{\sigma'}$$

(A13)

with $\delta_4 = m_5^2 - m_2^2$ and $\delta_5 = m_4^2 - t_1$. In case (i) σ' is taken to be s_2^+ while in cases (ii) and (iii) the correct value of s_2 is substituted. New boundaries for s_2 can be obtained by inverting equation (A12) after replacing t_2^{max} by the value of t_2 that has been taken between t_2^{max} and t_2^{min} and σ' by s_2:

$$s_2'^{\pm} = \Big\{ -(m_4^2 - t_1 - t_2)(\delta_4 - t_2) + 2t_2(t_1 + m_1^2)$$

$$\pm \lambda^{1/2}(m_4^2, t_1, t_2)\, \lambda^{1/2}(t_2, m_1^2, m_5^2)\Big\}\big/_{2t_2}$$

(A14)

$$s_2'^- s_2'^+ = \delta_3 \delta_6 + \frac{(\delta_3 - \delta_6)(\delta_3 m_5^2 - \delta_6 m_1^2)}{t_2}$$

(A15)

with $\delta_6 = m_4^2 - m_5^2$. In a computer program one should check which root is the stable one that can be evaluated using (A14). The boundaries of s_2 are now such that

$$s_2'^- < s_2 < s_2^+$$

(A16)

The limits $s_2'^\pm$ are needed for a stable evaluation of the quantity D_2 since:

$$D_2 = \tfrac{1}{4} t_2 (s_2 - s_2'^+)(s_2 - s_2'^-)$$

(A17)

$$\bullet \; \epsilon^{p_2 q_1 q_1 \mu} \epsilon_{p_2 q_2 q_1 \mu} \tag{A18}$$

The boundaries of Δ are obtained directly from the Gram determinant Δ_4, since Δ_4 can be rewritten as a quadratic polynomial in Δ:

$$\Delta_4 = a \Delta^2 + b \Delta + c \tag{A19}$$

with:

$$a = \frac{(p_2 . q_1)^2 - m_2^2 t_1}{(p_2 . q_1)^2} \tag{A20}$$

and:

$$b = \frac{2(p_1 . q_1 \; p_2 . q_1 - p_1 . p_2 \; t_1)(p_1 . q_1 \; p_2 . q_1 - q_1 . q_2 \; m_2^2)}{(p_2 . q_1)^2} \tag{A21}$$

The expression for c is more complicated but not really needed here as it is possible to rewrite the discriminant of equation (A19):

$$D = b^2 - 4ac = \frac{4 D_1 D_2}{(p_2 . q_1)^2} \tag{A22}$$

The 4-vector product $p_2 . q_1$ can be expressed in terms of the invariants via the relation $p_2 . q_1 = \frac{1}{2}(s_2 - t_1 - m_2^2)$. The boundaries of Δ are now:

$$-\frac{b}{2a} - \frac{\sqrt{D}}{2a} < \Delta < -\frac{b}{2a} + \frac{\sqrt{D}}{2a} \tag{A23}$$

and the Gram determinant can be written as $-\Delta_4 = a(\Delta^+ - \Delta)(\Delta - \Delta^-)$. To remove the singularities due to $(-\Delta_4)^{-\frac{1}{2}}$ one extra change of variables is called for:

$$\Delta = -\frac{b}{2a} + \cos(\pi x) \frac{\sqrt{D}}{2a} \tag{A24}$$

where $0 \leq x \leq 1$. The final form of the phase space integral now becomes:

$$\int\int\int\int_0^1 \frac{dx \; ds_2 \; dt_2 \; dt_1 \quad \pi^2}{4 \; \lambda^{1/2}(s, m_1^2, m_2^2) \; \lambda^{1/2}(s_2, t_1, m_2^2)} \tag{A25}$$

The Gram determinant, whose value is still needed in the rest of the kinematics (see appendix B) and often in the matrix element is now given by:

$$\Delta_4 = \frac{-4 \sin^2(\pi x) \; D_1 D_2}{\lambda(s_2, t_1, m_2^2)} \tag{A26}$$

