Announcements

Please turn in Assignment 1 and pick up Assignment 2
You can also email assignments to the TAs:

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Pawan Gupta (p.gupta@nikhef.nl)

If you haven't yet, please provide your email and affiliation.

Note minor amendments to online syllabus.

Upcoming Events

* Press conference this Wednesday on first result from Event Horizon Telescope: first-ever image of a black hole

General Relativity: A Summary

Lecture 2: Gravitational Waves MSc Course

- A Brief Introduction
- Some Terminology
- Spacetime
  - Metric of flat space: Newtonian
  - Metric of flat space: Special Relativity
  - Metric of curved space
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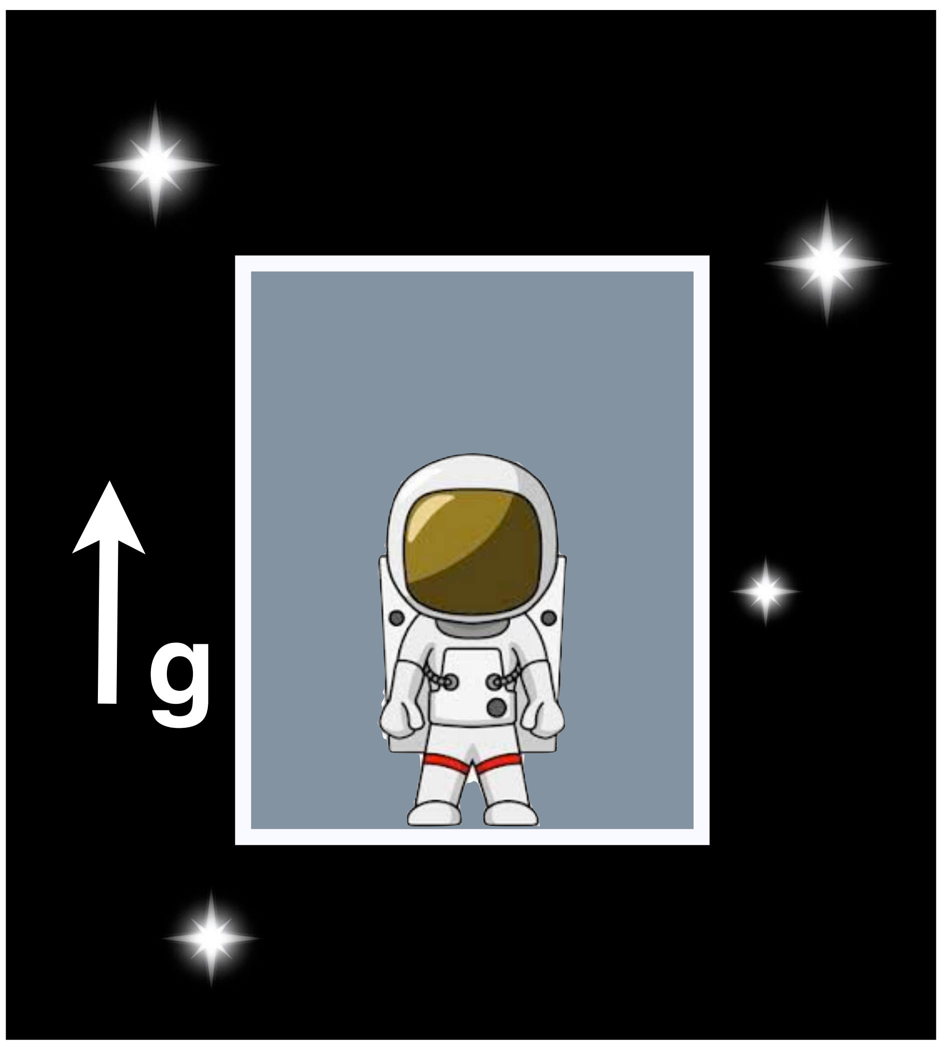
  - Parallel Transport
  - Curvature and the Riemann Tensor
- Motivating the Einstein Equations

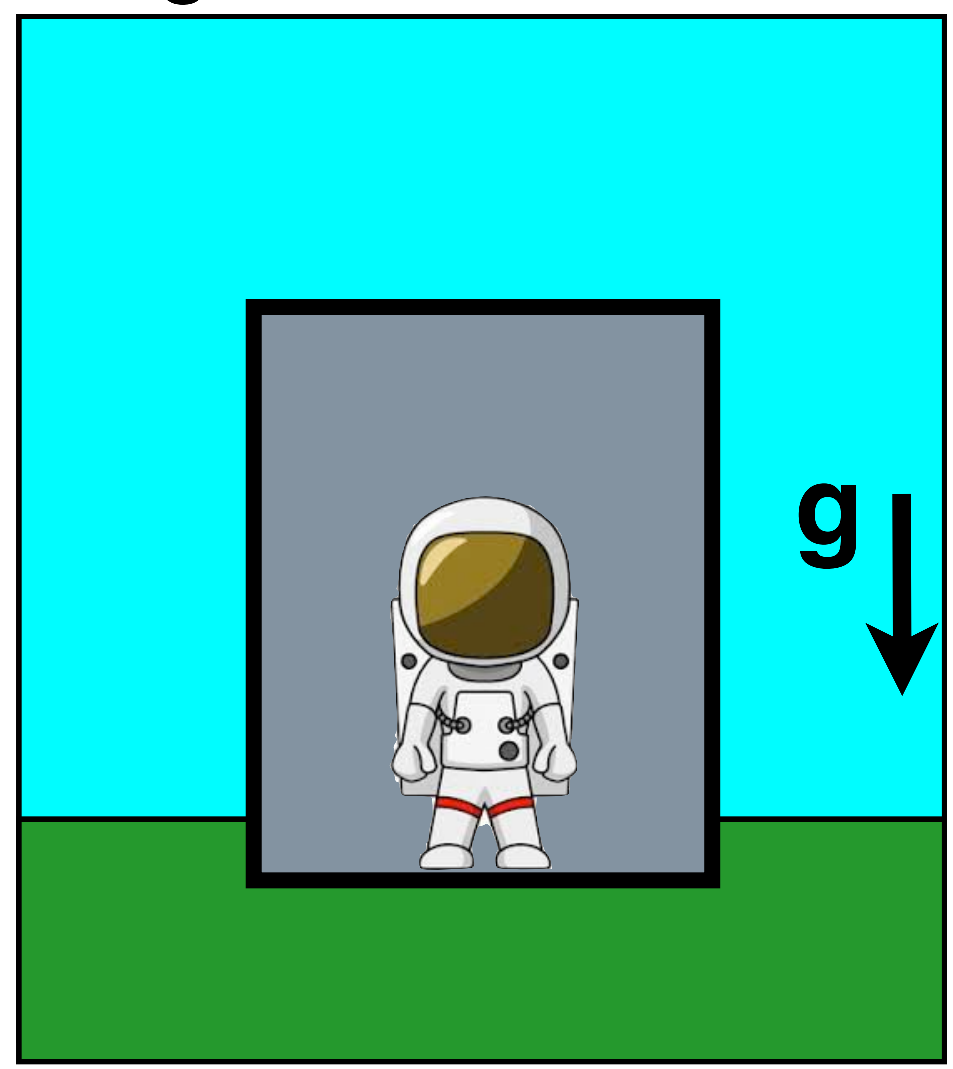
Principle of Equivalence

There is no experiment you can do that will distinguish between the following two experiments.

