

## Announcements

We will start using Piazza for communications, documents and HW upload: [piazza.com/university\\_of\\_amsterdam/spring2020/5354grwa3y](https://piazza.com/university_of_amsterdam/spring2020/5354grwa3y)

You should have received several emails about Piazza.

Turning in Assignment #1: **EITHER**

- email to myself and Khun Sang ([physarah@gmail.com](mailto:physarah@gmail.com),  
[k.s.phukon@nikhef.nl](mailto:k.s.phukon@nikhef.nl))
- **OR** send us an email with HW attached through Piazza.

Starting with Assignment #2: turn in HW through Piazza

Recordings for lectures will be available only through <https://www.nikhef.nl/~caudills/teaching.html>

Course documents will be available through either Piazza or my personal website.

# Final Project and Presentation

You should have received an email about this.

**Final project:** Pick a journal article, read it, and prepare a report (preferably using LaTeX) to answer a number questions:

- Instructions: [https://www.nikhef.nl/~caudills/coursedocs/final\\_project\\_2020.pdf](https://www.nikhef.nl/~caudills/coursedocs/final_project_2020.pdf)
- List of possible articles: <https://docs.google.com/document/d/15NeJOfyAqzFsMW-mST9W49LvEr8KkWFUOgyJEQBU/edit?usp=sharing>
- Note: You can choose an article from this list or find another one that you like. Either way, let me know so I can edit the document with your name for that article. There is only one student per article.

**Final presentation:** 5-minute slide presentation to provide background material and to answer a number of questions:

- Instructions: [https://www.nikhef.nl/~caudills/coursedocs/final\\_presentation\\_2020.pdf](https://www.nikhef.nl/~caudills/coursedocs/final_presentation_2020.pdf)

Due date for report and presentation: Friday, May 29. Presentations will take place on Zoom from 13:00-16:00.

Advice: There are 17 days between our last lecture and the due date. This should be plenty of time to prepare the report.

# General Relativity: A Summary

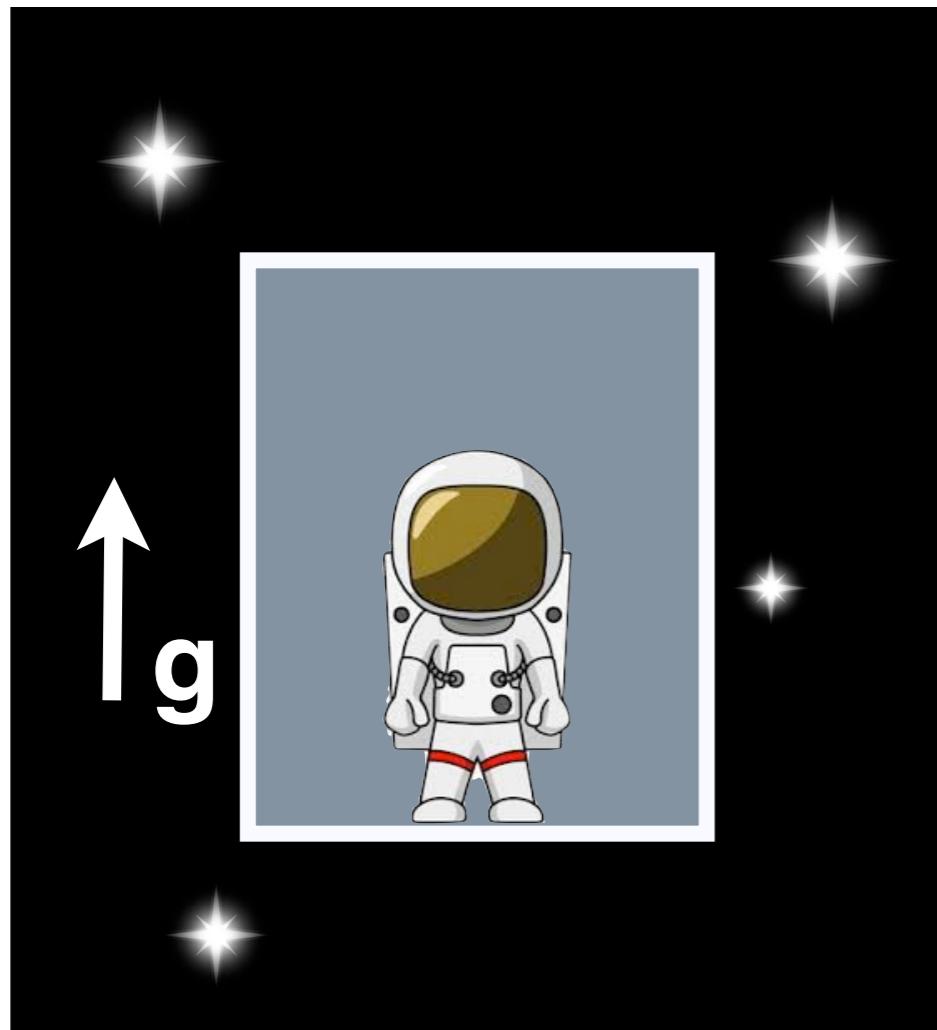
Lecture 2: Gravitational Waves MSc Course

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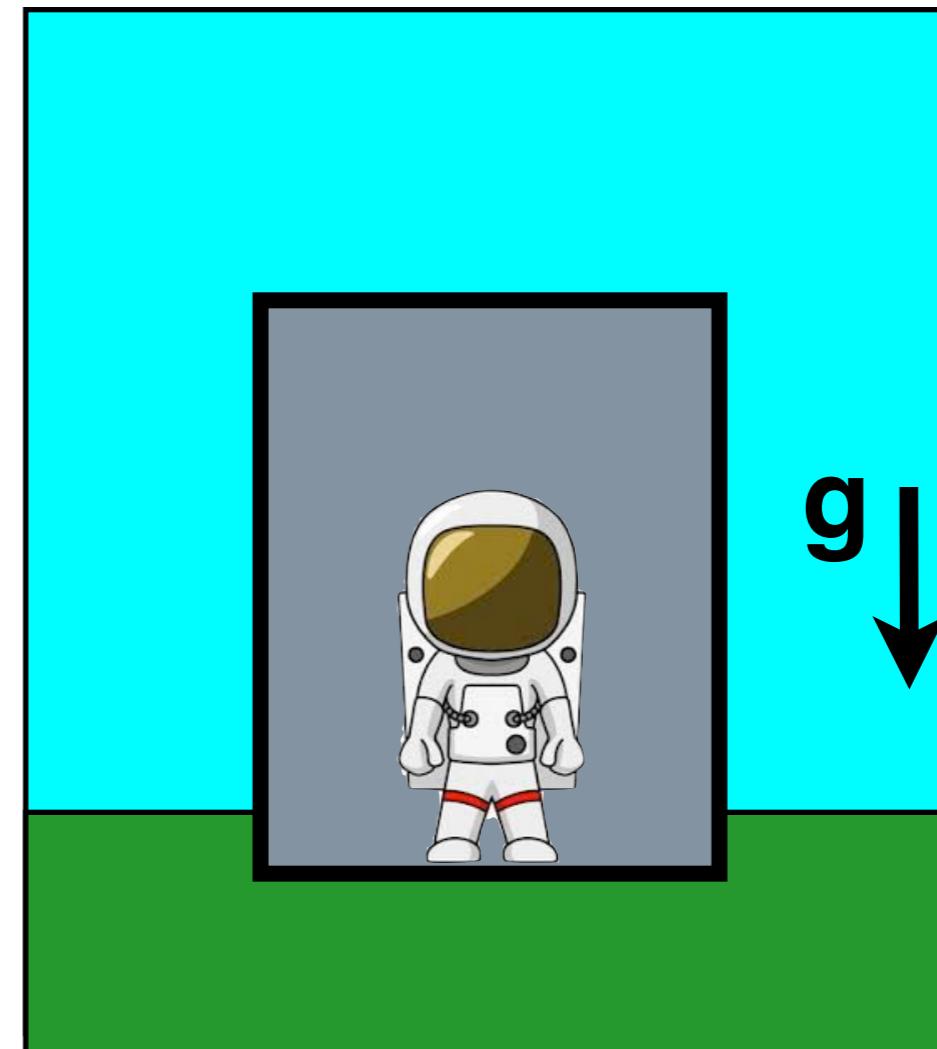
# Principle of Equivalence

There is no experiment you can do that will distinguish between the following two experiments.

