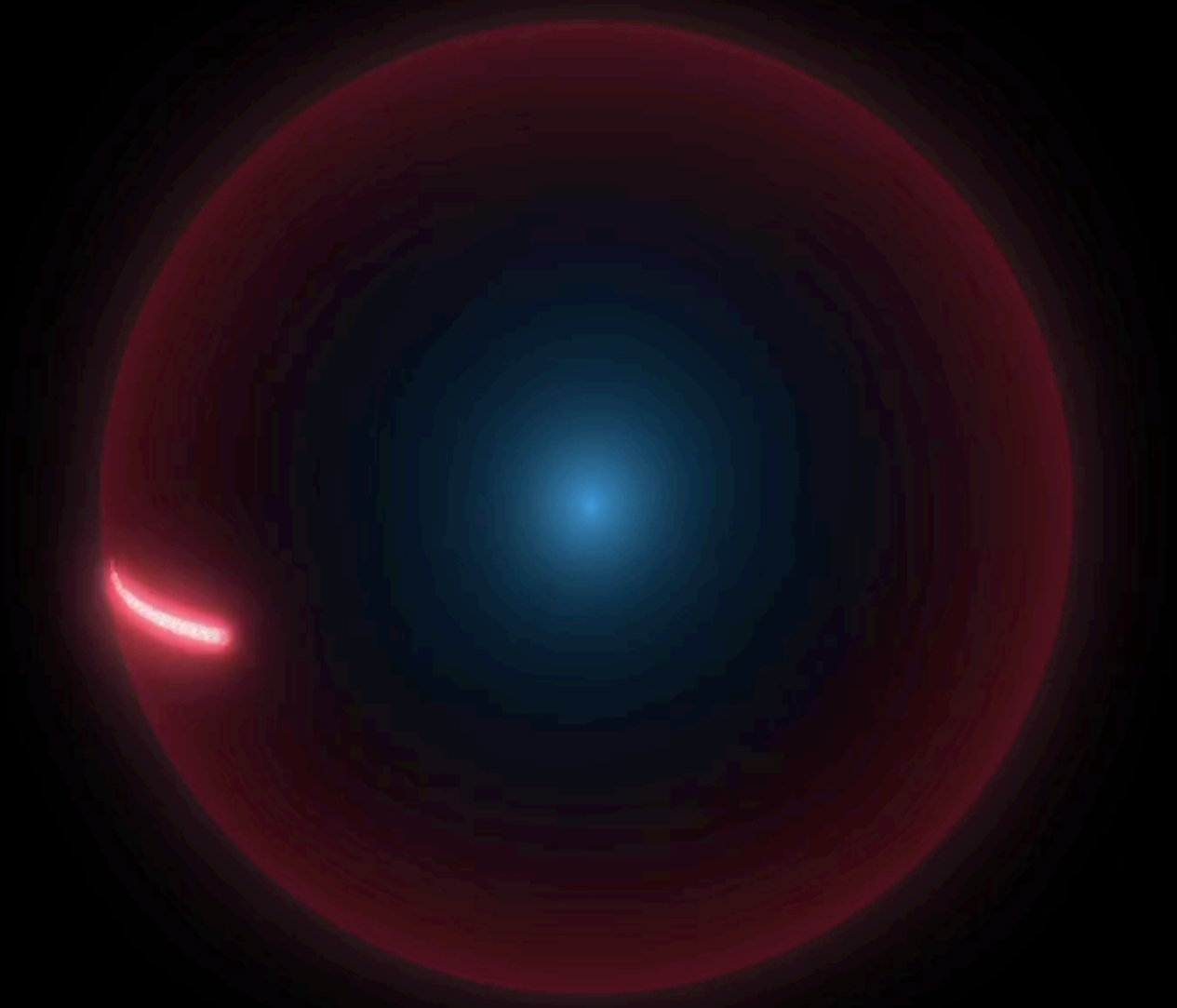
Review for Final

Also:

