The Multiple Zeta Value Data Mine

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Nikhef and Humboldt foundation

- Introduction
- Notations
- Multiple Zeta Values.
- Bases
- Relations
- The program
- Alternating Sums
- Non-Alternating Sums
- Data mining
- A push down basis
- Conclusions

Work with J. Blümlein and D. Broadhurst.

Introduction

The problem of studying Multiple Zeta Values (MZV's) has been around for a few centuries now. It ranges from finding relations between them, determining a basis into which they can all be expressed to trying to prove their trancendentality. It is the basis problem that we will address today.

Many insights find their origin in looking at data. Hence we will attack the problem with the application of brute force to generate as many results as technology will allow us.

To apply this brute force we have to derive some new equations and construct a relatively simple but yet powerful computer program.

The running of the programs gives us many Gigabytes of relations. We study the outputs and try to see some new patterns in them. We do find some.

This field is full of conjectures. We will add a few of our own.

For many results presented in this talk, we have made heavy use of symbolic computation. This has been done with the system FORM and its variety TFORM. TFORM is a parallel version which can use many threads simultaneously. The bigger runs were done on a computer with 8 Xeon cores, running about 7 times faster than on a single core. Also (T)FORM isn't much limited by the size of the CPU memory (which was 32 Gbytes) as it can use the disk rather efficiently (which was 4 Tbytes). Another advantage is that FORM has a very compact data representation and is (not only because of that) very fast. This comes in handy when you have to manipulate expressions of 10^9 terms.

Our worst example:

```
Time =    33207.54 sec    Generated terms =639841672647
              FF          Terms in output =  1156667274
substitution(6-sh)-44460 Bytes used      = 61958287204
```

Notations

Physicists define harmonic sums by:

$$
\begin{aligned}
S_m(N) &= \sum_{i=1}^{N} \frac{1}{i^m} \\
S_{-m}(N) &= \sum_{i=1}^{N} \frac{(-1)^i}{i^m} \\
S_{m,m_2,\cdots,m_p}(N) &= \sum_{i=1}^{N} \frac{1}{i^m} S_{m_2,\cdots,m_p}(i) \\
S_{-m,m_2,\cdots,m_p}(N) &= \sum_{i=1}^{N} \frac{(-1)^m}{i^m} S_{m_2,\cdots,m_p}(i)
\end{aligned}
$$

Mathematician use mostly $i-1$ for the argument of the S in the recursive formula. Those sums we call Z-sums.

Sums that involve negative indices we call alternating sums (AS) and the ones that have only positive indices we call non-alternating sums (NAS).

The S-sums and the Z-sums can be converted into each other. In any decent symbolic system this is easily programmable.

Related functions are the harmonic polylogarithms (Hpl) which are defined by:

$$\begin{aligned} H(0;x) &= \ln x \\ H(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x) \\ H(-1;x) &= \int_0^x \frac{dx'}{1+x'} = \ln(1+x) \end{aligned}$$

and the functions

$$f(0;x) = \tfrac{1}{x}, \qquad f(1;x) = \tfrac{1}{1-x}, \qquad f(-1;x) = \tfrac{1}{1+x}$$

If $\vec{a}_w$ is an array with W elements, all with value a, then:

$$\begin{aligned} H(\vec{0}_w;x) &= \frac{1}{W!}\ln^W x \\ H(a,\vec{m}_w;x) &= \int_0^x dx'\; f(a;x')\; H(\vec{m}_w;x') \end{aligned}$$

These functions are related by Mellin transforms:

$$M(f,N) = \int_0^1 dx\, x^N\, f(x)$$

This is how the harmonic sums enter in field theory, but in the context of this talk, that is not what we are interested in. What we ARE interested in is that the Hpl's in one are related to the harmonic sums in infinity. They are commonly referred to as Multiple Zeta Values (MZV's).
We can define a unified notation as in:

$$\begin{aligned} H_{0,0,1,0,-1} &= H_{3,-2} \\ S_{7,-2,1} &= S_{0,0,0,0,0,0,1,0,-1,1} \end{aligned}$$

The notation with the $0, 1, -1$ we call integral notation and the other notation we call sum notation. The number of indices in the integral notation is the weight, and the number of indices in the sum notation is the depth.
For non-alternating sums we have $Z_{\vec{p}}(\infty) = H_{\vec{p}}(1)$ With alternating sums there can be signs. Again: trivially programmable.

We will usually omit the argument of the S, Z and H functions. This means that they are taken in ∞, ∞ and one respectively.

Multiple Zeta Values

The sums in infinity (or the Hpl's in one) are called Multiple Zeta Values. Some of them also go by the name Euler-Zagier sums. They have become interesting to mathematicians in the 1990's, but then mainly the sums with positive indices only. We (=physicists) also need the sums that can have negative indices. Gastmans and Troost (1981) managed to give the relations for all weight 4 and a number of weight 5 sums. We need basically weight $= 2\times$ (number of loops).

The number of MZV's that exists is $2 \times 3^{W-1}$ for the alternating sums and 2^{W-1} for the non-alternating ones. If we remove the divergent sums these numbers become $4 \times 3^{W-2}$ and 2^{W-2} respectively. This means that for weight 6 there are 324 (16) constants that we have to determine.

Example of an inverse Mellin transform of a weight 6 harmonic sum. We do not consider relations between the MZV's.

```
#define SIZE "6"
#include- harmpol.h
Off statistics;
.global
Local F = S(R(-1,3,-2),N);
#call invmel(S,N,H,x)
Print +f +s;
.end
```

```
F =
   - sign_(N)*H(R(1,0,0),x)*Htab2(0,-1)*[1+x]^-1
   - sign_(N)*Htab5(0,-1,0,0,-1)*[1+x]^-1
   - sign_(N)*Htab5(0,-1,0,0,1)*[1+x]^-1
   + sign_(N)*Htab5(0,-1,1,0,0)*[1+x]^-1
   - 2*sign_(N)*Htab5(0,0,-1,0,1)*[1+x]^-1
   - 3*sign_(N)*Htab5(0,0,0,-1,1)*[1+x]^-1
   - 3*sign_(N)*Htab5(0,0,0,1,-1)*[1+x]^-1
```

```
- sign_(N)*Htab5(0,0,1,0,-1)*[1+x]^-1
+ sign_(N)*Htab5(0,1,-1,0,0)*[1+x]^-1
+ sign_(N)*Htab5(0,1,0,-1,0)*[1+x]^-1
+ sign_(N)*Htab5(0,1,0,0,-1)*[1+x]^-1
+ sign_(N)*Htab5(1,0,-1,0,0)*[1+x]^-1
+ 2*sign_(N)*Htab5(1,0,0,-1,0)*[1+x]^-1
+ 3*sign_(N)*Htab5(1,0,0,0,-1)*[1+x]^-1
- H(R(-1),x)*Htab4(0,0,-1,0)*[1-x]^-1
+ H(R(-1,-3,0),x)*[1-x]^-1
- H(R(-1,0),x)*Htab3(0,-1,0)*[1-x]^-1
- H(R(-1,0,0),x)*Htab2(-1,0)*[1-x]^-1
+ 6*Htab5(-1,-1,0,0,0)*[1-x]^-1
+ 5*Htab5(-1,0,-1,0,0)*[1-x]^-1
+ 3*Htab5(-1,0,0,-1,0)*[1-x]^-1
+ 4*Htab5(0,-1,-1,0,0)*[1-x]^-1
+ 3*Htab5(0,-1,0,-1,0)*[1-x]^-1
+ 2*Htab5(0,0,-1,-1,0)*[1-x]^-1
+ Htab5(0,0,-1,0,-1)*[1-x]^-1
+ Htab6(-1,0,-1,0,0,-1)
+ Htab6(-1,0,-1,0,0,1)
+ 2*Htab6(-1,0,0,-1,0,1)
+ 3*Htab6(-1,0,0,0,-1,1)
+ 3*Htab6(-1,0,0,0,1,-1)
```

```
 + Htab6(-1,0,0,1,0,-1)
 + 2*Htab6(0,-1,-1,0,0,-1)
 + 2*Htab6(0,-1,-1,0,0,1)
 + Htab6(0,-1,0,-1,0,-1)
 + 3*Htab6(0,-1,0,-1,0,1)
 + 2*Htab6(0,-1,0,0,-1,-1)
 + 5*Htab6(0,-1,0,0,-1,1)
 + 3*Htab6(0,-1,0,0,1,-1)
 + Htab6(0,-1,0,1,0,-1)
 + 4*Htab6(0,0,-1,-1,0,1)
 + 5*Htab6(0,0,-1,0,-1,1)
 + 3*Htab6(0,0,-1,0,1,-1)
 + Htab6(0,0,-1,1,0,-1)
 + 6*Htab6(0,0,0,-1,-1,1)
 + 3*Htab6(0,0,0,-1,1,-1)
;
```

The Htab objects are Hpl's in one in which for instance Htab6(0,0,0,-1, 1,-1) stands for $H_{-4,1,-1}(1)$. The numbers are related to the sums in infinity.

And now the same program, but now the MZV's are reduced to a set of independent objects.

```
F =
     - 51/32*[1-x]^-1*z5
     + 3/4*[1-x]^-1*z2*z3
     - 7/2*s6
     + 51/32*z5*ln2
     - 33/64*z3^2
     + 9/4*z2*z3*ln2
     + 121/840*z2^3
     - 51/32*sign_(N)*[1+x]^-1*z5
     + 3/4*sign_(N)*[1+x]^-1*z2*z3
     - 1/2*sign_(N)*H(R(1,0,0),x)*[1+x]^-1*z2
     + 21/20*H(R(-1),x)*[1-x]^-1*z2^2
     + H(R(-1,-3,0),x)*[1-x]^-1
     + 3/2*H(R(-1,0),x)*[1-x]^-1*z3
     + 1/2*H(R(-1,0,0),x)*[1-x]^-1*z2
    ;
```

It is clear that reducing the MZV's to an independent set gives a much shorter answer.

Unfortunately there is no known constructive way to take one of these constants and express it into a basis. Already there are problems in determining what constitutes a good basis.
The only two ways to express them in an independent set that are currently known are:

- Write down all algebraic relations for these objects and solve the system of equations. Then tabulate all MZV's and use table substitution afterwards.
- Guess a relation and fit the coefficients with a program like PSLQ after computing all objects in the relation numerically to a very large number of digits. Broadhurst has done much of this in the 1990's.

Bases

There are several conjectures about the size of a basis. The best known are the Zagier conjecture (NAS), the Broadhurst conjecture (AS) and the Broadhurst-Kreimer conjectures (NAS).
There even exist conjectures about how to construct some specific bases. We consider the following types of bases:

1. Fibonacci bases. In such a basis all elements have the same weight. Hence there are no products of lower weight objects.
2. Lyndon bases. In these bases we reduce the elements as much as possible to products of lower weight basis elements. What remains at the given weight we express in terms of MZV's of which the indices form a Lyndon word.
3. Mixed bases. Everything else.

Example number 1: An alternating basis.

It is a conjecture of Broadhurst that the total number of basis elements for the alternating sums follows a simple Fibonacci sequence:

$$N(W) = N(W-1) + N(W-2), W \geq 2$$

with $N(1) = 1$ and $N(2) = 2$. From this we conjecture that we can make a Fibonacci basis for the alternating sums that consists of all index combinations of -1's and -2's. We have verified this to weight 12.

If one takes only the elements of this Fibonacci basis of which the index field is a Lyndon word, one obtains a Lyndon basis. Again we have verified this to weight 12.

Example number 2: Non-alternating sums. The Hoffman basis.

It is the conjecture of Zagier that the number of basis elements in a Fibonacci basis of non-alternating sums follows the rule

$$N(W) = N(W-2) + N(W-3), W > 3$$

with $N(1) = 0, N(2) = N(3) = 1$. Out of this Hoffman conjectured that one can construct a basis for the non-alternating sums that consists of all index combinations of 2's and 3's. This is a Fibonacci basis. We have confirmed that indeed this is a basis to weight 22.

If one takes from the elements of Hoffmans Fibonacci basis only those of which the indices form a Lyndon word, one obtains a Lyndon basis. This is less trivial than it seems, but we have been able to verify this to weight 22 in rational arithmetic and to weight 24 in modular arithmetic.

Example number 3: Another set of bases for the alternating sums.

We found another Fibonacci basis for the alternating sums. The recipe is:

- Begin with $B(1) = H_{-1}$ and $B(2) = H_{0,-1}, H_{-1,-1}$.
- $B(W)$ is obtained by putting an index -1 in front of the index fields of the elements of $B(W-1)$ and putting two zeroes in front of the index fields of the elements of $B(W-2)$.

