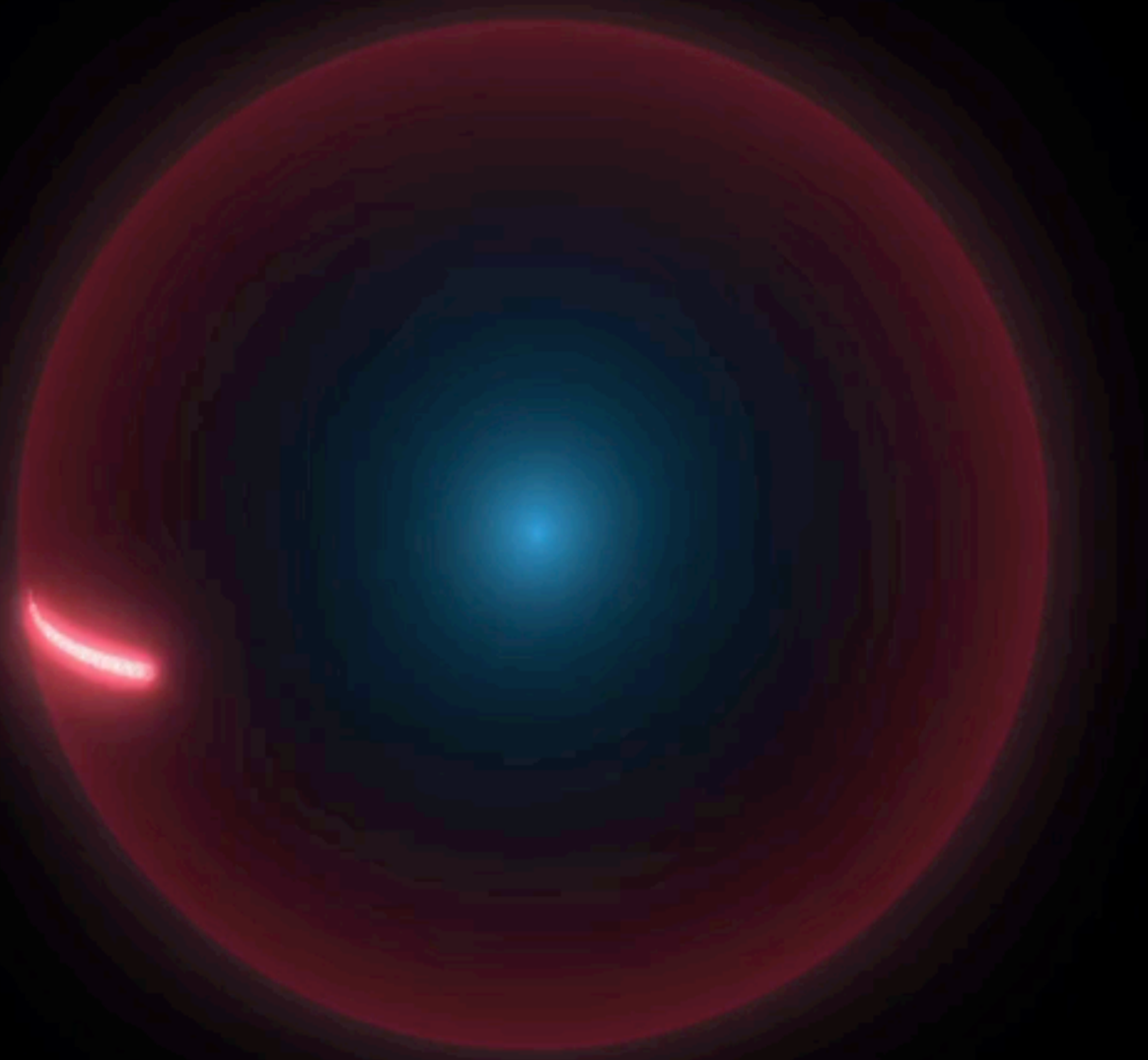
Review for Final

Also:

Review homework

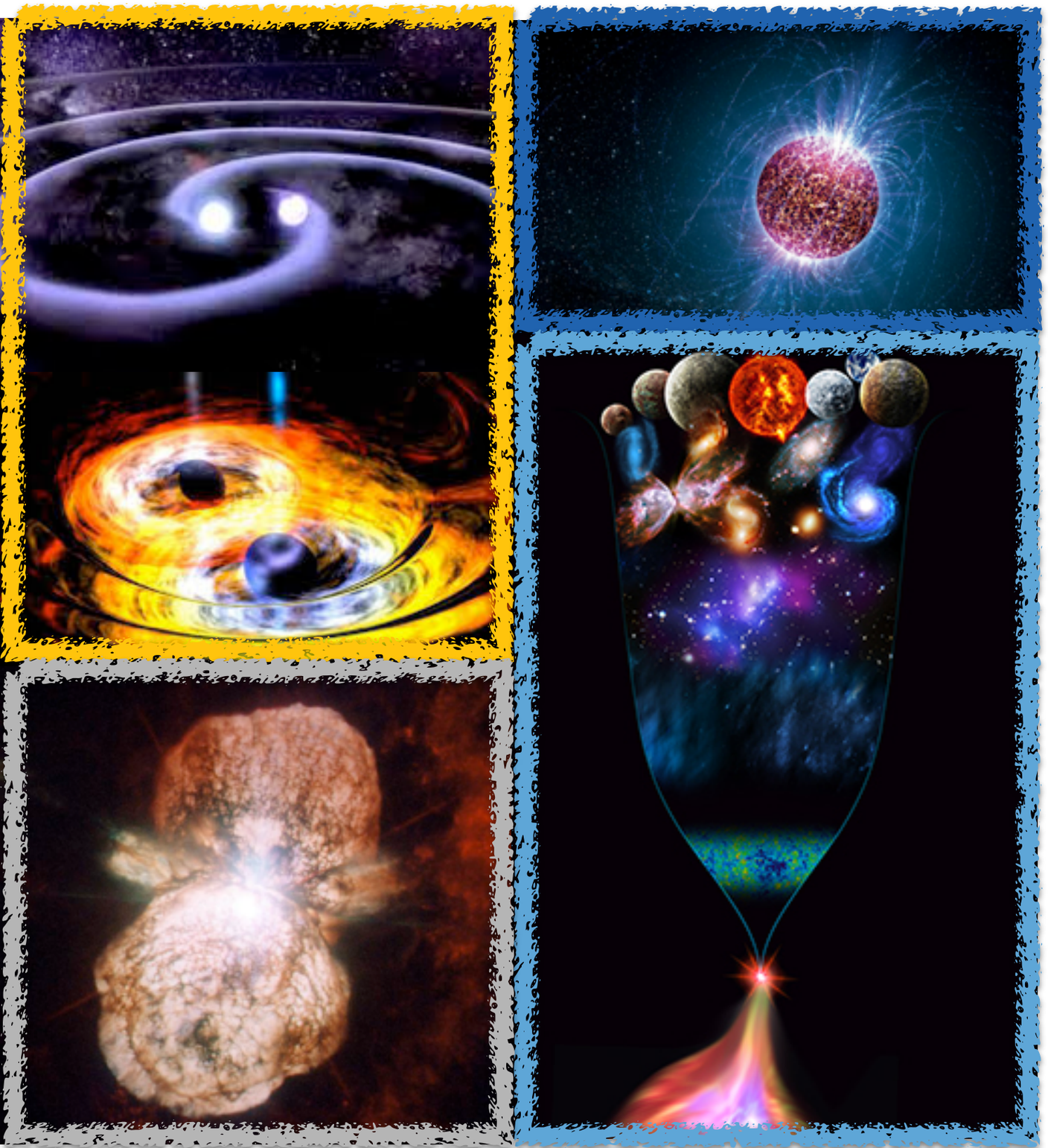
Review Lecture 9 slides

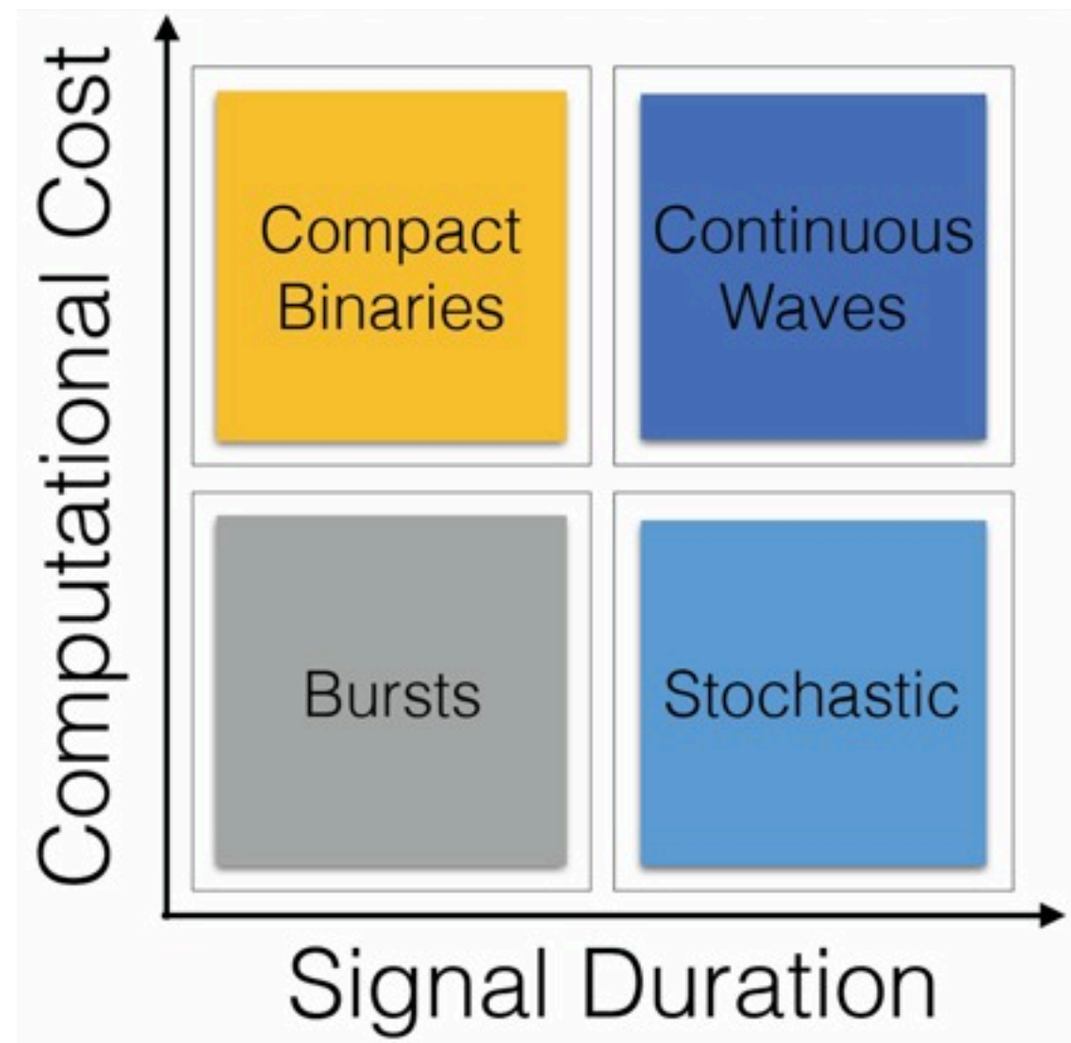
Example: Binary star system in Virgo cluster (16.5 Mpc away) would produce $h \sim 10^{-21}$. Over a distance of $L = 1$ AU, ΔL would be ~ 1 atomic diameter.



Credit: LIGO

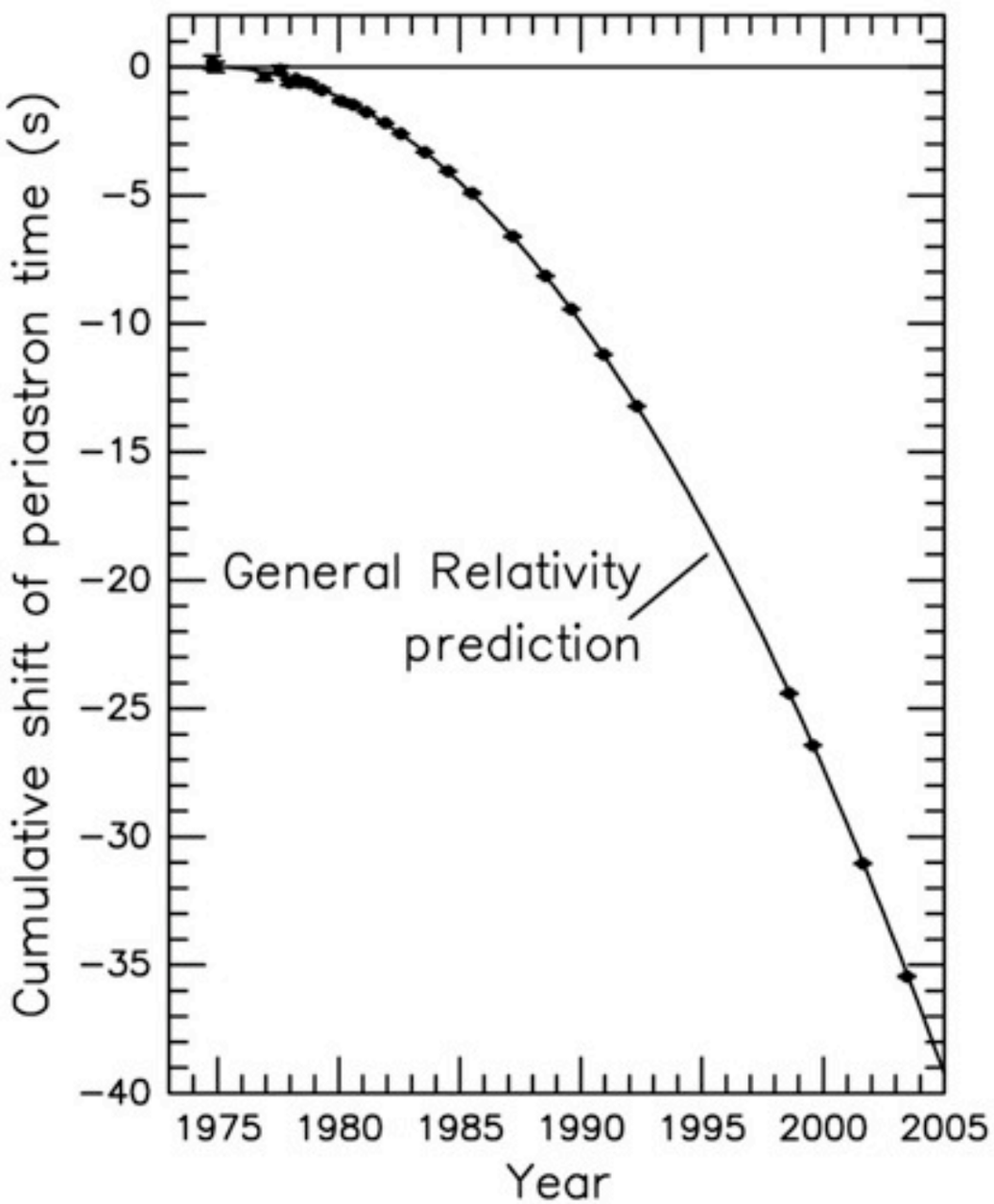
Gravitational-wave Sources for Ground-based Detectors