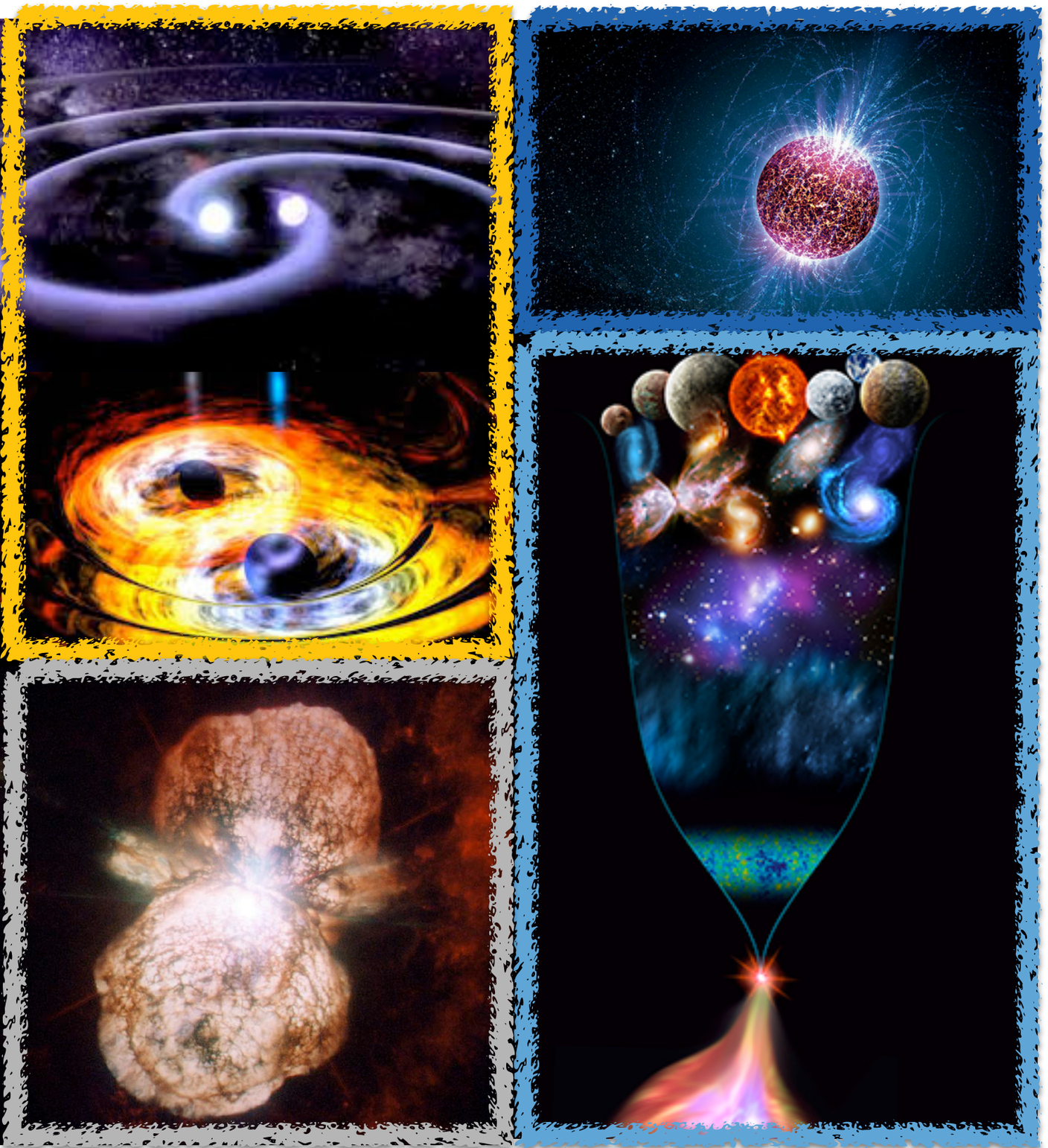
Review homework problems

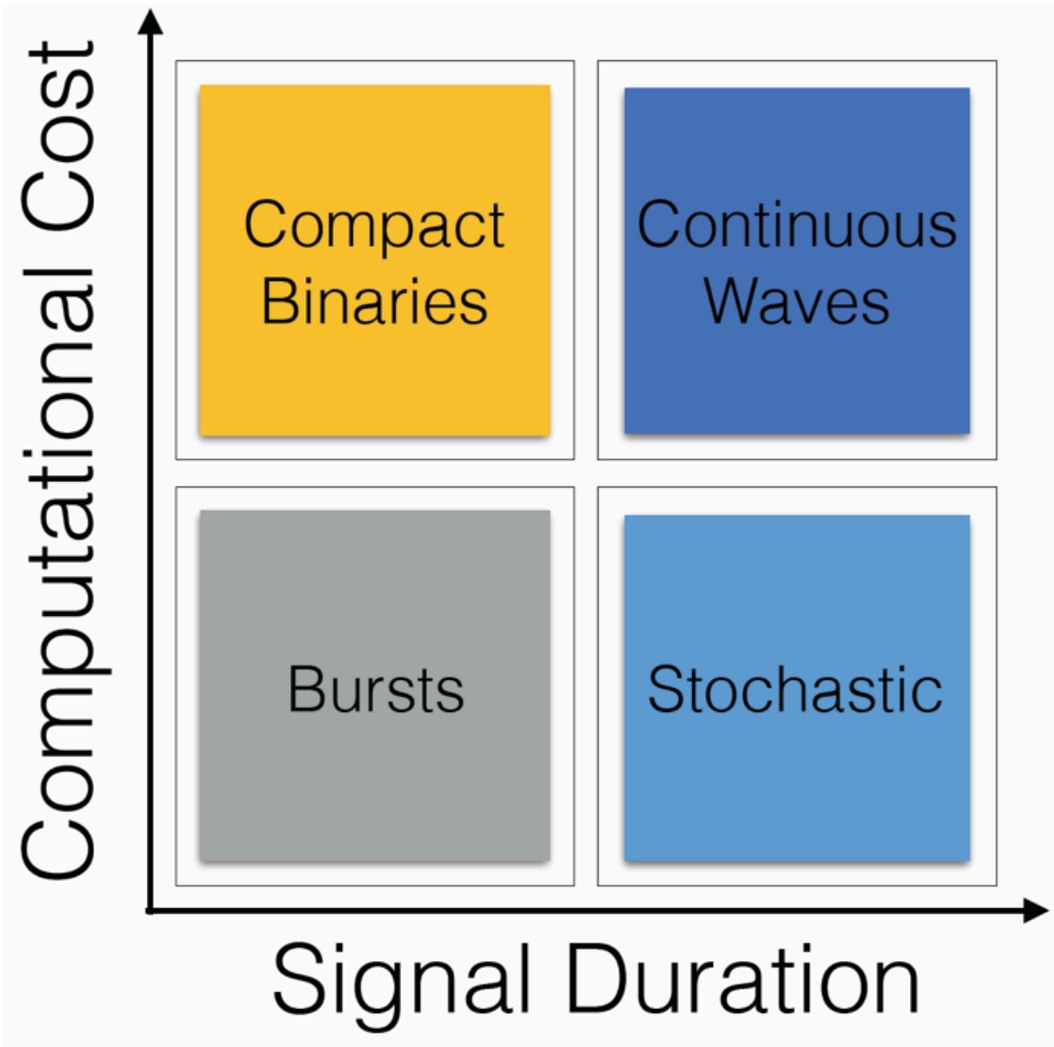
Example: Binary star system in Virgo cluster (16.5 Mpc away) would produce $h \sim 10^{-21}$. Over a distance of $L = 1$ AU, ΔL would be ~ 1 atomic diameter.