Other quantities that can be useful are the counterparts of the minors D_1 and D_2:
$D_3 = \varepsilon^{p_1 p_2 q_2 \mu} \varepsilon_{p_1 p_2 q_2 \mu}$ and $D_4 = \varepsilon^{p_1 q_1 q_2 \mu} \varepsilon_{p_1 q_1 q_2 \mu}$. Also
$D_5 = \varepsilon^{p_1 p_2 p_4 \mu} \varepsilon_{p_1 p_2 p_4 \mu}$ can be of use. The evaluation of D_3 and D_4 is similar to
(A6) - (A11), (A14) - (A17) respectively. It is only necessary to make the
exchanges $m_1 \leftrightarrow m_2$, $m_3 \leftrightarrow m_5$, $s_1 \leftrightarrow s_2$ and $t_1 \leftrightarrow t_2$. Of course s_1^+ and
$s_1^{'-}$ are not the real boundaries of s_1 because the order of integration has not
been changed. As a consequence the expressions for D_3 and D_4 are more stable
than one would expect naively. For most applications there are only "random"
instabilities, i.e. for 10^4 Monte Carlo points roughly one point would have the
property that $(s_1^+ - s_1)/s_1$ would be of order 10^{-4} which means a loss of
accuracy of 4 digits. To improve this situation requires much work as one needs
to express D_3 and D_4 in terms of the four quasi random variables that come from
the Monte Carlo integrator, and then rewrite all coefficients in such a way that
the whole formula becomes numerically stable. After this the formula is not
necessarily so compact.

The evaluation of D_5 can be done by substituting $p_4 = q_1 + q_2$ which gives:

$$D_5 = D_1 + D_3 + 2 \varepsilon^{p_1 p_2 q_1 \mu} \varepsilon_{p_1 p_2 q_2 \mu} \qquad (A27)$$

The final Levi-Civita term can be rewritten:

$$\varepsilon^{p_1 p_2 q_1 \mu} \varepsilon_{p_1 p_2 q_2 \mu} = \{ - \Delta(p_1 \cdot p_2 \, p_2 \cdot q_1 - m_1^2 \, p_1 \cdot q_1)$$
$$+ (p_1 \cdot p_2 \, p_2 \cdot q_1 - m_1^2 \, p_2 \cdot q_1)(p_2 \cdot q_2 \, p_2 \cdot q_1 - m_2^2 \, q_1 \cdot q_2)\} / p_2 \cdot q_1 \qquad (A28)$$

Normally when there is a t-channel structure like t_1 and t_2 there are no strong
cancellations in (A27). If however such cancellations do occur it is still possible
to use the triangle relation

$$\lambda(D_1, D_3, D_5) = 4 \Delta_4 \, s \, p_1^2 \qquad (A29)$$

to define D_5. A stable evaluation of (A29) involves $(D_1 - D_3)^2$. If this quantity
is persistently unstable it is necessary to write D_3 as a function of x and then
$D_1 - D_3$ as a function of s_2 and its boundaries. This can then be made stable but
the results are messy. Often the best solution is to go back to the beginning and
relabel the external momenta.

Finally the factor $\lambda^{-\frac{1}{2}}(s_2, t_1, m_2^2)$ in (A25) can be made to disappear by one
more change of variables. If a function $f(y)$ has to be "mapped" away while
integrating over y one calculates $F(x) = \int_{y^-}^x f(y) dy$ and defines the new variable
u by:

$$u = \frac{F(x)}{F(y^+)} \qquad (A30)$$

in which y^+ and y^- are the upper and lower boundaries of the integration over y.
This change of variable can be done analytically if (A30) can be inverted to give
x as a function of u. When $f(y)$ is given by $\lambda^{-\frac{1}{2}}(y, a, b)$ this can be done leading
to the relations:

$$y = a + b + \frac{1}{2} \left\{ (y^- - a - b + \alpha^-) \left(\frac{y^+ - a - b + \alpha^+}{y^- - a - b + \alpha^-} \right)^u \right.$$
$$\left. + \frac{4ab}{y^- - a - b + \alpha^-} \left(\frac{y^- - a - b + \alpha^-}{y^+ - a - b + \alpha^+} \right)^u \right\} \qquad (A31)$$

and

$$\frac{dy}{du} = \lambda^{\frac{1}{2}}(y, a, b) \, \ln\left(\frac{y^+ - a - b + \alpha^+}{y^- - a - b + \alpha^-} \right) \qquad (A32)$$

In these equations α^+ and α^- are given by

$$\alpha^{\pm} = \lambda^{1/2}(\gamma^{\pm}, a, b) \tag{A33}$$

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Appendix B

Once the integration over the 2 to 3 particles phase space has been expressed in terms of invariants and/or the variable Δ or x of appendix A, it is still necessary to translate these quantities into laboratory variables. Especially a Monte Carlo integration over an experimental acceptance needs this, since most of the experimental cuts are expressed in terms of laboratory angles and momenta. Again numerical stability turns out to be of great importance as careless evaluation can easily lead to a loss of 10 to 20 digits by subtracting a large number from another large number to obtain a result that is 10 to 20 orders of magnitude smaller. A good example of this is the calculation of the quantity $E - p$ for a 50 GeV electron. If the final result is obtained via the relation

$$E - p = \frac{m^2}{E + p} \tag{B1}$$

there is no loss of accuracy and it is seen that the answer is about 2.5×10^{-9} GeV. The original terms E and p were 50 GeV so the relative loss of accuracy due to the subtraction $E - p$ would be about 5×10^{-11} or more than 10 digits.

The necessary formulae are derived for the CM frame and can almost all be found in ref. [18]. In the case that the lab frame is not the same as the CM frame a simple boost along the z-axis is usually sufficient. This does not cause any severe problems as the perpendicular components of the 4-vectors which are usually responsible for the numerical instabilities, are not affected by such a boost.

In the CM frame the energies of all particles can be found from the relation $\sqrt{s} \, E_i = p_1 \cdot p_i + p_2 \cdot p_i$. The righthand side 4-vector products can be expressed in terms of invariants and masses leading to the equations

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}$$

$$E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$

$$E_3 = E_1 - \frac{(s_2 - m_3^2 + m_1^2 - m_2^2)}{2\sqrt{s}}$$

$$E_5 = E_2 - \frac{(s_1 - m_5^2 + m_2^2 - m_1^2)}{2\sqrt{s}}$$ (B2)

$$E_4 = E_1 - E_3 + E_2 - E_5$$

Sometimes is is useful to define $\zeta_3 = E_1 - E_3$ and $\zeta_5 = E_2 - E_5$ because these energy transfers can occasionally be small. The values of the three momenta can be easily obtained from (B2).

The θ angles (angles with the positive z-axis which is chosen here as the direction of p_1) can be rather close to $0°$ or $180°$ whenever the energies get large. It is therefore better to calculate $\sin\theta$ and the sign of $\cos\theta$ rather than to use the relation $p_1 \cdot p_i = E_1 E_i - |p_1||p_i|\cos\theta_i$. This can be done via the relations (see also ref. [18]

$$D_1 = s|\vec{p_1}|^2|\vec{p_2}|^2 \sin^2\vartheta_3$$

$$D_3 = s|\vec{p_1}|^2|\vec{p_5}|^2 \sin^2\vartheta_5$$ (B3)

$$D_5 = s|\vec{p_1}|^2|\vec{p_4}|^2 \sin^2\vartheta_4$$

Since $\cos\theta_i = (E_1 E_i - p_1 \cdot p_i) / (|p_1||p_i|)$ the sign of $\cos\theta_i$ can be obtained by comparing $E_1 E_i$ with $p_1 \cdot p_i$. For reasons of convenience particle 4 is chosen to define the xz plane so that $p_4 = |p_4|(\sin\theta_4, 0, \cos\theta_4)$. This is the most convenient choice for 2γ physics since "particle" 4 is the one which decays. The y components of p_3 is equal in magnitude to the y component p_5 but opposite in sign because they are the only two three vectors with an y component. They are found via the relation:

$$-\Delta_4 = -\epsilon^{p_1 p_2 p_3 p_4} \epsilon_{p_1 p_2 p_3 p_4}$$

$$= s|\vec{p_1}|^2|\vec{p_4}|^2 \sin^2\vartheta_4 |\vec{p_3}|^2 \sin^2\vartheta_3 \sin^2\varphi_3$$ (B4)

The sign of $\sin\phi_3$ is ambiguous as it is not determined by the values of the invariants or the 4-vector products. It can therefore be taken randomly. This leaves only the signs of $\cos\phi_3$ and $\cos\phi_5$ to be fixed. They can be determined by imposing conservation of three momentum in the x-direction. This gives that $p_{4x} + p_{3x} + p_{5x} = 0$. As p_{4x} is known and also the absolute values of p_{3x} and p_{5x} it is rather easy to find the combination of signs that satisfies this conservation law.

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Appendix C

To obtain equation (III.5) several reduction formulae were needed. Their derivation is mainly dependent on the use of the so-called Schouten identity for Levi-Civita tensors:

$$q_a \epsilon_{bcda} = q_b \epsilon_{acde} + q_c \epsilon_{bade} + q_d \epsilon_{bcae} + q_e \epsilon_{bcda} \tag{C1}$$

Nevertheless at least one of the identities (C11) is far from trivial and it is much easier to have a computer program like Schoonschip[19] check it by comparing the left- and righthand sides after contracting the Levi-Civita tensors rather than to derive the relation by hand. For illustrative purposes one of the relations is derived below. This will indicate some of the techniques that are useful.

$$Q \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot R \doteq \epsilon^{q_1 e_1 Q \mu} \epsilon_{q_2 e_1 \alpha \beta} \epsilon^{q_1 e_1 \beta \gamma} \epsilon_{q_1 e_1 \gamma R} \tag{C2}$$

To reduce this term one exchanges the index α of the first Levi-Civita tensor with the indices of the third, using the Schouten identity:

$$Q \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot R = 0 + 0 + \epsilon^{q_1 e_1 Q \beta} \epsilon_{q_1 e_1 \alpha \beta} \epsilon^{q_1 e_1 \alpha \gamma} \epsilon_{q_1 e_1 \gamma R}$$

$$+ \epsilon^{q_1 e_1 Q \gamma} \epsilon_{q_1 e_1 \alpha \beta} \epsilon^{q_1 e_1 \beta \alpha} \epsilon_{q_2 e_1 \gamma R} \tag{C3}$$

$$= -Q \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot R + Tr[1 \cdot 2] Q \cdot 1 \cdot 2 \cdot R$$

The final result is now:

$$Q \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot R = \frac{1}{2} Tr[1 \cdot 2] Q \cdot 1 \cdot 2 \cdot R \tag{C4}$$

This result is independent of the 4-vectors Q and R. Relation (C9) is derived by noticing that

$$\epsilon^{q_1 e_1 q_1 e_1} \epsilon_{q_1 e_1 q_1 e_1} = \frac{1}{16} \epsilon^{Qq_1 \alpha \beta} \epsilon^{q_1 e_1 \alpha \beta} \epsilon_{\alpha \beta \gamma \delta} \epsilon_{q_1 e_1 \kappa \lambda} \epsilon_{q_2 e_1 \mu \nu} \epsilon^{\kappa \lambda \mu \nu} \tag{C5}$$

