Gravitational Wave Derivation

Lecture 3: Gravitational Waves MSc Course

- Solving the Einstein Equations
  - Linearized Theory
  - Vacuum Solution
  - Solution with Source Term
- Generation of Gravitational Waves
- Effect of Gravitational Waves on Matter

The Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Given the source distribution $T_{\mu\nu}$, one can solve this set of 10 coupled nonlinear partial differential equations for the metric $g_{\mu\nu}(x)$

Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity (next time)
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

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Linearized Theory of Metric Field

Consider the Minkowski metric - a combination of three dimensional Euclidean space and time into four dimensions.

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider a small perturbation $h_{\mu\nu}$ on flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

so that higher orders of $h_{\mu\nu}$ can be neglected when substituting in Einstein Field Equations (EFE)

Linearized Theory of Metric Field

Can we make coordinate transformations under such systems? Yes, from one slightly curved one to another, aka “Background Lorentz transformation”

So EFE are invariant under general coordinate transformations **but** invariance is broken as a result of background.

$h_{\mu\nu}$ is an as yet unknown perturbation on flat space. We can make small changes in coordinates that leave $\eta_{\mu\nu}$ unchanged but make small changes in $h_{\mu\nu}$

We can only consider a sufficiently large specific reference frame where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ holds.

In other words, we’re restricted in how much we can change the coordinates.

Linearized Theory of Metric Field

We are restricted to a limited set of coordinate transformations called "gauge transformations"

$$x^{\mu} \rightarrow x'^{\mu} + \xi\left(x^{\mu}\right)$$

If we transform the metric under this change of coordinates we find that the metric has the same form but with new perturbations given by

$$h_{\mu\nu}\left(x\right) \rightarrow h'_{\mu\nu}\left(x'\right) = h_{\mu\nu}\left(x\right) - \left(\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}\right)$$

Linearized Theory of Metric Field

We can stream line some calculations by an appropriate choice of gauge conditions.

We require a coordinate system in which **Lorentz gauge** (or harmonic gauge) holds

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

where we've defined the trace-reversed perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$$

such that the trace has opposite sign:

$$\bar{h}^\mu_\mu \equiv \bar{h}_{\mu\nu} = -h$$

Linearized Theory of Metric Field

The Riemann curvature tensor

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\beta g_{\nu\alpha}\right)$$

for a flat metric with a perturbation will become

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right)$$

Then substituting the trace-reversed perturbation, EFE takes form:

$$\partial_\mu\partial^\mu \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

If we define the d'Alembertian operator: $\Box \equiv \partial_\mu\partial^\mu$

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

Linearized Theory of Metric Field

And impose the harmonic gauge, then the last three terms in previous equation vanish and we end up with the **Linearized Einstein Equations**

$$\boxed{\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}}$$

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Solution in a Vacuum

What happens outside the source, where $T_{\mu\nu} = 0$?

Then, the EFE reduces to

$$\Box \bar{h}_{\mu\nu} = 0$$

$$\left(-\frac{1}{c^2}\partial t^2 + \nabla^2\right)\bar{h}_{\mu\nu} = 0$$

Wave equation for waves propagating at speed of light c!

Solutions to wave equation can be written as superpositions of plane waves traveling with wave vectors $\vec{k}$ and frequency

$$\omega = c\left|\vec{k}\right|$$

Solution in a Vacuum

Plane wave solution:

$$h(t) = A_{\mu\nu} \cos\left(\omega t - \vec{k} \cdot \vec{x}\right)$$

Implications: Spacetime has dynamics of its own, independent of matter. Even though matter generated the solution, it can still exist far away from the source where $T_{\mu\nu} = 0$

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Solution with Source

Now allow for source. What would cause the waves to be generated?

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Solve using retarded Green's function assuming no incoming radiation from infinity. The solution is

$$\bar{h}_{\mu\nu}\left(t, \vec{x}\right) = \frac{4G}{c^4} \int d^3x' \, \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right)$$

Solution with Source

We can utilize an additional gauge freedom by imposing the radiation gauge:

$$h = 0, \;\; h_{0i} = 0$$

Combining the harmonic gauge and this radiation gause, we can write the solution in the **transverse traceless** (TT) gauge

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} T_{kl}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right)$$

$\vec{n}$ - direction of propagation of GW

$\Lambda_{ij,kl}(\hat{n})$ is a tool to bring $h_{\mu\nu}$ outside the source in the TT gauge.