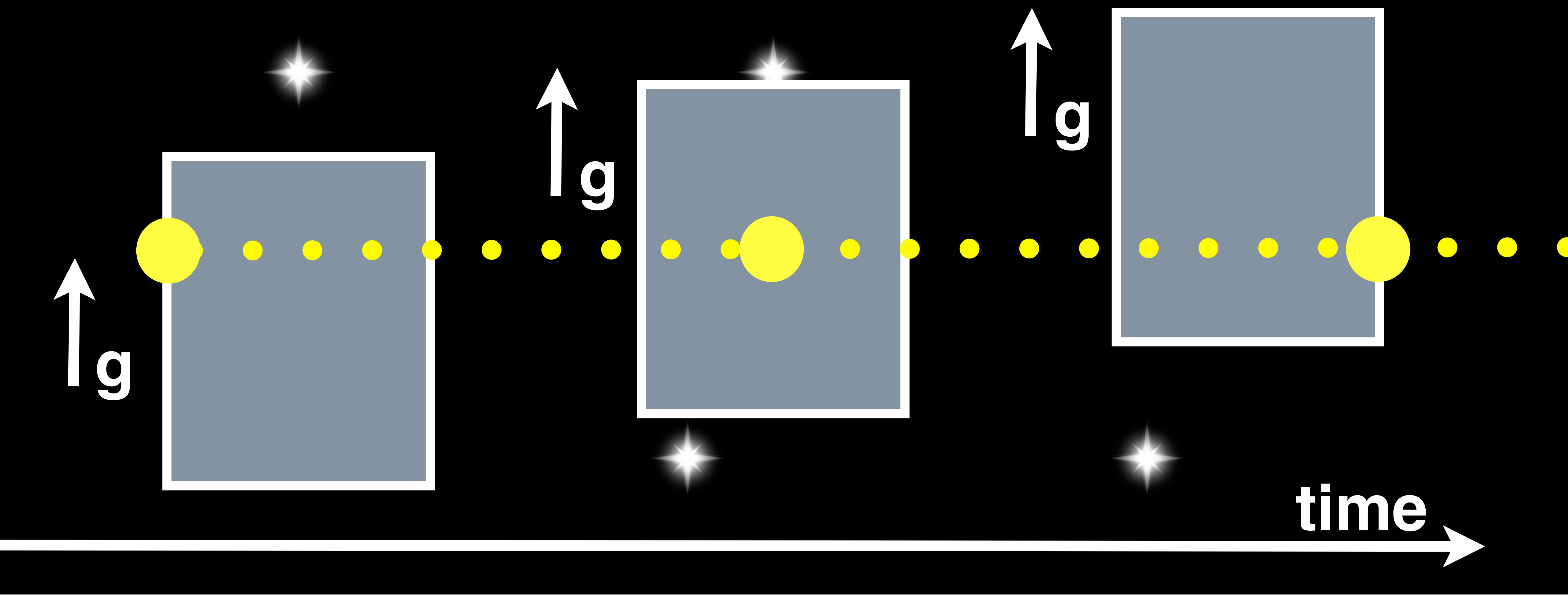
Accelerating at **g**

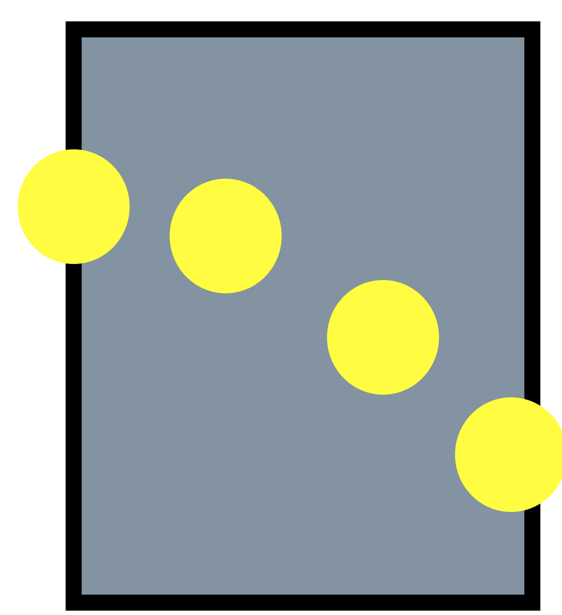


Stationary but subject to gravitational force



Light Bends in a Gravitational Field





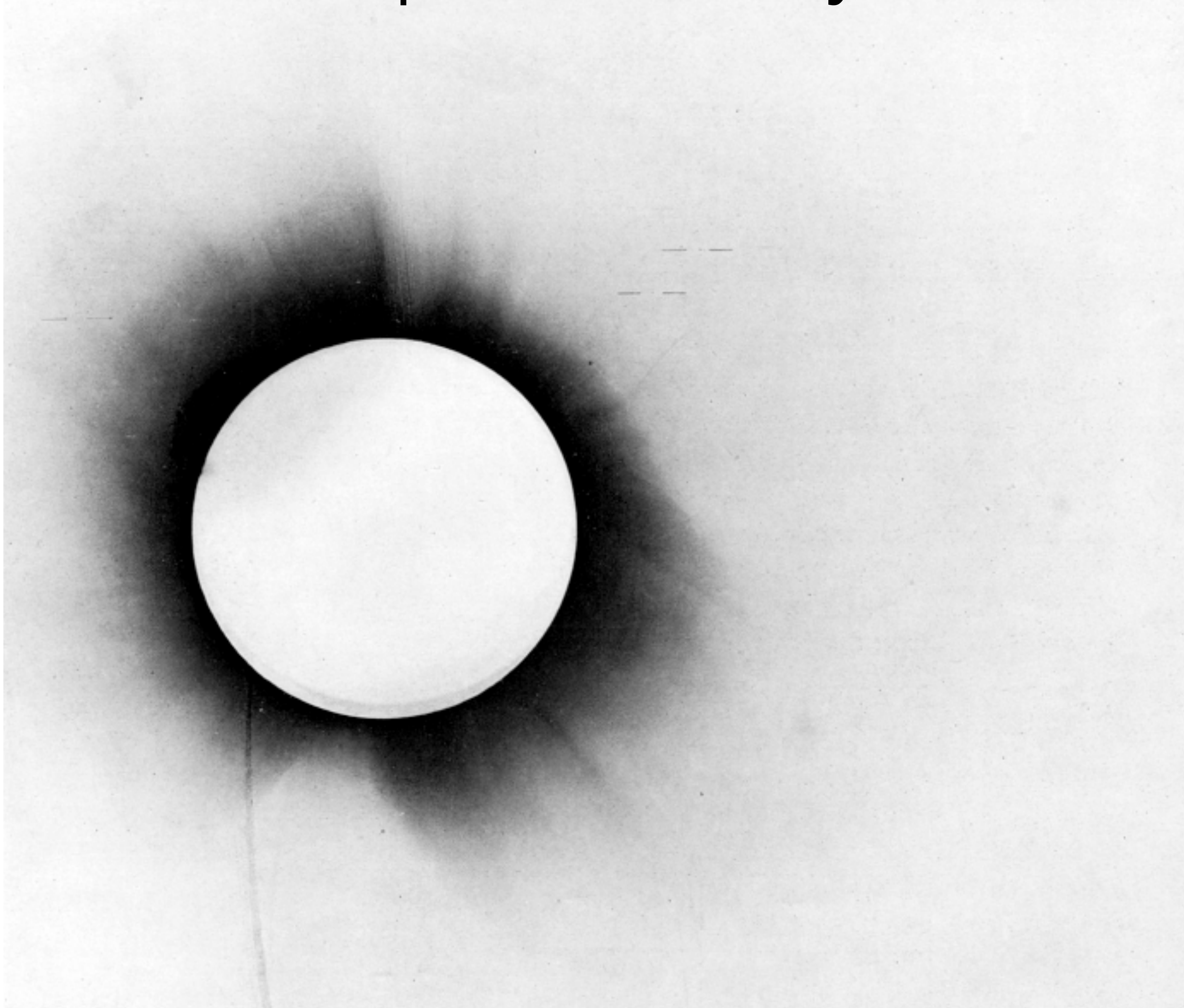
Light has followed a curve.

Light appears to curve when you are accelerating through space with acceleration **g**.

By principle of equivalence, accelerating with acceleration **g** is equivalent to being stationary subject to acceleration **g**.

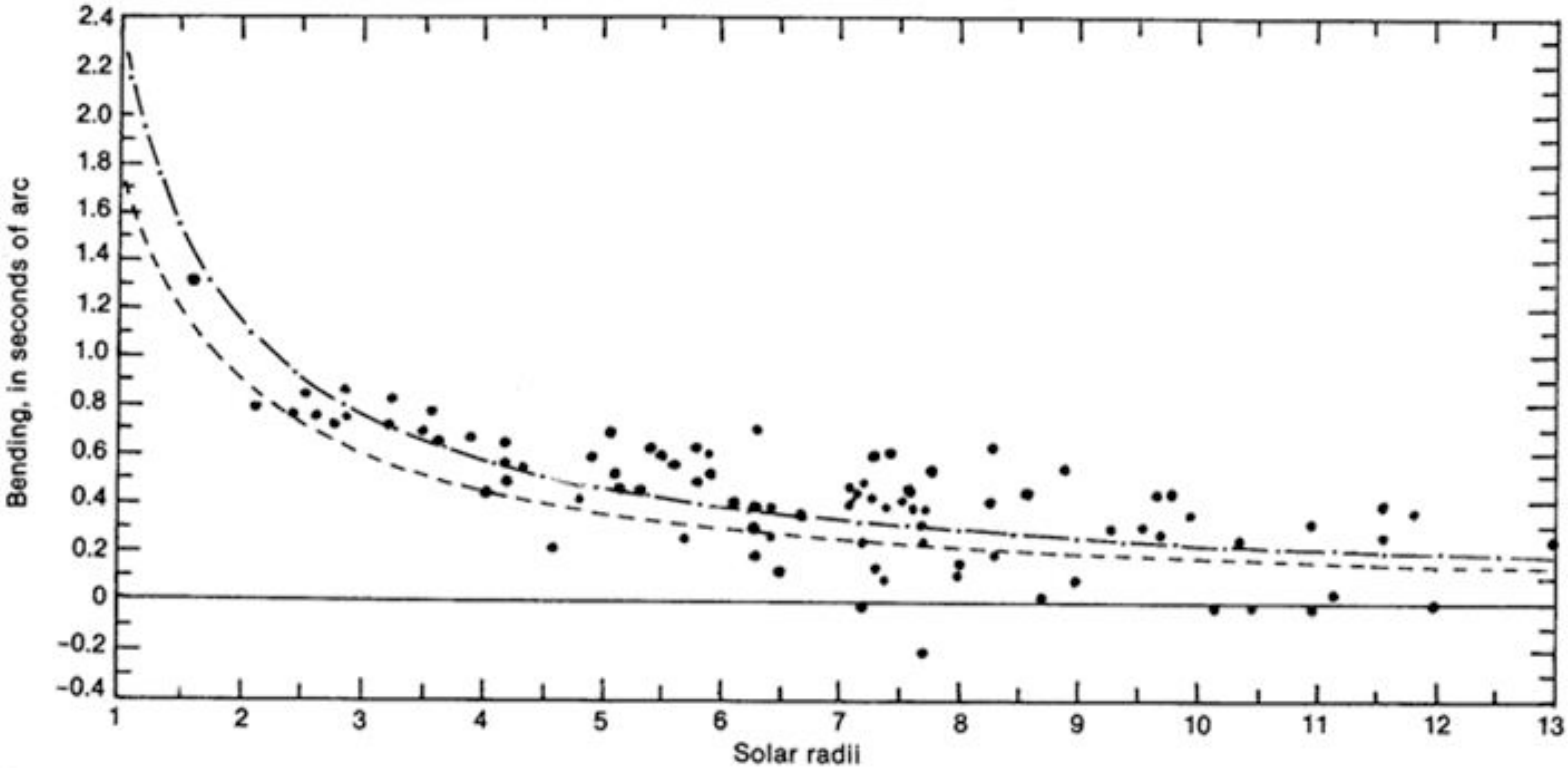
Then, light should also appear to curve in a gravitational field.