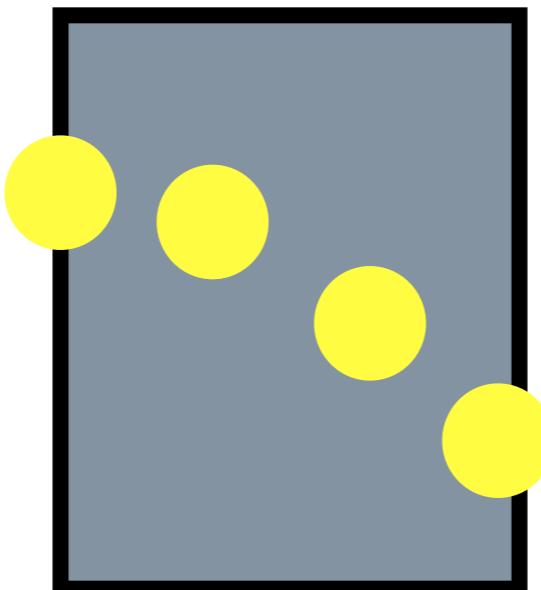
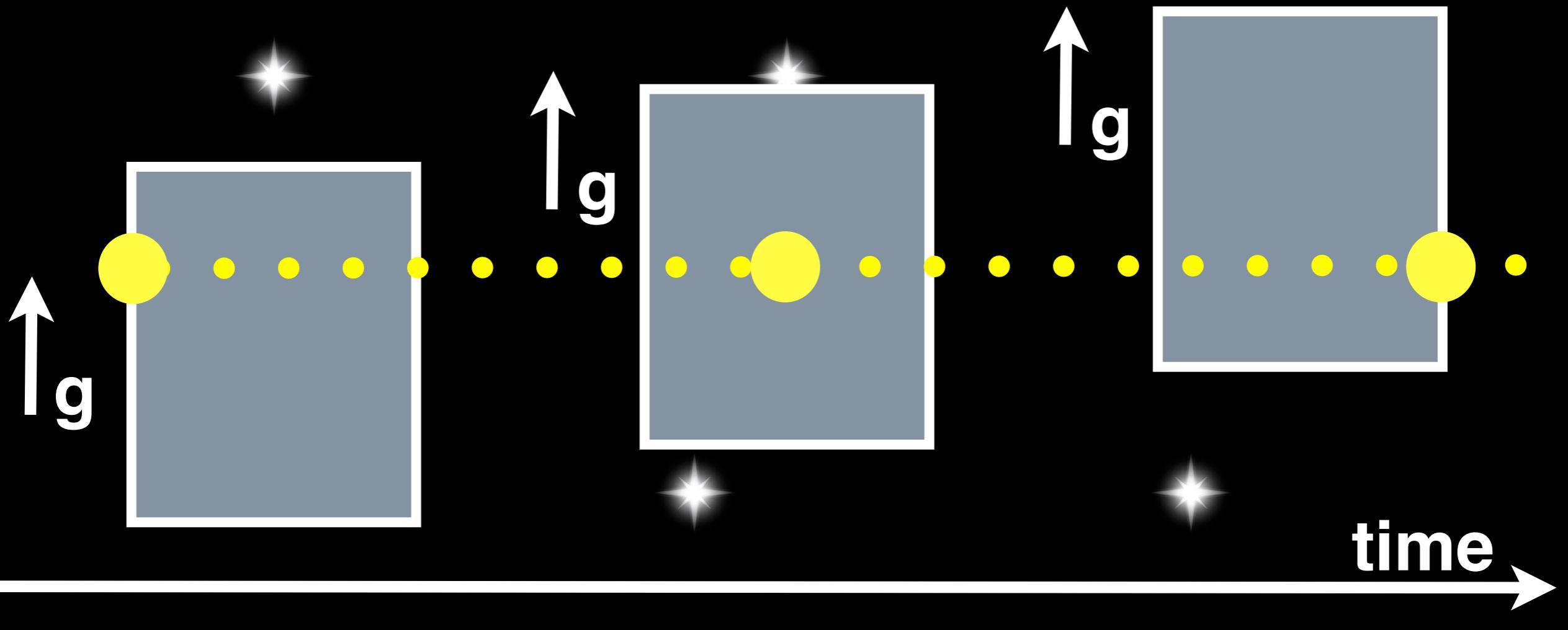
Accelerating at **g**



Stationary but subject  
to gravitational force



# Light Bends in a Gravitational Field



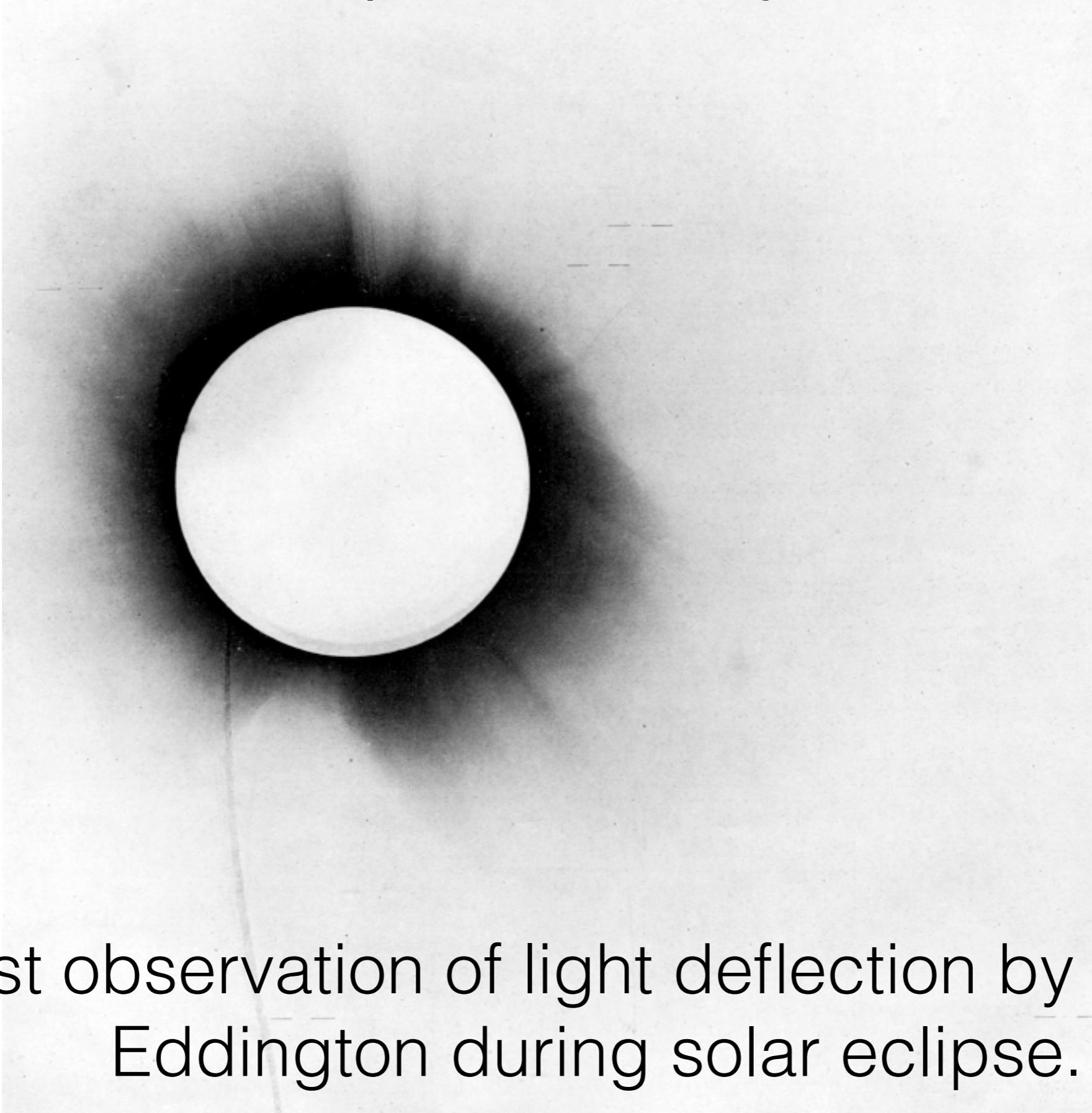
Light has followed a curve.

Light appears to curve when you are accelerating through space with acceleration **g**.

By principle of equivalence, accelerating with acceleration **g** is equivalent to being stationary subject to acceleration **g**.

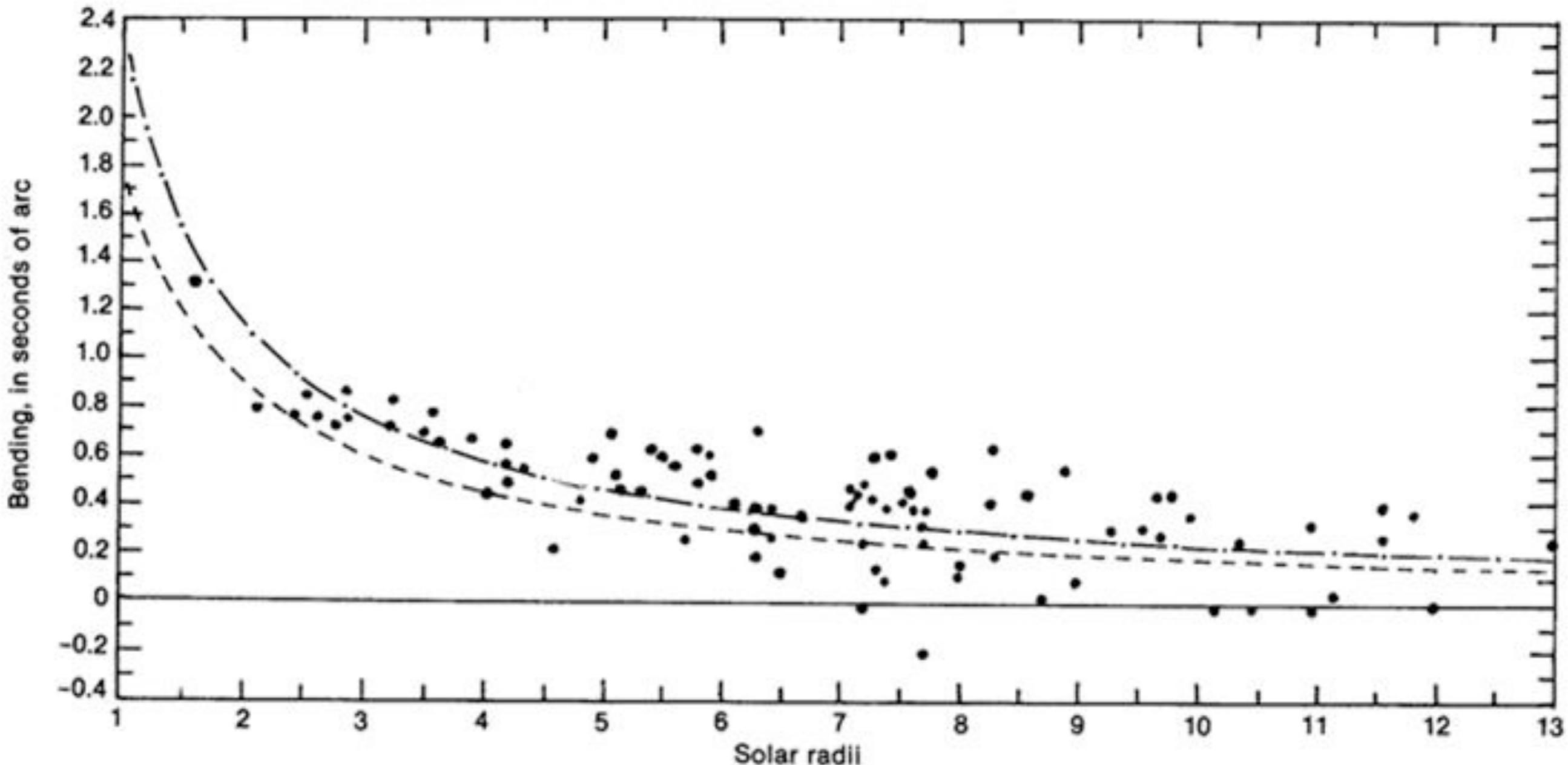
Then, light should also appear to curve in a gravitational field.