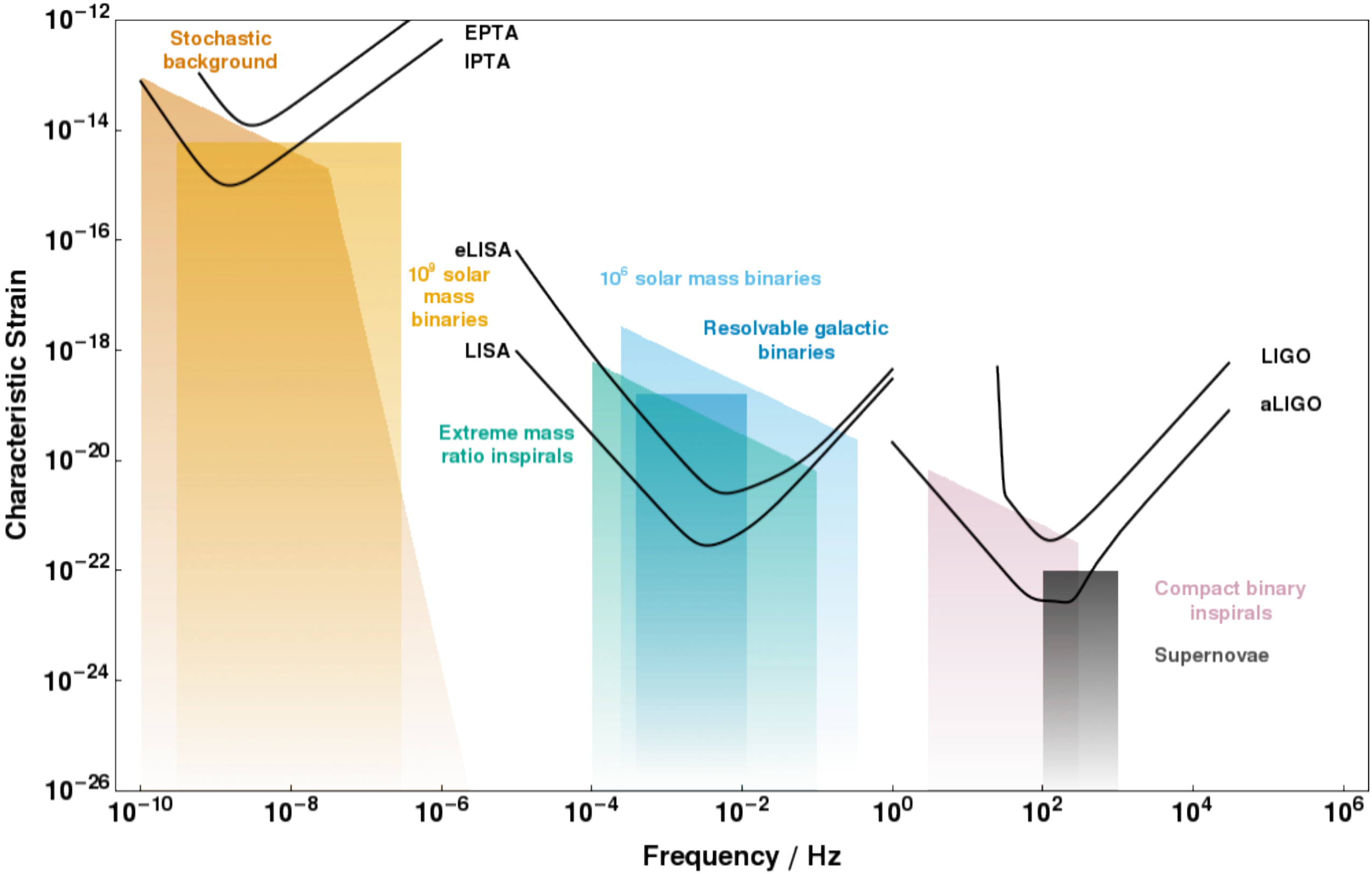
Credit: LIGO

Gravitational-wave Sources for Ground-based Detectors





Gravitational-wave Spectrum



What is a tensor?

Scalar - tensor rank 0, magnitude, ex: Temperature
Vector - tensor rank 1, magnitude and direction, ex: Force
Tensor - combination of vectors where there is a fixed relationship, independent of coordinate system; ex: Dot product, work

$$T^{mn} = A^m B^n$$

Principle of relativity - “Physics equations should be covariant under coordinate transformation.”

To ensure that this is automatically satisfied, write physics equations in terms of tensors.

Einstein Summation Convention

Repeated indices imply summation.