Solar Eclipse of May 29, 1919



First observation of light deflection by Arthur Eddington during solar eclipse.

Later Eclipse Measurements



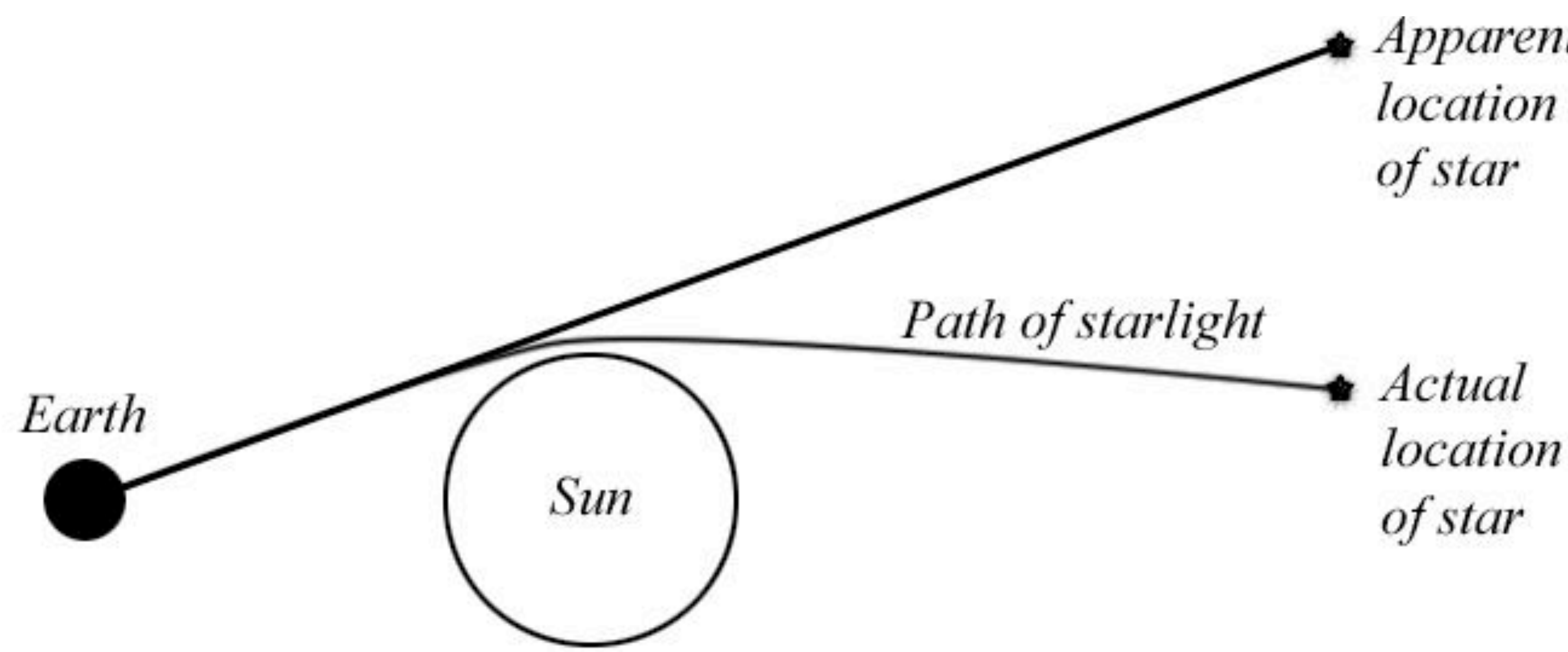
Results from later eclipse experiments in 1922 and 1929. Dashed line is Einstein's prediction. Dot dashed is least-squared fit of actual data.

Why do these measurements imply spacetime is curved?

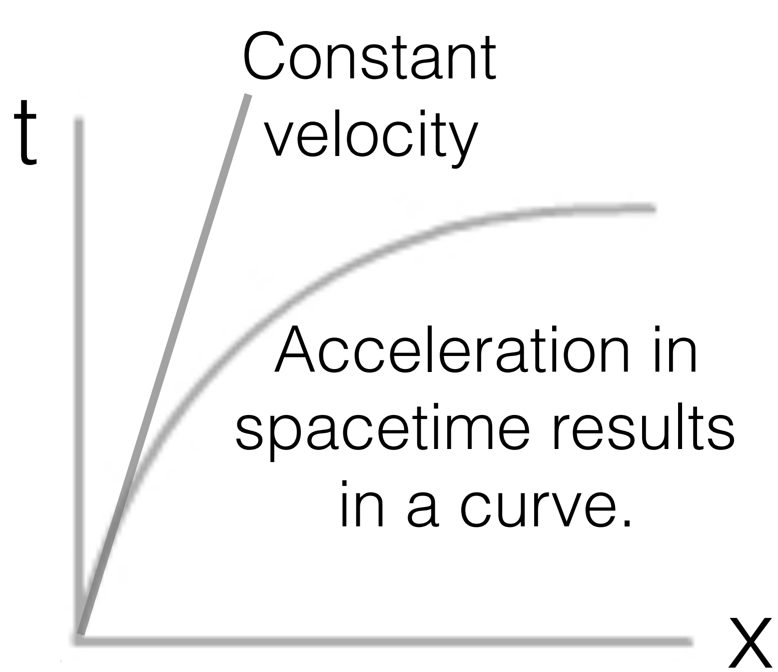
Newton's law of gravitation

$$F = \frac{GMm}{r^2}$$

$$m_{\text{photon}} = 0$$



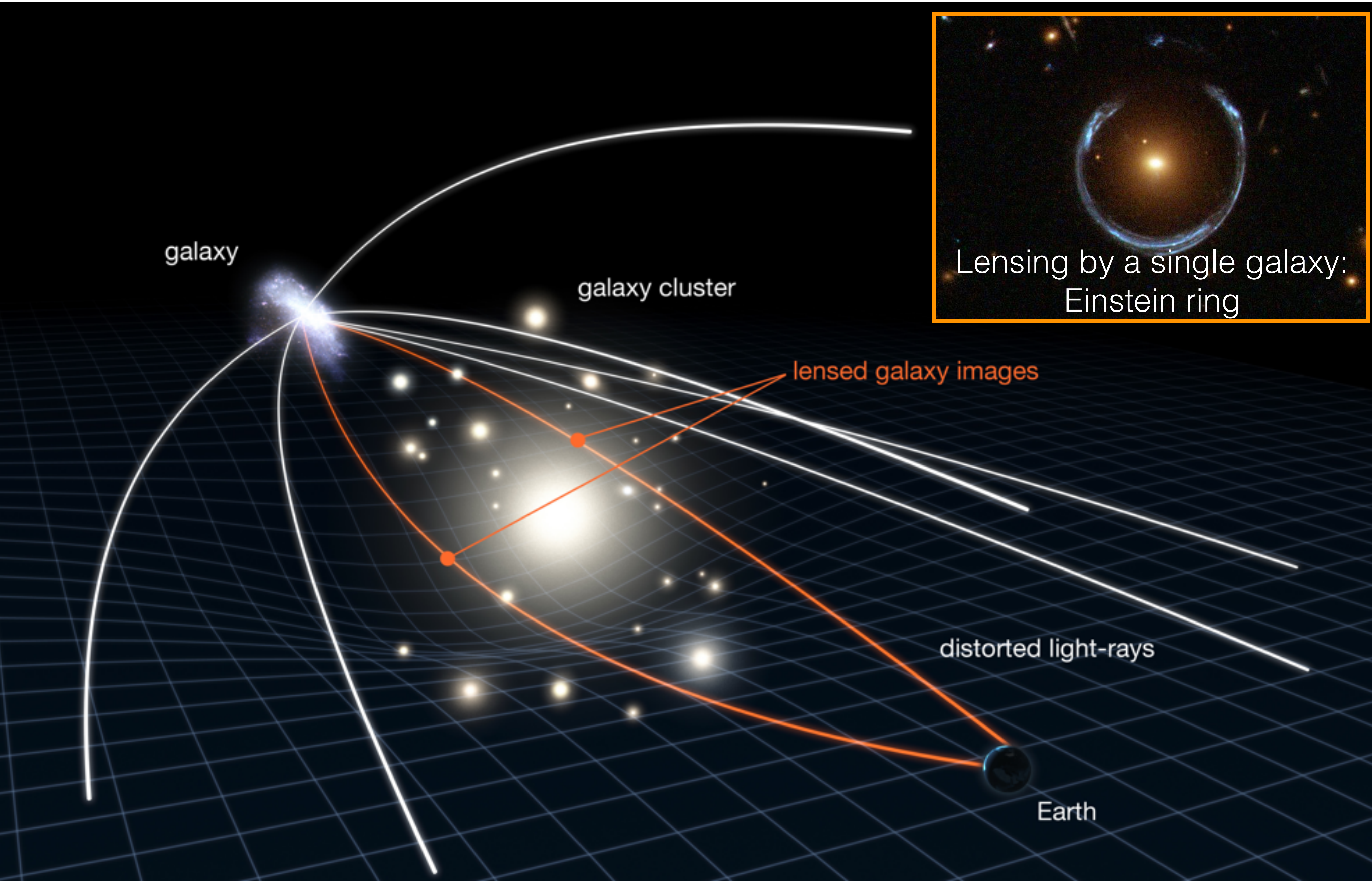
This form of Newton's law doesn't work for light!



"Gravity" causes acceleration.

Therefore, **spacetime** must be curved if it is creating an acceleration.

Gravitational Lensing



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3 Essential Ideas Underlying General Relativity

1. Spacetime may be described as curved, 4-D mathematical structure called pseudo-Riemannian manifold

2. At every spacetime point, there exist locally inertial reference frames, corresponding to locally flat coordinates carried by freely falling observers: Einstein's strong equivalence principle.