# Solar Eclipse of May 29, 1919



First observation of light deflection by Arthur Eddington during solar eclipse.

# Later Eclipse Measurements



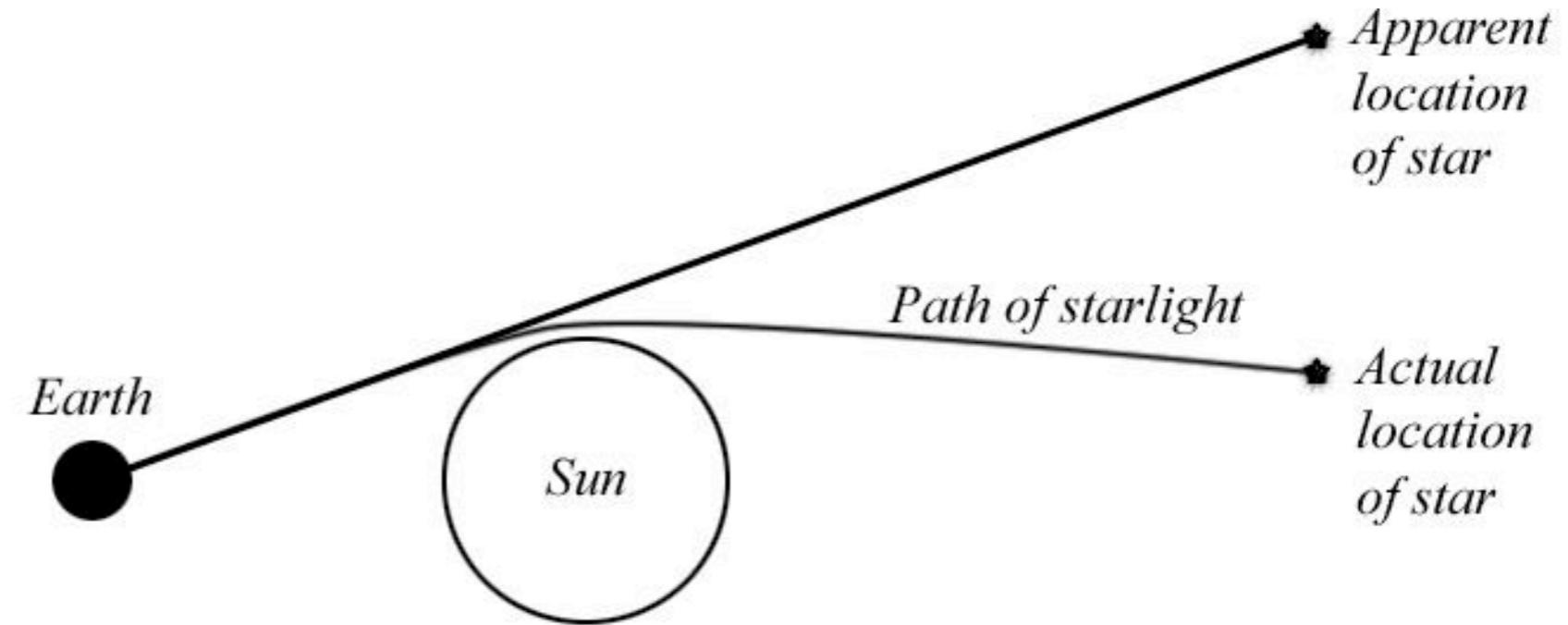
Results from later eclipse experiments in 1922 and 1929. Dashed line is Einstein's prediction. Dot dashed is least-squared fit of actual data.

# Why do these measurements imply spacetime is curved?

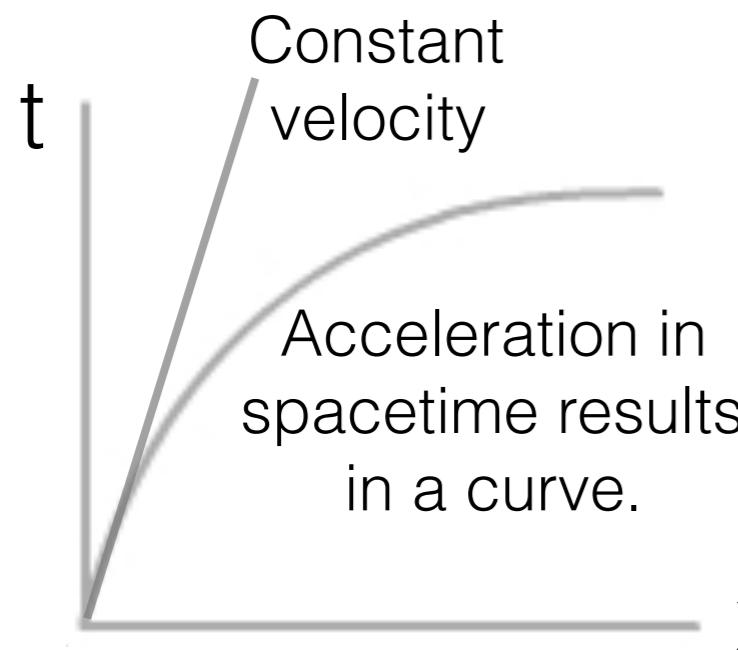
Newton's law of gravitation

$$F = \frac{GMm}{r^2}$$

$$m_{\text{photon}} = 0$$



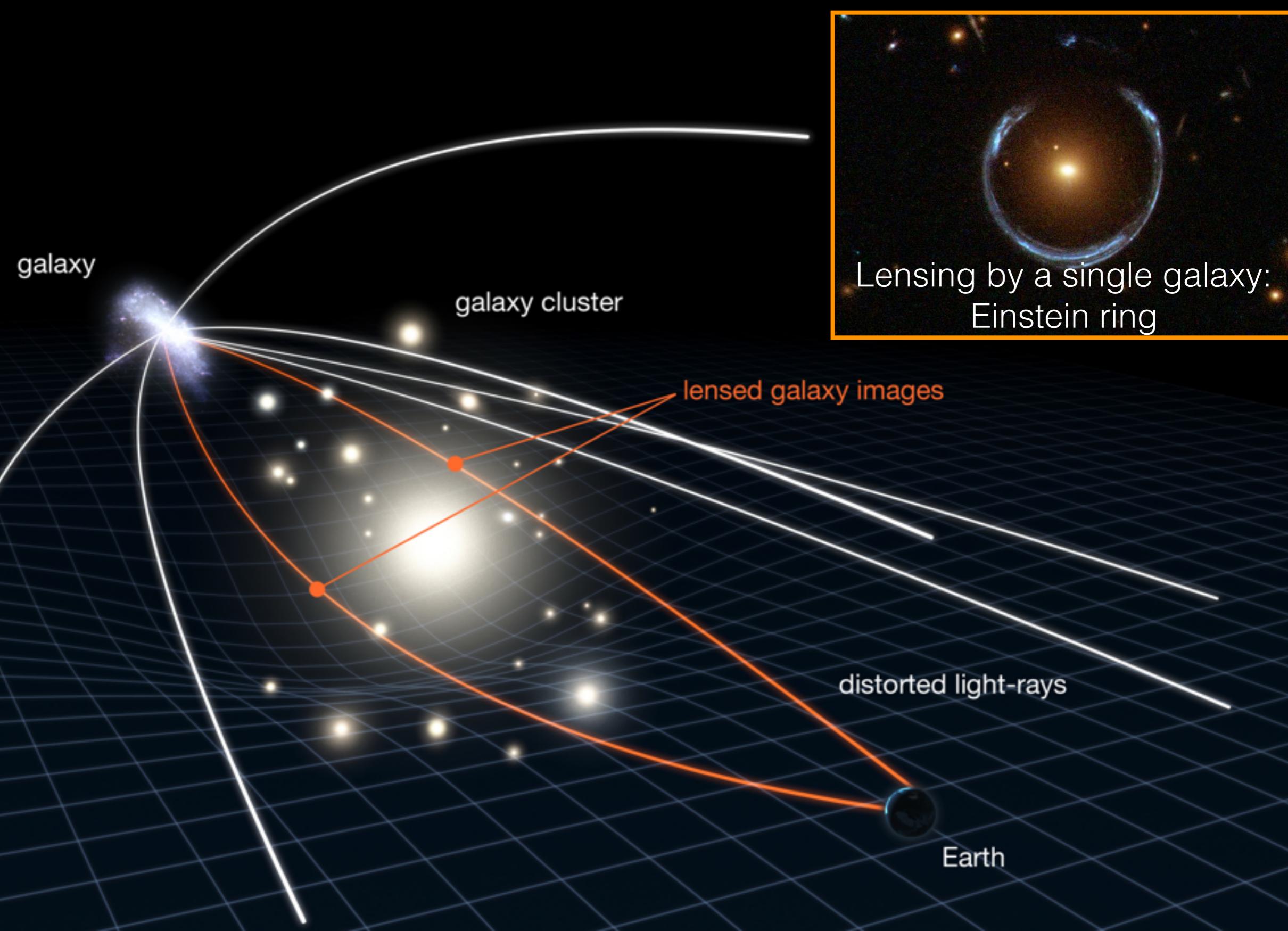
This form of Newton's law doesn't work for light!