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu \ = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$= [B_0 \ B_1 \ B_2 \ B_3] \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix}$$

Free index - appears exactly once in every term of equation

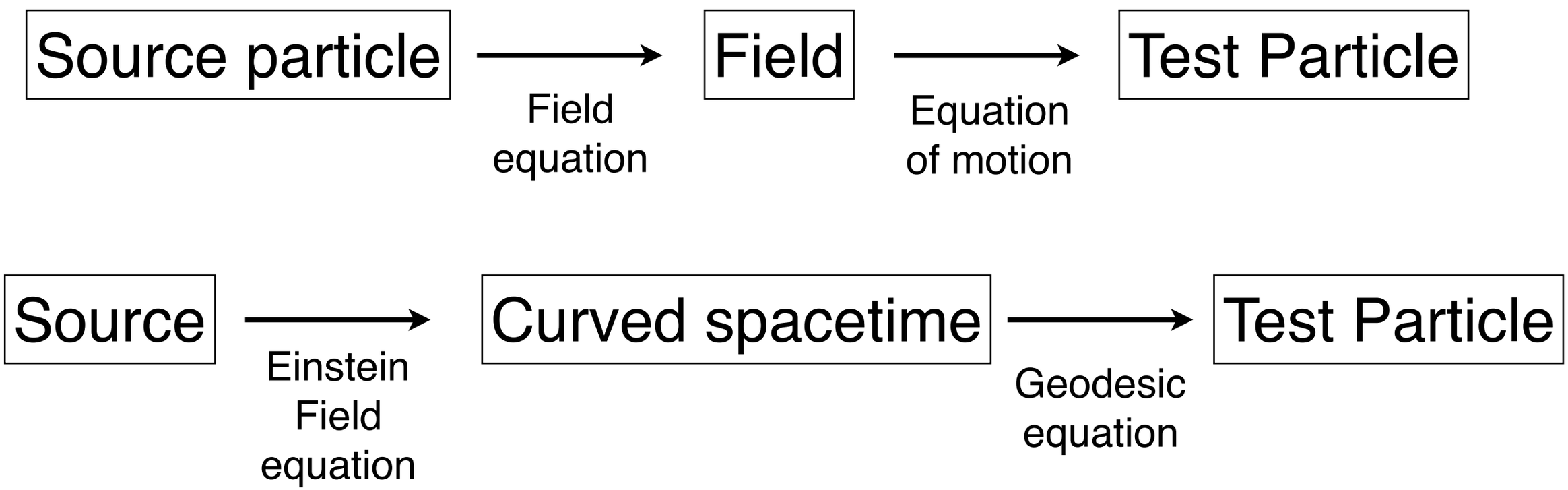
Dummy index - appears exactly twice in one given term of equation but only once in equation

The Metric of Curved Space: General Relativity

General relativity as a geometric theory of gravity posits that matter and energy cause spacetime to warp so that

$$g_{\mu\nu} \neq \eta_{\mu\nu}$$

Thus gravitational phenomena are just effects of a curved spacetime on a test particle.



Tensor Calculus: Covariant Derivative

Ordinary derivatives of tensor components are not tensors. The combination $\partial_\nu A^\mu$ does not transform properly.

$$\partial_\nu A^\mu \to \partial'_\nu A'^\mu \neq \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\partial_\lambda A^\rho$$

We seek a covariant derivative ∇_ν to be used in covariant physics equations. Such a differentiation is constructed so that when acting on tensor components it still yields a tensor.

$$\nabla_\nu A^\mu \to \nabla'_\nu A'^\mu = \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\nabla_\lambda A^\rho$$

In order to produce the covariant derivative, the ordinary derivative must be supplemented by another term:

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda \qquad \nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$$

The Geodesic

Mathematical expression for parallel transport of vector components is

$$\nabla A^\mu = dA^\mu + \Gamma^\mu_{\nu\lambda} A^\nu dx^\lambda = 0$$

The process of parallel transporting a vector A^μ along a curve $x^\mu\left(\sigma\right)$ can be expressed according to:

$$\frac{dA^\mu}{d\sigma} + \Gamma^\mu_{\nu\lambda} A^\nu \frac{dx^\lambda}{d\sigma} = 0$$