Hence:

$$\begin{aligned} B(3) &= H_{-1,0,-1}, H_{-1,-1,-1}, H_{0,0,-1} \\ B(4) &= H_{-1,-1,0,-1}, H_{-1,-1,-1,-1}, H_{-1,0,0,-1}, H_{0,0,0,-1}, H_{0,0,-1,-1} \end{aligned}$$

If we take the Lyndon words from this basis, and we omit all elements that (in sum notation) have even indices with the exception of H_{-2} (which is related to ζ_2) we have a Lyndon basis. This is exactly the basis proposed by Broadhurst. We have tested this basis to weight 12 and for restricted values of the depth to much higher values of the weight. (as in $W = 18, D \leq 6$). We use this basis for all alternating sums.

Later in this talk we will construct another Lyndon basis for all the non-alternating MZV's we determine in our programs. Unfortunately we don't have a complete recipe. It looks however that this basis contains more information. We call it the push down basis.

Relations

The harmonic sums obey a 'stuffle' algebra which is based on properties of sums:

$$\begin{aligned} S_{a,b}(N)S_{c,d}(N) &= S_{a,b,c,d}(N) + S_{a,c,b,d}(N) + S_{a,c,d,b}(N) \\ &+S_{c,a,b,d}(N) + S_{c,a,d,b}(N) + S_{c,d,a,b}(N) \\ &-S_{a+c,b,d}(N) - S_{a,c+b,d}(N) - S_{a,c,b+d}(N) \\ &-S_{c,a,b+d}(N) - S_{c,a+d,b}(N) + S_{a+c,b+d}(N) \end{aligned}$$

For the Z-sums the minus signs should be replaced by plus signs.
The harmonic polylogarithms obey a 'shuffle' algebra as in

$$\begin{aligned} H_{a,b}H_{c,d} &= H_{a,b,c,d} + H_{a,c,b,d} + H_{a,c,d,b} \\ &+H_{c,a,b,d} + H_{c,a,d,b} + H_{c,d,a,b} \end{aligned}$$

When we take the limit $N \longrightarrow \infty$ or $x \longrightarrow 1$ the sums and the Hpl's can be expressed into each other and we obtain MZV's. Hence the MZV's obey both relations. It should be noted that when there are negative indices the sum of the indices becomes a bit more complicated:

$$a + b \;\longrightarrow\; \sigma_a\sigma_b(|a| + |b|) = \sigma_a b + \sigma_b a$$

in which σ_a is the sign of a.

For the alternating sums there are more relations.

$$S_{n_1,\cdots,n_p}(N) \;=\; 2^{n_1+\cdots+n_p-p}\sum_{\pm} S_{\pm n_1,\cdots,\pm n_p}(2N)$$

They are called the doubling relations. When $n \to \infty$ and the sums are finite, this gives useful relations.

For the alternating sums there is yet another category of relations which we call the generalized doubling relations (GDR's). They are based on similar principles but we have only a computer algorithm to generate them. No closed formula.

The construction and its derivation are described in the (forthcoming) paper. These equations can be lengthy.

It will be necessary to take divergent sums into account. The divergences are rather mild and hence not difficult to regularize. They pose no special problems.

A simple example of what happens without the GDR's:

$$Z_{-4,-2} \;=\; -H_{-4,2} = \frac{97}{420}\zeta_2^3 - \frac{3}{4}\zeta_3^2$$

To derive this equation with shuffles and stuffles alone the number of relations one needs is enormous:

depth	shuffles	stuffles
2	2	8
3	52	19
4	72	41

Using the GDR's one needs only relations involving depth 2 (or lower) objects (and there are only 6 GDR's at depth 2, weight 6).

These GDR's can be very complicated as is seen by the next example of $W = 12, D = 4$. There are 492 terms in the formula.

$$
\begin{aligned}
-\frac{321}{256}H_{8,2,1,1} \quad = \quad & -\frac{488}{3}H_{-12}+146H_{-11,-1}-34H_{-11,1}-16H_{-10,-2}+32H_{-10,-1,-1}+32H_{-10,-1,1}-36H_{-10,1,1}-32H_{-10,2}+\frac{2}{3}H_{-9,-3}-2H_{-9,-2,-1}\\
&-2H_{-9,-2,1}-2H_{-9,-1,-2}+4H_{-9,-1,-1,-1}+4H_{-9,-1,-1,1}-4H_{-9,-1,2}-4H_{-9,1,1,1}-2H_{-9,1,2}+2H_{-9,3}+\frac{1}{2}H_{-8,2,1,1}+\frac{1}{2}H_{-8,3,1}+76H_{2,-10}\\
&+30H_{2,-9,-1}+22H_{2,-9,1}-2H_{2,-8,-2}+4H_{2,-8,-1,-1}+4H_{2,-8,-1,1}-\frac{1}{2}H_{2,-8,1,1}-4H_{2,-8,2}-8H_{2,1,-9}-\frac{1}{2}H_{2,1,-8,1}-\frac{1}{2}H_{2,1,1,-8}\\
&+\frac{127}{256}H_{2,1,1,8}-\frac{1}{2}H_{2,1,2,-7}+\frac{63}{128}H_{2,1,2,7}-\frac{1}{2}H_{2,1,3,-6}+\frac{31}{64}H_{2,1,3,6}-\frac{1}{2}H_{2,1,4,-5}+\frac{15}{32}H_{2,1,4,5}-\frac{1}{2}H_{2,1,5,-4}+\frac{7}{16}H_{2,1,5,4}-\frac{1}{2}H_{2,1,6,-3}\\
&+\frac{3}{8}H_{2,1,6,3}-\frac{1}{2}H_{2,1,7,-2}+\frac{1}{4}H_{2,1,7,2}-\frac{1}{2}H_{2,1,8,-1}+\frac{127}{256}H_{2,1,8,1}+\frac{511}{64}H_{2,1,9}+20H_{2,2,-8}+4H_{2,2,-7,-1}+\frac{7}{2}H_{2,2,-7,1}-\frac{1}{2}H_{2,2,1,-7}\\
&+\frac{63}{128}H_{2,2,1,7}+\frac{7}{2}H_{2,2,2,-6}-\frac{217}{64}H_{2,2,2,6}+\frac{7}{2}H_{2,2,3,-5}-\frac{105}{32}H_{2,2,3,5}+\frac{7}{2}H_{2,2,4,-4}-\frac{49}{16}H_{2,2,4,4}+\frac{7}{2}H_{2,2,5,-3}-\frac{21}{8}H_{2,2,5,3}+\frac{7}{2}H_{2,2,6,-2}\\
&-\frac{7}{4}H_{2,2,6,2}+\frac{7}{2}H_{2,2,7,-1}+\frac{63}{128}H_{2,2,7,1}-\frac{5137}{256}H_{2,2,8}+18H_{2,3,-7}+4H_{2,3,-6,-1}+\frac{7}{2}H_{2,3,-6,1}-\frac{1}{2}H_{2,3,1,-6}+\frac{31}{64}H_{2,3,1,6}+\frac{7}{2}H_{2,3,2,-5}\\
&-\frac{105}{32}H_{2,3,2,5}+\frac{7}{2}H_{2,3,3,-4}-\frac{49}{16}H_{2,3,3,4}+\frac{7}{2}H_{2,3,4,-3}-\frac{21}{8}H_{2,3,4,3}+\frac{7}{2}H_{2,3,5,-2}-\frac{7}{4}H_{2,3,5,2}+\frac{7}{2}H_{2,3,6,-1}+\frac{31}{64}H_{2,3,6,1}-\frac{1159}{64}H_{2,3,7}\\
&+18H_{2,4,-6}+4H_{2,4,-5,-1}+\frac{7}{2}H_{2,4,-5,1}-\frac{1}{2}H_{2,4,1,-5}+\frac{15}{32}H_{2,4,1,5}+\frac{7}{2}H_{2,4,2,-4}-\frac{49}{16}H_{2,4,2,4}+\frac{7}{2}H_{2,4,3,-3}-\frac{21}{8}H_{2,4,3,3}+\frac{7}{2}H_{2,4,4,-2}\\
&-\frac{7}{4}H_{2,4,4,2}+\frac{7}{2}H_{2,4,5,-1}+\frac{15}{32}H_{2,4,5,1}-\frac{1159}{64}H_{2,4,6}+18H_{2,5,-5}+4H_{2,5,-4,-1}+\frac{7}{2}H_{2,5,-4,1}-\frac{1}{2}H_{2,5,1,-4}+\frac{7}{16}H_{2,5,1,4}+\frac{7}{2}H_{2,5,2,-3}\\
&-\frac{21}{8}H_{2,5,2,3}+\frac{7}{2}H_{2,5,3,-2}-\frac{7}{4}H_{2,5,3,2}+\frac{7}{2}H_{2,5,4,-1}+\frac{7}{16}H_{2,5,4,1}-18H_{2,5,5}+18H_{2,6,-4}+4H_{2,6,-3,-1}+\frac{7}{2}H_{2,6,-3,1}-\frac{1}{2}H_{2,6,1,-3}\\
&+\frac{3}{8}H_{2,6,1,3}+\frac{7}{2}H_{2,6,2,-2}-\frac{7}{4}H_{2,6,2,2}+\frac{7}{2}H_{2,6,3,-1}+\frac{3}{8}H_{2,6,3,1}-\frac{281}{16}H_{2,6,4}+18H_{2,7,-3}+4H_{2,7,-2,-1}+\frac{7}{2}H_{2,7,-2,1}-\frac{1}{2}H_{2,7,1,-2}\\
&+\frac{1}{4}H_{2,7,1,2}+\frac{7}{2}H_{2,7,2,-1}+\frac{1}{4}H_{2,7,2,1}-\frac{65}{4}H_{2,7,3}+18H_{2,8,-2}+4H_{2,8,-1,-1}+\frac{7}{2}H_{2,8,-1,1}-\frac{1}{2}H_{2,8,1,-1}+\frac{127}{256}H_{2,8,1,1}-\frac{689}{64}H_{2,8,2}\\
&+22H_{2,9,-1}+\frac{511}{64}H_{2,9,1}-\frac{9727}{128}H_{2,10}+\frac{58}{3}H_{3,-9}+28H_{3,-8,-1}+13H_{3,-8,1}-2H_{3,-7,-2}+4H_{3,-7,-1,-1}+4H_{3,-7,-1,1}-H_{3,-7,1,1}\\
&-4H_{3,-7,2}-15H_{3,1,-8}-H_{3,1,-7,1}-H_{3,1,1,-7}+\frac{63}{64}H_{3,1,1,7}-H_{3,1,2,-6}+\frac{31}{32}H_{3,1,2,6}-H_{3,1,3,-5}+\frac{15}{16}H_{3,1,3,5}-H_{3,1,4,-4}\\
&+\frac{7}{8}H_{3,1,4,4}-H_{3,1,5,-3}+\frac{3}{4}H_{3,1,5,3}-H_{3,1,6,-2}+\frac{1}{2}H_{3,1,6,2}-H_{3,1,7,-1}+\frac{63}{64}H_{3,1,7,1}+\frac{239}{16}H_{3,1,8}+11H_{3,2,-7}+4H_{3,2,-6,-1}\\
&+3H_{3,2,-6,1}-H_{3,2,1,-6}+\frac{31}{32}H_{3,2,1,6}+3H_{3,2,2,-5}-\frac{45}{16}H_{3,2,2,5}+3H_{3,2,3,-4}-\frac{21}{8}H_{3,2,3,4}+3H_{3,2,4,-3}-\frac{9}{4}H_{3,2,4,3}+3H_{3,2,5,-2}\\
&-\frac{3}{2}H_{3,2,5,2}+3H_{3,2,6,-1}+\frac{31}{32}H_{3,2,6,1}-\frac{715}{64}H_{3,2,7}+9H_{3,3,-6}+4H_{3,3,-5,-1}+3H_{3,3,-5,1}-H_{3,3,1,-5}+\frac{15}{16}H_{3,3,1,5}+3H_{3,3,2,-4}-\frac{21}{8}H_{3,3,2,4}\\
&+3H_{3,3,3,-3}-\frac{9}{4}H_{3,3,3,3}+3H_{3,3,4,-2}-\frac{3}{2}H_{3,3,4,2}+3H_{3,3,5,-1}+\frac{15}{16}H_{3,3,5,1}-\frac{149}{16}H_{3,3,6}+9H_{3,4,-5}+4H_{3,4,-4,-1}+3H_{3,4,-4,1}\\
&-H_{3,4,1,-4}+\frac{7}{8}H_{3,4,1,4}+3H_{3,4,2,-3}-\frac{9}{4}H_{3,4,2,3}+3H_{3,4,3,-2}-\frac{3}{2}H_{3,4,3,2}+3H_{3,4,4,-1}+\frac{7}{8}H_{3,4,4,1}-\frac{151}{16}H_{3,4,5}+9H_{3,5,-4}\\
&+4H_{3,5,-3,-1}+3H_{3,5,-3,1}-H_{3,5,1,-3}+\frac{3}{4}H_{3,5,1,3}+3H_{3,5,2,-2}-\frac{3}{2}H_{3,5,2,2}+3H_{3,5,3,-1}+\frac{3}{4}H_{3,5,3,1}-\frac{19}{2}H_{3,5,4}+9H_{3,6,-3}\\
&+4H_{3,6,-2,-1}+3H_{3,6,-2,1}-H_{3,6,1,-2}+\frac{1}{2}H_{3,6,1,2}+3H_{3,6,2,-1}+\frac{1}{2}H_{3,6,2,1}-\frac{37}{4}H_{3,6,3}+9H_{3,7,-2}+4H_{3,7,-1,-1}+3H_{3,7,-1,1}\\
&-H_{3,7,1,-1}+\frac{63}{64}H_{3,7,1,1}-\frac{193}{32}H_{3,7,2}+13H_{3,8,-1}+\frac{239}{16}H_{3,8,1}-\frac{619}{32}H_{3,9}-\frac{23}{2}H_{4,-8}+28H_{4,-7,-1}+7H_{4,-7,1}-2H_{4,-6,-2}\\
&+4H_{4,-6,-1,-1}+4H_{4,-6,-1,1}-\frac{3}{2}H_{4,-6,1,1}-4H_{4,-6,2}-21H_{4,1,-7}-\frac{3}{2}H_{4,1,-6,1}-\frac{3}{2}H_{4,1,1,-6}+\frac{93}{64}H_{4,1,1,6}-\frac{3}{2}H_{4,1,2,-5}+\frac{45}{32}H_{4,1,2,5}
\end{aligned}
$$