"Gravity" causes acceleration.

Therefore, **spacetime** must be curved if it is creating an acceleration.

# Gravitational Lensing



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# 3 Essential Ideas Underlying General Relativity

1. Spacetime may be described as curved, 4-D mathematical structure called pseudo-Riemannian manifold
2. At every spacetime point, there exist locally inertial reference frames, corresponding to locally flat coordinates carried by freely falling observers: Einstein's strong equivalence principle.
3. Mass and mass/momentum flux curves spacetime in a way described by Einstein's tensor field equations.

# What is a tensor?

Scalar - tensor rank 0, magnitude, ex: temperature.

Vector - tensor rank 1, magnitude and direction, ex: force.

Tensor - combination of vectors where there is a fixed relationship, independent of coordinate system; ex: dot product, work.

$$T^{mn} = A^m B^n$$

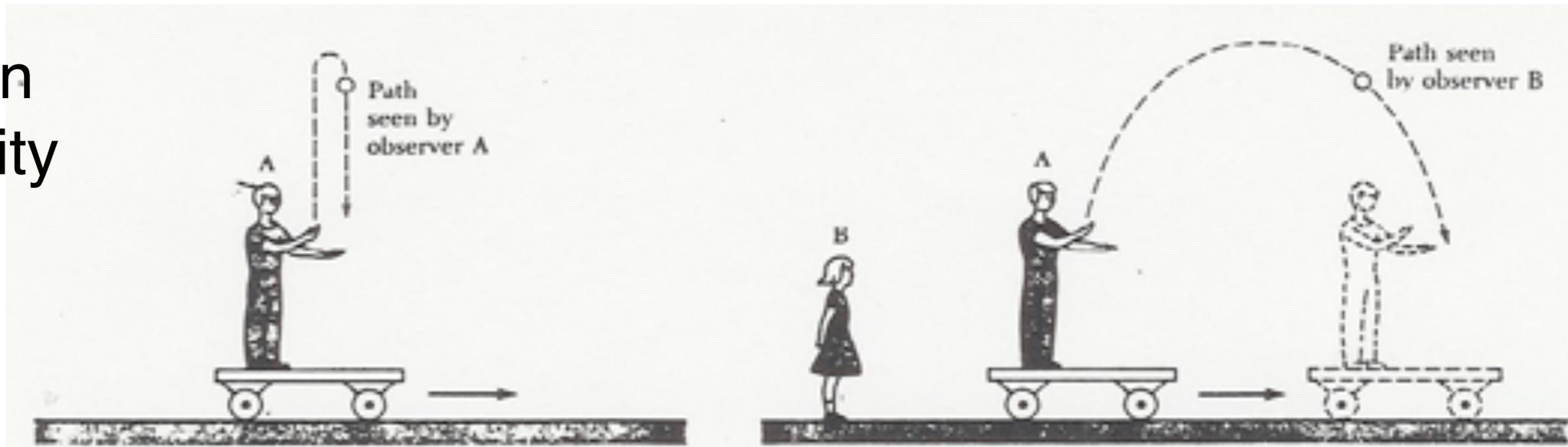
Principle of relativity - “Physics equations should be covariant under coordinate transformation.”

To ensure that this is automatically satisfied, write physics equations in terms of tensors.

# Inertial Frame of Reference

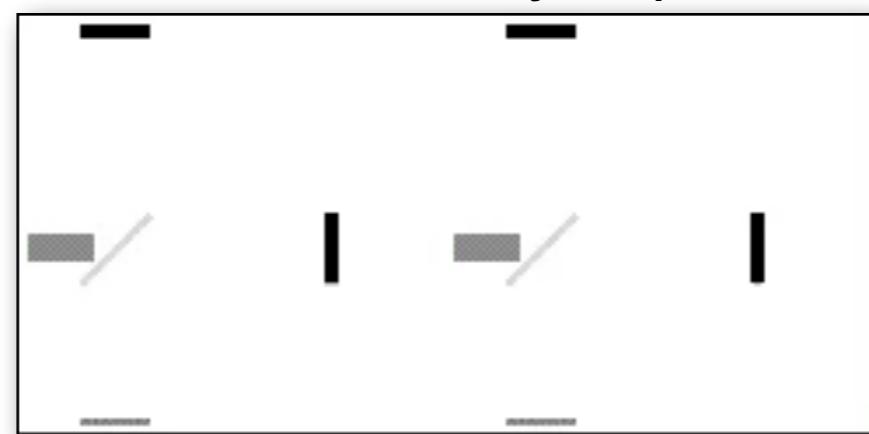
Coordinate systems in which a particle will, if no external force acts, continue it's state of motion with constant velocity. Physics descriptions are simplest here.

Galilean  
Relativity



Special Relativity -  
constancy of light  
speed

Michelson-Morley Experiment



Famous null  
experiment - motion  
through aether does  
not cause a  
differential phase  
shift

# Coordinate Symmetry Transformations

• Galilean transformation	<p>Classical, non-relativistic mechanics.</p> <p>Valid for <math>v \ll c</math></p>
• Lorentz transformation	<p>Revealed by Special Relativity, i.e. Maxwell equations.</p> <p>Valid for <math>v \leq c</math></p>
• General coordinate transformation	<p>Needed for General Relativity.</p> <p>Physics should be covariant under general transformations between frames of reference.</p> <p>Valid for <math>v \leq c</math> and accelerating frames.</p>

# Einstein Summation Convention

Repeated indices imply summation.

$$\begin{aligned} A^\mu B_\mu &= \sum_{\mu=0}^3 A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \\ &= [B_0 \ B_1 \ B_2 \ B_3] \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} \end{aligned}$$

Free index - appears exactly once in every term of equation

Dummy index - appears exactly twice in one given term of equation but only once in equation

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# Spacetime

Spacetime points (events) can be labeled by coordinate system:

$$x^\mu = (x^0, x^1, x^2, x^3)$$

which has no intrinsic meaning.

May be described as curved, 4-D mathematical structure called pseudo-Riemannian differentiable manifold.

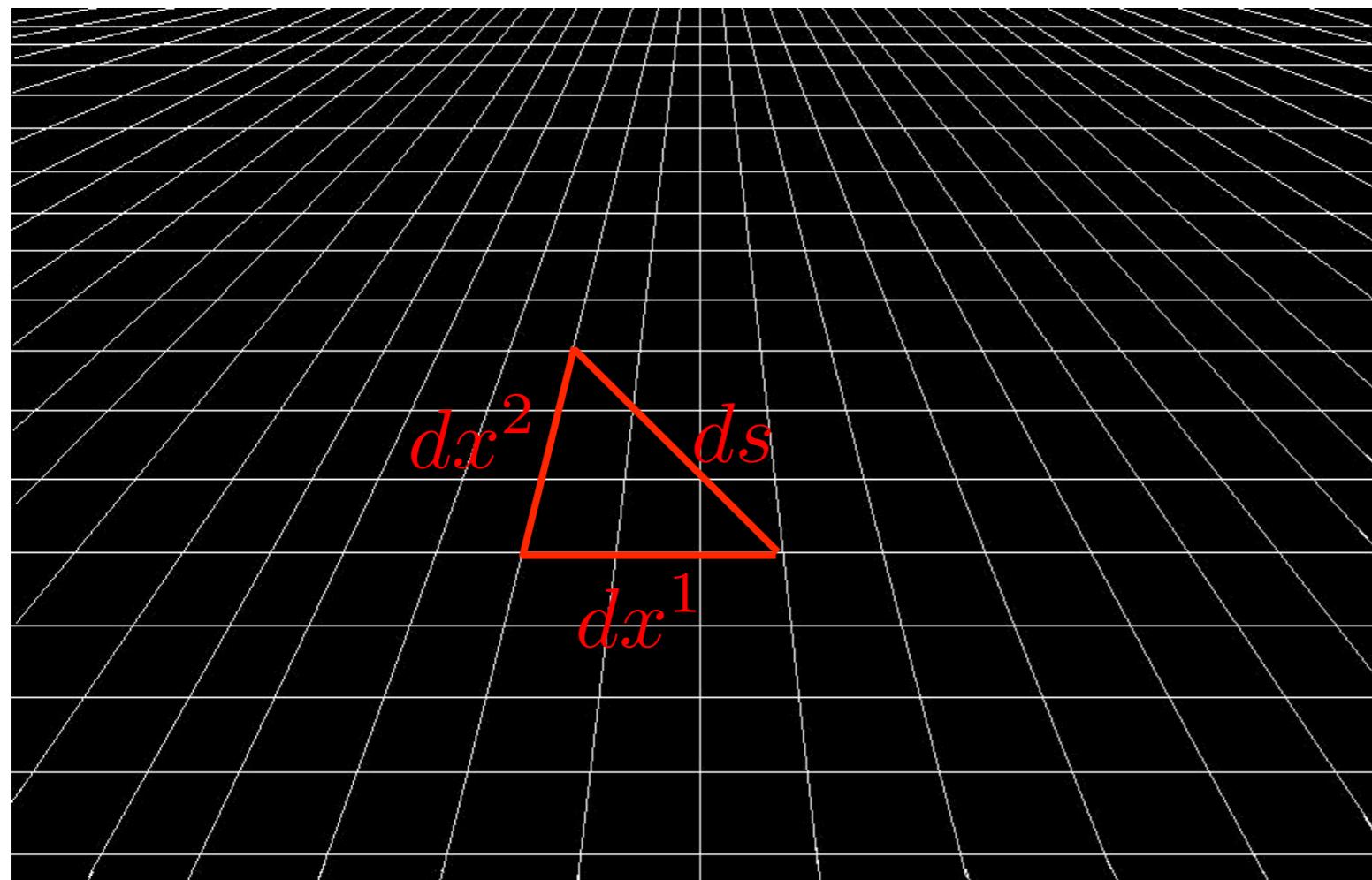
Distances between nearby events are calculated using a metric  $g_{\mu\nu}$