First Observational Evidence

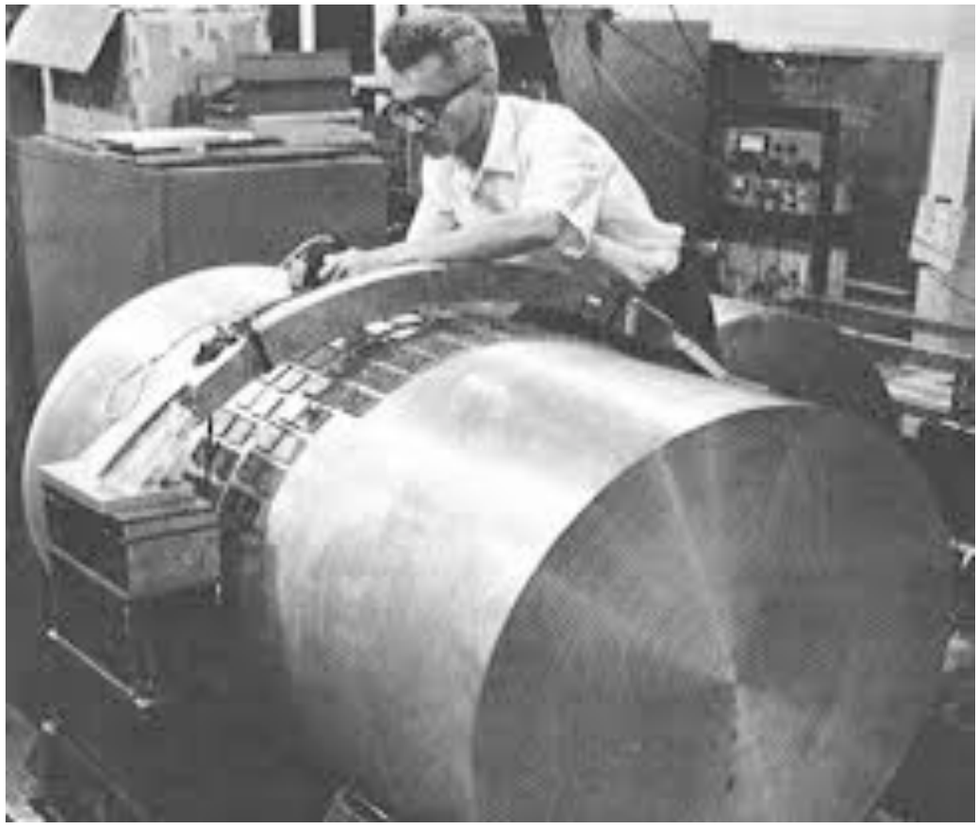
Hulse-Taylor Binary Pulsar



Pulsar-Neutron Star System

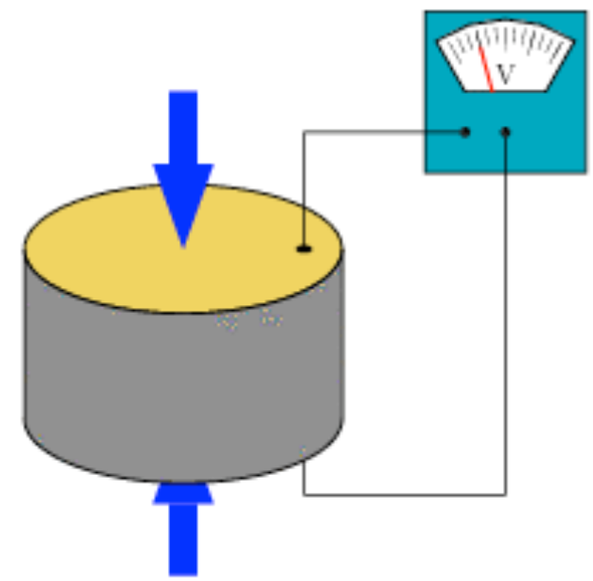
- Period: 7.75 hours
- Discovered by R. Hulse and J. Taylor
- Awarded 1993 Nobel prize

Bar Detectors

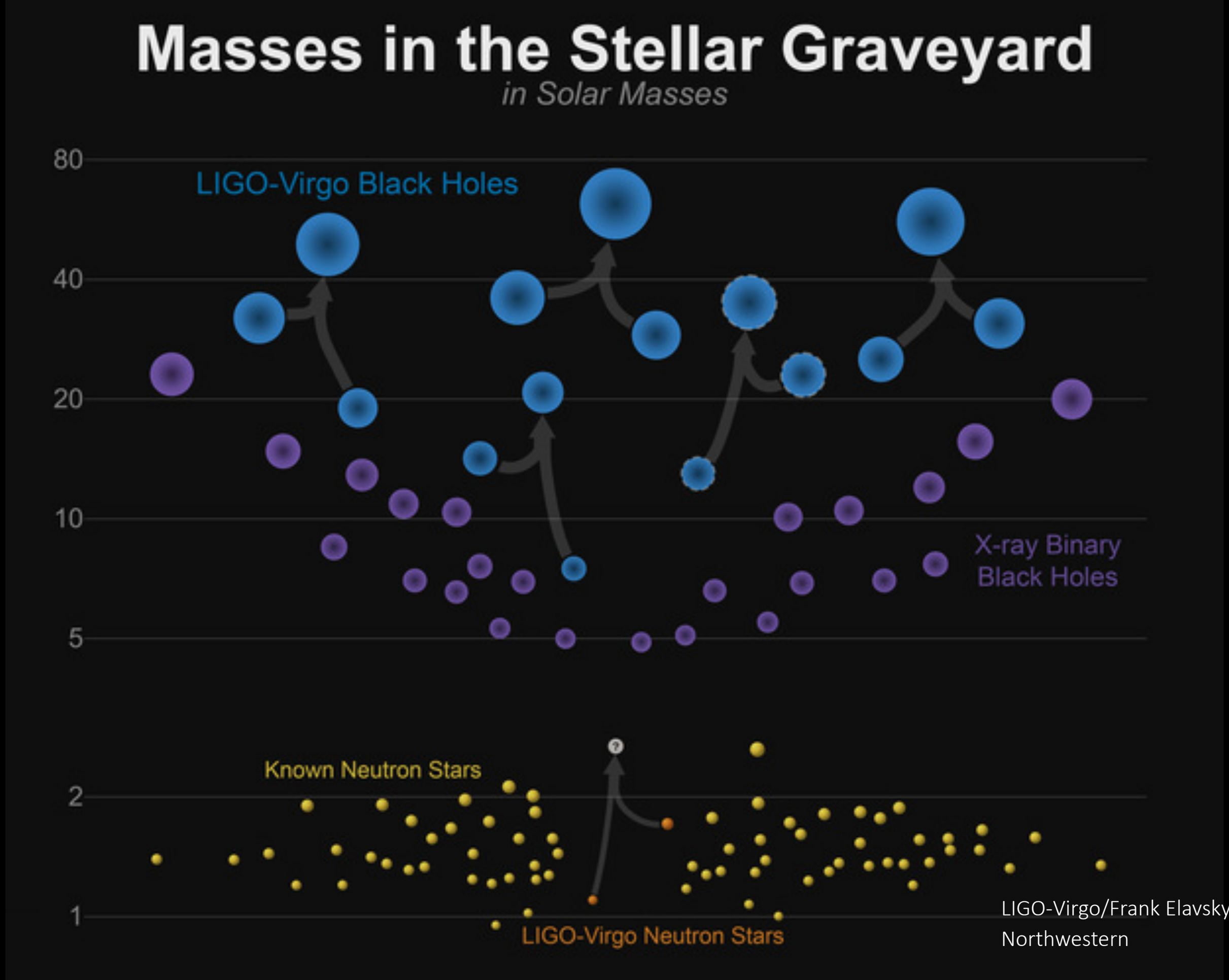


Joseph Webber - University of Maryland 1961: proposed to use resonant bar detectors to detect GWs