$$-\frac{3}{2}H_{4,1,3,-4}+\frac{21}{16}H_{4,1,3,4}-\frac{3}{2}H_{4,1,4,-3}+\frac{9}{8}H_{4,1,4,3}-\frac{3}{2}H_{4,1,5,-2}+\frac{3}{4}H_{4,1,5,2}-\frac{3}{2}H_{4,1,6,-1}+\frac{93}{64}H_{4,1,6,1}+\frac{333}{16}H_{4,1,7}+5H_{4,2,-6}$$
$$+4H_{4,2,-5,-1}+\frac{5}{2}H_{4,2,-5,1}-\frac{3}{2}H_{4,2,1,-5}+\frac{45}{32}H_{4,2,1,5}+\frac{5}{2}H_{4,2,2,-4}-\frac{35}{16}H_{4,2,2,4}+\frac{5}{2}H_{4,2,3,-3}-\frac{15}{8}H_{4,2,3,3}+\frac{5}{2}H_{4,2,4,-2}-\frac{5}{4}H_{4,2,4,2}$$
$$+\frac{5}{2}H_{4,2,5,-1}+\frac{45}{32}H_{4,2,5,1}-\frac{343}{64}H_{4,2,6}+3H_{4,3,-5}+4H_{4,3,-4,-1}+\frac{5}{2}H_{4,3,-4,1}-\frac{3}{2}H_{4,3,1,-4}+\frac{21}{16}H_{4,3,1,4}+\frac{5}{2}H_{4,3,2,-3}-\frac{15}{8}H_{4,3,2,3}$$
$$+\frac{5}{2}H_{4,3,3,-2}-\frac{5}{4}H_{4,3,3,2}+\frac{5}{2}H_{4,3,4,-1}+\frac{21}{16}H_{4,3,4,1}-\frac{59}{16}H_{4,3,5}+3H_{4,4,-4}+4H_{4,4,-3,-1}+\frac{5}{2}H_{4,4,-3,1}-\frac{3}{2}H_{4,4,1,-3}+\frac{9}{8}H_{4,4,1,3}$$
$$+\frac{5}{2}H_{4,4,2,-2}-\frac{5}{4}H_{4,4,2,2}+\frac{5}{2}H_{4,4,3,-1}+\frac{9}{8}H_{4,4,3,1}-\frac{65}{16}H_{4,4,4}+3H_{4,5,-3}+4H_{4,5,-2,-1}+\frac{5}{2}H_{4,5,-2,1}-\frac{3}{2}H_{4,5,1,-2}+\frac{3}{4}H_{4,5,1,2}$$
$$+\frac{5}{2}H_{4,5,2,-1}+\frac{3}{4}H_{4,5,2,1}-\frac{9}{2}H_{4,5,3}+3H_{4,6,-2}+4H_{4,6,-1,-1}+\frac{5}{2}H_{4,6,-1,1}-\frac{3}{2}H_{4,6,1,-1}+\frac{93}{64}H_{4,6,1,1}-\frac{45}{16}H_{4,6,2}+7H_{4,7,-1}$$
$$+\frac{333}{16}H_{4,7,1}+\frac{729}{64}H_{4,8}-34H_{5,-7}+28H_{5,-6,-1}+2H_{5,-6,1}-2H_{5,-5,-2}+4H_{5,-5,-1,-1}+4H_{5,-5,-1,1}-2H_{5,-5,1,1}-4H_{5,-5,2}$$
$$-26H_{5,1,-6}-2H_{5,1,-5,1}-2H_{5,1,1,-5}+\frac{15}{8}H_{5,1,1,5}-2H_{5,1,2,-4}+\frac{7}{4}H_{5,1,2,4}-2H_{5,1,3,-3}+\frac{3}{2}H_{5,1,3,3}-2H_{5,1,4,-2}+H_{5,1,4,2}$$
$$-2H_{5,1,5,-1}+\frac{15}{8}H_{5,1,5,1}+\frac{51}{2}H_{5,1,6}+4H_{5,2,-4,-1}+2H_{5,2,-4,1}-2H_{5,2,1,-4}+\frac{7}{4}H_{5,2,1,4}+2H_{5,2,2,-3}-\frac{3}{2}H_{5,2,2,3}+2H_{5,2,3,-2}$$
$$-H_{5,2,3,2}+2H_{5,2,4,-1}+\frac{7}{4}H_{5,2,4,1}-\frac{3}{4}H_{5,2,5}-2H_{5,3,-4}+4H_{5,3,-3,-1}+2H_{5,3,-3,1}-2H_{5,3,1,-3}+\frac{3}{2}H_{5,3,1,3}+2H_{5,3,2,-2}$$
$$-H_{5,3,2,2}+2H_{5,3,3,-1}+\frac{3}{2}H_{5,3,3,1}+\frac{1}{2}H_{5,3,4}-2H_{5,4,-3}+4H_{5,4,-2,-1}+2H_{5,4,-2,1}-2H_{5,4,1,-2}+H_{5,4,1,2}+2H_{5,4,2,-1}$$
$$+H_{5,4,2,1}-\frac{1}{2}H_{5,4,3}-2H_{5,5,-2}+4H_{5,5,-1,-1}+2H_{5,5,-1,1}-2H_{5,5,1,-1}+\frac{15}{8}H_{5,5,1,1}-\frac{1}{8}H_{5,5,2}+2H_{5,6,-1}+\frac{51}{2}H_{5,6,1}$$
$$+\frac{1075}{32}H_{5,7}-49H_{6,-6}+28H_{6,-5,-1}-2H_{6,-5,1}-2H_{6,-4,-2}+4H_{6,-4,-1,-1}+4H_{6,-4,-1,1}-\frac{5}{2}H_{6,-4,1,1}-4H_{6,-4,2}-30H_{6,1,-5}$$
$$-\frac{5}{2}H_{6,1,-4,1}-\frac{5}{2}H_{6,1,1,-4}+\frac{35}{16}H_{6,1,1,4}-\frac{5}{2}H_{6,1,2,-3}+\frac{15}{8}H_{6,1,2,3}-\frac{5}{2}H_{6,1,3,-2}+\frac{5}{4}H_{6,1,3,2}-\frac{5}{2}H_{6,1,4,-1}+\frac{35}{16}H_{6,1,4,1}+\frac{115}{4}H_{6,1,5}$$
$$-4H_{6,2,-4}+4H_{6,2,-3,-1}+\frac{3}{2}H_{6,2,-3,1}-\frac{5}{2}H_{6,2,1,-3}+\frac{15}{8}H_{6,2,1,3}+\frac{3}{2}H_{6,2,2,-2}-\frac{3}{4}H_{6,2,2,2}+\frac{3}{2}H_{6,2,3,-1}+\frac{15}{8}H_{6,2,3,1}+\frac{39}{16}H_{6,2,4}$$
$$-6H_{6,3,-3}+4H_{6,3,-2,-1}+\frac{3}{2}H_{6,3,-2,1}-\frac{5}{2}H_{6,3,1,-2}+\frac{5}{4}H_{6,3,1,2}+\frac{3}{2}H_{6,3,2,-1}+\frac{5}{4}H_{6,3,2,1}+\frac{11}{4}H_{6,3,3}-6H_{6,4,-2}+4H_{6,4,-1,-1}$$
$$+\frac{3}{2}H_{6,4,-1,1}-\frac{5}{2}H_{6,4,1,-1}+\frac{35}{16}H_{6,4,1,1}+2H_{6,4,2}-2H_{6,5,-1}+\frac{115}{4}H_{6,5,1}+\frac{765}{16}H_{6,6}-58H_{7,-5}+28H_{7,-4,-1}-5H_{7,-4,1}$$
$$-2H_{7,-3,-2}+4H_{7,-3,-1,-1}+4H_{7,-3,-1,1}-3H_{7,-3,1,1}-4H_{7,-3,2}-33H_{7,1,-4}-3H_{7,1,-3,1}-3H_{7,1,1,-3}+\frac{9}{4}H_{7,1,1,3}-3H_{7,1,2,-2}$$
$$+\frac{3}{2}H_{7,1,2,2}-3H_{7,1,3,-1}+\frac{9}{4}H_{7,1,3,1}+30H_{7,1,4}-7H_{7,2,-3}+4H_{7,2,-2,-1}+H_{7,2,-2,1}-3H_{7,2,1,-2}+\frac{3}{2}H_{7,2,1,2}+H_{7,2,2,-1}$$
$$+\frac{3}{2}H_{7,2,2,1}+\frac{15}{4}H_{7,2,3}-9H_{7,3,-2}+4H_{7,3,-1,-1}+H_{7,3,-1,1}-3H_{7,3,1,-1}+\frac{9}{4}H_{7,3,1,1}+\frac{7}{2}H_{7,3,2}-5H_{7,4,-1}+30H_{7,4,1}$$
$$+\frac{439}{8}H_{7,5}-\frac{125}{2}H_{8,-4}+28H_{8,-3,-1}-7H_{8,-3,1}-2H_{8,-2,-2}+4H_{8,-2,-1,-1}+4H_{8,-2,-1,1}-\frac{7}{2}H_{8,-2,1,1}-4H_{8,-2,2}-35H_{8,1,-3}$$
$$-\frac{7}{2}H_{8,1,-2,1}-\frac{7}{2}H_{8,1,1,-2}+\frac{7}{4}H_{8,1,1,2}-\frac{7}{2}H_{8,1,2,-1}+\frac{7}{4}H_{8,1,2,1}+28H_{8,1,3}-9H_{8,2,-2}+4H_{8,2,-1,-1}+\frac{1}{2}H_{8,2,-1,1}-\frac{7}{2}H_{8,2,1,-1}$$
$$+\frac{13}{4}H_{8,2,2}-7H_{8,3,-1}+\frac{7041}{256}H_{8,3,1}+\frac{219}{4}H_{8,4}-64H_{9,-3}+28H_{9,-2,-1}-8H_{9,-2,1}-2H_{9,-1,-2}+4H_{9,-1,-1,-1}+4H_{9,-1,-1,1}$$
$$-4H_{9,-1,1,1}-4H_{9,-1,2}-36H_{9,1,-2}-4H_{9,1,-1,1}-4H_{9,1,1,-1}+4H_{9,1,1,1}+20H_{9,1,2}-6H_{9,2,-1}+\frac{1281}{64}H_{9,2,1}+\frac{2827}{64}H_{9,3}$$
$$-66H_{10,-2}+32H_{10,-1,-1}-4H_{10,-1,1}-36H_{10,1,-1}+36H_{10,1,1}+\frac{4481}{128}H_{10,2}-34H_{11,-1}+180H_{11,1}+\frac{20821}{128}H_{12}$$

Equipped with the above relations we want to construct a computer program that generates all possible equations and then solves for the MZV's, leaving us in the end with a minimal set as remaining unknowns.