Greek indices for components  $\mu, \nu \in \{0, 1, 2, 3\}$

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# The Metric of Flat Space: Newtonian Mechanics

## Pythagorean Theorem in 2-D Euclidean Space



$$\begin{aligned} ds^2 &= (dx^1)^2 + (dx^2)^2 \\ &= \sum_{mn} dx^m dx^n \delta_{mn} \\ &= \delta_{mn} \sum_{mn} dx^m dx^n \end{aligned}$$

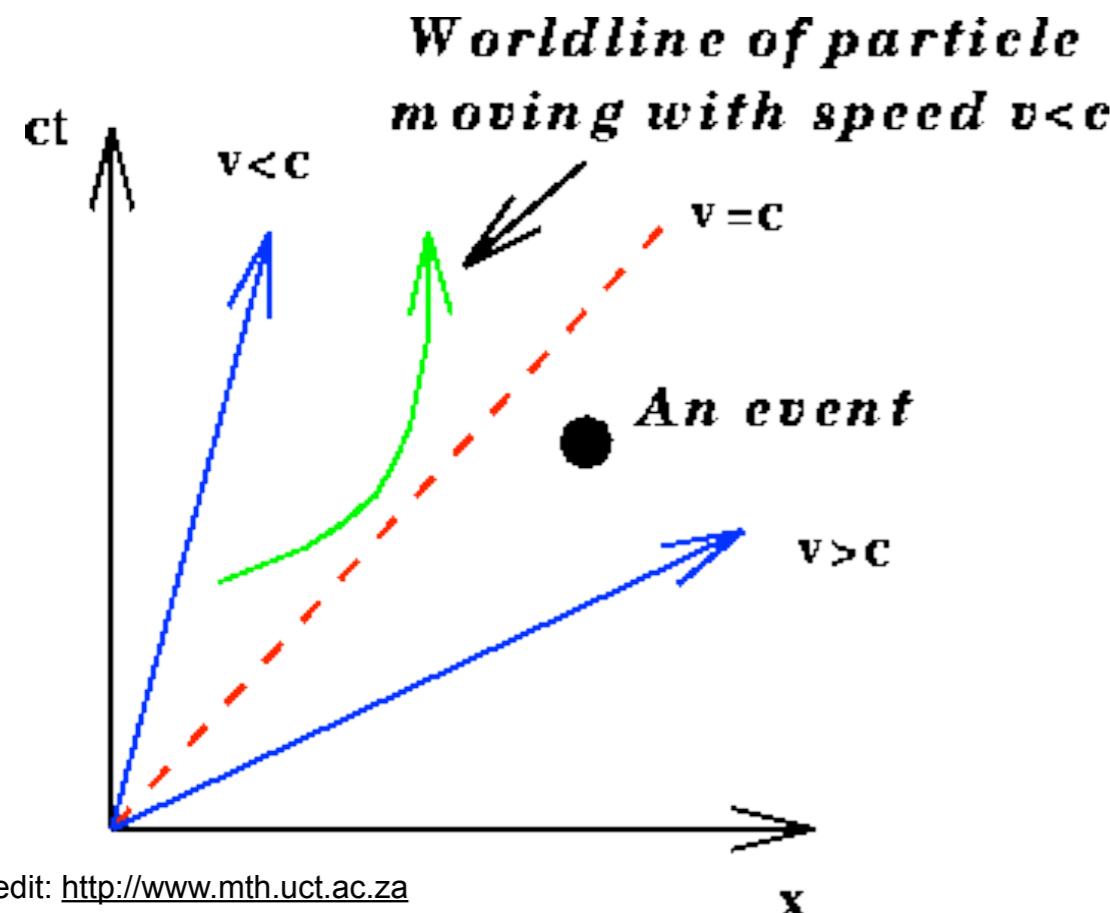
Kronecker delta is metric tensor in flat space.

$$g_{mn} = \delta_{mn} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Position-  
independent metric

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# The Metric of Flat Space: Special Relativity



Units in which  $c = 1$

Spacetime interval in flat 4D spacetime

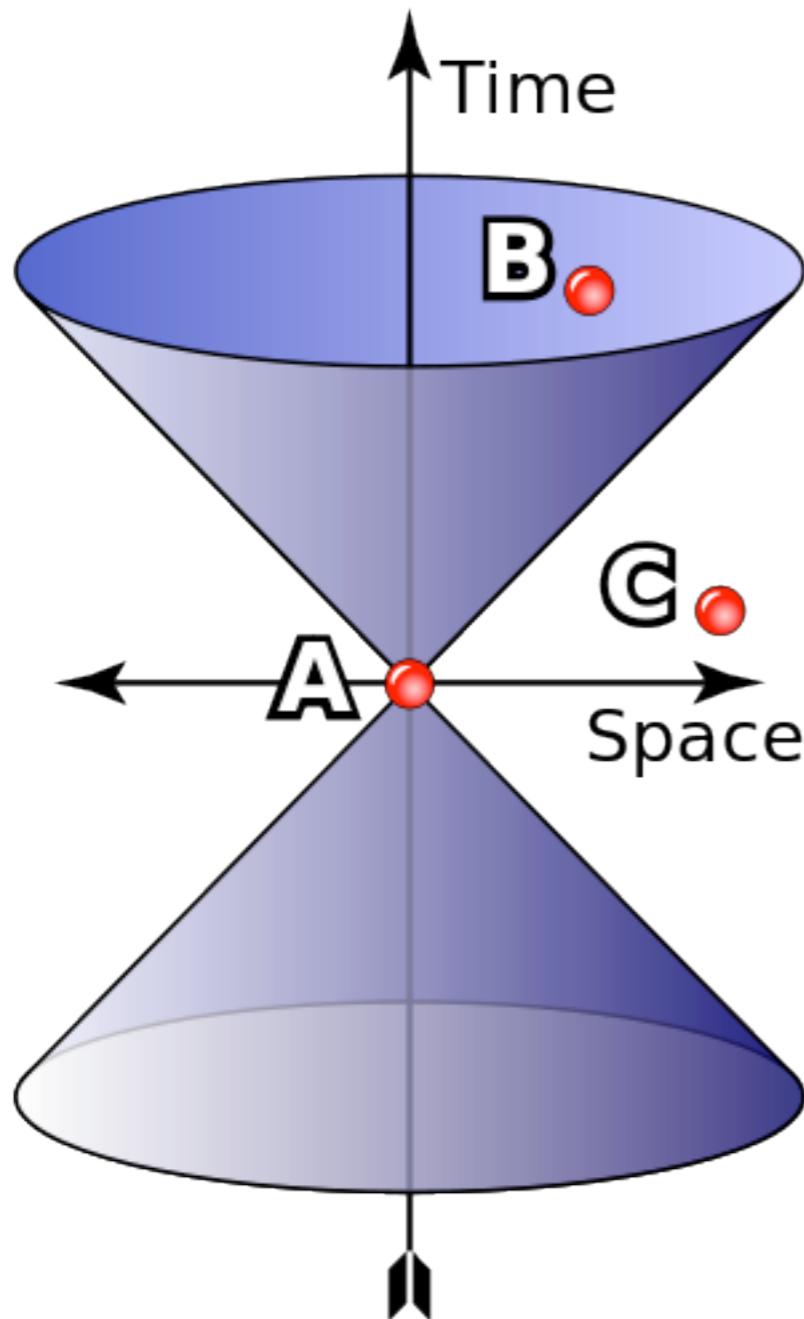
$$\begin{aligned}ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\&= \eta_{\mu\nu}x^\mu x^\nu\end{aligned}$$

Minkowski metric is metric tensor in flat 4D spacetime.

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{diag}(-1, 1, 1, 1)$$

Position-independent metric

# Spacetime Intervals



Light cone

B. Timelike separation - causally connected to A

$$ds^2 < 0$$

A. Lightlike (Null) separation

$$ds^2 = 0$$

C. Spacelike separation - cannot exchange signals between A and C

$$ds^2 > 0$$

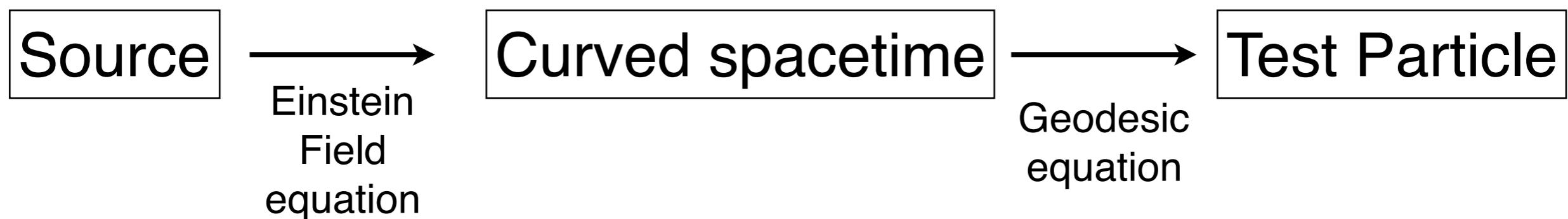
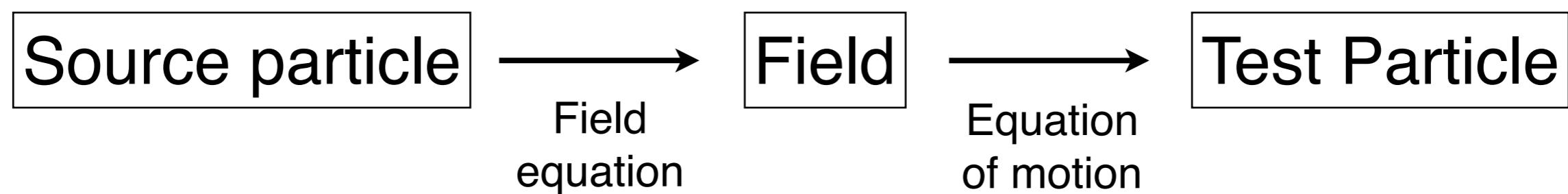
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# The Metric of Curved Space: General Relativity

General relativity as a geometric theory of gravity posits that matter and energy cause spacetime to warp so that

$$g_{\mu\nu} \neq \eta_{\mu\nu}$$

Thus gravitational phenomena are just effects of a curved spacetime on a test particle.