3. Mass and mass/momentum flux curves spacetime in a way described by Einstein's tensor field equations.

What is a tensor?

Scalar - tensor rank 0, magnitude, ex: temperature.
Vector - tensor rank 1, magnitude and direction, ex: force.
Tensor - combination of vectors where there is a fixed relationship, independent of coordinate system; ex: dot product, work.

$$T^{mn} = A^m B^n$$

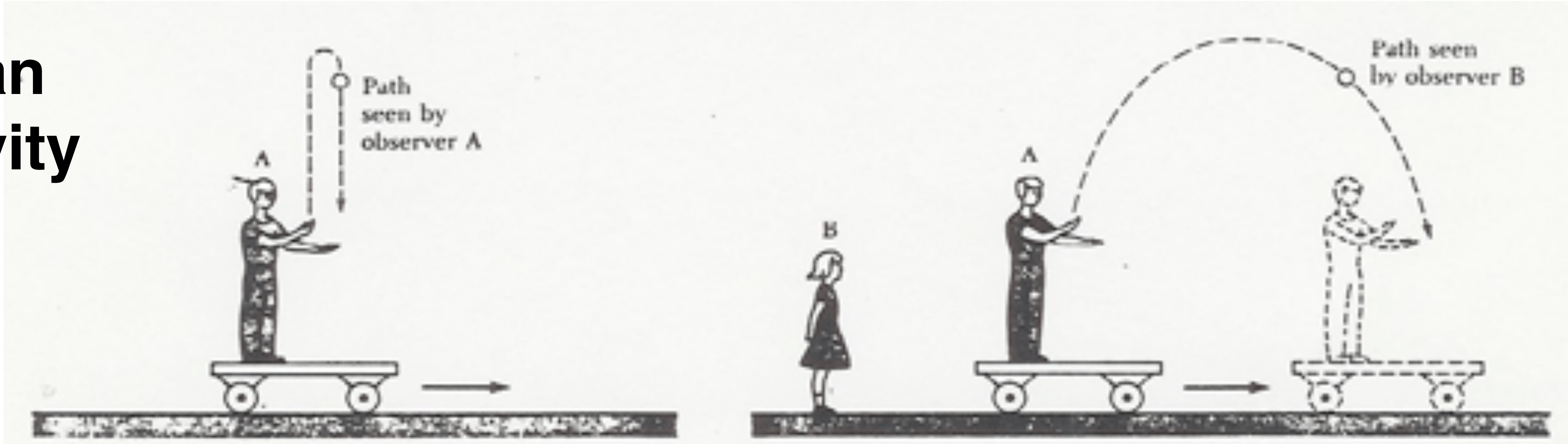
Principle of relativity - “Physics equations should be **covariant** under coordinate transformation.”

To ensure that this is automatically satisfied, write physics equations in terms of tensors.

Inertial Frame of Reference

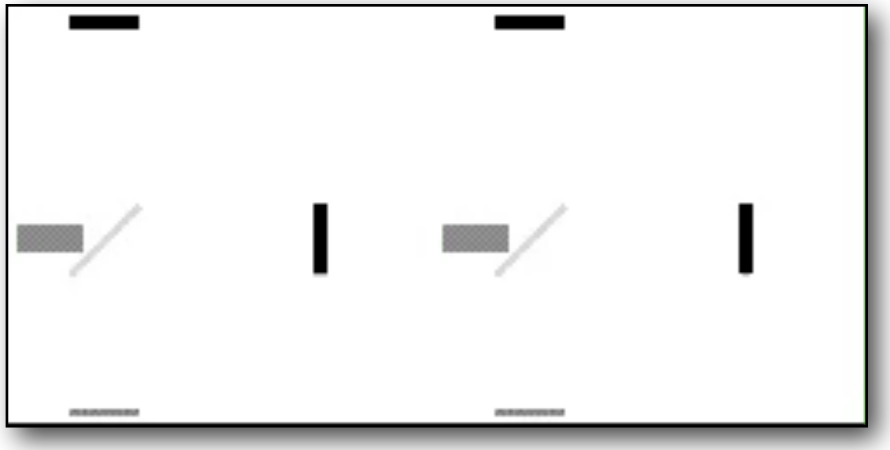
Coordinate systems in which a particle will, if no external force acts, continue it's state of motion with constant velocity. Physics descriptions are simplest here.

Galilean Relativity



Special Relativity - constancy of light speed

Michelson-Morley Experiment



Famous null experiment - motion through aether does not cause a differential phase shift

Coordinate Symmetry Transformations

• Galilean transformation	Classical, non-relativistic mechanics. Valid for $v \ll c$
• Lorentz transformation	Revealed by Special Relativity, i.e. Maxwell equations. Valid for $v \leq c$
• General coordinate transformation	Needed for General Relativity. Physics should be covariant under general transformations between frames of reference. Valid for $v \leq c$ and accelerating frames.

Einstein Summation Convention

Repeated indices imply summation.

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu \;\; = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$= [B_0 \;\; B_1 \;\; B_2 \;\; B_3] \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix}$$

Free index - appears exactly once in every term of equation

Dummy index - appears exactly twice in one given term of equation but only once in equation

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Spacetime

Spacetime points (events) can be labeled by coordinate system:

$$x^\mu = \left(x^0, x^1, x^2, x^3\right)$$

which has no intrinsic meaning.

May be described as curved, 4-D mathematical structure called pseudo-Riemannian differentiable manifold.

Distances between nearby events are calculated using a metric $g_{\mu\nu}$

Greek indices for components $\mu,\ \nu \in \{0, 1, 2, 3\}$

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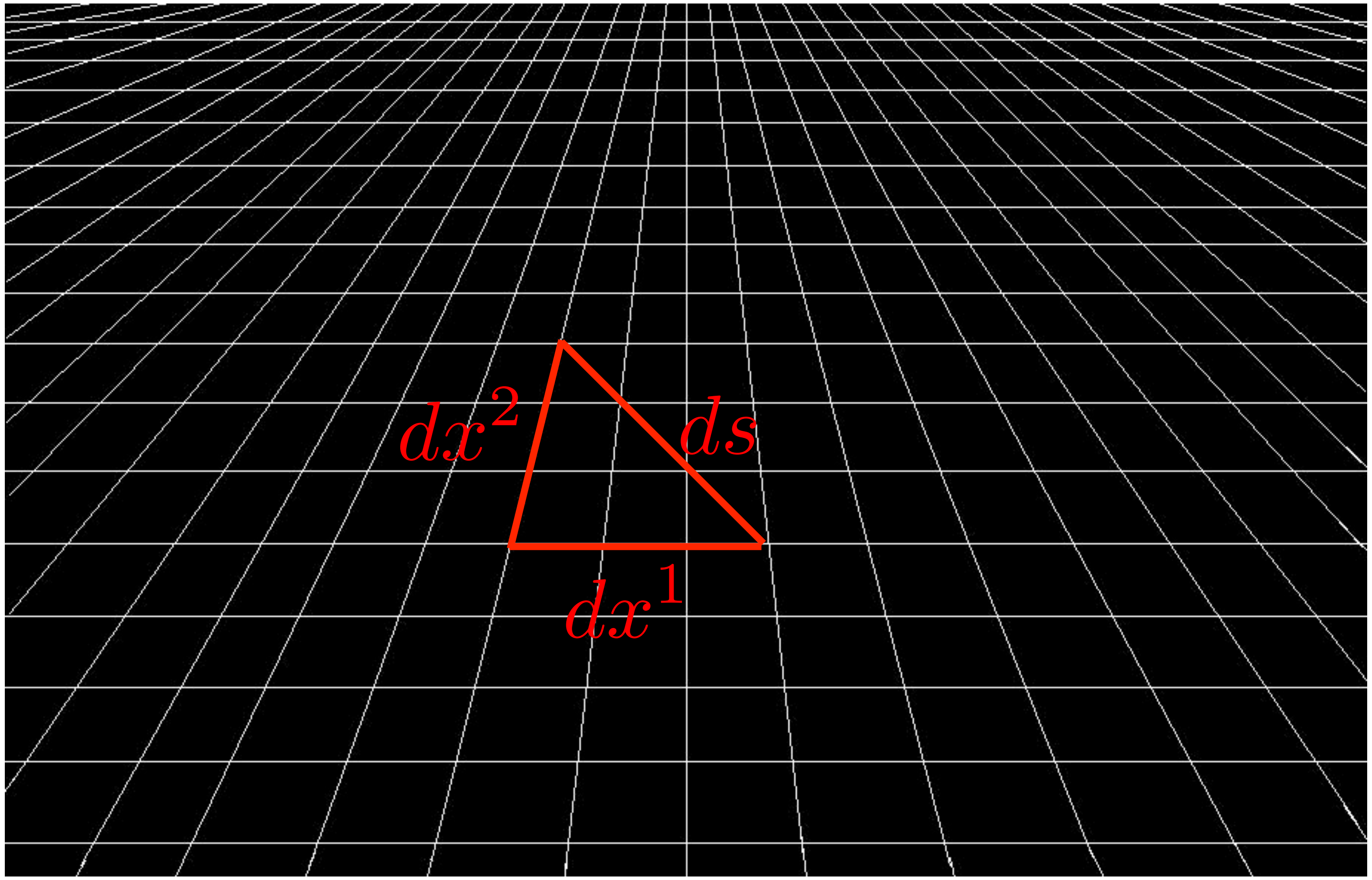
The Metric of Flat Space: Newtonian Mechanics

Pythagorean Theorem in 2-D Euclidean Space



$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2$$
$$= \sum_{mn} dx^m dx^n \delta_{mn}$$
$$= \delta_{mn} \sum_{mn} dx^m dx^n$$

Kronecker delta is metric tensor in flat space.

$$g_{mn} = \delta_{mn} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Position-independent metric