What are we up against?

For the non-alternating sums there are 2^{W-3} objects to be determined (there is a duality relation that cuts the number down by (roughly) a factor 2).

We would like to go beyond what M. Kaneko, M. Noro and K. Tsurumaki managed. They treated this as a matrix problem (with a size of $2^{W-3} \times 2^{W-4}$) and went to W=20. Using calculus modulus a 15 bits prime they needed about 18 Gbytes of memory and could not go beyond this.

W	size	time
16	72M	150
17	288M	880
18	1.2G	5000
19	4.6G	33000
20	18G	245000

Parameters of the Kaneko et al program on an 8 core computer.

The program managed to determine the size of a Fibonacci basis. The size was according to the Zagier conjecture.
It should be noticed that the matrix is sparse. In our program the weight 20 expression has at its worst 4158478 terms (100 Mbytes) which means that only one in 2000 entries of the matrix would not be zero.

For the alternating sums one needs to calculate $4\times 3^{W-2}$ objects. Results have been reported in the past for $W=8$ by the Lille group and $W=8,9,10$ by JV. The results up to $W=9$ have been available in the FORM distribution.

To $W=7$ the stuffles and the shuffles suffice. At $W=8,9,10$ it is sufficient to add the doubling relations. Starting at $W=11$ the generalized doubling relations are needed to obtain a minimal basis that is in accordance with the Broadhurst conjecture.

For $W=12$ there will be 236196 MZV's to be determined.

The program

Trying to solve large systems of equations can be quite a challenge. And because we want to reach the limits of what is possible we need the most powerful program we can lay our hands on. Of course we have FORM, but there is the newer TFORM that can make use of multi-core machines. This gives added power.
We use the MZV program as a test under extreme conditions for TFORM. This has enabled us to

- Test and improve TFORM.
- Improve the program for solving sets of equations.
- Get more results on MZV's.

The first thing to consider is that it may not be possible to have all equations in memory simultaneously. Hence we should select a method that doesn't need this.

So how do we solve 5×10^6 equations with 2×10^6 unknowns?

We start generating a master expression which contains one term for each sum that we want to compute. For the non-alternating sums of weight 4 this expression looks in computer terms like

```
FF =
   +E(0,0,0,1)*(H(0,0,0,1))
   +E(0,0,1,1)*(H(0,0,1,1))
   +E(0,1,0,1)*(H(0,1,0,1));
```

We have used already that we will only compute the finite elements and that there is a duality that allows us to eliminate all elements with a depth greater than half the weight. When the depth is exactly half the weight we choose from a sum and its dual the element that comes first lexicografically.

We pull the function E outside brackets. The contents of a bracket is what we know about the object indicated by the indices of the function E. At first this is all trivial knowledge.
Assume now that we generate the stuffle relation

$$H_{0,1}H_{0,1} \;=\; H_{0,0,0,1} + 2H_{0,1,0,1}$$

The left hand side can be substituted from the tables for the lower weight MZV's. Hence it becomes ζ_2^2. The right hand side objects are replaced by the contents of the corresponding E brackets in the master expression. These are for now trivial substitutions. From the result we generate the substitution

```
id H(0,1,0,1) = z2^2/2-H(0,0,0,1)/2;
```

We apply this to the master expression. Hence the master expression becomes

```
FF =
   +E(0,0,0,1)*(H(0,0,0,1))
   +E(0,0,1,1)*(H(0,0,1,1))
   +E(0,1,0,1)*(z2^2/2-H(0,0,0,1)/2);
```

Let us now generate the corresponding shuffle relation:

$$H_{0,1}H_{0,1} \;=\; 4H_{0,0,1,1} + 2H_{0,1,0,1}$$

and replace the right hand side objects by the contents of the corresponding E bracket in the master expression. This gives

$$\zeta_2^2 \;=\; 4H_{0,0,1,1} + \zeta_2^2 - H_{0,0,0,1}$$

which leads to the substitution

```
id H(0,0,1,1) = H(0,0,0,1)/4;
```

We obtain

```
FF =
    +E(0,0,0,1)*(H(0,0,0,1))
    +E(0,0,1,1)*(H(0,0,0,1)/4)
    +E(0,1,0,1)*(z2^2/2-H(0,0,0,1)/2);
```

We also need the divergent shuffles and stuffles. This is done by including the shuffles involving the basic divergent object and breaking down the multiple divergent sums with the stuffle relations as in:

$$\begin{aligned} H_1 H_{0,0,1} &= 2H_{0,0,1,1} + H_{0,1,0,1} + H_{1,0,0,1} \\ &= -H_{0,0,0,1} + H_{0,0,1,1} + H_{0,1,0,1} + H_1 H_{0,0,1}; \end{aligned}$$

Substituting from the master expression we obtain the relation

$$0 = -\frac{5}{4}H_{0,0,0,1} + \frac{1}{2}\zeta_2^2$$

Hence the substitution

```
id H(0,0,0,1) = z2^2*2/5;
```

and finally the master expression becomes

```
FF =
   +E(0,0,0,1)*(z2^2*2/5)
   +E(0,0,1,1)*(z2^2/10)
   +E(0,1,0,1)*(z2^2*3/10);
```

Now we can read off the values of all MZV's of weight 4 that we went to compute. All other elements (dual or divergent) can be obtained from these by trivial operations that involve the use of one or two relations only.

In practise we are a bit more sophisticated. It is noticed that the master expression can become rather big and hence to make a single substitution on it for each equation gives much sorting overhead. Therefore we group the equations and first diagonalize this group as much as possible. Then we substitute the results of the entire group into the master expression. When the group has 1000 elements this would give 1000 substitution statements. The result would be 1000 pattern matchings per term. This is solved by using tables which FORM can use internally in a binary tree search. The result is a rather fast program.

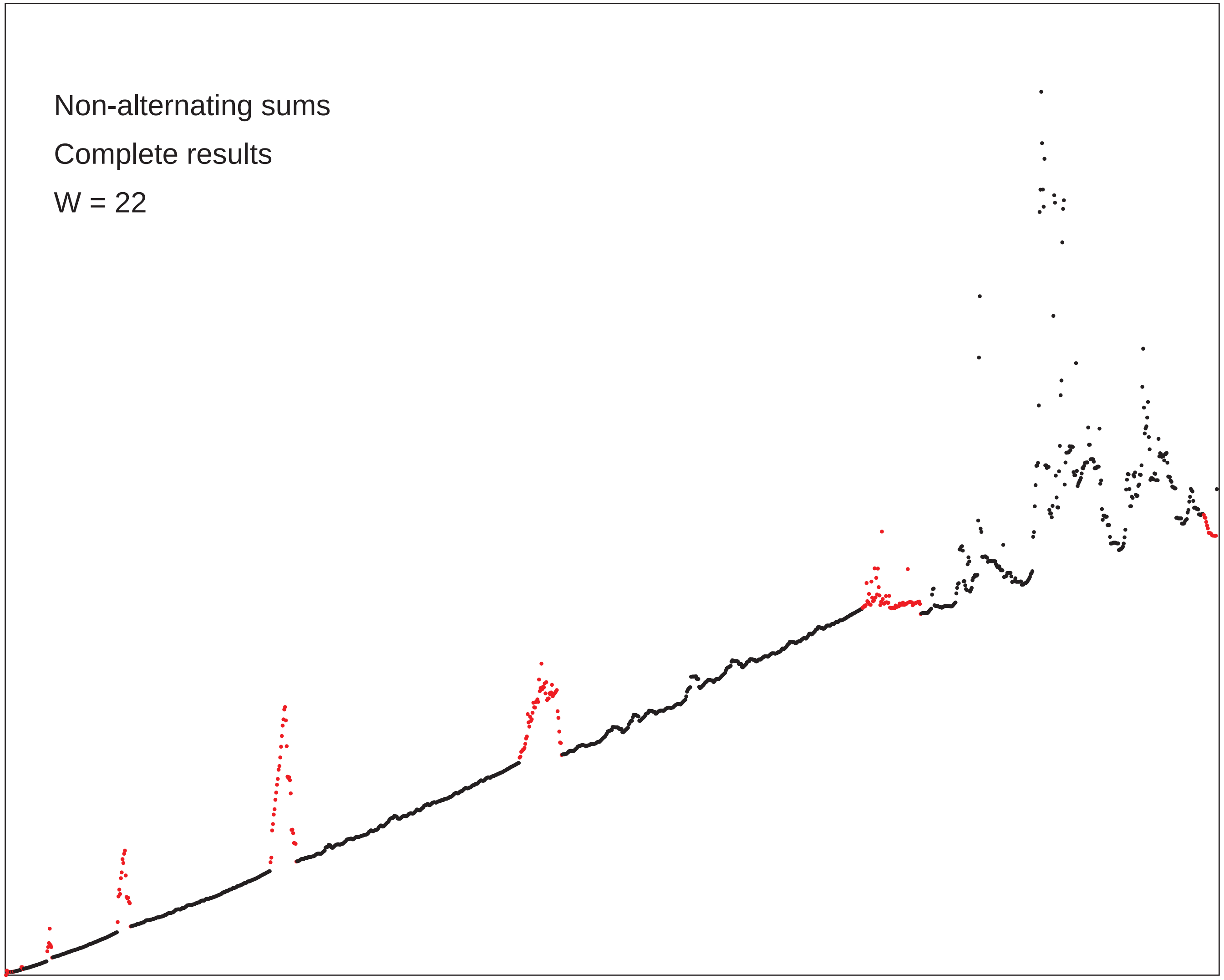
The size of the group is a function of the size of the problem. The optimal value is more or less related to the square root of the number of variables.

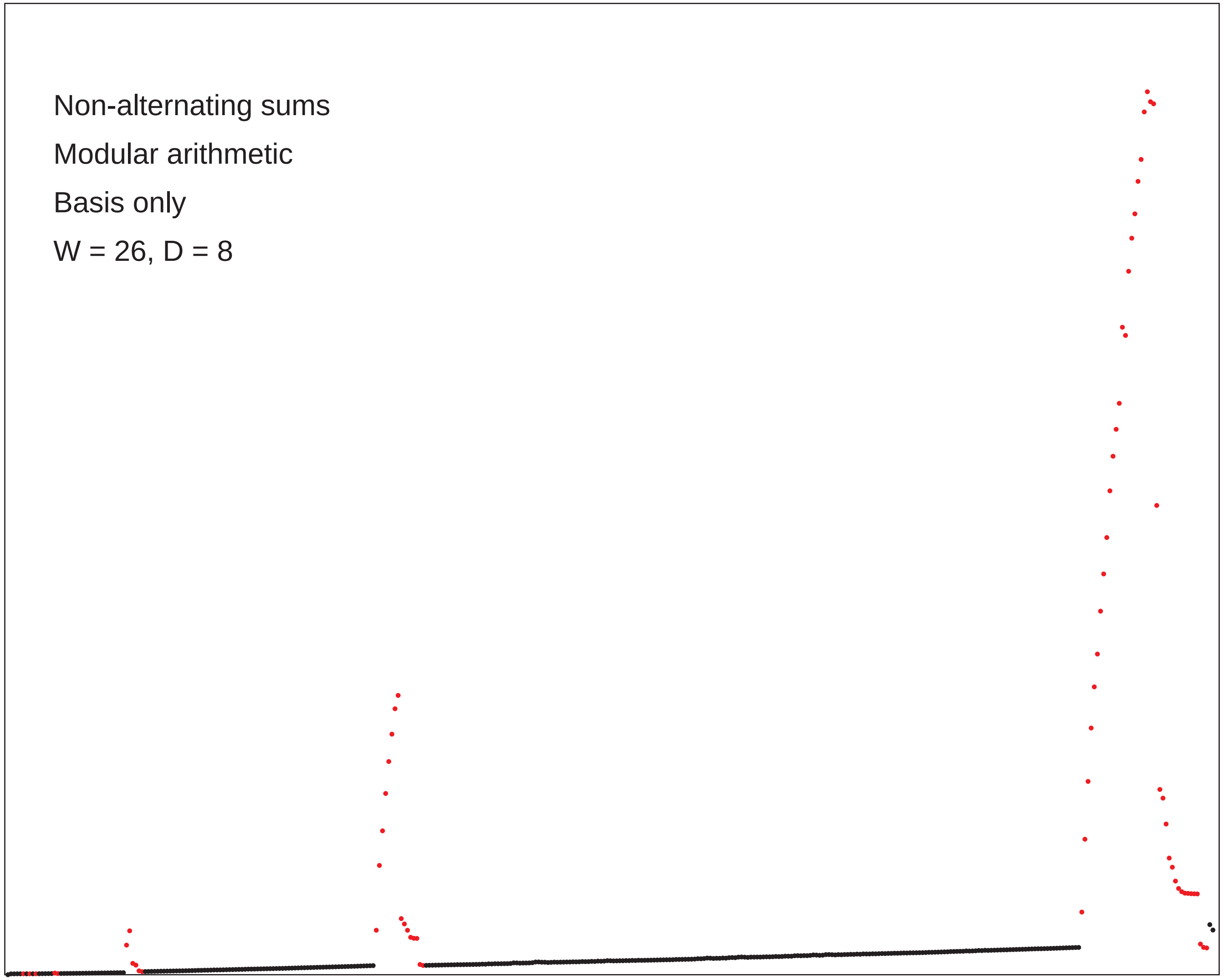
The whole program for the non-alternating sums is only about 600 lines including commentary (400 lines in a stripped version). For the alternating sums it is a few hundred lines longer (the procedure for the generalized doubling relations is almost 200 lines).