Piezoelectric sensors

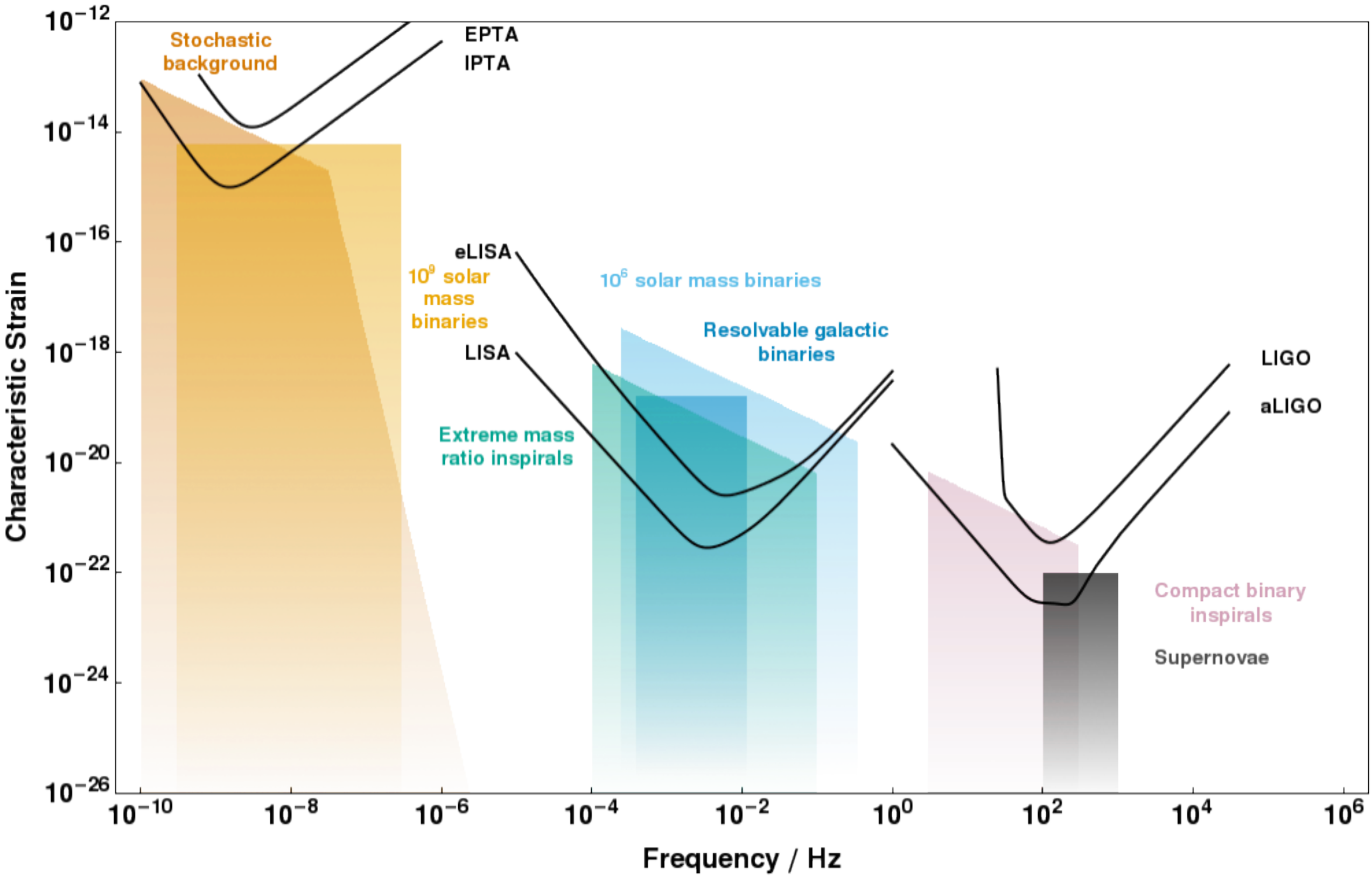


Only sensitive over very narrow range of frequencies



LIGO-Virgo/Frank Elavsky/
Northwestern

Gravitational-wave Spectrum



What is a tensor?

Scalar - tensor rank 0, magnitude, ex: Temperature
Vector - tensor rank 1, magnitude and direction, ex: Force
Tensor - combination of vectors where there is a fixed relationship, independent of coordinate system; ex: Dot product, work

$$T^{mn} = A^m B^n$$

Principle of relativity - “Physics equations should be **covariant** under coordinate transformation.”

To ensure that this is automatically satisfied, write physics equations in terms of tensors.

Einstein Summation Convention

Repeated indices imply summation.

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu \ = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$= [B_0 \ B_1 \ B_2 \ B_3] \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix}$$

Free index - appears exactly once in every term of equation

Dummy index - appears exactly twice in one given term of equation but only once in equation

The Metric of Curved Space: General Relativity

General relativity as a geometric theory of gravity posits that matter and energy cause spacetime to warp so that $g_{\mu\nu} \neq \eta_{\mu\nu}$

Thus gravitational phenomena are just effects of a curved spacetime on a test particle.

$$\boxed{\text{Source particle}} \xrightarrow[\text{equation}]{\text{Field}} \boxed{\text{Field}} \xrightarrow[\text{of motion}]{\text{Equation}} \boxed{\text{Test Particle}}$$

$$\boxed{\text{Source}} \xrightarrow[\text{Field equation}]{\text{Einstein}} \boxed{\text{Curved spacetime}} \xrightarrow[\text{equation}]{\text{Geodesic}} \boxed{\text{Test Particle}}$$

Tensor Calculus: Covariant Derivative

Ordinary derivatives of tensor components are not tensors. The combination $\partial_\nu A^\mu$ does not transform properly.

$$\partial_\nu A^\mu \to \partial'_\nu A'^\mu \neq \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\partial_\lambda A^\rho$$

We seek a covariant derivative ∇_ν to be used in covariant physics equations. Such a differentiation is constructed so that when acting on tensor components it still yields a tensor.

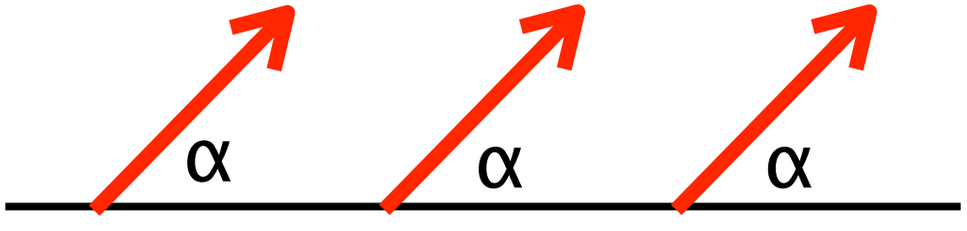
$$\nabla_\nu A^\mu \to \nabla'_\nu A'^\mu = \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\nabla_\lambda A^\rho$$

In order to produce the covariant derivative, the ordinary derivative must be supplemented by another term:

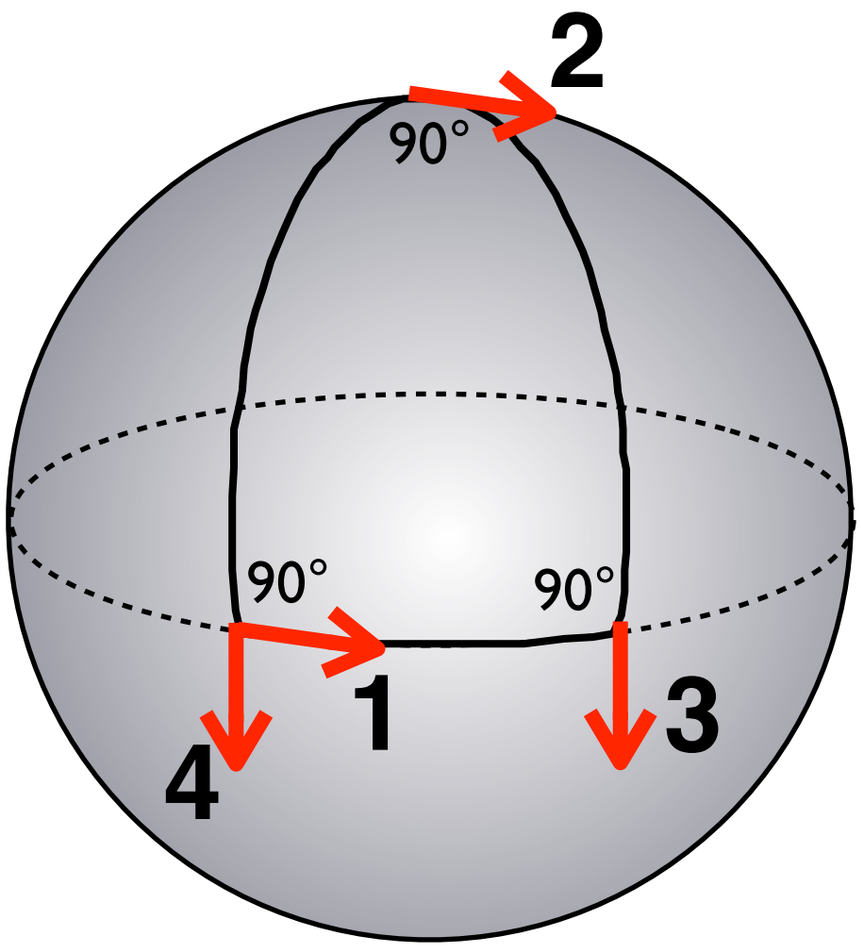
$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda \qquad \nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$$

Parallel Transport and Geodesics

Consider a vector transported along a curve. A difference in the vector could be caused by either:
1. change of the vector itself
2. coordinate change



Thus, if we move a vector (tensor) without changing itself, then the only change in components is due to coordinate changes.



Curvature and the Riemann Tensor

Local Lorentz Frame: effects of curvature become noticeable when taking second derivatives.

$$[\nabla_\alpha, \nabla_\beta] A^\mu = \nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu \equiv R^\mu_{\lambda\alpha\beta} A^\lambda$$

$$R^\mu_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\lambda\alpha}$$

In Local Lorentz Frame:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\partial_\mu \partial_\alpha g_{\nu\beta} - \partial_\nu \partial_\alpha g_{\mu\beta} + \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\mu \partial_\beta g_{\nu\alpha}\right)$$

Form of Riemann Tensor: $R = d\Gamma + \Gamma\Gamma \longrightarrow \partial^2 g + (\partial g)^2$

In flat space , the first and second derivatives of the metric vanish.

$R^\mu_{\lambda\alpha\beta} = 0$ implies flat space.

Motivating Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \xrightarrow{\quad ? \quad} R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We need the spacetime curvature term on the left. Einstein thought it should be the Ricci curvature tensor. But there is a problem.