But the geodesic is a curve for which the tangent vector parallel transports itself, i.e.:

$$A^\mu = \frac{dx^\mu}{d\sigma}$$

Thus, the geodesic equation is: $\boxed{\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0}$

Curvature and the Riemann Tensor

Local Lorentz Frame: effects of curvature become noticeable when taking second derivatives.

$$[\nabla_\alpha, \nabla_\beta] A^\mu = \nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu \equiv R^\mu_{\lambda\alpha\beta} A^\lambda$$

$$R^\mu_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\lambda\alpha}$$

In Local Lorentz Frame:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\partial_\mu \partial_\alpha g_{\nu\beta} - \partial_\nu \partial_\alpha g_{\mu\beta} + \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\mu \partial_\beta g_{\nu\alpha}\right)$$

Form of Riemann Tensor: $R = d\Gamma + \Gamma\Gamma \longrightarrow \partial^2 g + (\partial g)^2$

In flat space , the first and second derivatives of the metric vanish.

$R^\mu_{\lambda\alpha\beta} = 0$ implies flat space.

Motivating Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \xrightarrow{\quad ? \quad} R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We need the spacetime curvature term on the left. Einstein thought it should be the Ricci curvature tensor. But there is a problem.

Due to energy conservation:

$$\nabla T_{\mu\nu} = 0$$

But the derivative of Ricci tensor does not equal zero as can be seen with the Bianchi Identities. Instead, what is found is

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

Einstein tensor

Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity (next time)
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

Linearized Theory of Metric Field

And impose the harmonic gauge, then the last three terms in previous equation vanish and we end up with the **Linearized Einstein Equations**

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

Solution in a Vacuum

What happens outside the source, where $T_{\mu\nu} = 0$?

Then, the EFE reduces to

$$\Box \bar{h}_{\mu\nu} = 0$$

$$\left(-\frac{1}{c^2} \partial t^2 + \nabla^2 \right) \bar{h}_{\mu\nu} = 0$$

Wave equation for waves propagating at speed of light c!

Solutions to wave equation can be written as superpositions of plane waves traveling with wave vectors $\vec{k}$ and frequency

$$\omega = c \left| \vec{k} \right|$$

Solution with Source

Now allow for source. What would cause the waves to be generated?

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Solve using retarded Green's function assuming no incoming radiation from infinity. The solution is

$$\bar{h}_{\mu\nu}\left(t, \vec{x}\right) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right)$$

Generation of Gravitational Waves

To leading order in v/c, we can eliminate the multipole moments in favor of the mass moments to get a solution of the form:

$$\left[h_{ij}^{\mathrm{TT}}(t,\vec{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t - r/c\right)$$

where we have used: $S^{ij} = \frac{1}{2}\ddot{M}^{ij}$

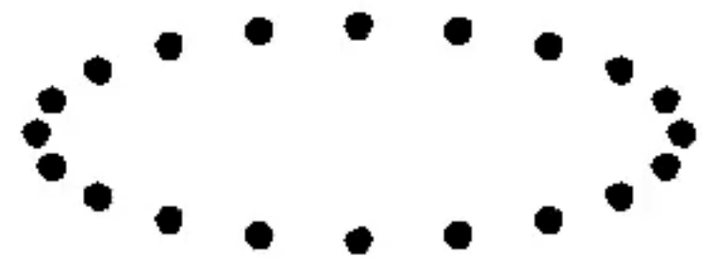
Mass quadrupole radiation!

Effect of Gravitational Waves on Matter

h_+ polarization

$$\delta x(t) = \frac{h_+}{2} x_0 \cos(\omega t)$$

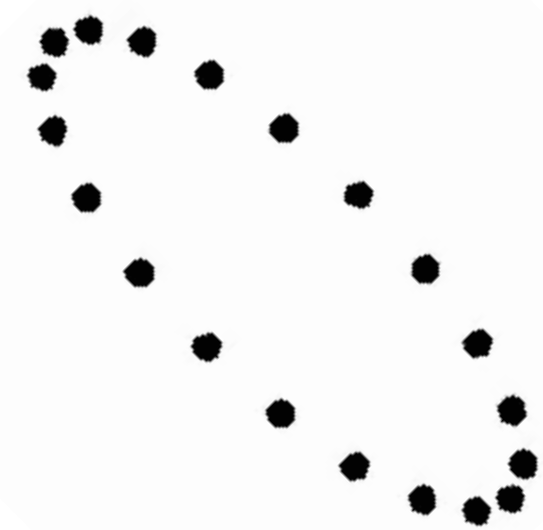
$$\delta y(t) = -\frac{h_+}{2} y_0 \cos(\omega t)$$