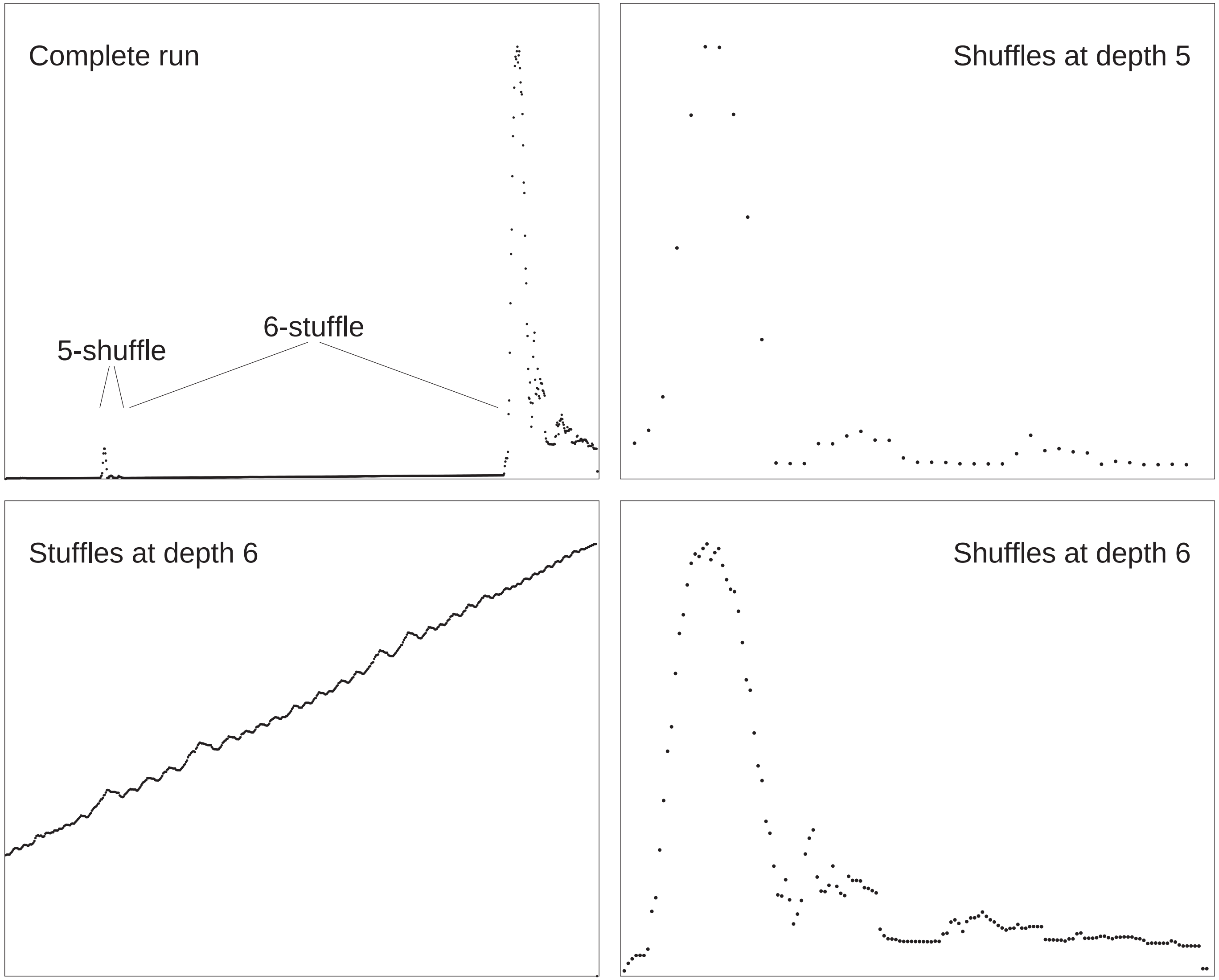
The major problem in the program is the order in which we feed in the equations. This can make a big difference in both the execution time and the space used (orders of magnitude!).

The stuffles don't cause too many troubles, but the shuffles are rather difficult to control. It is extremely hard to "block diagonalize" this system.

We have a heuristic ordering of the equations that we couldn't improve upon. Yet it still contains inefficiencies. This is shown in the following graphs in which we have on the x-axis the number of the module and on the y-axis the size of the output expression in that module.


Non-alternating sums
Complete results
W = 22


Non-alternating sums
Modular arithmetic
Basis only
W = 26, D = 8



Size of output expression for each module during phases of the alternating W=18,D=6 run.

We will run three types of programs.

1. A full expression of all MZV's in a minimal (Lyndon) basis.
2. An expression of all MZV's in a minimal (Lyndon) basis modulus a prime number. We drop all terms that are products of lower weight objects.
3. An expression of all MZV's in a minimal (Lyndon) basis modulus a prime number. We drop all terms that are products of lower weight objects. We consider only elements up to a given depth D.

We run most of our programs on the computer of the theory group in Karlsruhe. This machine has 24 nodes, each node has 8 Xeon cores at 3 GHz with 32 Gbytes of memory and a 4Tbyte disk. One of these nodes has been reserved for development work with TFORM and was hence used most of the time for this project. The other nodes were just running unrelated programs.

We also used some of the blade computers at DESY Zeuthen (8 Xeon cores at a somewhat lower frequency and 16 Gbytes of memory) and the main development machine for TFORM at Nikhef which has 4 Opterons at 2.3 GHz, 16 Gbytes of memory and a 1.5 Tbytes disk.

The last computer has also been used to compose the data mine which is more or less waiting for disk space in the public disks of Nikhef so that it may be made publicly available.

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Alternating Sums

The alternating sums need the doubling ($W \geq 8$) and the generalized doubling ($W \geq 11$) formulas. They are also needed if we want to obtain results up to a given depth. Details are in the paper.

W	variables	eqns	remaining	size	output	time
4	36	57	1	4.3K	2.0K	0.06
5	108	192	2	21K	8.9K	0.12
6	324	665	2	98K	42K	0.37
7	972	2205	4	472K	219K	1.71
8	2916	7313	5	2.25M	1.15M	7.78
9	8748	23909	8	11M	6.3M	50
10	26244	77853	11	58M	36M	353
11	78732	251565	18	360M	213M	3266
12	236196	809177	25	3.1G	1.29G	47311

The size of the outputs becomes a bigger problem than the running time.

We have also runs with restricted depth. The most important ones are where we limit the depth to 6 or less. In this case we have used modular arithmetic and dropped all terms that are products of lower weight objects in an all out attempt to obtain $W = 18, D \leq 6$.

weight	constants	remaining	running time [sec]	output [Mbyte]
13	56940	22	2611	
14	90564	37	12716	51
15	138636	35	55204	87
16	205412	66	206951	214
17	295916	55	789540	288
18	416004	109	2622157	711

The last run was rather impressive. It took one month on an 8 core Xeon machine, working its way through a combined total of more than 7×10^{12} terms or 7 TeraTerms!

Runs to depth 5 are to weight 21 and runs to depth 4 are to weight 30.

Non-Alternating Sums

In the first sequence of programs we try to see how far we can get. We use a 31 bits prime (2147479273) and try to determine a Lyndon basis. We drop all terms that are products of lower weight objects. We want expressions for all MZV's of the given weights in terms of the basis.

W	Group	size	output	CPU	time	Eff.
16	128	1.7M	1.2M	300	57	5.25
17	256	5.6M	3.2M	713	134	5.32
18	256	14.4M	7.2M	2706	465	5.82
19	512	39M	19M	6901	1206	5.72
20	512	104M	45M	30097	4819	6.25
21	1024	239M	114M	75302	12379	6.08
22	1024	767M	280M	449202	65644	6.84
23	2048	2.17G	734M	992431	151337	6.56
24	2048	8.04G	1.77G	9251325	1268247	7.29

At this point we noticed that all basis elements had a depth that fulfilled $D \leq W/3$. Hence assuming that this will be always the case we made a few more runs. And in addition we made some 'incomplete' runs.

W	D	size	output	CPU	real	Eff.
23	7	1.55G	89M	61447	9579	6.41
24	8	673M	380M	536921	72991	7.36
25	7	6.37G	244M	369961	50197	7.37
26	8	38.3G	1160M	4786841	651539	7.35
27	7	12.7G	914M	2152321	277135	7.77
28	6	2.88G	314M	235972	30960	7.62
29	7	41.0G	3007M	8580364	1112836	7.71
30	6	6.27G	658M	829701	106353	7.80

It shouldn't come as a great surprise that all the results of the above runs are in agreement with the Zagier and Broadhurst-Kreimer conjectures.

More later.....

We also made complete runs. That is: over the rationals and including products of lower weight objects. This gave the following:

W	size	output	num	CPU	real	Eff.	Rat.
16	10.9M	10.6M	21	254	59	4.29	1.05
17	30M	29M	19	690	149	4.62	1.11
18	86M	77M	25	3491	700	4.98	1.51
19	218M	205M	27	9460	1855	5.10	1.54
20	756M	552M	31	65640	11086	5.92	2.30
21	1.63G	1.55G	39	165561	27771	5.96	2.24
22	8.05G	4.00G	36	2276418	326489	6.97	4.97