Due to energy conservation:

$$\nabla T_{\mu\nu} = 0$$

But the derivative of Ricci tensor does not equal zero as can be seen with the Bianchi Identities. Instead, what is found is

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

Einstein tensor

Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity (next time)
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

Linearized Theory of Metric Field

And impose the harmonic gauge, then the last three terms in previous equation vanish and we end up with the **Linearized Einstein Equations**

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

Solution in a Vacuum

What happens outside the source, where $T_{\mu\nu} = 0$?

Then, the EFE reduces to

$$\Box \bar{h}_{\mu\nu} = 0$$

$$\left(-\frac{1}{c^2} \partial t^2 + \nabla^2 \right) \bar{h}_{\mu\nu} = 0$$

Wave equation for waves propagating at speed of light c!

Solutions to wave equation can be written as superpositions of plane waves traveling with wave vectors $\vec{k}$ and frequency

$$\omega = c \left| \vec{k} \right|$$

Solution with Source

Now allow for source. What would cause the waves to be generated?

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Solve using retarded Green's function assuming no incoming radiation from infinity. The solution is

$$\bar{h}_{\mu\nu}\left(t, \vec{x}\right) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right)$$

Generation of Gravitational Waves

To leading order in v/c, we can eliminate the multipole moments in favor of the mass moments to get a solution of the form:

$$\left[h_{ij}^{\mathrm{TT}}(t,\vec{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t - r/c\right)$$

where we have used: $S^{ij} = \frac{1}{2}\ddot{M}^{ij}$

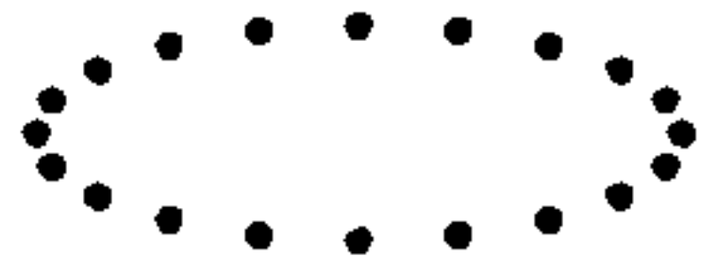
Mass quadrupole radiation!

Effect of Gravitational Waves on Matter

h_+ polarization

$$\delta x(t) = \frac{h_+}{2} x_0 \cos(\omega t)$$

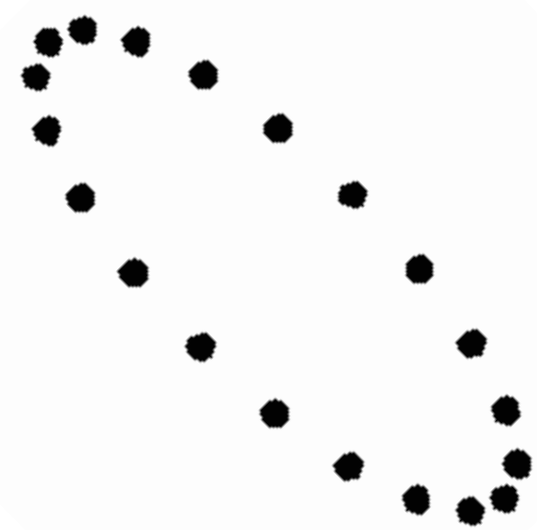
$$\delta y(t) = -\frac{h_+}{2} y_0 \cos(\omega t)$$



$h_\times$ polarization

$$\delta x(t) = -\frac{h_\times}{2} y_0 \cos(\omega t)$$

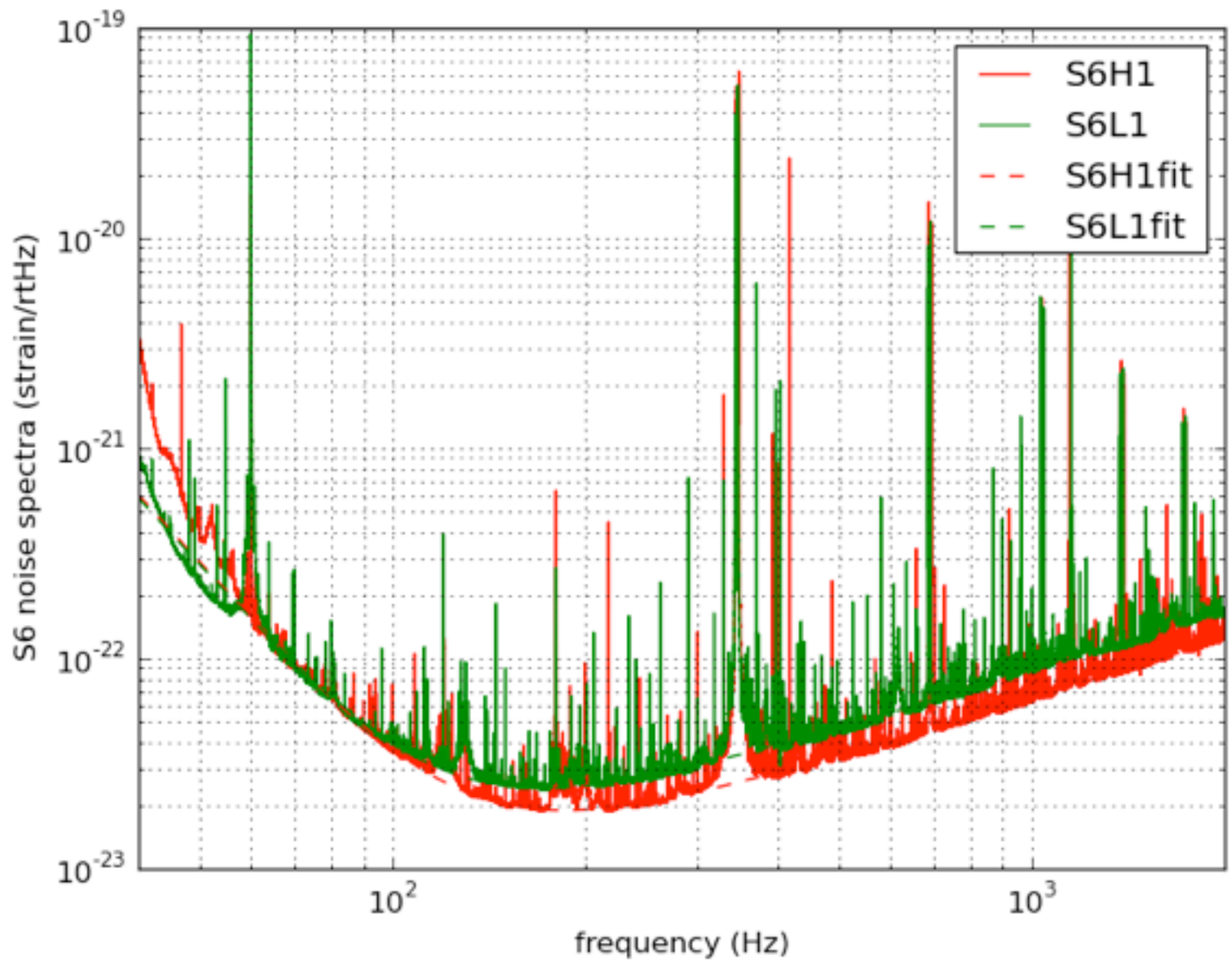
$$\delta y(t) = -\frac{h_\times}{2} x_0 \cos(\omega t)$$



Noise spectral density

$S_n(f)$ is the noise spectral density (aka noise spectral sensitivity or noise power spectrum):

$$\langle n^2(t) \rangle = \int_0^\infty df\, S_n(f)$$

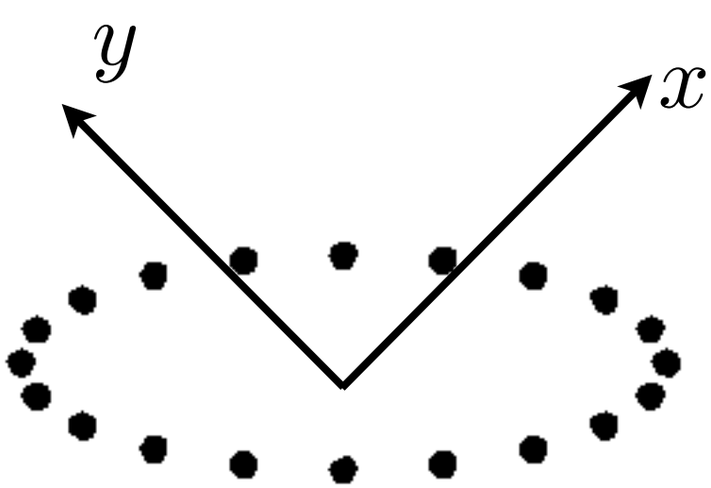


Interferometric GW Detector Pattern Functions

$$F_+(\theta, \phi; \psi = 0) = \frac{1}{2}\left(1 + \cos^2\theta\right)\cos 2\phi$$

$$F_\times(\theta, \phi; \psi = 0) = \cos\theta \sin 2\phi$$

Thus GW interferometers have blind directions. For instance, for a GW with plus polarization,



$\phi = \pi/4$ and $F_+ = 0$

This wave produces the same displacement in the y and x arm.