By contracting the two Levi-Civita tensors that are not contracted with q_1, q_2, e_1 or e_2 one can obtain equation (C9). The only direct derivation of (C11) known to the author involves the introduction of extra Levi-Civita tensors via the relation:

$$q_1^2 Q^\alpha Q_\beta = -\epsilon^{Qq_1 \alpha \gamma} \epsilon_{Qq_1 \beta \gamma} + (Q^2 q_1^2 - (Q.q_1)^2) g^\alpha_\beta$$

$$- q_1^\alpha q_{1\beta} Q^2 + Q^\alpha q_{1\beta} Q.q_1 + Q_\beta q_1^\alpha Q.q_1 \tag{C6}$$

If this expression is contracted with the tensor $_\alpha 1 \cdot 2 \cdot 2 \cdot 1^\beta$ the final three terms give zero and the g^α_β gives $[1 \cdot 2 \cdot 2 \cdot 1]$. Now the Schouten identity has to be used several times to obtain the desired result. The relations needed for equation (III.5) are:

$$Q \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot R = \frac{1}{2} Tr[1 \cdot 2] Q \cdot 1 \cdot 2 \cdot R \tag{C7}$$

$$Tr[1 \cdot 2 \cdot 1 \cdot 2] = \frac{1}{2} Tr[1 \cdot 2] Tr[1 \cdot 2] \tag{C8}$$

$$Tr[1 \cdot 2 \cdot 2 \cdot 1] = \frac{1}{4} Tr[1 \cdot 1] Tr[2 \cdot 2] + \frac{1}{4} Tr[1 \cdot 2] Tr[1 \cdot 2]$$
$$- \epsilon^{q_1 e_1 q_1 e_1} \epsilon_{q_1 e_1 q_1 e_1} \tag{C9}$$

$$P \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot P = -q_2^2 \epsilon^{q_1 e_1 q_1 e_1} \epsilon_{q_1 e_1 q_1 e_1} + \frac{1}{2} Tr[2 \cdot 2] P \cdot 1 \cdot 1 \cdot P \tag{C10}$$

$$q_i^2 Q.1.2.2.1.Q = q_1^2 Tr[1.2] Q.1.2.Q - Q.q_1 Tr[1.2] Q.1.2.P$$

$$+ (Q.1.2.P)^2 + \tfrac{1}{4} Tr[1.2] Tr[1.2] ((Q.q_1)^2 - Q^2 q_1^2)$$

$$- \tfrac{1}{8} Tr[1.1] \left(\epsilon^{q_1 e_2 \alpha \beta} \epsilon_{Q q_1 \alpha \beta} \right)^2 \tag{C11}$$

In these relations Q and R are arbitrary 4-vectors and $P = q_1 + q_2$. One can of course also substitute the 4-vector P for Q in relation (C11). In that case the following identity is rather useful:

$$\left(\epsilon^{q_1 e_2 \alpha \beta} \epsilon_{P q_1 \alpha \beta} \right)^2 = 2 Tr[2.2] ((q_1.q_2)^2 - q_1^2 q_2^2) + 4 q_2^2 P.2.2.P \tag{C12}$$

To obtain the quantities M_{12} and M_{21} in equation (III.11) one needs a few additional relations:

$$\epsilon^{q_1 e_1 Q q_2} \epsilon_{q_1 e_1 Q q_2} = \tfrac{1}{4} \left(\epsilon^{q_1 e_1 \alpha \beta} \epsilon_{Q q_1 \alpha \beta} \right)^2 - \tfrac{1}{2} Tr[1.1] ((Q.q_1)^2 - Q^2 q_1^2)$$

$$- Q^2 P.1.1.P - q_1^2 Q.1.1.Q + 2 Q.q_1 Q.1.1.P \tag{C13}$$

$$2 Q.q_1 q_1.q_2 Q.1.1.P = (Q.q_1)^2 P.1.1.P + (q_1.q_2)^2 Q.1.1.Q$$

$$+ Tr[1.1] \left\{ Q.q_1 Q.q_2 q_1.q_2 - \tfrac{1}{2} (Q.q_1)^2 q_2^2 - \tfrac{1}{2} Q^2 (q_1.q_2)^2 \right\} \tag{C14}$$

$$- \tfrac{1}{4} q_2^2 \left(\epsilon^{q_1 e_1 \alpha \beta} \epsilon_{Q q_1 \alpha \beta} \right)^2$$