It should become clear by now that the size of the output becomes a major obstacle. To store millions of expressions, each of them with quite a number of terms, will take Gigabytes.

```
Fill htable22(0,0,1,0,1,0,1,0,1,0,1,0,0,0,0,0,0,1,1,0,1,1)=229121/1728*
        z14z3z1z1z2z1+173609/576*z14z3z1z2z1z1+15692195/31104*
        z14z3z2z1z1z1+3726961/31104*z14z4z1z1z1z1-56339/1152*z14z5z1z2
        -3378973/13824*z14z5z2z1+1007419717/2488320*z14z6z1z1-3423/16*
        z15z2z1z2z1z1+2073365/1296*z15z3z1z1z1z1-307559/216*z15z4z1z2-
        666657535/165888*z15z4z2z1+2485272541/1658880*z15z5z1z1-
        502565387/31104*z16z2z1z1z1z1-8240323/1728*z16z3z1z2-
        50468588359/3317760*z16z3z2z1-4457267917/829440*z16z4z1z1-
        188177646093889/8599633920*z16z6+193151925403/19906560*
        z17z3z1z1+6998148491689/13271040*z18z2z1z1+5830492751924959/
        6879707136*z18z4-64399622164350811/1911029760*z20z2-1415173/
        43200*z5z3z3^2-141/4*z7z3*z8z2z1z1+15765715/62208*z7z3*z9z3+
        108/25*z5z3*z5z3z3z3+2332219/48600*z5z3*z9z5+654535363/5702400
        *z5z3*z11z3-30606548603/921600*z11*z5z3z3+4674331597474072633/
        57330892800*z11^2+3646960903267/217728000*z9*z5z5z3-
        26283756319/1451520*z9*z7z3z3+227618177777097021133/
        1504935936000*z9*z13-54161081/10368*z7*z10z2z1z1z1-14895806515/
        4644864*z7*z7z5z3+1810659173/497664*z7*z9z3z3+
        144492042468201625557/120394874880*z7*z15+7516571189/1126400*
        z7^2*z5z3-4571/5*z5*z5z3z3z3z3-27702313/5184*z5*z12z2z1z1z1+
        1897913010697639/388949299200*z5*z7z5z5-737558452534697/
        155579719680*z5*z7z7z3-8678023289443/13891046400*z5*z9z5z3+
        65728422985853/11112837120*z5*z11z3z3+185458251647/136857600*
        z5*z9*z5z3+655173768451/34836480*z5*z7*z7z3+
        89804940812290198429 21/134420877803520*z5*z17-3819/4*z5^2*
        z8z2z1z1-271512575762737/20836569600*z5^2*z9z3-
        1538354691225468 1/55564185600*z5^3*z7-2969/8*z3*z12z4z1z1z1+
        126/25*z3*z5z3z5z3z3-163253/400*z3*z5z5z3z3z3+5677/16*z3*
        z7z3z3z3z3-69740687/10368*z3*z14z2z1z1z1-374706432302269505/
        41015642443776*z3*z7z7z5+5592579619608285 67/109863327974400*z3
        *z9z5z5-675929428026804667/219726655948800*z3*z9z7z3+
        472645097440330207/97656291532800*z3*z11z5z3+17405218743810383/
        2048733388800*z3*z13z3z3+186/25*z3*z5z3*z5z3z3+560126822557/
        8294400*z3*z11*z5z3+241944929861/4976640*z3*z9*z7z3-48533/32*
        z3*z7*z8z2z1z1+3258424132907/44789760*z3*z7*z9z3+32205/16*z3*
        z5*z5z3z3z3+62730931353098707/4069012147200*z3*z5*z9z5-
        211693794294616819/4882814576640*z3*z5*z11z3-117303745103293/
        164229120*z3*z5*z7^2-3785404660891098517/4394533118976*z3*z5^2
        *z9-97948194466623147428643 71/109375046516736000*z3*z19-150567/
        1120*z3^2*z5z5z3z3+37369/224*z3^2*z7z3z3z3-76731/64*z3^2*
        z12z2z1z1+5836777489/4257792*z3^2*z11z5-631656298061/56609280*
        z3^2*z13z3-24/5*z3^2*z5z3^2-1476536914610227/4269957120*z3^2*
        z7*z9-940205/1728*z3^2*z5*z5z3z3-63798454917713/181149696*z3^2
        *z5*z11+89314457/907200*z3^3*z5z5z3-4391335/36288*z3^3*z7z3z3-
        102881298198157/1045094400*z3^3*z13+2015873/25920*z3^3*z5*z5z3
        +4771/112*z3^4*z7z3+178901285/1306368*z3^4*z5^2+129247787/
        466560*z3^5*z7-188/5*z2*z5z3z3z3z3z3-838*z2*z14z2z1z1z1z1-
        400090555909/130636800*z2*z7z3z5z5-860982225443/104509440*z2*
        z7z7z3z3-410971121201/87091200*z2*z7z5z5z3+432991955441/
        55987200*z2*z9z3z5z3-5561422085/1119744*z2*z9z5z3z3+
        30038614163/2488320*z2*z11z3z3z3+12317476820806379/11287019520
        *z2*z13z7-4814984387/46656*z2*z16z2z1z1-26973572103166541417/
        3386105856000*z2*z15z5+6506289659937159453 53/11512759910400*z2
        *z17z3+2703067/16128*z2*z7z3^2+15297217/51840*z2*z5z3*z9z3-
        1967338523/116640*z2*z9*z5z3z3+443971105937439694 5289/
        3837586636800*z2*z9*z11+203331234901/16329600*z2*z7*z5z5z3-
        2245163981/163296*z2*z7*z7z3z3+172861806934439936513/
        213199257600*z2*z7*z13-2530*z2*z5*z10z2z1z1z1+221934828641/
        37324800*z2*z5*z7z5z3-185137871143/18662400*z2*z5*z9z3z3+
        235685777058450464 4547037/6120950685696000*z2*z5*z15-
        8784777689/466560*z2*z5*z7*z5z3-29339484871/12441600*z2*z5^2*
        z7z3-946617250799/97977600*z2*z5^4+4388/5*z2*z3*z5z3z3z3z3-
        2050*z2*z3*z12z2z1z1z1-2515919247697/1620622080*z2*z3*z7z5z5-
        5508608353973/1620622080*z2*z3*z7z7z3-65616653437/19293120*z2*
        z3*z9z5z3+4317757951/602910*z2*z3*z11z3z3+2459401/2880*z2*z3*
        z9*z5z3-5826659/2268*z2*z3*z7*z7z3+168589792847478366952 3733/
        19824227181158400*z2*z3*z17-3112*z2*z3*z5*z8z2z1z1-
        1913867931511/347276160*z2*z3*z5*z9z3-12126144556601/
        2083656960*z2*z3*z5^2*z7-1086/5*z2*z3^2*z5z3z3z3-4867384441/
        1088640*z2*z3^2*z9z5+71577340969/3991680*z2*z3^2*z11z3+
        11050634658317/143700480*z2*z3^2*z7^2+449759798507/4490640*z2*
        z3^2*z5*z9+128*z2*z3^3*z5z3z3-5793264895/139968*z2*z3^3*z11-
        207/5*z2*z3^4*z5z3-162/5*z2*z3^5*z5+27/5*z2^2*z12z2z1z1z1z1-
        984359/75600*z2^2*z7z3z5z3+2137981343/2721600*z2^2*z5z5z5z3-
        11370756889/1814400*z2^2*z7z5z3z3+1301016437/233280*z2^2*
        z9z3z3z3-7911180517/155520*z2^2*z14z2z1z1+336721679218271/
        4528742400*z2^2*z13z5-6306214666487 8129/62705664000*z2^2*z15z3
        +6644509/43200*z2^2*z5z3*z7z3-38514635023952878361/
        1630347264000*z2^2*z9^2-15429815879/1944000*z2^2*z7*z5z3z3-
        8274399031910863279/271724544000*z2^2*z7*z11+4208229059/544320
        *z2^2*z5*z5z5z3-658253387/77760*z2^2*z5*z7z3z3-
        8720289305450158267/952528896000*z2^2*z5*z13-50810851429/
        5443200*z2^2*z5^2*z5z3+999/5*z2^2*z3*z10z2z1z1z1-45306816419/
        2268000*z2^2*z3*z7z5z3+571783303/30375*z2^2*z3*z9z3z3+
        98747576355234045376 2817/127441460450304000*z2^2*z3*z15-
        670666193/72000*z2^2*z3*z7*z5z3-131835349/25920*z2^2*z3*z5*
        z7z3+73744749319/6531840*z2^2*z3*z5^3+1593/10*z2^2*z3^2*
        z8z2z1z1-5617847/40320*z2^2*z3^2*z9z3+113181386863/2177280*
        z2^2*z3^2*z5*z7+186543726721/6531840*z2^2*z3^3*z9-951/100*z2^2
        *z3^6+24711581/15120*z2^3*z5z5z3z3-234965329/136080*z2^3*
        z7z3z3z3-146515315/6048*z2^3*z12z2z1z1-435261786095987/
        7185024000*z2^3*z11z5+2456425078110467/7547904000*z2^3*z13z3+
        12415031/252000*z2^3*z5z3^2+117865176559161139/1046139494400*
        z2^3*z7*z9-226177577/45360*z2^3*z5*z5z3z3+
        5393961686980635863 69/212366317363200*z2^3*z5*z11+1568719081/
        661500*z2^3*z3*z5z5z3-811187497/317520*z2^3*z3*z7z3z3-
        26840936328970507766 81/953087845248000*z2^3*z3*z13-6731243/
        2800*z2^3*z3*z5*z5z3+1080509/15120*z2^3*z3^2*z7z3+2009725/168*
        z2^3*z3^2*z5^2+570093989/52920*z2^3*z3^3*z7+428519309/105000*
        z2^4*z5z3z3z3+20548647742626947/411505920000*z2^4*z9z5-
        9101449727910540 17/6035420160000*z2^4*z11z3-
        137357515583841561 49/12070840320000*z2^4*z7^2-
        946886957134260991 27/58342394880000*z2^4*z5*z9-141084539/78750
        *z2^4*z3*z5z3z3+140544106016863793716739/2601929817527040000*
        z2^4*z3*z11+17966741/252000*z2^4*z3^2*z5z3+5233954847/13608000
        *z2^4*z3^3*z5-89747783/12474*z2^5*z8z2z1z1+42587330003873/
        2235340800*z2^5*z9z3+19746145461233683237/53480528640000*z2^5*
        z5*z7+128732393599968680184 7583/3066560142085440000*z2^5*z3*z9
        +1323224553841/1571724000*z2^5*z3^4+196664555715971051/
        22884301440000*z2^6*z7z3+689800062898138493 23/
        11355698914560000*z2^6*z5^2+94971440713063356192982873/
        10464636484866564 00000*z2^6*z3*z7+313619248788976309/
        44951306400000*z2^7*z5z3+909871564554223072 79/1064596773240000
        *z2^7*z3*z5+21641573024873924687/3863315055600000*z2^8*z3^2-
        288994255199496099205383627006427/16273221799745710800000000*
        z2^11;
```

We have of course more results when we restrict the depth. They are less interesting from the viewpoint of this talk.

Data mining

The results of all the runs we made have been put together in a place that will be publicly accessible. We call it the MZV datamine.

The format of the files is text (but in a notation that is most suited for FORM). In some cases there may be binary FORM files.

There are also FORM programs that help to read the files. And there are example files that show how one can manipulate the data. In particular there are some programs that show how to change bases.

One should keep in mind that one needs more than the average laptop to manipulate some of these files. Putting a 4 Gbyte file in an editor is rather stressful for a computer.

The FORM binary files (extension is .sav) are easier to manipulate. Even laptops may do in many cases.

For the bigger tables 32-bits processors may not work. FORM has some restrictions there.

Of course, FORM and TFORM are freely available.

The first things we look up in the datamine are some relations that Broadhurst discovered in the 1990's with the use of PSLQ. Now we can obtain 'formal' proof of them. They are so-called push down relations in which an object that has at least depth D as a non-alternating sum, can be expressed in terms of depth $D-2$ alternating sums.

The simplest example of such a push down relation is the following:

$$\begin{aligned} H_{8,2,1,1} = & -\frac{1593344}{47475}H_{-11,-1} + \frac{10624}{28485}H_{-9,-3} + \frac{56896}{712125}H_{-7,-5} \\ & +\frac{64}{243}H_{-3}^{4} + \frac{194772992}{2421225}H_{-9}H_{-3} + \frac{56203264}{712125}H_{-7}H_{-5} \\ & +\frac{21504}{1583}\zeta_2 H_{-9,-1} - \frac{768}{1583}\zeta_2 H_{-7,-3} - \frac{8660992}{299187}\zeta_2 H_{-7}H_{-3} \\ & -\frac{529216}{39575}\zeta_2 H_{-5}^{2} + \frac{512}{171}\zeta_2^2 H_{-7,-1} - \frac{512}{2565}\zeta_2^2 H_{-5,-3} \\ & -\frac{98624}{12825}\zeta_2^2 H_{-5}H_{-3} - \frac{352}{315}\zeta_2^3 H_{-3}^{2} - \frac{5975591045926246}{18760001932546875}\zeta_2^6 \end{aligned}$$

The next one at $W = 15$ becomes already rather bad.