$h_\times$ polarization

$$\delta x(t) = -\frac{h_\times}{2} y_0 \cos(\omega t)$$

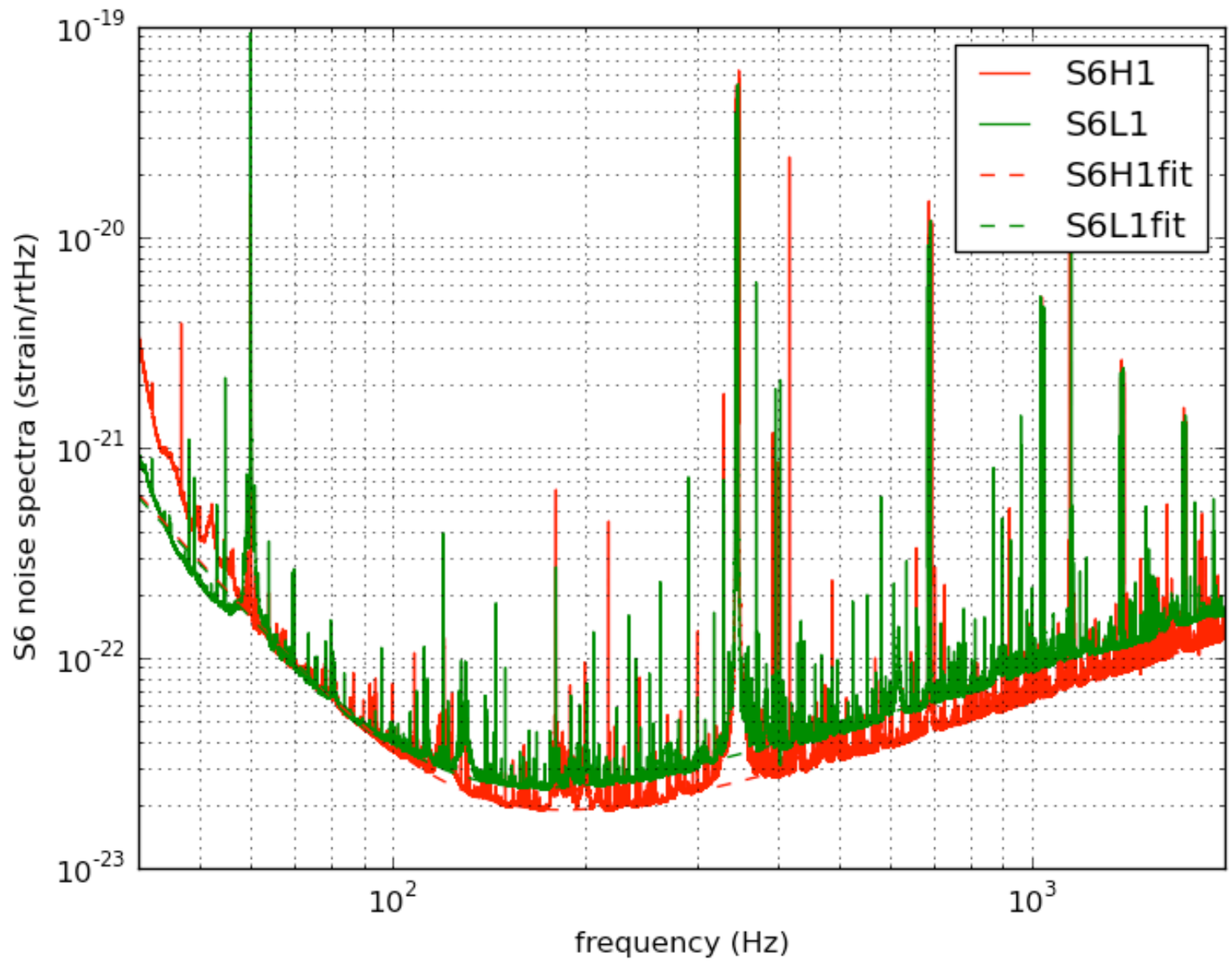
$$\delta y(t) = -\frac{h_\times}{2} x_0 \cos(\omega t)$$