The procedure to obtain M_{12} and M_{21} is to first execute the contraction in which for instance both e_2's are replaced by the same index μ. Then the two Levi-Civita tensors containing this μ are contracted after which the reduction formulae (C13) and (C14) can be used. This leads to M_{21}. To obtain M_{12} one contracts the e_1's by replacing them by the same index ν. Then one needs the versions of (C13) and (C14) in which the indices 1 and 2 have been interchanged.

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Figure captions

1. The general structure of a two photon process.

2. The reaction $e^+e^- \to e^+e^-\pi^0$. The four-vectors e_1 and e_2 are the polarization vectors of the two photons.

3. The diagrams of the reaction $\gamma^*\gamma^* \to \mu^+\mu^-$. The polarization vectors of the two photons are represented by e_1 and e_2.

4. Differential cross sections with respect to the dimuon mass under the assumption that both muons pass the cuts. The solid lines are for the purely elastic processes. They are from top to bottom:
(i) e^+e^- at $\sqrt{s} = 100$ GeV, (ii) ep at HERA energies, (iii) pp at $\sqrt{s} = 540$ GeV. The dashed curves represent processes in which at least one proton is scattered inelastically. The top curve is for HERA and the bottom curve for pp at $\sqrt{s} = 540$ GeV. The systematic uncertainty of these dashed curves is much greater due to the structure functions (see text).

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Table captions

1. Observable dimuon cross sections. It is assumed that both muons are seen inside the standard detector defined in the text.