$$
\begin{aligned}
H_{6,2,5,1,1} \;=\; & -\frac{2800918270496177337699639890311817494218475426579852912259 6}{305651913521473711081726272715815595332022071566091290625}\zeta_2^6 H_{-3} \\
& -\frac{6868723880789436171485501864576122208348106977850627944}{38707190153725780323875000478239018538890298220085625}\zeta_2^5 H_{-5} \\
& -\frac{3526208994483592359567080506287829836788447453426 56}{10136380124102082253309022120299197412125409744 65}\zeta_2^4 H_{-7} \\
& -\frac{450346189502746275947949624113680029363879966160832}{10796896122163872074326652634403907017928068023 75}\zeta_2^3 H_{-9} \\
& +\frac{2176}{945}\zeta_2^3 H_{-3}^3 - \frac{2037950288768}{2234346324525}\zeta_2^2 H_{-3}^2 H_{-5} + \frac{176193784832}{29791284327}\zeta_2^2 H_{-3} H_{-7,-1} \\
& -\frac{19599298746371297483193212289321032985913744503680}{4725229832288188719587664447056768718434401535 1}\zeta_2^2 H_{-11} \\
& -\frac{172882684928}{446869264905}\zeta_2^2 H_{-3} H_{-5,-3} - \frac{25300992}{8296097}\zeta_2^2 H_{-9,-1,-1} + \frac{74885120}{174218037}\zeta_2^2 H_{-7,-3,-1} \\
& +\frac{18508800}{58072679}\zeta_2^2 H_{-7,-1,-3} - \frac{111818752}{871090185}\zeta_2^2 H_{-5,-5,-1} - \frac{224668672}{7839811665}\zeta_2^2 H_{-5,-3,-3} \\
& -\frac{22126906767952017266176}{6122114344816491010 5}\zeta_2 H_{-3}^2 H_{-7} - \frac{30664508461328784676096}{4372938817726065007 5}\zeta_2 H_{-3} H_{-5}^2 \\
& +\frac{3632939862491022991 36}{323921393905634445}\zeta_2 H_{-3} H_{-9,-1} - \frac{4369910014768059392}{107973797968544815}\zeta_2 H_{-3} H_{-7,-3}
\end{aligned}
$$

$$+\frac{1644070289092638208}{1841625107486235}\zeta_2 H_{-5}H_{-7,-1}-\frac{3361783780336378888}{5524875322458705}\zeta_2 H_{-5}H_{-5,-3}$$
$$+\frac{8536274697078583916151006789674894498122216 96}{5971326016876880366312289810238865388772 5}\zeta_2 H_{-13}$$
$$-\frac{58973326655000576}{40925002388583}\zeta_2 H_{-11,-1,-1}+\frac{11777430067486720}{122775007165749}\zeta_2 H_{-9,-3,-1}$$
$$+\frac{20405818414364672}{613875035828745}\zeta_2 H_{-9,-1,-3}-\frac{84062945969500 16}{613875035828745}\zeta_2 H_{-7,-5,-1}$$
$$+\frac{11529790700 87168}{368325021497247}\zeta_2 H_{-7,-3,-3}+\frac{122738670251837 44}{613875035828745}\zeta_2 H_{-7,-1,-5}$$
$$-\frac{2873606698310656}{1841625107486235}\zeta_2 H_{-5,-5,-3}-\frac{1792}{3645}H^5_{-3}$$
$$-\frac{425689628884887186442759975705 6}{345082792925864909648655961 65}H^2_{-3}H_{-9}$$
$$+\frac{390750819618975077368265702232899584}{712288099409986982475670367743425}H_{-3}H_{-5}H_{-7}$$
$$-\frac{1208984451017729087145407744}{375907181836454149944069675}H_{-3}H_{-7,-5}$$
$$+\frac{3840626217263581248362959360}{1353265854611234939798650 83}H_{-3}H_{-9,-3}$$
$$-\frac{40937844638235531233520436428 8}{676632927305617469899325415}H_{-3}H_{-11,-1}$$
$$+\frac{22436065292082513617347371398041 6}{11421788288297549873999631738 75}H^3_{-5}$$

$$
\begin{aligned}
&-\frac{66613761278338041301228507648 0}{1015270070070893322133300599}H_{-5}H_{-9,-1}\\
&+\frac{8793800151761933528704002 56}{3760259518781086378271483 7}H_{-5}H_{-7,-3}\\
&-\frac{2844342500576392653874336768 0}{853006439162531976271187 49}H_{-7}H_{-7,-1}\\
&+\frac{568868500115278530774867353 6}{2559019317487595928813562 47}H_{-7}H_{-5,-3}\\
&-\frac{21125334595108151477529198769507 84}{15761057698646347406673907498 5}H_{-15}\\
&+\frac{8529416561599079443949977 6}{712620249926927298472 17}H_{-13,-1,-1}\\
&-\frac{174907949900455846422691 84}{2137860749780781895416 51}H_{-11,-3,-1}-\frac{125855319359428327200389 12}{2137860749780781895416 51}H_{-11,-1,-3}\\
&+\frac{467182771000149178765312 0}{2137860749780781895416 51}H_{-9,-5,-1}+\frac{487242448068471372021555 2}{1924074674802703705874859}H_{-9,-3,-3}\\
&+\frac{862712257577949234710528}{712620249926927298472 17}H_{-9,-1,-5}-\frac{510117151499171079299072}{712620249926927298472 17}H_{-7,-7,-1}\\
&-\frac{474464980999666928489984}{1924074674802703705874859}H_{-7,-5,-3}-\frac{247377046826432734064128}{641358224934234568624953}H_{-7,-3,-5}
\end{aligned}
$$

It just gives some more respect for Broadhurst who located these relations with the help of PSLQ in the 90's.

Verifying push downs isn't necessarily a trivial lookup in the tables. For example there are two non-alternating sums at weight 17 and depth 5. There should be one push down. It is however a linear combination of the two that obtains the push down as in

$$H_{5,3,5,2,2} - 52 H_{5,3,3,3,3} \longrightarrow (D \leq 3)$$

or

$$H_{6,4,5,1,1} + \frac{72}{5} H_{5,3,3,3,3} \longrightarrow (D \leq 3)$$

We do not show here the right hand sides as they involve 99 terms each.

$$-\frac{391637561921020510388495527693101233498239312730472192870557734150375516560722487938377037680200651787224018403770884822305866731511488}{12096842033646879193852836812120840799898022503305835922565953025114624797521762549901601984894859006780341916995765114718351715625}\zeta_2^7 H_{-3}$$

This means that checks of the more complicated push downs require quite an amount of algebra first to get the 'non-push downs' out of the way.

These push downs seem to exist because of the doubling and the generalized doubling relations.

We checked this for the only system that we have complete control over: $W = 12$. Here we have the object $H_{8,2,1,1}$.

If we omit the doubling and the generalized doubling relations, there are three extra undetermined objects. Two of depth 4 and one of depth 2. The push down doesn't take place.

If we use the doubling relations and we omit the generalized doubling relations there is only one extra undetermined object of depth 4. And the push down does take place.

Unfortunately we cannot test other push downs. The next one is at $W = 15$ and if we omit the GDR's we have to run nearly all depths.

Without the GDR's many relations at a given depth are only obtained by combining many relations at a greater depth!

A push down basis

Broadhurst and Kreimer gave a conjecture for the number of basis elements for each weight and depth for non-alternating sums. They also gave a conjecture for each weight and depth when the non-alternating sums are expressed in terms of alternating sums. These conjectures are given on the next page. In red are the numbers we explicitly verified.

From them one can see that there should be NA basis elements that have fewer indices when expressed in terms of alternating basis elements as we have seen before. The push downs.

From the tables one can derive how many there should be, under the assumption that a push down is only from D to $D-2$.

W/D	1	2	3	4	5	6	7	8	9	10
1										
2	1									
3	1									
4										
5	1									
6		0								
7	1									
8		1								
9	1		0							
10		1								
11	1		1							
12		1		1						
13	1		2							
14		2		1						
15	1		2		1					
16		2		3						
17	1		4		2					
18		2		5		1				
19	1		5		5					
20		3		7		3				
21	1		6		9		1			
22		3		11		7				
23	1		8		15		4			
24		3		16		14		1		
25	1		10		23		11			
26		4		20		27		5		
27	1		11		36		23		2	
28		4		27		45		16		
29	1		14		50		48		7	
30		4		35		73		37		2

W/D	1	2	3	4	5	6	7	8	9	10
1										
2	1									
3	1									
4										
5	1									
6										
7	1									
8		1								
9	1									
10		1								
11	1		1							
12		2								
13	1		2							
14		2		1						
15	1		3							
16		3		2						
17	1		5		1					
18		3		5						
19	1		7		3					
20		4		8		1				
21	1		9		7					
22		4		14		3				
23	1		12		14		1			
24		5		20		9				
25	1		15		25		4			
26		5		30		20		1		
27	1		18		42		12			
28		6		40		42		4		
29	1		22		66		30		1	
30		6		55		75		15		

In determining a nice basis for the non-alternating sums we noticed that the number of elements for each weight followed a prescription. They were equal to the number of elements one obtains when making all Lyndon words out of odd integers ≥ 3 in which the integers add up to the weight. Let us call this set L_W. The number of elements of a given weight and given depth in this construction follows exactly the second Broadhurst-Kreimer table!

Next we tried to write as many basis elements as possible in terms of elements of this set.

This would not cover the whole set. The remaining elements could be obtained by allowing two even integers (say the first two indices) and making the last two indices equal to one. These elements would match the missing elements of our set if one would take away the ones and add them to the even integers. We call such a basis P_W.

Example: $W = 12$.

$$\begin{array}{llll} L_{12}: & H_{9,3} & H_{7,5} \\ P_{12}: & H_{9,3} & H_{6,4,1,1} \end{array}$$

Example: $W = 18$.

$$\begin{array}{lllllllll} L_{18}: & H_{15,3} & H_{13,5} & H_{11,7} & H_{9,3,3,3} & H_{7,5,3,3} & H_{7,3,5,3} & H_{7,3,3,5} & H_{5,5,5,3} \\ P_{18}: & H_{15,3} & H_{13,5} & H_{10,6,1,1} & H_{9,3,3,3} & H_{7,5,3,3} & H_{7,3,5,3} & H_{6,2,3,5,1,1} & H_{5,5,5,3} \end{array}$$

The interesting thing is that each of these special elements seems to be connected to a push down relation.

This is why we needed the run for alternating sums at $W = 18, D = 6$.

$$H_{10,6,1,1}+46630979\ H_{5,5,5,3}+122713096\ H_{7,5,3,3}+1002156999\ H_{9,3,3,3}$$
$$\rightarrow 672686306\ H_{-17,-1}+72010179\ H_{-15,-3}-705663559\ H_{-13,-5}$$
$$+817296192\ H_{-11,-7}+\cdots$$

The complete recipe is:

1. Write basis elements always with the lowest depth possible.
2. Generate the set L_W of all Lyndon words of odd-only ≥ 3 indices.
3. Starting at the lowest depth D, write as many elements of the basis as elements of L_W. Keep the remaining elements.
4. At the next depth D+2 write as many elements of the basis as elements of L_W. Extend the elements of L_W that remained at D according to prescription A_1 and write as many basis elements as possible as these 'extended' elements.
5. Do the same at $D+4$, fill with elements at $D+2$, extended with A_1 and possibly with elements still remaining from D, extended according to prescription A_2
6. Keep raising the depth till there is no more and a complete basis has been obtained.

Prescription A_n: Of a list of indices, subtract one from the first $2n$ elements and add $2n$ ones to the end of the list.

Note 1: it may be necessary to backtrack. The selections in the steps 3-5 are not unique and one may have to alter the selection when at a later stage things don't work out.
Note 2: the result of prescription A_n should be a Lyndon word. If not, this element is not eligible for extension and note 1 applies.

Conjecture: It is always possible, with a suitable choice of steps 3 and following, to obtain a basis.

Conjecture: The elements with added pairs of ones correspond to push downs and the number of ones indicate the units in depth that the push down corresponds to.

Example, $W = 26$:

The basis, as determined by the computer program has a depth distribution of (4,20,27,5) for D=(2,4,6,8). The depth distribution of set L_{26} is (5,30,20,1).

We start with $D = 2$ and see that we have one element left in L_{26}.

Next at $D = 4$ we can write 19 basis elements as elements of L_{26}. This means that there is one element still to be determined. We take the element that remained at $D = 2$ and extend it with A_1 to depth 4. This gives us for instance the element $H_{14,10,1,1}$. If the 19 other elements have been selected properly this one completes the $D = 4$ part of P_{26}. There are 11 elements of L_{26} remaining at $D = 4$.

Next we try the same at $D = 6$. 16 elements can be written as elements of L_{26}. For the remaining 11 we can take the A_1-extended elements we had left at $D = 4$. It is very unlikely that this 'fits' immediately and one may have to go back to the previous step to make a different selection for the 19 elements of L_{26} at the onset of that step. Eventually it fits. There are 4 elements left at $D = 6$ in L_{26}.