Differential phase shift vanishes!

Define the signal-to-noise ratio...

Using this scalar product definition, we have:

$$\frac{S}{N} = \frac{(u|h)}{(u|u)^{1/2}} \quad \text{where} \quad \tilde{u}(f) = \frac{1}{2} S_n(f) \tilde{K}(f)$$

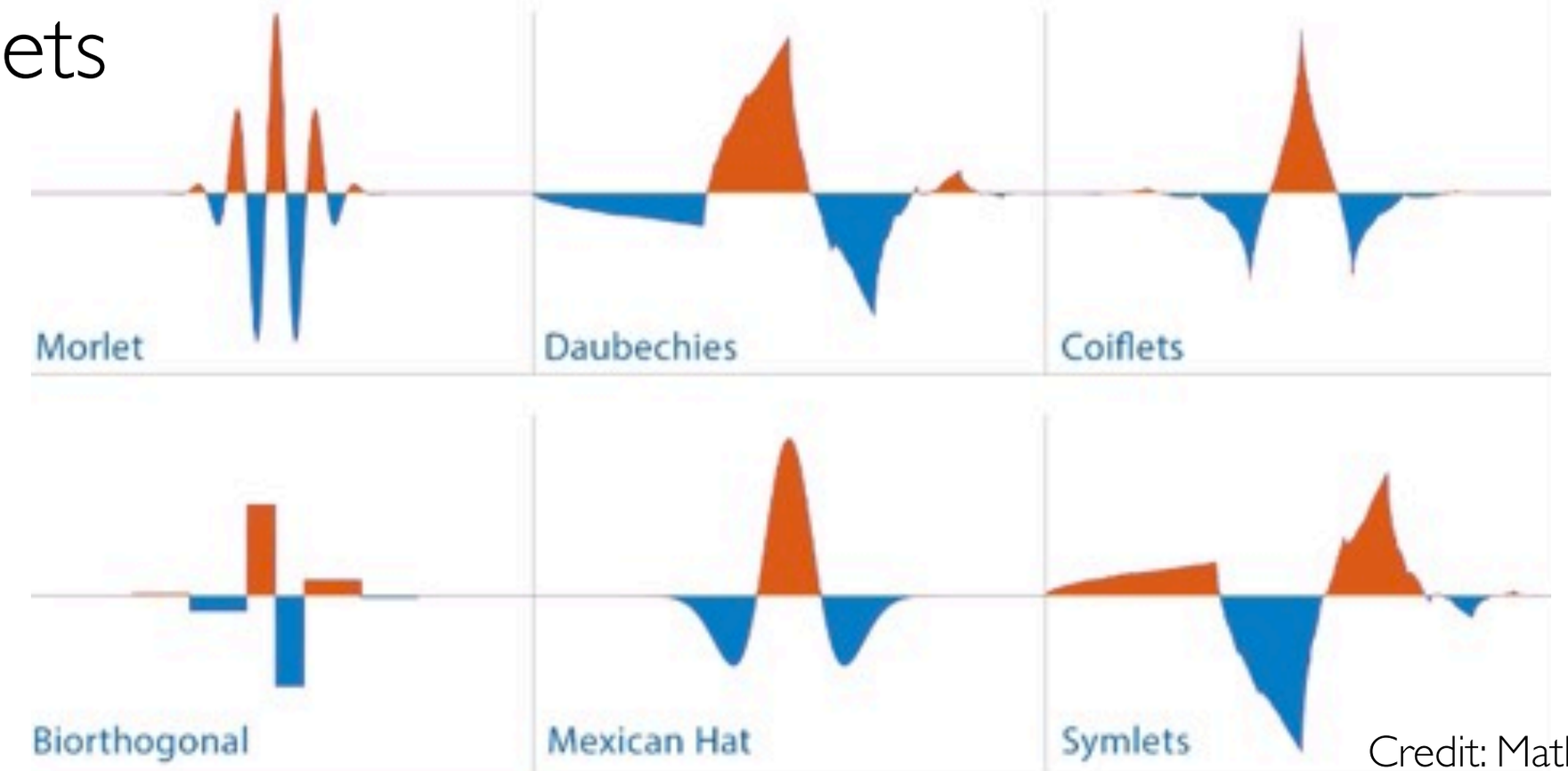
We are searching for vector $u/(u|u)^{1/2}$ such that its scalar product with vector $\boldsymbol{h}$ is maximum.

They should be parallel (i.e. proportional): $\boxed{\tilde{K}(f) = \mathrm{const.} \frac{\tilde{h}(f)}{S_n(f)}}$

This is the **Wiener filter** (aka matched filter).

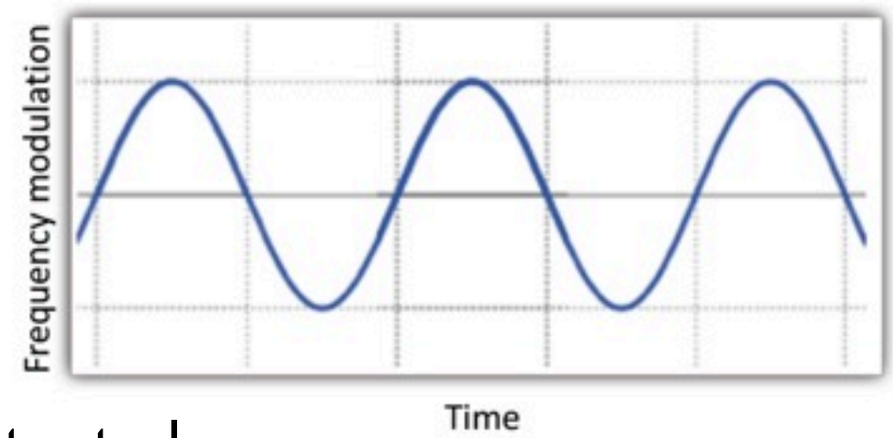
Burst Analysis with Wavelets

- Wavelets are waveforms of limited duration and bandwidth
- GW bursts can be described as superposition of wavelets



Credit: MathWorks

The Continuous Wave from an Isolated NS



- The source should emit a nearly monochromatic sinusoidal wave
- Limit on observation comes from total available observation time
- But the detector will see a modified signal

$$\vec{\mathcal{A}} = \{f, \dot{f}, \ddot{f}, \ldots, \alpha, \delta, h_0, \cos\iota, \psi, \phi_0\}$$

  - Four phase evolution parameters
  - Four amplitude parameters

What is a stochastic background?

- Stochastic (random) background of gravitational radiation

- Can arise from superposition of large number of unresolved GW sources
  1. Cosmological origin
  2. Astrophysical origin

- Strength of background measured as gravitational wave energy density ρ_{GW}

Detecting Stochastic Backgrounds

The filter function has the form:

$$\tilde{Q}(f) = N \frac{\gamma(f)\Omega_{\mathrm{GW}}(f)H_0^2}{f^3 P_1(f) P_2(f)}$$

overlap reduction function: $\gamma(f)$

power law template for GW spectrum: $\Omega_{\mathrm{GW}}(f) = \Omega_\alpha \left(f/100\,\mathrm{Hz}\right)^\alpha$