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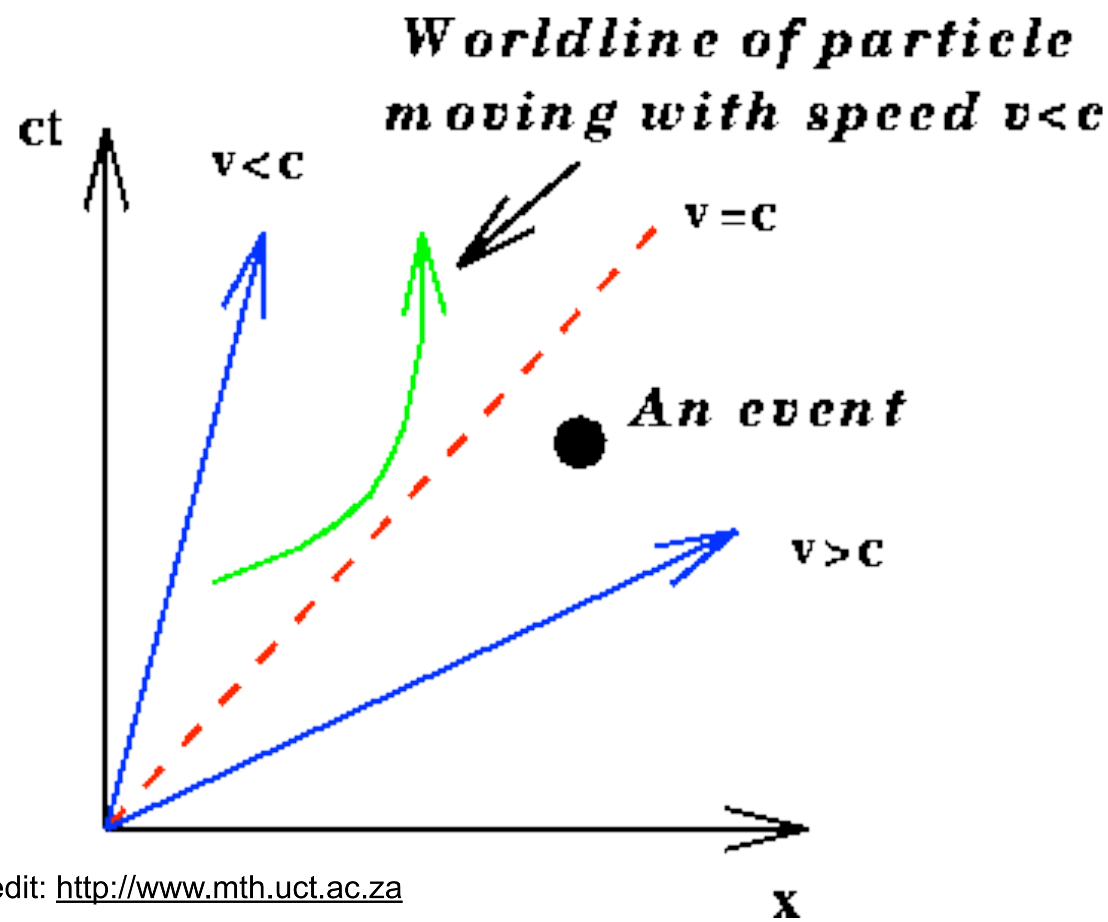
The Metric of Flat Space: Special Relativity



Credit: http://www.mth.uct.ac.za

Units in which $c = 1$

Spacetime interval in flat 4D spacetime

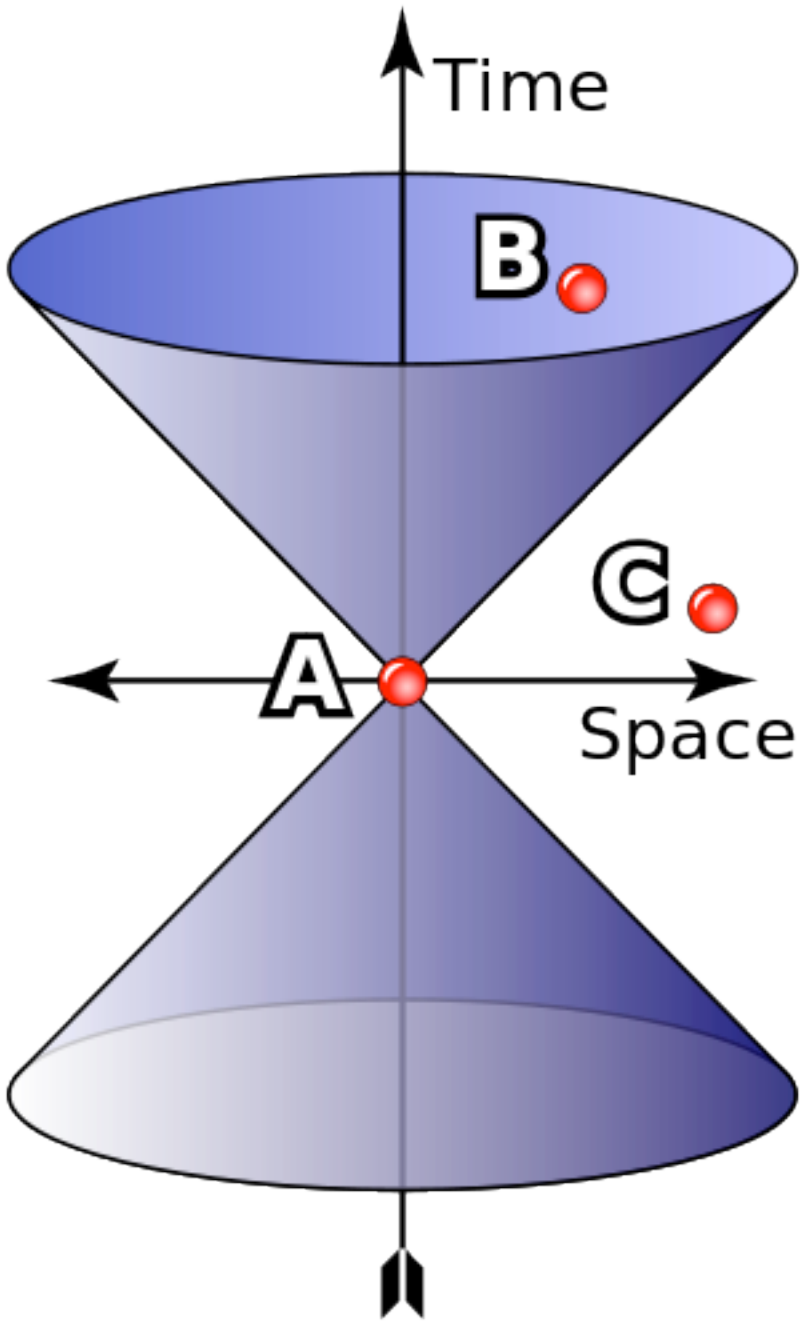
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$= \eta_{\mu\nu} x^\mu x^\nu$$

Minkowski metric is metric tensor in flat 4D spacetime.

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}\,(-1,\,1,\,1,\,1)$$

Position-independent metric

Spacetime Intervals



Light cone

B. Timelike separation - **causally connected** to A

$$ds^2 < 0$$

A. Lightlike (Null) separation

$$ds^2 = 0$$

C. Spacelike separation - cannot exchange signals between A and C

$$ds^2 > 0$$

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The Metric of Curved Space: General Relativity

General relativity as a geometric theory of gravity posits that matter and energy cause spacetime to warp so that

$$g_{\mu\nu} \neq \eta_{\mu\nu}$$

Thus gravitational phenomena are just effects of a curved spacetime on a test particle.

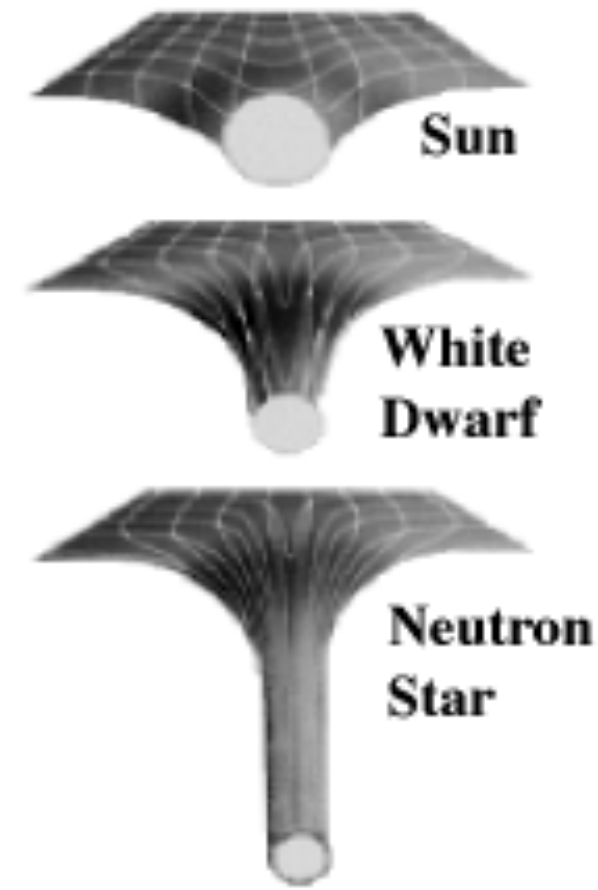
$$\boxed{\text{Source particle}} \xrightarrow[\text{Field equation}]{} \boxed{\text{Field}} \xrightarrow[\text{Equation of motion}]{} \boxed{\text{Test Particle}}$$

$$\boxed{\text{Source}} \xrightarrow[\text{Einstein Field equation}]{} \boxed{\text{Curved spacetime}} \xrightarrow[\text{Geodesic equation}]{} \boxed{\text{Test Particle}}$$