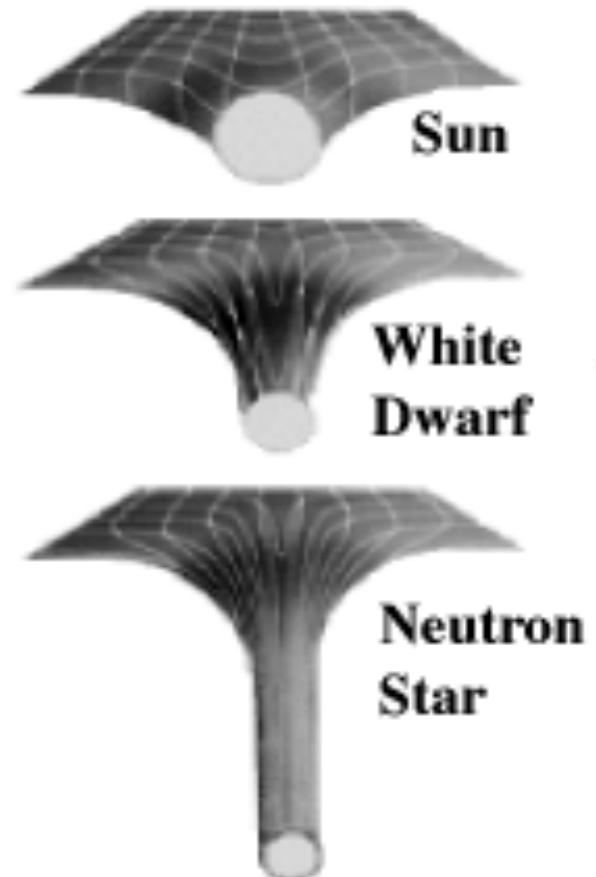


# The Metric of Curved Space: General Relativity

Some facts about the warped manifold of space and time:

1. It has a position-dependent metric  $g_{\mu\nu}(x)$

- Metric describes gravitational field completely
- Metric plays role of relativistic gravitational potentials

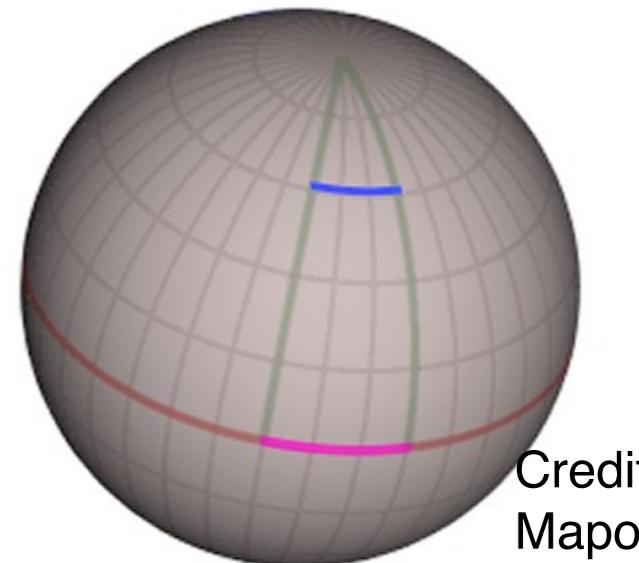


2. It has non-Euclidean relations

- In curved space, Euclidean relations no longer hold
- Ex: sum of interior angles of triangle on sphere deviates 180°

3. It will have a locally flat metric and locally inertial frame

- A small local region can always be described approximately as flat space (Flatness theorem)
- In this region, because of the absence of gravity, Special Relativity is valid and the metric is flat Minkowski; local lightcone structure



# Local Inertial Frames or Local Lorentz Frame

Local properties of curved spacetime should be indistinguishable from those of flat spacetime.

Given a metric  $g_{\alpha\beta}$  in one system of coordinates, at each point  $\mathcal{P}$  it is possible to introduce new coordinates such that

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta}$$

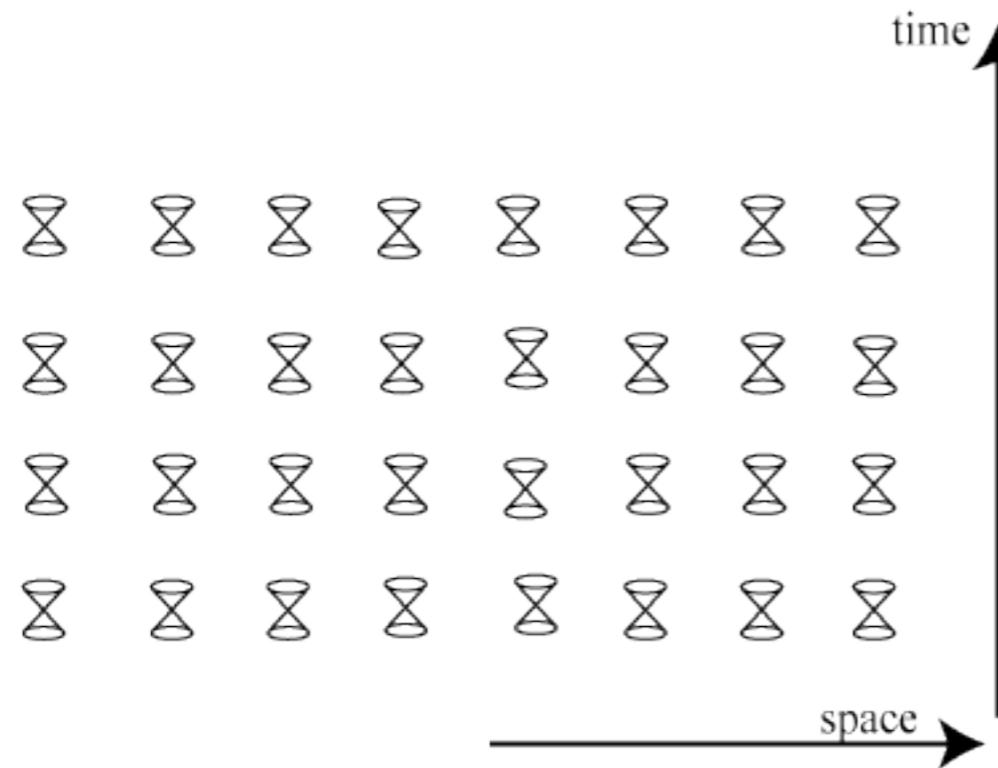
It is not possible to find coordinates in which the metric is flat over the whole of curved spacetime.

At every spacetime point, one can construct a free-fall frame in which gravity is transformed away. However, in a finite-sized region, one can detect the residual tidal force which are second derivatives of the gravitational potential. It is the curvature of spacetime.

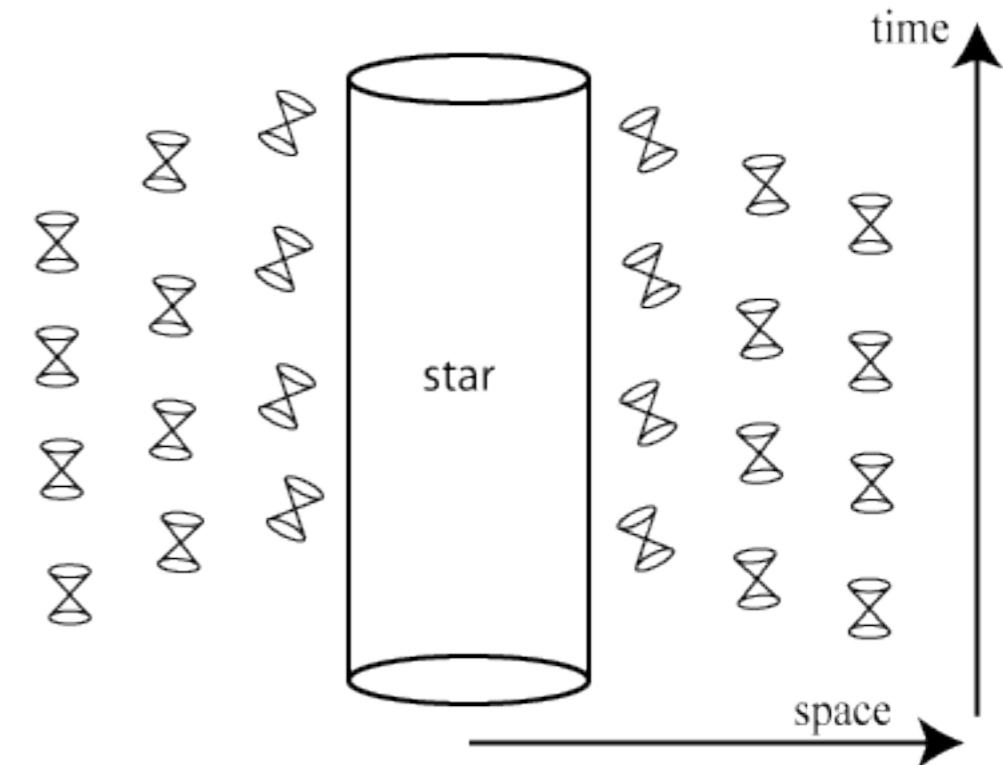
$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(\mathcal{P}) = 0$$

$$\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(\mathcal{P}) \neq 0$$

# Local Spacetime Intervals



Flat spacetime - all light cones oriented in same direction



Curved spacetime - tilted light cones to reflect change in causal structure

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# The Metric Tensor

Greek indices for components  $\mu, \nu \in \{0, 1, 2, 3\}$

Basis vectors in a general coordinate system are not necessarily mutually orthogonal or of unit length

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu \equiv g_{\mu\nu} \neq \delta_{\mu\nu}$$

But we can define an inverse basis such that

$$\mathbf{e}_\mu \cdot \mathbf{e}^\nu = \delta_\mu^\nu$$

As an example, in a four dimensional Cartesian coordinate system:

Basis

vectors:

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ etc.}$$

Inverse basis vectors

(One-forms):

$$\mathbf{e}^0 = [1 \quad 0 \quad 0 \quad 0],$$

$$\mathbf{e}^1 = [0 \quad 1 \quad 0 \quad 0],$$

etc.