Finally at depth $D = 8$ there is one element in L_{26} and the A_1-extension of the 4 elements that were left in the previous step complete the 5 basis elements that we need.

$$
\begin{aligned}
P_{26} \;=\; & H_{17,9}, H_{19,7}, H_{21,5}, H_{23,3}, H_{7,7,7,5}, H_{9,5,9,3}, H_{11,3,9,3}, H_{11,5,3,7}, \\
& H_{11,5,5,5}, H_{11,5,7,3}, H_{11,7,3,5}, H_{11,7,5,3}, H_{11,9,3,3}, H_{13,3,3,7}, H_{13,3,5,5}, \\
& H_{13,3,7,3}, H_{13,5,3,5}, H_{13,5,5,3}, H_{13,7,3,3}, H_{15,3,3,5}, H_{15,3,5,3}, H_{15,5,3,3}, \\
& H_{17,3,3,3}, H_{14,10,1,1}, H_{5,5,5,3,5,3}, H_{5,5,5,5,3,3}, H_{7,3,3,5,5,3}, H_{7,3,5,3,5,3}, \\
& H_{7,3,5,5,3,3}, H_{7,3,7,3,3,3}, H_{7,5,3,3,5,3}, H_{7,5,3,5,3,3}, H_{7,5,5,3,3,3}, \\
& H_{7,7,3,3,3,3}, H_{9,3,3,3,3,5}, H_{9,3,3,3,5,3}, H_{9,3,3,5,3,3}, H_{9,3,5,3,3,3}, \\
& H_{9,5,3,3,3,3}, H_{11,3,3,3,3,3}, H_{8,2,7,7,1,1}, H_{8,4,5,7,1,1}, H_{8,4,7,5,1,1}, \\
& H_{8,6,3,7,1,1}, H_{8,6,5,5,1,1}, H_{8,6,7,3,1,1}, H_{8,8,3,5,1,1}, H_{8,8,5,3,1,1}, \\
& H_{10,2,3,9,1,1}, H_{10,2,5,7,1,1}, H_{10,2,7,5,1,1}, H_{5,3,3,3,3,3,3,3}, \\
& H_{6,2,3,3,5,5,1,1}, H_{6,2,3,5,3,5,1,1}, H_{6,2,5,3,3,5,1,1}, H_{6,4,3,3,3,5,1,1}
\end{aligned}
$$

Similarly one obtaines for P_{27}:

$$H_{27}, H_{11,7,9}, H_{13,11,3}, H_{15,3,9}, H_{15,5,7}, H_{15,7,5}, H_{15,9,3}, H_{17,5,5}, H_{17,7,3},$$
$$H_{19,3,5}, H_{19,5,3}, H_{21,3,3}, H_{7,5,5,7,3}, H_{7,5,7,3,5}, H_{7,7,3,7,3}, H_{7,7,7,3,3},$$
$$H_{9,3,9,3,3}, H_{9,5,3,5,5}, H_{9,5,3,7,3}, H_{9,5,5,3,5}, H_{9,5,5,5,3}, H_{9,5,7,3,3}, H_{9,7,3,3,5},$$
$$H_{9,7,3,5,3}, H_{9,7,5,3,3}, H_{9,9,3,3,3}, H_{11,3,3,3,7}, H_{11,3,3,5,5}, H_{11,3,3,7,3}, H_{11,3,5,3,5},$$
$$H_{11,3,5,5,3}, H_{11,3,7,3,3}, H_{11,5,3,3,5}, H_{11,5,3,5,3}, H_{11,5,5,3,3}, H_{11,7,3,3,3}, H_{13,3,3,3,5},$$
$$H_{13,3,3,5,3}, H_{13,3,5,3,3}, H_{13,5,3,3,3}, H_{15,3,3,3,3}, H_{10,8,7,1,1}, H_{10,10,5,1,1},$$
$$H_{12,2,11,1,1}, H_{12,4,9,1,1}, H_{12,6,7,1,1}, H_{12,8,5,1,1}, H_{16,2,7,1,1},$$
$$H_{5,3,5,3,5,3,3}, H_{5,5,3,3,3,5,3}, H_{5,5,3,3,5,3,3}, H_{5,5,3,5,3,3,3}, H_{5,5,5,3,3,3,3},$$
$$H_{7,3,3,3,3,3,5}, H_{7,3,3,3,3,5,3}, H_{7,3,3,3,5,3,3}, H_{7,3,3,5,3,3,3}, H_{7,3,5,3,3,3,3},$$
$$H_{9,3,3,3,3,3,3}, H_{6,4,5,5,5,1,1}, H_{6,6,3,5,5,1,1}, H_{6,6,5,3,5,1,1}, H_{6,6,5,5,3,1,1},$$
$$H_{8,2,3,5,7,1,1}, H_{8,2,3,7,5,1,1}, H_{8,2,5,3,7,1,1}, H_{8,2,5,5,5,1,1}, H_{8,2,5,7,3,1,1},$$
$$H_{8,2,7,3,5,1,1}, H_{8,2,7,5,3,1,1}, H_{8,4,3,3,7,1,1},$$
$$H_{7,5,7,5,3} \longrightarrow ?H_{6,4,6,4,3,1,1,1,1}, H_{7,5,3,3,3,3,3} \longrightarrow ?H_{6,4,3,3,3,3,3,1,1}$$

The last two elements are guessed from the remaining odds-only elements.

One seems to indicate a double push down!

$$
\begin{aligned}
P_7 &= H_7 \\
P_8 &= H_{5,3} \\
P_9 &= H_9 \\
P_{10} &= H_{7,3} \\
P_{11} &= H_{11}, H_{5,3,3} \\
P_{12} &= H_{9,3}, H_{6,4,1,1} \\
P_{13} &= H_{13}, H_{7,3,3}, H_{5,5,3} \\
P_{14} &= H_{11,3}, H_{9,5}, H_{5,3,3,3} \\
P_{15} &= H_{15}, H_{7,5,3}, H_{9,3,3}, H_{6,2,5,1,1} \\
P_{16} &= H_{11,5}, H_{13,3}, H_{5,5,3,3}, H_{7,3,3,3}, H_{8,6,1,1} \\
P_{17} &= H_{17}, H_{7,7,3}, H_{9,3,5}, H_{9,5,3}, H_{11,3,3}, H_{5,3,3,3,3}, H_{6,4,5,1,1} \\
P_{18} &= H_{13,5}, H_{15,3}, H_{5,5,5,3}, H_{7,3,5,3}, H_{7,5,3,3}, H_{9,3,3,3}, H_{10,6,1,1}, H_{6,2,3,5,1,1} \\
P_{19} &= H_{19}, H_{9,5,5}, H_{9,7,3}, H_{11,3,5}, H_{11,5,3}, H_{13,3,3}, \\
&\quad H_{5,3,5,3,3}, H_{5,5,3,3,3}, H_{7,3,3,3,3}, H_{6,6,5,1,1}, H_{8,2,7,1,1}
\end{aligned}
$$

$$
\begin{aligned}
P_{20} \;=\;& H_{13,7}, H_{15,5}, H_{17,3}, H_{7,5,5,3}, H_{7,7,3,3}, H_{9,3,3,5}, H_{9,3,5,3},\\
& H_{9,5,3,3}, H_{11,3,3,3}, H_{10,8,1,1}, H_{5,3,3,3,3,3}, H_{6,2,5,5,1,1}, H_{6,4,3,5,1,1}\\
P_{21} \;=\;& H_{21}, H_{11,3,7}, H_{9,9,3}, H_{11,7,3}, H_{13,3,5}, H_{13,5,3}, H_{15,3,3},\\
& H_{5,5,3,5,3}, H_{5,5,5,3,3}, H_{7,3,3,5,3}, H_{7,3,5,3,3}, H_{7,5,3,3,3}, H_{9,3,3,3,3},\\
& H_{8,4,7,1,1}, H_{8,6,5,1,1}, H_{10,4,5,1,1}, H_{6,2,3,3,5,1,1}\\
P_{22} \;=\;& H_{15,7}, H_{17,5}, H_{19,3}, H_{7,5,7,3}, H_{9,3,5,5}, H_{9,3,7,3}, H_{9,5,3,5},\\
& H_{9,5,5,3}, H_{9,7,3,3}, H_{11,3,3,5}, H_{11,3,5,3}, H_{11,5,3,3}, H_{13,3,3,3}, H_{12,8,1,1},\\
& H_{5,3,5,3,3,3}, H_{5,5,3,3,3,3}, H_{7,3,3,3,3,3}\\
& H_{6,4,5,5,1,1}, H_{6,6,3,5,1,1}, H_{6,6,5,3,1,1}, H_{8,2,3,7,1,1},\\
P_{23} \;=\;& H_{23}, H_{11,7,5}, H_{11,9,3}, H_{13,3,7}, H_{13,5,5}, H_{13,7,3}, H_{15,3,5}, H_{15,5,3},\\
& H_{17,3,3}, H_{5,5,5,5,3}, H_{7,3,7,3,3}, H_{7,3,5,5,3}, H_{7,5,3,5,3}, H_{7,5,5,3,3},\\
& H_{7,7,3,3,3}, H_{9,3,3,3,5}, H_{9,3,3,5,3}, H_{9,3,5,3,3}, H_{9,5,3,3,3}, H_{11,3,3,3,3},\\
& H_{8,6,7,1,1}, H_{8,8,5,1,1}, H_{10,2,9,1,1}, H_{10,4,7,1,1},\\
& H_{5,3,3,3,3,3,3} H_{6,2,3,5,5,1,1}, H_{6,2,5,3,5,1,1}, H_{6,4,3,3,5,1,1},
\end{aligned}
$$

$$
\begin{aligned}
P_{24} = \; & H_{17,7}, H_{19,5}, H_{21,3}, H_{7,7,7,3}, H_{9,7,3,5}, H_{9,7,5,3}, H_{9,9,3,3}, \\
& H_{11,3,3,7}, H_{11,3,5,5}, H_{11,3,7,3}, H_{11,5,3,5}, H_{11,5,5,3}, H_{11,7,3,3}, H_{13,3,3,5}, \\
& H_{13,3,5,3}, H_{13,5,3,3}, H_{15,3,3,3}, H_{12,10,1,1}, H_{14,8,1,1}, H_{5,5,3,3,5,3}, \\
& H_{5,5,3,5,3,3}, H_{5,5,5,3,3,3}, H_{7,3,3,3,5,3}, H_{7,3,3,5,3,3}, H_{7,3,5,3,3,3}, H_{7,5,3,3,3,3}, \\
& H_{9,3,3,3,3,3}, H_{6,6,5,5,1,1}, H_{8,2,5,7,1,1}, H_{8,2,7,5,1,1}, H_{8,4,3,7,1,1}, H_{8,4,5,5,1,1}, \\
& H_{8,4,7,3,1,1}, H_{6,2,3,3,3,5,1,1} \\
P_{25} = \; & H_{25}, H_{11,11,3}, H_{13,5,7}, H_{13,7,5}, H_{13,9,3}, H_{15,3,7}, H_{15,5,5}, H_{15,7,3}, \\
& H_{17,3,5}, H_{17,5,3}, H_{19,3,3}, H_{7,3,7,3,5}, H_{7,5,3,7,3}, H_{7,5,7,3,3}, \\
& H_{9,3,3,3,7}, H_{9,3,3,5,5}, H_{9,3,3,7,3}, H_{9,3,5,3,5}, H_{9,3,5,5,3}, H_{9,3,7,3,3}, \\
& H_{9,5,3,3,5}, H_{9,5,3,5,3}, H_{9,5,5,3,3}, H_{9,7,3,3,3}, H_{11,3,3,3,5}, H_{11,3,3,5,3}, \\
& H_{11,3,5,3,3}, H_{11,5,3,3,3}, H_{13,3,3,3,3}, H_{8,8,7,1,1}, H_{10,4,9,1,1}, \\
& H_{10,6,7,1,1}, H_{10,8,5,1,1}, H_{12,2,9,1,1}, H_{5,3,3,5,3,3,3}, H_{5,3,5,3,3,3,3}, \\
& H_{5,5,3,3,3,3,3}, H_{7,3,3,3,3,3,3}, H_{6,2,5,5,5,1,1}, H_{6,4,3,5,5,1,1}, H_{6,4,5,3,5,1,1}, \\
& H_{6,4,5,5,3,1,1}, H_{6,6,3,3,5,1,1}, H_{6,6,3,5,3,1,1}, H_{6,6,5,3,3,1,1}
\end{aligned}
$$

Currently we don't know how to work the prescription backward. Starting from the complete set of Lyndon words we don't have a way of telling which elements should be selected to remain as they are and which ones should be extended, and by how much.

This means that we cannot predict the complete basis and we need the computer runs.

Yet it looks like progress.

It seems important to have more insight in the embedding of the non-alternating sums in the alternating sums. The role that the doubling and the GDR's play here is crucial and should be understood.

Conclusions

We have now complete and partial tables for the MZV's that will cover many more values of the weight and depth than were previously available.
These results will be publicly accessible soon under the name "MZV Datamine". It should be linked in the FORM pages and the home pages of the authors of the paper (http://www.nikhef.nl/∼form).
FORM programs to allow one to manipulate this data are available in the Datamine as well. So is FORM. One is advised to use computers with a 64-bits architecture for this.
We have conjectured a new type of basis which seems to be connected to an embedding of the non-alternating sums in the alternating sums.
The holy grail in the field of MZV's is an algorithm to express each MZV into an unique basis in a constructive way. That way we would have a (hopefully) small procedure rather than giant tables. Thus far this has not been found.