Solution with Source

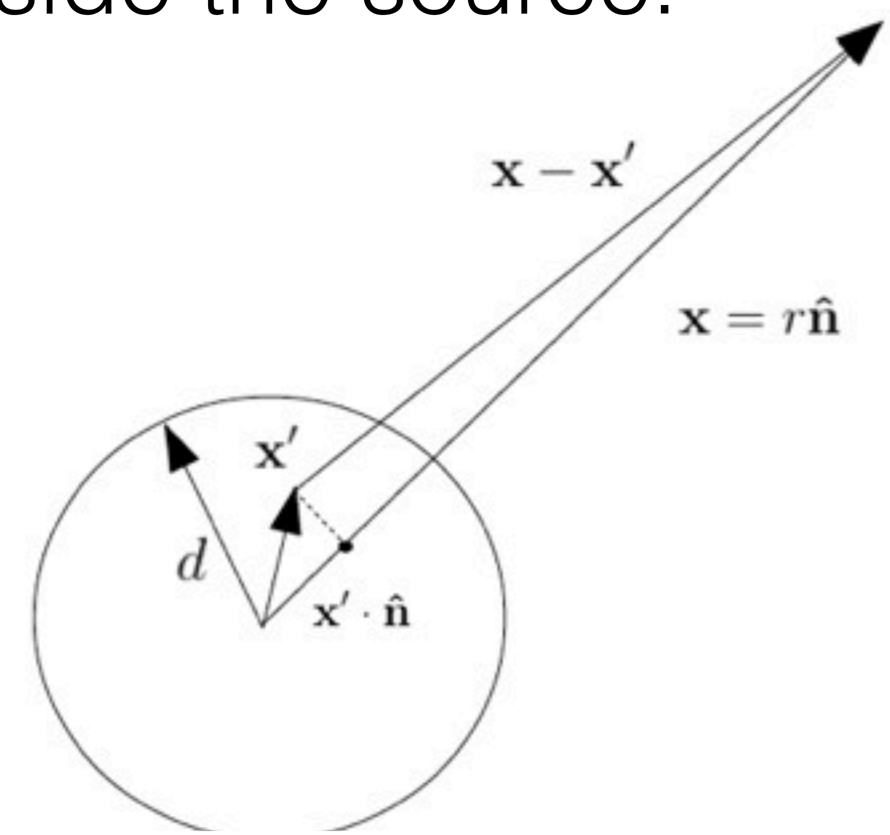
$\Lambda_{ij,kl}(\hat{n})$ is a tool to bring $h_{\mu\nu}$ outside the source in the TT gauge.

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$P_{ij} \equiv \delta_{ij} - n_i n_j$$

Then the perturbation $h_{ij}^{TT}(t, \vec{x})$ can be evaluated outside the source at $\vec{x}$ while $\vec{x}'$ is a point inside the source.

$$T_{kl}\left(t - |\vec{x} - \vec{x}'|/c, \vec{x}'\right) \neq 0$$



We're looking at a distance r that is much larger than the size of the source d. Then we can expand

$$\Delta\vec{x} = r - \vec{x}' \cdot \hat{n} + \mathcal{O}\left(d^2/r\right)$$

Solution with Source

Then we can write the TT solution as

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \frac{1}{|r - \vec{x}' \cdot \hat{n}|} T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{n}}{c}, \vec{x}'\right)$$

If the source is non-relativistic, $\mathrm{v/c} << 1$, then we can expand

$$T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{n}}{c}, \vec{x}'\right) = T_{kl}\left(t - \frac{r}{c}, \vec{x}'\right) + \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_0^2 T_{kl} + ...$$

We can substitute this for T_{kl} in the TT solution to get the **multipole expansion**

$$h_{ij}^{\mathrm{TT}}(t, \vec{x}) = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \ldots\right]_{\mathrm{ret}}$$

where ret is the retarded time $t - r/c$

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Generation of Gravitational Waves

Multipole moments of stress tensor T^{ij}

$$S^{ij} = \int d^3x T^{ij}\left(t, \vec{x}\right)$$

$$S^{ij,k} = \int d^3x T^{ij}\left(t, \vec{x}\right) x^k$$

$$S^{ij,kl} = \int d^3x T^{ij}\left(t, \vec{x}\right) x^k x^l$$

...

Multipole moments of the stress energy tensor are not physically intuitive.

Generation of Gravitational Waves

We can express the multipole moments in terms of the mass moments and the momentum multipoles.

Mass moments: momenta of energy density T^{00}/c^2

$$M = \frac{1}{c^2}\int d^3x T^{00}\left(t,\vec{x}\right)$$

$$M^i = \frac{1}{c^2}\int d^3x T^{00}\left(t,\vec{x}\right)x^i$$

$$M^{ij} = \frac{1}{c^2}\int d^3x T^{00}\left(t,\vec{x}\right)x^i x^j$$

...

Generation of Gravitational Waves

We can express the multipole moments in terms of the mass moments and the momentum multipoles.

Momenta of momentum density T^{0i}/c

$$P^{i} = \frac{1}{c}\int d^{3}x T^{0i}\left(t,\vec{x}\right)$$

$$P^{i,j} = \frac{1}{c}\int d^{3}x T^{0i}\left(t,\vec{x}\right)x^{j}$$

$$P^{i,jk} = \frac{1}{c}\int d^{3}x T^{0i}\left(t,\vec{x}\right)x^{j}x^{k}$$

...