# The Metric Tensor

Metric:  $\mathbf{e}_\mu \cdot \mathbf{e}_\nu \equiv g_{\mu\nu}$

Inverse Metric:  $\mathbf{e}^\mu \cdot \mathbf{e}^\nu \equiv g^{\mu\nu}$

Metric matrices are  
inverse to each other:  $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$

Because there are two sets of coordinate basis vectors,  
there are two possible expansions for vector A:

Contravariant components:  $\mathbf{A} = A^\mu \mathbf{e}_\mu \quad A^\mu = \mathbf{A} \cdot \mathbf{e}^\mu$

Covariant components:  $\mathbf{A} = A_\mu \mathbf{e}^\mu \quad A_\mu = \mathbf{A} \cdot \mathbf{e}_\mu$

# Using the Metric

Scalar product of two vectors:  $\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu$

$$= g^{\mu\nu} A_\mu B_\nu$$

Angle between two vectors:  $\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$

In curved spacetime, the metric only determines the infinitesimal length:

$$ds = \sqrt{g_{ab} dx^a dx^b}$$

For a finite length, perform the line integration

$$s = \int ds = \int \frac{ds}{d\lambda} d\lambda = \int \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} d\lambda$$

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# Tensor Calculus: Coordinate Transformations

Recall the chain rule differentiation relation using the gradient:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

Transformation for contravariant vector:

$$A^\mu \rightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

Transformation for covariant vector (1-form):

$$A_\mu \rightarrow A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

Transformation of tensor with mixed indices:

$$T_\nu^\mu \rightarrow T'_\nu^\mu = \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\rho} T_\lambda^\rho$$

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# Tensor Calculus: Covariant Derivative

Ordinary derivatives of tensor components are not tensors. The combination  $\partial_\nu A^\mu$  does not transform properly.

$$\partial_\nu A^\mu \rightarrow \partial'_\nu A'^\mu \neq \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\rho} \partial_\lambda A^\rho$$

We seek a covariant derivative  $\nabla_\nu$  to be used in covariant physics equations. Such a differentiation is constructed so that when acting on tensor components it still yields a tensor.

$$\nabla_\nu A^\mu \rightarrow \nabla'_\nu A'^\mu = \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\rho} \nabla_\lambda A^\rho$$

In order to produce the covariant derivative, the ordinary derivative must be supplemented by another term:

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma_{\nu\lambda}^\mu A^\lambda$$

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma_{\nu\mu}^\lambda A_\lambda$$

# Covariant Derivative and Metric Tensor

Metric tensor is position-dependent but it is a constant with respect to covariant differentiation:

$$\partial g \neq 0 \quad \nabla g = 0 \quad \nabla_\lambda g_{\mu\nu} = 0$$

We can use this relationship to find an expression for the coefficients in the extra term. These coefficients are known as Christoffel symbols - the first derivative of the metric tensor, i.e. “the fundamental theorem of Riemannian geometry”.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} [\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}]$$

In the special case of a Local Lorentz Frame, the Christoffel symbols vanish.

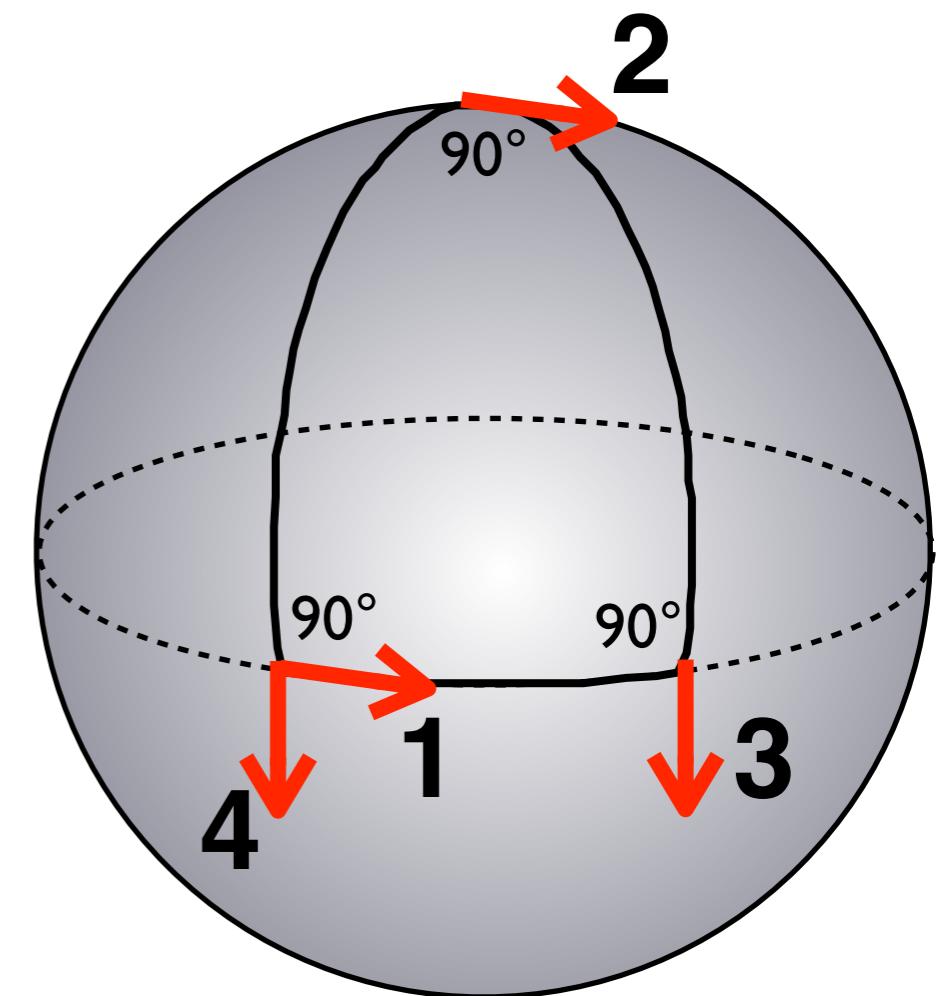
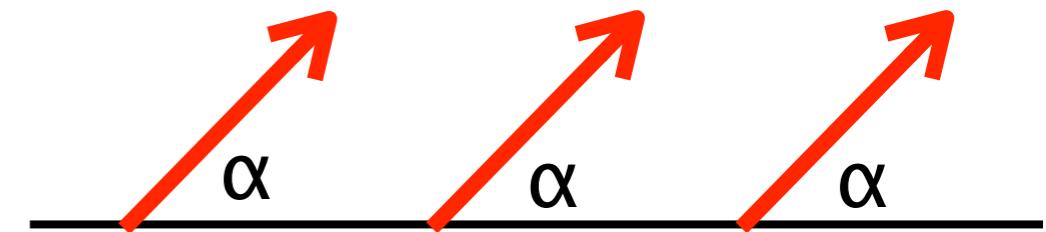
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# Parallel Transport and Geodesics

Consider a vector transported along a curve. A difference in the vector could be caused by either:

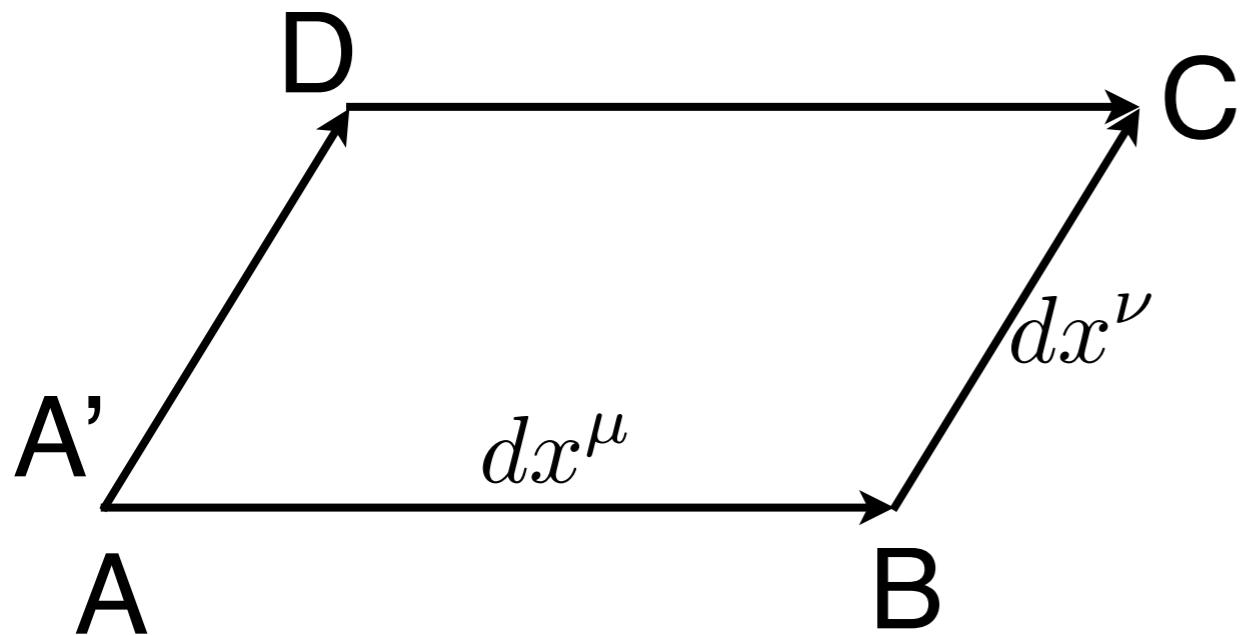
1. change of the vector itself
2. coordinate change

Thus, if we move a vector (tensor) without changing itself, then the only change in components is due to coordinate changes.