The Metric of Curved Space: General Relativity

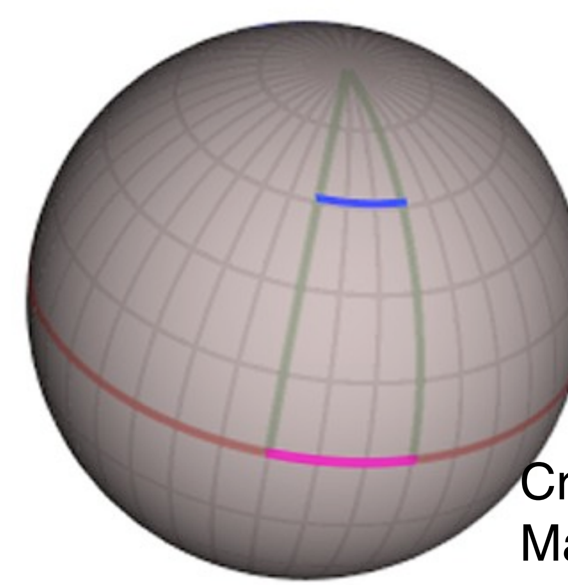
Some facts about the warped manifold of space and time:

1. It has a position-dependent metric $g_{\mu\nu}(x)$
- Metric describes gravitational field completely
- Metric plays role of relativistic gravitational potentials



2. It has non-Euclidean relations
- In curved space, Euclidean relations no longer hold
- Ex: sum of interior angles of triangle on sphere deviates 180°

3. It will have a locally flat metric and locally inertial frame
- A small local region can always be described approximately as flat space (**Flatness theorem**)
- In this region, because of the absence of gravity, Special Relativity is valid and the metric is flat Minkowski; local lightcone structure



Credit: Mapos

Local Inertial Frames or Local Lorentz Frame

Local properties of curved spacetime should be indistinguishable from those of flat spacetime.

Given a metric $g_{\alpha\beta}$ in one system of coordinates, at each point $\mathcal{P}$ it is possible to introduce new coordinates such that

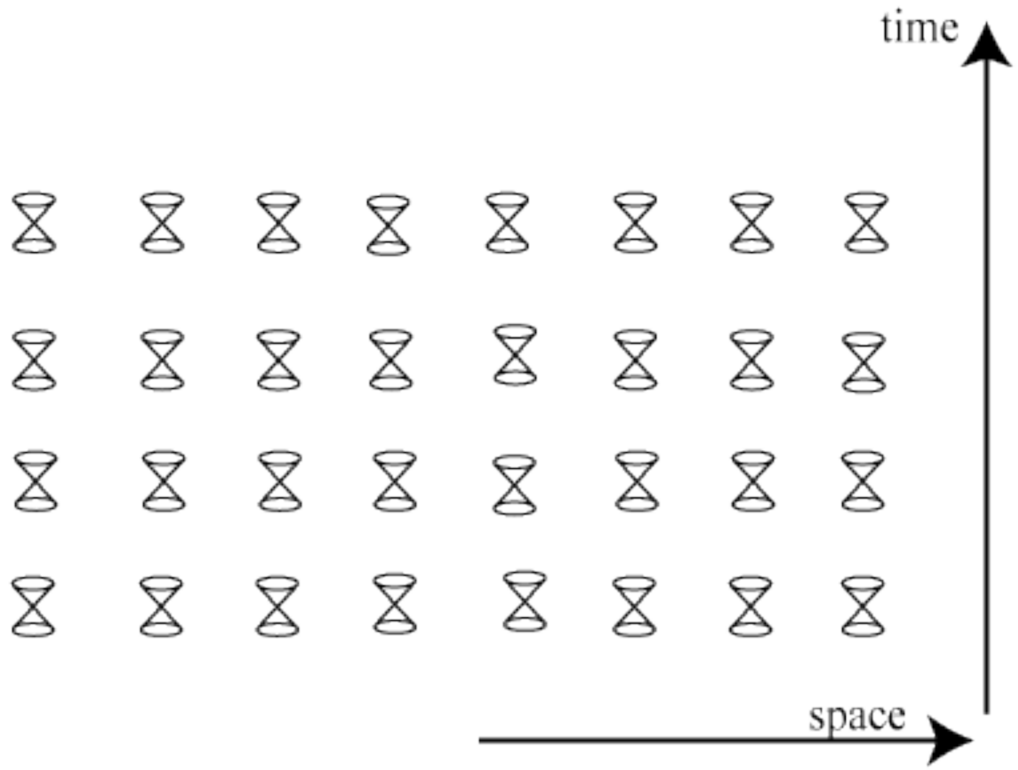
$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta}$$

It is not possible to find coordinates in which the metric is flat over the whole of curved spacetime.

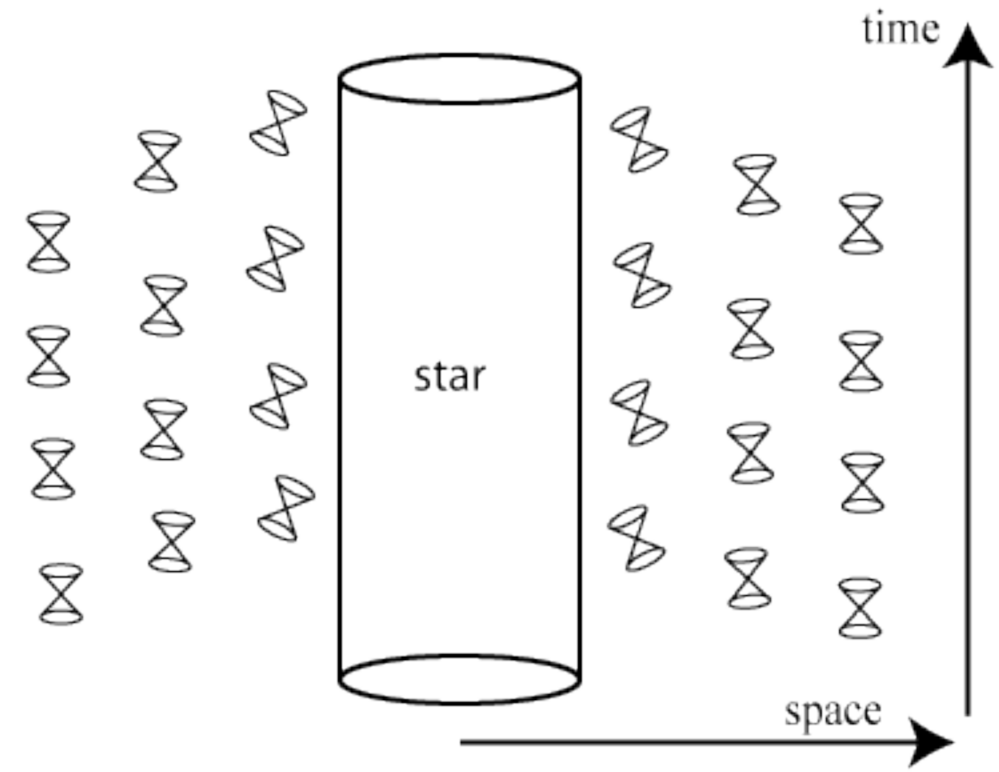
At every spacetime point, one can construct a free-fall frame in which gravity is transformed away. However, in a finite-sized region, one can detect the residual tidal force which are second derivatives of the gravitational potential. It is the curvature of spacetime.

$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(\mathcal{P}) = 0 \qquad\qquad \frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(\mathcal{P}) \neq 0$$

Local Spacetime Intervals



Flat spacetime - all light cones oriented in same direction



Curved spacetime - tilted light cones to reflect change in causal structure

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The Metric Tensor

Greek indices for components $\mu,\ \nu \in \{0, 1, 2, 3\}$

Basis vectors in a general coordinate system are not necessarily mutually orthogonal or of unit length

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu \equiv g_{\mu\nu} \neq \delta_{\mu\nu}$$

But we can define an inverse basis such that

$$\mathbf{e}_\mu \cdot \mathbf{e}^\nu = \delta_\mu^\nu$$

As an example, in a four dimensional Cartesian coordinate system:

Basis vectors:

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ etc.}$$

Inverse basis vectors (One-forms):

$$\mathbf{e}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{e}^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

etc.

The Metric Tensor

Metric: $\mathbf{e}_\mu \cdot \mathbf{e}_\nu \equiv g_{\mu\nu}$

Inverse Metric: $\mathbf{e}^\mu \cdot \mathbf{e}^\nu \equiv g^{\mu\nu}$

Metric matrices are inverse to each other: $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$

Because there are two sets of coordinate basis vectors, there are two possible expansions for vector **A**:

Contravariant components: $\mathbf{A} = A^\mu \mathbf{e}_\mu \quad A^\mu = \mathbf{A} \cdot \mathbf{e}^\mu$

Covariant components: $\mathbf{A} = A_\mu \mathbf{e}^\mu \quad A_\mu = \mathbf{A} \cdot \mathbf{e}_\mu$

Using the Metric

Scalar product of two vectors:
$$\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^{\mu} B^{\nu}$$
$$= g^{\mu\nu} A_{\mu} B_{\nu}$$

Angle between two vectors:
$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

In curved spacetime, the metric only determines the infinitesimal length:
$$ds = \sqrt{g_{ab} \mathrm{d}x^a \mathrm{d}x^b}$$

For a finite length, perform the line integration
$$s = \int ds = \int \frac{ds}{d\lambda} d\lambda = \int \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} d\lambda$$

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Tensor Calculus: Coordinate Transformations

Recall the chain rule differentiation relation using the gradient:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$

Transformation for **contravariant** vector:

$$A^{\mu} \rightarrow A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

Transformation for **covariant** vector (1-form):

$$A_{\mu} \rightarrow A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}$$

Transformation of tensor with mixed indices:

$$T^{\mu}_{\nu} \rightarrow T'^{\mu}_{\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} T^{\rho}_{\lambda}$$

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Tensor Calculus: Covariant Derivative

Ordinary derivatives of tensor components are not tensors. The combination $\partial_\nu A^\mu$ does not transform properly.

$$\partial_\nu A^\mu \to \partial'_\nu A'^\mu \neq \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\partial_\lambda A^\rho$$

We seek a covariant derivative ∇_ν to be used in covariant physics equations. Such a differentiation is constructed so that when acting on tensor components it still yields a tensor.

$$\nabla_\nu A^\mu \to \nabla'_\nu A'^\mu = \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\nabla_\lambda A^\rho$$

In order to produce the covariant derivative, the ordinary derivative must be supplemented by another term:

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda \qquad \nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$$

Covariant Derivative and Metric Tensor

Metric tensor is position-dependent but it is a constant with respect to covariant differentiation:

$$\partial \mathbf{g} \neq 0 \qquad \nabla \mathbf{g} = 0 \qquad \nabla_\lambda g_{\mu\nu} = 0$$

We can use this relationship to find an expression for the coefficients in the extra term. These coefficients are known as **Christoffel symbols** - the first derivative of the metric tensor, i.e. “the fundamental theorem of Riemannian geometry”.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right]$$

In the special case of a Local Lorentz Frame, the Christoffel symbols vanish.