present value of Hubble parameter: H_0

noise in detector 1: $P_1(f)$

noise in detector 2: $P_2(f)$

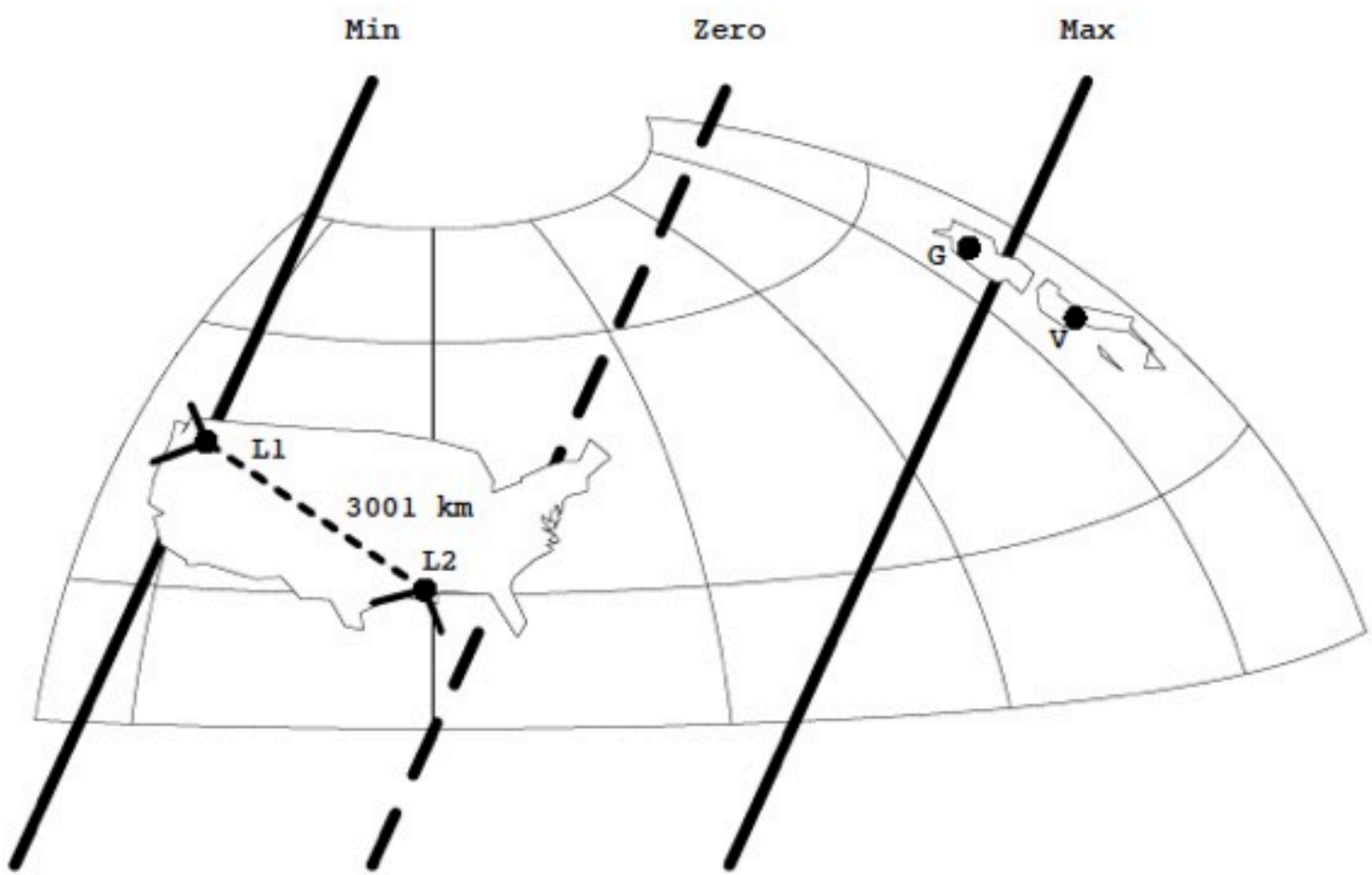
Purpose: Enhance SNR at frequencies where signal is strong and suppress SNR at frequencies where detector noise is large.

Overlap Reduction Function

Signal in two detectors will not be exactly the same because:

i) time delay between detectors

ii) non-alignment of detector



Bayes' Theorem

Given:

$$P(A \cap B) = P(A|B)P(B)$$

$$P(B \cap A) = P(B|A)P(A)$$

$$A \cap B = B \cap A$$

We can derive **Bayes' Theorem**:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

A = hypothesis (or parameters or theory)
B = data

P(hypothesis|data) $\propto$ P(data|hypothesis) P(hypothesis)

More on Bayes' Theorem

Initial Understanding + New Observation = Updated Understanding

Likelihood function

Prior probability

Posterior probability

$$p\left(h'|d\right) = \frac{p\left(d|h'\right)p\left(h'\right)}{p\left(d\right)}$$

Evidence

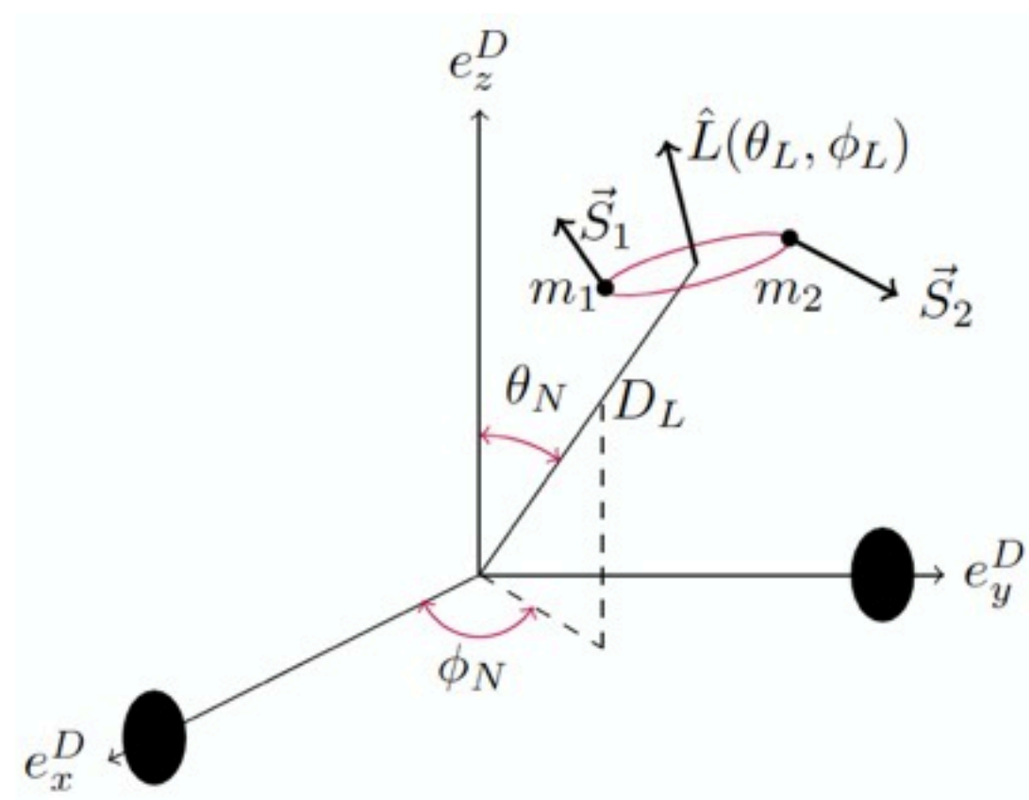
The likelihood function: the data

$$h_t \equiv h(\theta_t)$$

$$\Lambda(s|\theta_t) = \mathcal{N}\exp\left\{(h_t|s) - \frac{1}{2}(h_t|h_t) - \frac{1}{2}(s|s)\right\}$$

In this form, information might not be very manageable.

For binary coalescence there could be more than 15 parameters θ^i



The evidence: model selection

$$p(h'|d, M) = \frac{p(d|h', M)p(h'|M)}{p(d|M)}$$

M: any overall assumption or model (e.g. the signal is a GW, the binary black hole is spin-precessing, the binary components are neutron stars)

Odds Ratio: Compare competing models, for example "GW170817 was a BNS" vs "GW170817 was a BBH":

$$\mathcal{O}_{ij} = \frac{p(M_i|d)}{p(M_j|d)}$$

$$= \frac{p(M_i)p(d|M_i)}{p(M_j)p(d|M_j)}$$

What is the most probable value of the parameters, θ_t?