Generation of Gravitational Waves

To leading order in v/c, we can eliminate the multipole moments in favor of the mass moments to get a solution of the form:

$$\left[h_{ij}^{\mathrm{TT}}(t,\vec{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\,(t - r/c)$$

where we have used: $S^{ij} = \frac{1}{2}\ddot{M}^{ij}$

Mass quadrupole radiation!

Generation of Gravitational Waves

$$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\vec{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t-r/c\right)}$$

No Monopole Radiation

$$\dot{M} = \frac{1}{c}\int_V d^3x\partial_0 T^{00}$$

$$= -\frac{1}{c}\int_V d^3x\partial_i T^{0i}$$

$$= -\frac{1}{c}r^2\int_S d\Omega T^{0i}$$

$$= 0$$

No Dipole Radiation

Mass dipole M^i zero (i.e. constant) in center of mass frame

No momentum monopole contribution

$$\dot{P}^i = 0$$

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Effect of Gravitational Waves on Matter

The best way to understand the effect of gravitational waves on matter is to consider two neighboring free-falling particles at $x^\mu(\tau)$ and $x^\mu(\tau) + \zeta^\mu(\tau)$

Consider the geodesic equations for each particle:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

$$\frac{d^2 (x^\mu + \zeta^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x + \zeta) \frac{d(x^\mu + \zeta^\mu)}{d\tau} \frac{d(x^\mu + \zeta^\mu)}{d\tau} = 0$$

Take the difference of the two and expand to leading order in ζ^μ:

$$\frac{d^2 \zeta^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{d\zeta^\rho}{d\tau} + \zeta^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

Effect of Gravitational Waves on Matter

$$\frac{d^2\zeta^\mu}{d\tau^2} + \cancel{2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}}\frac{d\zeta^\rho}{d\tau} + \zeta^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

Transform into a Local Lorentz Frame such that:

$$g_{\mu\nu}(\mathcal{P}) = \eta_{\mu\nu}\,,\ \ \partial_\rho g_{\mu\nu} = 0 \longrightarrow \Gamma^\rho_{\mu\nu} = 0$$

Assume the particles are moving non-relativistically:

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}\,, \quad \frac{dx^0}{d\tau} \simeq c$$

Relate $\partial_\sigma\Gamma^\sigma_{00}$ to the Riemann tensor: $\displaystyle\frac{d^2\zeta^i}{d\tau^2} = -c^2 R^i_{0j0}\zeta^j$

Effect of Gravitational Waves on Matter

The components of the Riemann tensor may be calculated in any frame due to its invariance in linearized theory. We can use the TT frame:

$$R^i_{0j0} = R_{i0j0} = -\frac{1}{2c^2}\ddot{h}^{\mathrm{TT}}_{ij}$$

Now we see how the geodesic deviation between two particles is related to the perturbation caused by a passing GW:

$$\ddot{\zeta}^i = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\zeta^j$$

A tidal effect!

Effect of Gravitational Waves on Matter

Gravitational wave in the z-direction:

$$h_{ij}^{\mathrm{TT}} = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} \cos\left(\omega t - zt/c\right), \qquad \omega = c|\vec{k}|$$

Relative displacements of particles in (x, y) plane:

$\boxed{h_\times = 0}$

$$\delta\ddot{x} = -\frac{h_+}{2}\left(x_0 + \delta x\right)\omega^2 \cos(\omega t) \qquad \delta x(t) = \frac{h_+}{2} x_0 \cos(\omega t)$$

$$\delta\ddot{y} = \frac{h_+}{2}\left(y_0 + \delta y\right)\omega^2 \cos(\omega t) \longrightarrow \delta y(t) = -\frac{h_+}{2} y_0 \cos(\omega t)$$

$\boxed{h_+ = 0}$

$$\delta\ddot{x} = \frac{h_\times}{2}\left(y_0 + \delta y\right)\omega^2 \cos(\omega t) \qquad \delta x(t) = -\frac{h_\times}{2} y_0 \cos(\omega t)$$

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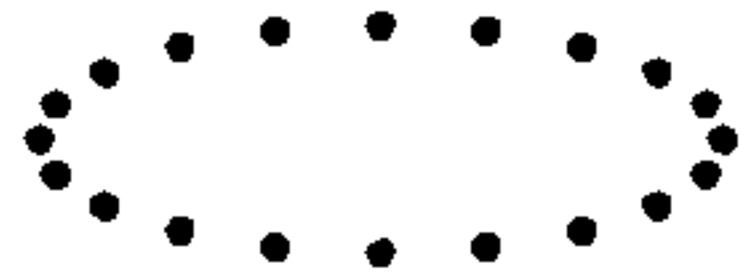
Effect of Gravitational Waves on Matter

h_+ polarization

$$\delta x(t) = \frac{h_+}{2} x_0 \cos(\omega t)$$

$$\delta y(t) = -\frac{h_+}{2} y_0 \cos(\omega t)$$



$h_\times$ polarization

$$\delta x(t) = -\frac{h_\times}{2} y_0 \cos(\omega t)$$

$$\delta y(t) = -\frac{h_\times}{2} x_0 \cos(\omega t)$$