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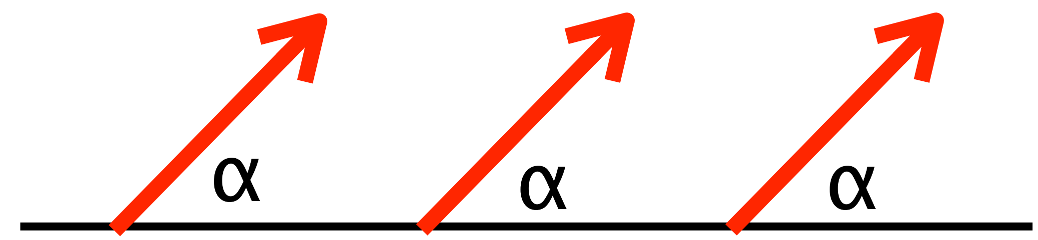
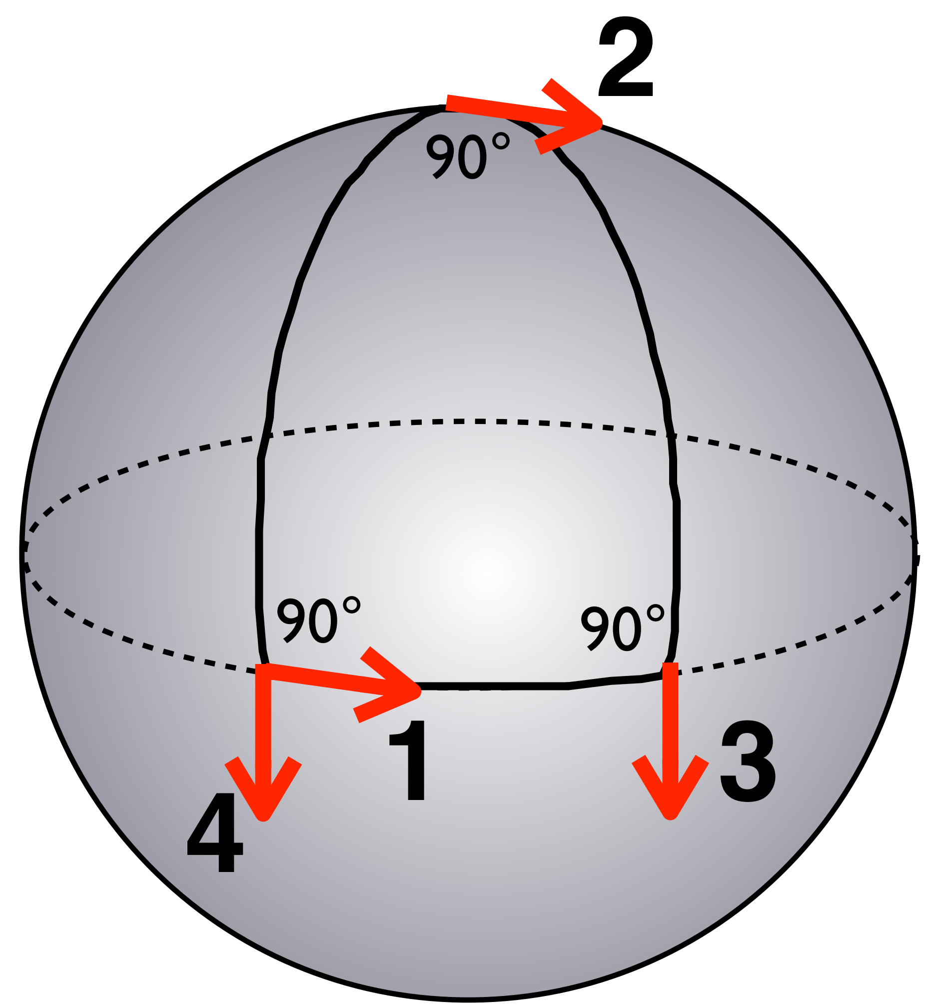
Parallel Transport and Geodesics

Consider a vector transported along a curve. A difference in the vector could be caused by either:
1. change of the vector itself
2. coordinate change

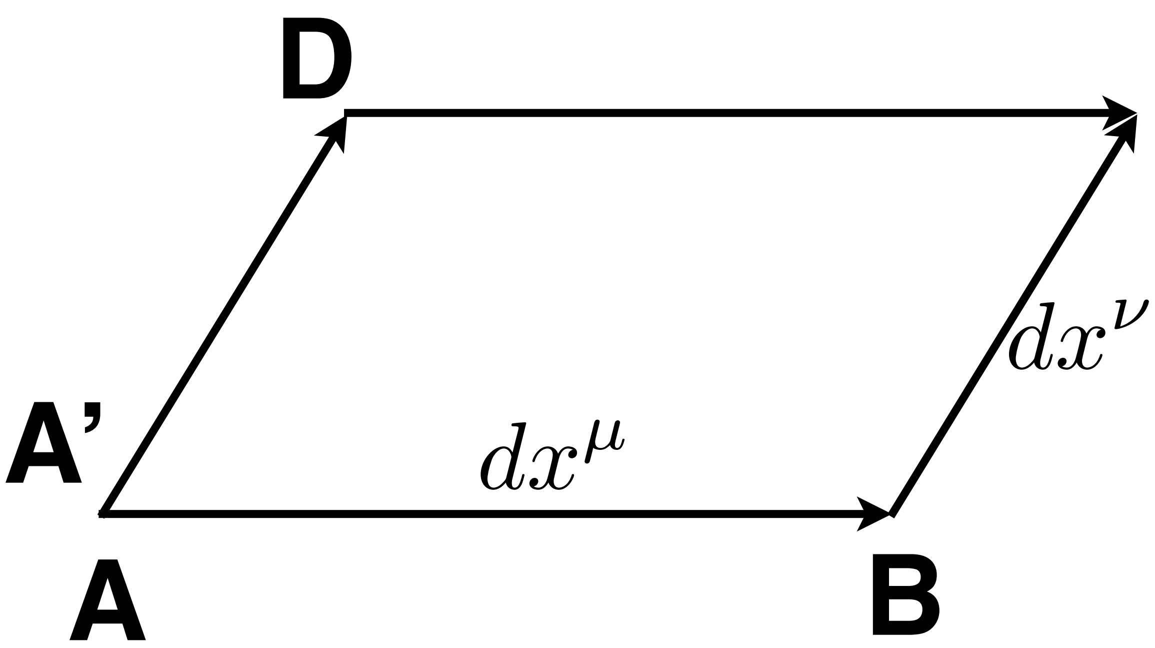




Thus, if we move a vector (tensor) without changing itself, then the only change in components is due to coordinate changes.

Parallel Transport and Geodesics

difference in direction dx^μ — difference in direction dx^ν



$$(V_C - V_D) - (V_B - V_A)$$

$$-\left[(V_C - V_B) - (V_D - V_{A'})\right]$$

$$V_A - V_{A'} = dV$$

$$dA^\mu = [\Delta A^\mu]_{\text{total}} = [\Delta A^\mu]_{\text{true}} + [\Delta A^\mu]_{\text{coord}}$$

$$[\Delta A^\mu]_{\text{true}} = (\nabla_\nu A^\mu)\, dx^\nu$$

$$[\Delta A^\mu]_{\text{coord}} = -\Gamma^\mu_{\nu\lambda} A^\nu dx^\lambda$$

Parallel transport - Only change due to coordinate changes

$$[\Delta A^\mu]_{\text{true}} = dA^\mu - [\Delta A^\mu]_{\text{coord}} = 0$$

The Geodesic

Mathematical expression for parallel transport of vector components is

$$\nabla A^{\mu} = dA^{\mu} + \Gamma^{\mu}_{\nu\lambda} A^{\nu} dx^{\lambda} = 0$$

The process of parallel transporting a vector A^{μ} along a curve $x^{\mu}(\sigma)$ can be expressed according to:

$$\frac{dA^{\mu}}{d\sigma} + \Gamma^{\mu}_{\nu\lambda} A^{\nu} \frac{dx^{\lambda}}{d\sigma} = 0$$

But the geodesic is a curve for which the tangent vector parallel transports itself, i.e.:

$$A^{\mu} = \frac{dx^{\mu}}{d\sigma}$$

Thus, the geodesic equation is: $\boxed{\frac{d^2 x^{\mu}}{d\sigma^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\sigma} \frac{dx^{\lambda}}{d\sigma} = 0}$

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Curvature and the Riemann Tensor

Local Lorentz Frame: effects of curvature become noticeable when taking second derivatives.

$$[\nabla_\alpha, \nabla_\beta]\, A^\mu = \nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu \equiv R^\mu_{\lambda\alpha\beta} A^\lambda$$

$$R^\mu_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\lambda\alpha}$$

In Local Lorentz Frame:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\partial_\mu \partial_\alpha g_{\nu\beta} - \partial_\nu \partial_\alpha g_{\mu\beta} + \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\mu \partial_\beta g_{\nu\alpha}\right)$$

Form of Riemann Tensor: $R = d\Gamma + \Gamma\Gamma \longrightarrow \partial^2 g + (\partial g)^2$

In flat space, the first and second derivatives of the metric vanish.