2. Observable dimuon cross sections with a 20 mrad electron tag. Both muons are seen inside the standard detector of the text and at least one electron is seen at an angle of at least 20 mrad.
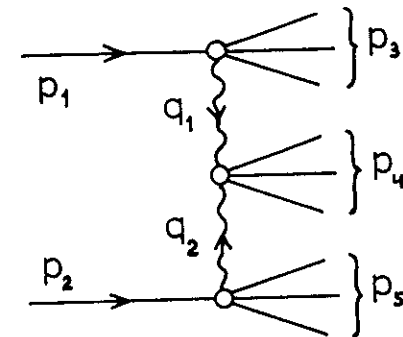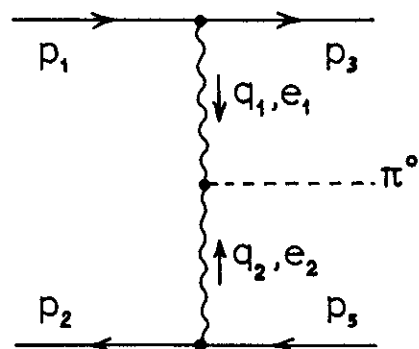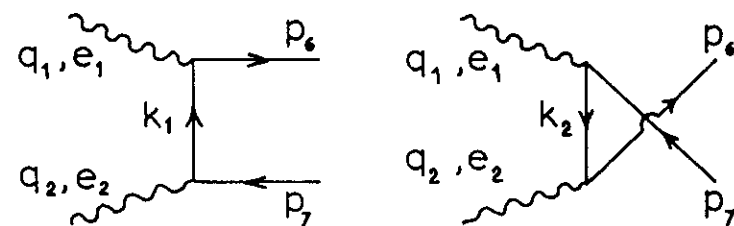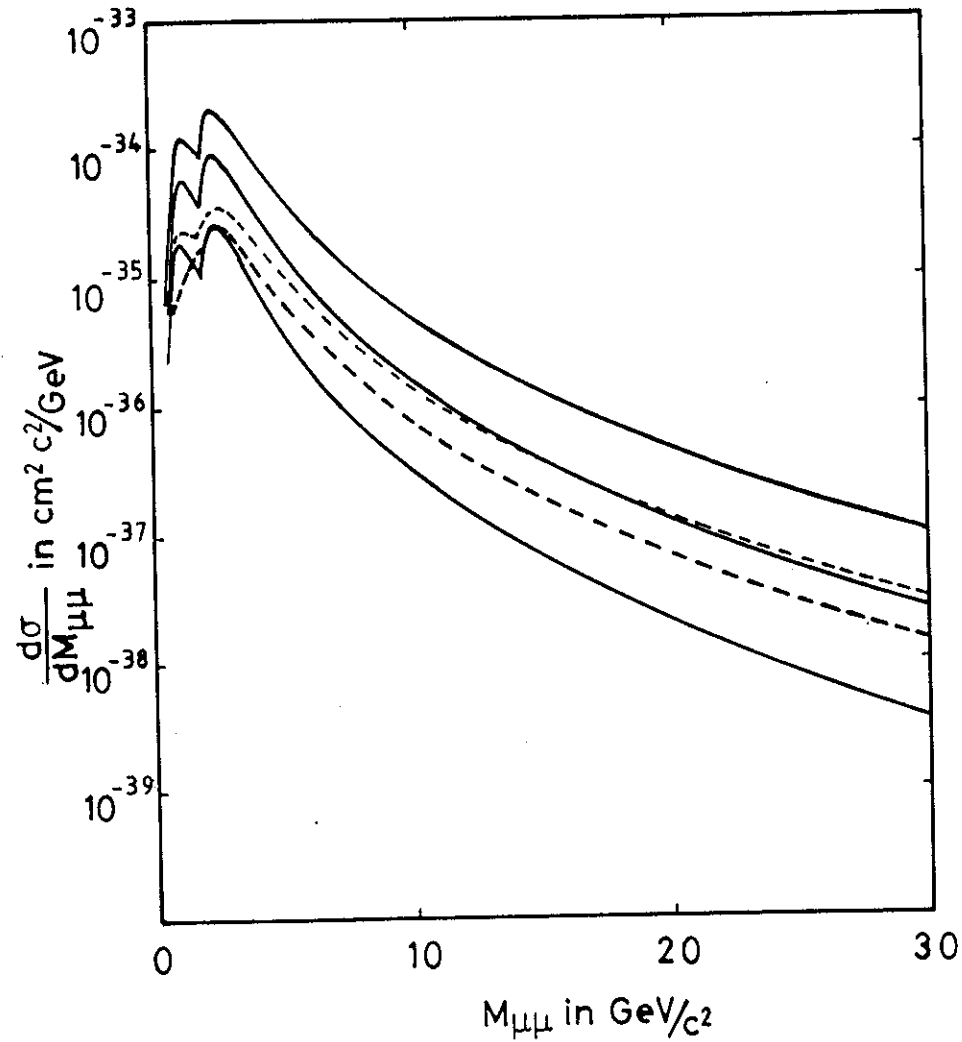
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Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Beams	Configuration	Energies (GeV)	Cross section (pb)
$e^+ e^-$	—	50 on 50	588.5 ± 1.6
$e^+ e^-$	—	70 on 70	644.1 ± 1.9
$e^- p$	elastic	30 on 820	239.7 ± 0.7
$e^- p$	inelastic	30 on 820	125.7 ± 0.5
p p	elastic-elastic	270 on 270	67.5 ± 0.3
p p	el.-inel.+inel.-el.	270 on 270	66.8 ± 0.6
p p	inelastic-inelastic	270 on 270	19.7 ± 0.2

Table 2

Beams	Configuration	Energies (GeV)	Cross section (pb)
$e^+ e^-$	single tag	50 on 50	87.7 ± 0.4
$e^+ e^-$	double tag	50 on 50	5.8 ± 0.1
$e^- p$	elastic	30 on 820	29.6 ± 0.2
$e^- p$	inelastic	30 on 820	18.0 ± 0.1