A rule for assigning the most probable value is called an estimator. Choices of estimators include:

1. Maximum likelihood estimator

2. Maximum posterior probability

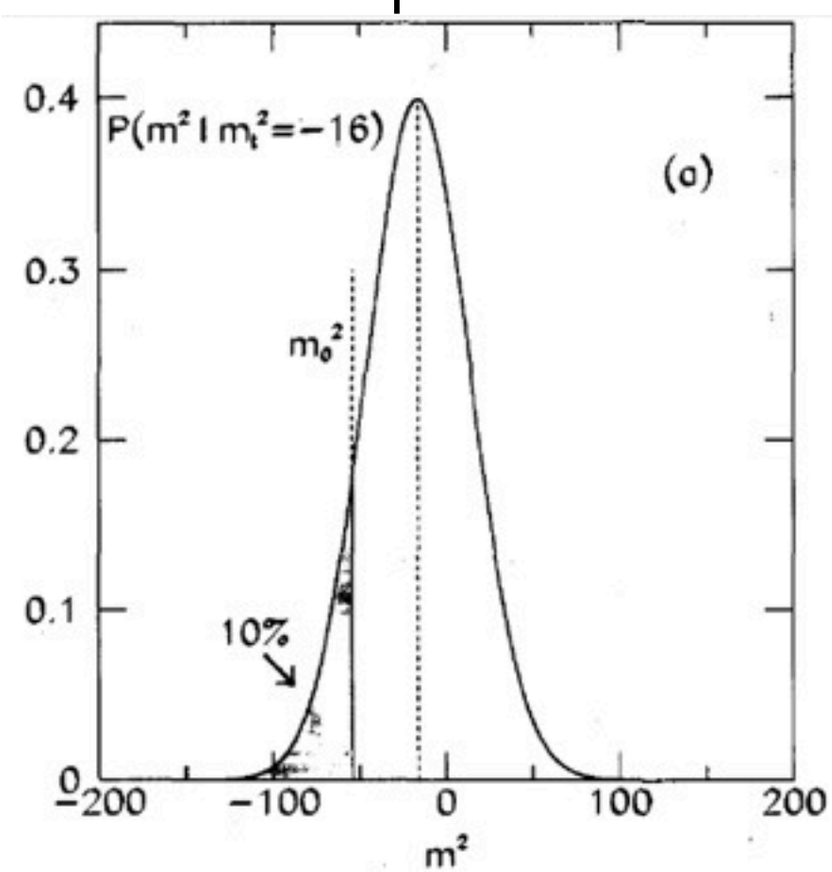
3. Bayes estimator

Confidence versus Credibility

Consider variable with bounded domain like a mass or rate. We can accommodate the physical constraint with a prior.

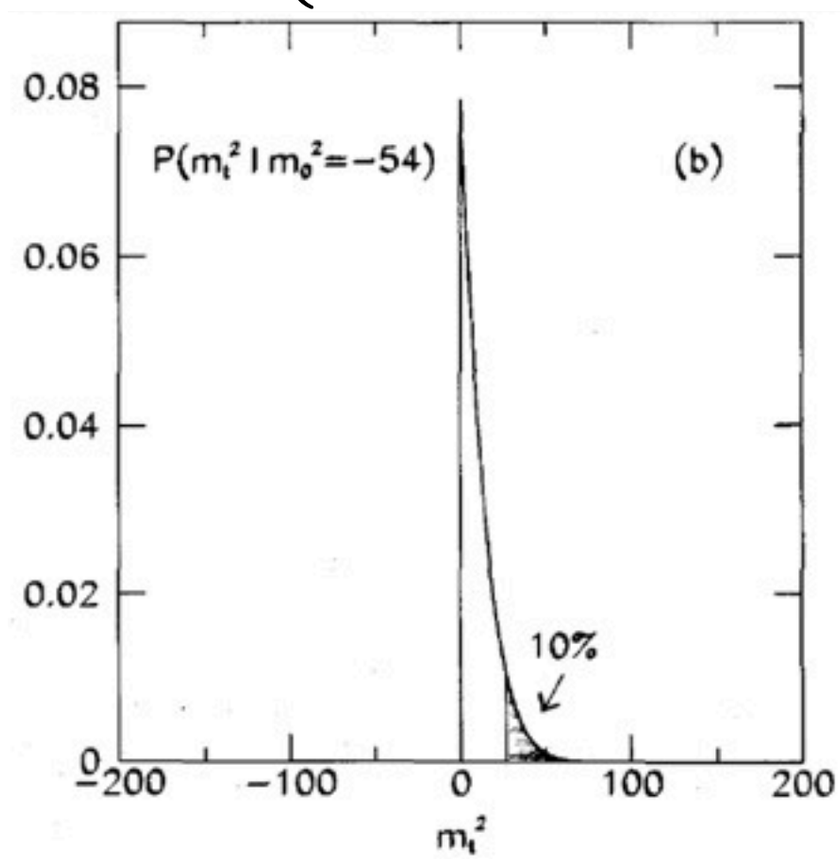
Example: square of mass of electron neutrino

No prior



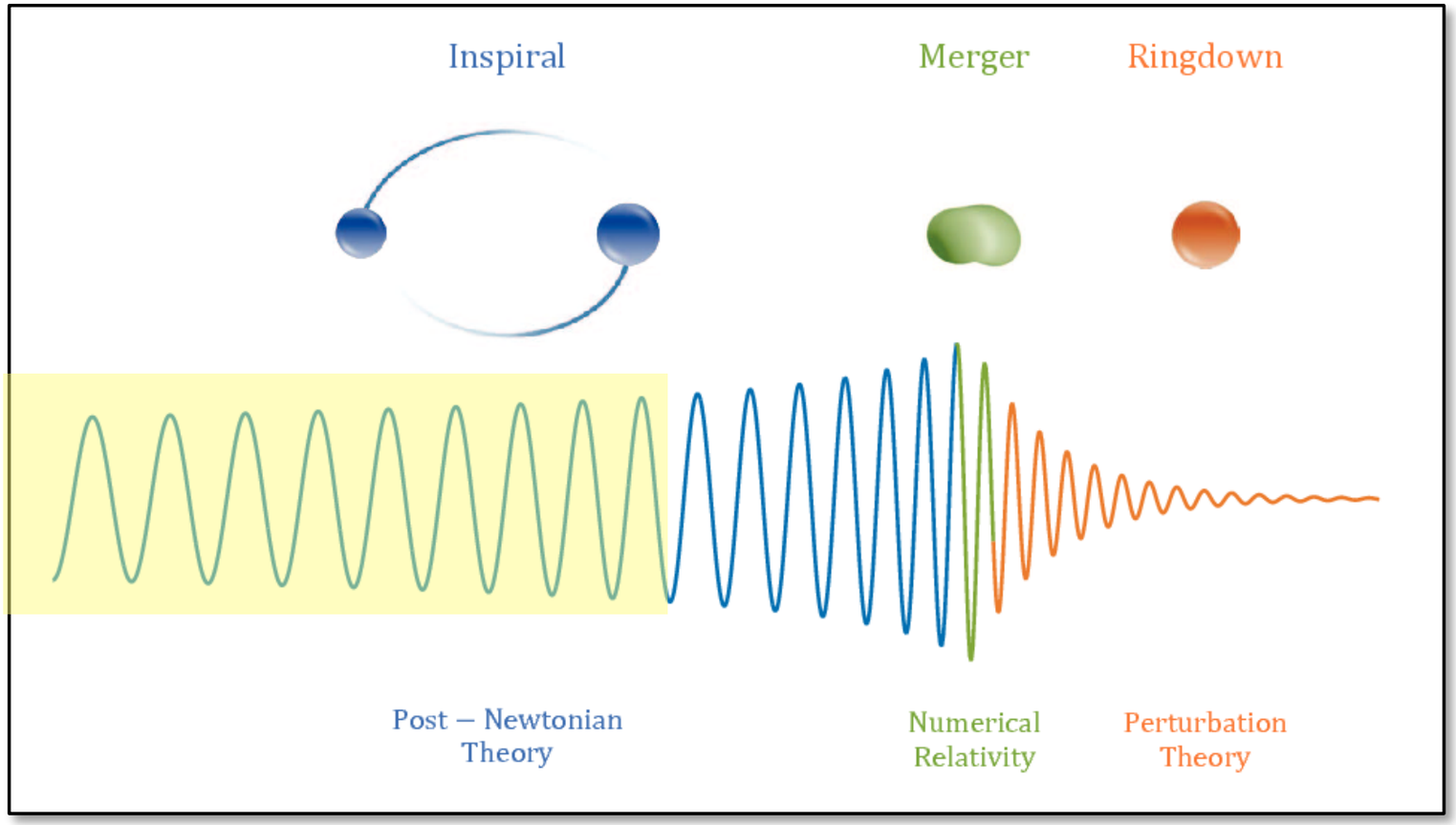
$$m^2 = (-54 \pm 30)\mathrm{eV}^2$$

$$P(m^2) = \begin{cases} 0 & m^2 < 0 \\ \text{uniform} & m^2 \geq 0 \end{cases}$$



$$m^2 < 26.6\mathrm{eV}^2$$

FD Cousins (1995)



During inspiral, phase evolution $\phi_{\rm GW}(t; m_{1,2}, \mathbf{S}_{1,2})$ can be computed with PN-theory in powers of v/c.

leading order

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{M^{1/5}}$$

$$\simeq \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3}\dot{f}\right]^{3/5}$$

higher order

$$q = \frac{m_2}{m_1} \leq 1$$

$$\mathbf{S}_{1,2} \parallel \mathbf{L}$$

even higher order

$$S_{1x}, S_{1y}, S_{1z}$$

$$S_{2x}, S_{2y}, S_{2z}$$