$R^\mu_{\lambda\alpha\beta} = 0$ implies flat space.

The Riemann Tensor

$$R^{\mu}_{\lambda\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\lambda\beta} - \partial_\beta \Gamma^{\mu}_{\lambda\alpha} + \Gamma^{\mu}_{\nu\alpha}\Gamma^{\nu}_{\lambda\beta} - \Gamma^{\mu}_{\nu\beta}\Gamma^{\nu}_{\lambda\alpha}$$

Symmetries:

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$$

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$$

$$R_{\mu\nu\alpha\beta} = +R_{\alpha\beta\mu\nu}$$

Ricci tensor: $R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\beta}_{\mu\beta\nu}$

Ricci scalar: $R \equiv g^{\alpha\beta} R_{\alpha\beta} = R^{\beta}_{\beta}$

Bianchi identity: $\nabla_\mu R_{\alpha\beta\gamma\delta} + \nabla_\gamma R_{\alpha\beta\delta\mu} + \nabla_\delta R_{\alpha\beta\mu\gamma} = 0$

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Motivating Einstein Equations

$$\Gamma^{\mathrm{a}}_{\mathrm{bc}}(x) = \frac{1}{2} g^{\mathrm{ad}} \left(\frac{\partial g_{\mathrm{dc}}}{\partial x^{\mathrm{b}}} + \frac{\partial g_{\mathrm{ab}}}{\partial x^{\mathrm{c}}} - \frac{\partial g_{\mathrm{bc}}}{\partial x^{\mathrm{d}}} \right)$$

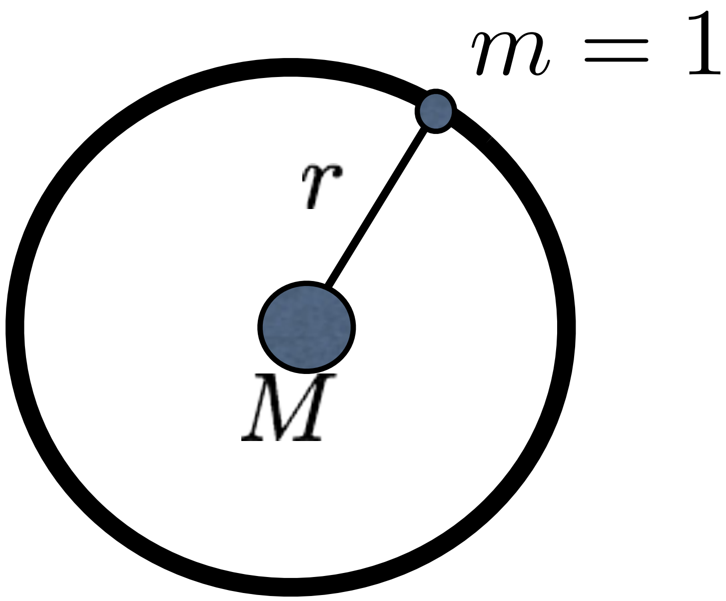
Consider low gravity, low speed, ordinary flat space. Then, GR must reduce to Newtonian gravity.

Only term with any significance is derivative of time component g_{00}. All other derivatives go to zero and g goes to 1.

$$\Gamma = \frac{1}{2} \frac{\partial g_{00}}{\partial x} \equiv F$$

$$F = -\frac{\partial \phi}{\partial x} \qquad F = -\nabla \phi \qquad \longrightarrow \qquad g_{00} = 2\phi + C$$

Motivating Einstein Equations



$$F = -\frac{GMm}{r^2} \longrightarrow F = -\frac{GM}{r^2}$$

Consider force capability across whole sphere

$$\int F \cdot \mathrm{d}A = \int -\frac{GM}{r^2} \cdot \mathrm{d}A = -\frac{GM}{r^2} 4\pi r^2 = -4\pi GM$$

Divergence theorem

$$\int_{\mathrm{area}} F \cdot \mathrm{d}A = \int_{\mathrm{vol}} \nabla \cdot F \mathrm{d}V$$

$$\rho = \frac{M}{V} \qquad M = \int \rho dV$$

$$-4\pi G \int \rho \mathrm{d}V = \int \nabla \cdot F \mathrm{d}V$$

$$-4\pi G\rho = \nabla \cdot F = \nabla \cdot (-\nabla \phi)$$

$$\nabla^2 \phi = 4\pi G \rho$$

Motivating Einstein Equations

$$\nabla^2 \left(\frac{1}{2} g_{00} \right) = 4\pi G \rho$$

$$\nabla^2 g_{00} = 8\pi G \rho$$

But this is not a tensor equation and for general relativity, we need tensor equations.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Now we have tensors on both left and right hand side. **Einstein tensor** is on the left. Instead of mass density on the right, we have a **stress-energy-momentum tensor** with all mass-energy-stress-pressure terms that you can have.

Stress-Energy-Momentum Tensor

Momentum 4-vector

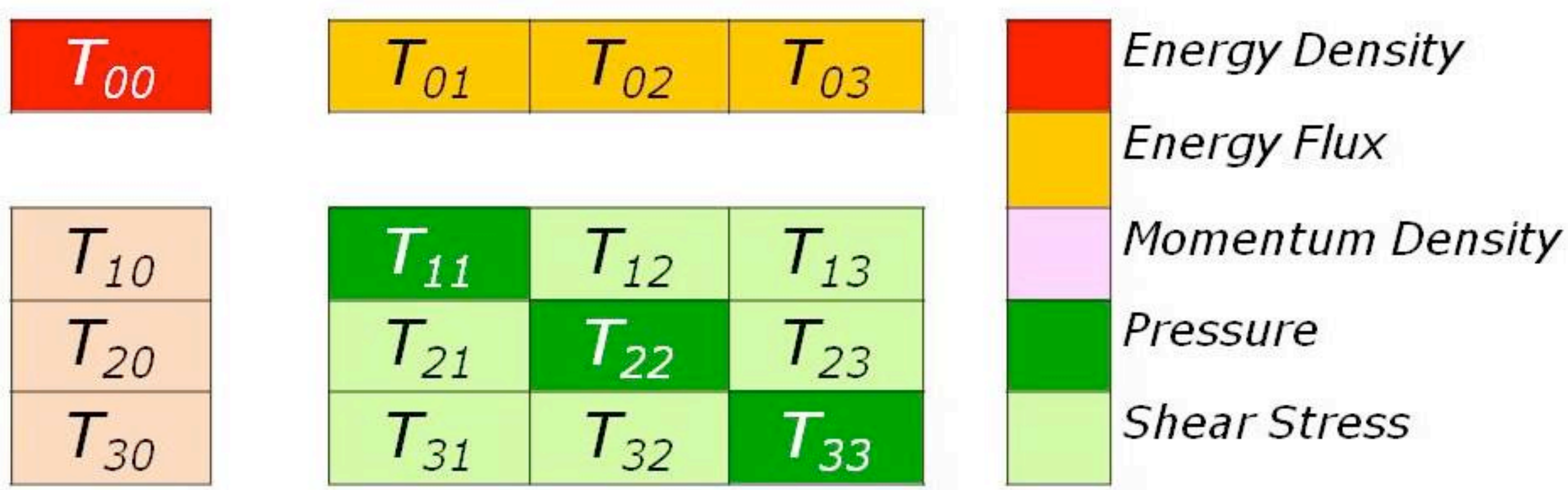
$$\vec{p} = m\left(\frac{x_0}{\tau}, \frac{x_1}{\tau}, \frac{x_2}{\tau}, \frac{x_3}{\tau}\right)$$

But we need a tensor!

$$\frac{E}{V} = \frac{W}{V} = \frac{F \times L}{L^3} = \frac{F}{L^2} = \frac{F}{A}$$

$T_{\mu\nu}$ has 0 to 3 indices.

- 00 - time component / energy part.
- Along top - energy flow
- Along side - momentum density
- 9 middle components - momentum flux-stress energy part



Motivating Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \xrightarrow{?} \quad R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We need the spacetime curvature term on the left. Einstein thought it should be the Ricci curvature tensor. But there is a problem.

Due to energy conservation:

$$\nabla T_{\mu\nu} = 0$$

But the derivative of Ricci tensor does not equal zero as can be seen with the Bianchi Identities. Instead, what is found is

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

Einstein tensor

Motivating Einstein Equations

Thus, the equation could have the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein thought he forgot something because it is also true that

$$\nabla g_{\mu\nu} = 0$$

Then we can add the metric tensor term with a constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Λ is the cosmological constant for space in math terms. It is often left out except for major cosmological scales.

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Indices μ, ν represent dimensions of spacetime.

Combinations of μ, ν mean there are 16 variations of this equation.

6 equations are duplicates.

Total of 10 Einstein Field Equations.

Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity (next time)
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

Online Resources

Sean Carroll lecture notes on General Relativity:
https://arxiv.org/abs/gr-qc/9712019

Leonard Susskind GR lectures on youtube:
https://www.youtube.com/watch?v=JRZgW1YjCKk