# Parallel Transport and Geodesics

difference in direction  $dx^\mu$  — difference in direction  $dx^\nu$



$$\frac{(V_C - V_D) - (V_B - V_A) - [(V_C - V_B) - (V_D - V_{A'})]}{V_A - V_{A'}} = dV$$

$$dA^\mu = [\Delta A^\mu]_{\text{total}} = [\Delta A^\mu]_{\text{true}} + [\Delta A^\mu]_{\text{coord}}$$

$$[\Delta A^\mu]_{\text{true}} = (\nabla_\nu A^\mu) dx^\nu$$

$$[\Delta A^\mu]_{\text{coord}} = -\Gamma_{\nu\lambda}^\mu A^\nu dx^\lambda$$

Parallel transport - Only change due to coordinate changes

$$[\Delta A^\mu]_{\text{true}} = dA^\mu - [\Delta A^\mu]_{\text{coord}} = 0$$

# The Geodesic

Mathematical expression for parallel transport of vector components is

$$\nabla A^\mu = dA^\mu + \Gamma_{\nu\lambda}^\mu A^\nu dx^\lambda = 0$$

The process of parallel transporting a vector  $A^\mu$  along a curve  $x^\mu(\sigma)$  can be expressed according to:

$$\frac{dA^\mu}{d\sigma} + \Gamma_{\nu\lambda}^\mu A^\nu \frac{dx^\lambda}{d\sigma} = 0$$

But the geodesic is a curve for which the tangent vector parallel transports itself, i.e.:

$$A^\mu = \frac{dx^\mu}{d\sigma}$$

Thus, the geodesic equation is:

$$\boxed{\frac{d^2x^\mu}{d\sigma^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0}$$

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# Curvature and the Riemann Tensor

Local Lorentz Frame: effects of curvature become noticeable when taking second derivatives.

$$[\nabla_\alpha, \nabla_\beta] A^\mu = \nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu \equiv R_{\lambda\alpha\beta}^\mu A^\lambda$$

$$R_{\lambda\alpha\beta}^\mu = \partial_\alpha \Gamma_{\lambda\beta}^\mu - \partial_\beta \Gamma_{\lambda\alpha}^\mu + \Gamma_{\nu\alpha}^\mu \Gamma_{\lambda\beta}^\nu - \Gamma_{\nu\beta}^\mu \Gamma_{\lambda\alpha}^\nu$$

In Local Lorentz Frame:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} (\partial_\mu \partial_\alpha g_{\nu\beta} - \partial_\nu \partial_\alpha g_{\mu\beta} + \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\mu \partial_\beta g_{\nu\alpha})$$

Form of Riemann Tensor:  $R = d\Gamma + \Gamma\Gamma \longrightarrow \partial^2 g + (\partial g)^2$

In flat space, the first and second derivatives of the metric vanish.

$R_{\lambda\alpha\beta}^\mu = 0$  implies flat space.

# The Riemann Tensor

$$R_{\lambda\alpha\beta}^{\mu} = \partial_{\alpha}\Gamma_{\lambda\beta}^{\mu} - \partial_{\beta}\Gamma_{\lambda\alpha}^{\mu} + \Gamma_{\nu\alpha}^{\mu}\Gamma_{\lambda\beta}^{\nu} - \Gamma_{\nu\beta}^{\mu}\Gamma_{\lambda\alpha}^{\nu}$$

Symmetries:  $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$$

$$R_{\mu\nu\alpha\beta} = +R_{\alpha\beta\mu\nu}$$

Ricci tensor:  $R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R_{\mu\beta\nu}^{\beta}$

Ricci scalar:  $R \equiv g^{\alpha\beta} R_{\alpha\beta} = R_{\beta}^{\beta}$

Bianchi identity:  $\nabla_{\mu}R_{\alpha\beta\gamma\delta} + \nabla_{\gamma}R_{\alpha\beta\delta\mu} + \nabla_{\delta}R_{\alpha\beta\mu\gamma} = 0$

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# Motivating Einstein Equations

$$\Gamma_{bc}^a(x) = \frac{1}{2}g^{ad} \left( \frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right)$$

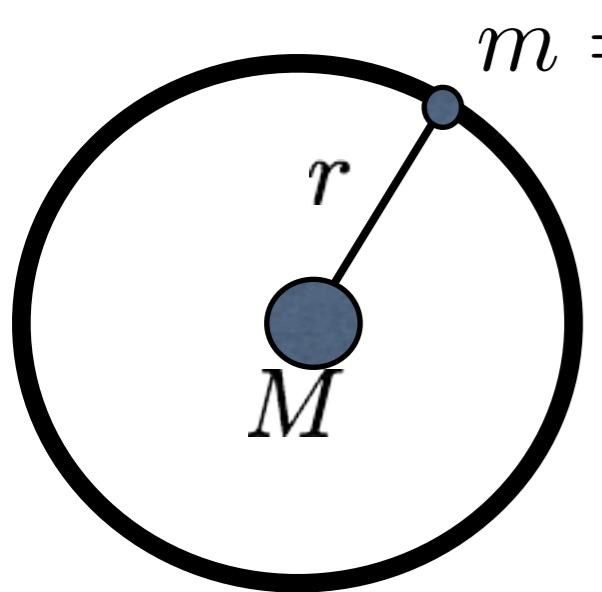
Consider low gravity, low speed, ordinary flat space. Then, GR must reduce to Newtonian gravity.

Only term with any significance is derivative of time component  $g_{00}$ .  
All other derivatives go to zero and  $g$  goes to 1.

$$\Gamma = \frac{1}{2} \frac{\partial g_{00}}{\partial x} \equiv F$$

$$F = -\frac{\partial \phi}{\partial x} \quad F = -\nabla \phi \quad \longrightarrow \quad g_{00} = 2\phi + C$$

# Motivating Einstein Equations



$$F = -\frac{GMm}{r^2} \longrightarrow F = -\frac{GM}{r^2}$$

Consider force capability across whole sphere

$$\int F \cdot dA = \int -\frac{GM}{r^2} \cdot dA = -\frac{GM}{r^2} 4\pi r^2 = -4\pi GM$$

Divergence theorem

$$\int_{\text{area}} F \cdot dA = \int_{\text{vol}} \nabla \cdot F dV$$

$$\rho = \frac{M}{V} \quad M = \int \rho dV$$

$$-4\pi G \int \rho dV = \int \nabla \cdot F dV$$
$$-4\pi G \rho = \nabla \cdot F = \nabla \cdot (-\nabla \phi)$$

$$\nabla^2 \phi = 4\pi G \rho$$

# Motivating Einstein Equations

$$\nabla^2 \left( \frac{1}{2} g_{00} \right) = 4\pi G \rho$$

$$\nabla^2 g_{00} = 8\pi G \rho$$

But this is not a tensor equation and for general relativity, we need tensor equations.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Now we have tensors on both left and right hand side. Einstein tensor is on the left. Instead of mass density on the right, we have a stress-energy-momentum tensor with all mass-energy-stress-pressure terms that you can have.

# Stress-Energy-Momentum Tensor

Momentum 4-vector

$$\vec{p} = m \left( \frac{x_0}{\tau}, \frac{x_1}{\tau}, \frac{x_2}{\tau}, \frac{x_3}{\tau} \right)$$

But we need a tensor!

$$\frac{E}{V} = \frac{W}{V} = \frac{F \times L}{L^3} = \frac{F}{L^2} = \frac{F}{A}$$

$T_{\mu\nu}$  has 0 to 3 indices.

- 00 - time component / energy part.
- Along top - energy flow
- Along side - momentum density
- 9 middle components - momentum flux-stress energy part

$T_{00}$
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$T_{01}$	$T_{02}$	$T_{03}$
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$T_{10}$
$T_{20}$
$T_{30}$

$T_{11}$	$T_{12}$	$T_{13}$
$T_{21}$	$T_{22}$	$T_{23}$
$T_{31}$	$T_{32}$	$T_{33}$

# Motivating Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \xrightarrow{?} R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We need the spacetime curvature term on the left. Einstein thought it should be the Ricci curvature tensor. But there is a problem.

Due to energy conservation:

$$\nabla T_{\mu\nu} = 0$$

But the derivative of Ricci tensor does not equal zero as can be seen with the Bianchi Identities. Instead, what is found is

$$\nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Einstein tensor

# Motivating Einstein Equations

Thus, the equation could have the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein thought he forgot something because it is also true that

$$\nabla g_{\mu\nu} = 0$$

Then we can add the metric tensor term with a constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$\Lambda$  is the cosmological constant for space in math terms. It is often left out except for major cosmological scales.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

# Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Indices  $\mu, \nu$  represent dimensions of spacetime.

Combinations of  $\mu, \nu$  mean there are 16 variations of this equation.

6 equations are duplicates.

Total of 10 Einstein Field Equations.

# Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

# Online Resources

Sean Carroll lecture notes on General Relativity:

<https://arxiv.org/abs/gr-qc/9712019>

Leonard Susskind GR lectures on youtube:

<https://www.youtube.com/watch?v=JRZgW1YjCKk>