Noise spectral density

$S_n(f)$ is the noise spectral density (aka noise spectral sensitivity or noise power spectrum):

$$\langle n^2(t) \rangle = \int_0^\infty df\, S_n(f)$$

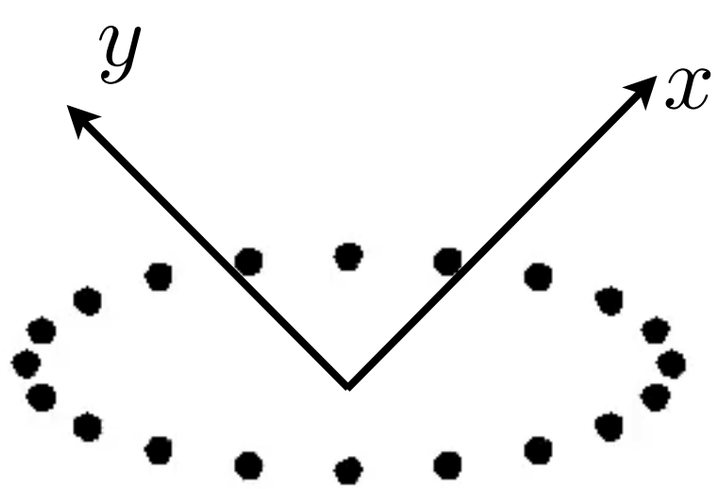


Interferometric GW Detector Pattern Functions

$$F_+(\theta, \phi; \psi = 0) = \frac{1}{2}\left(1 + \cos^2\theta\right)\cos 2\phi$$

$$F_\times(\theta, \phi; \psi = 0) = \cos\theta \sin 2\phi$$

Thus GW interferometers have blind directions. For instance, for a GW with plus polarization,



$\phi = \pi/4$ and $F_+ = 0$

This wave produces the same displacement in the y and x arm.

Differential phase shift vanishes!

Define the signal-to-noise ratio...

Using this scalar product definition, we have:

$$\frac{S}{N} = \frac{(u|h)}{(u|u)^{1/2}} \quad \text{where} \quad \tilde{u}(f) = \frac{1}{2} S_n(f) \tilde{K}(f)$$

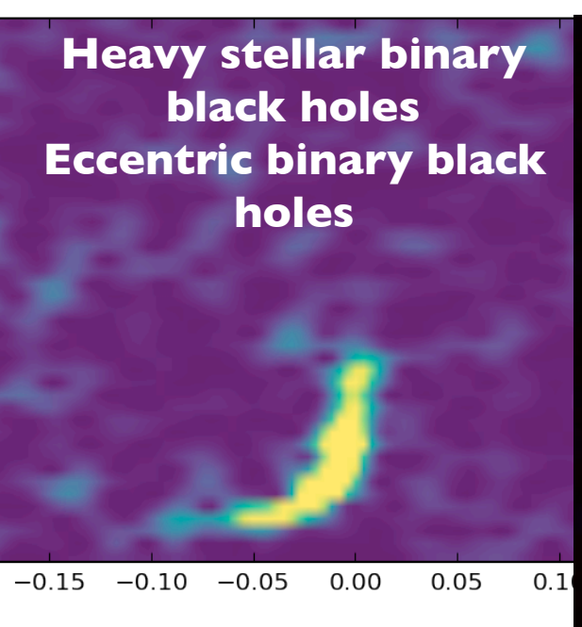
We are searching for vector $u/(u|u)^{1/2}$ such that its scalar product with vector $\boldsymbol{h}$ is maximum.

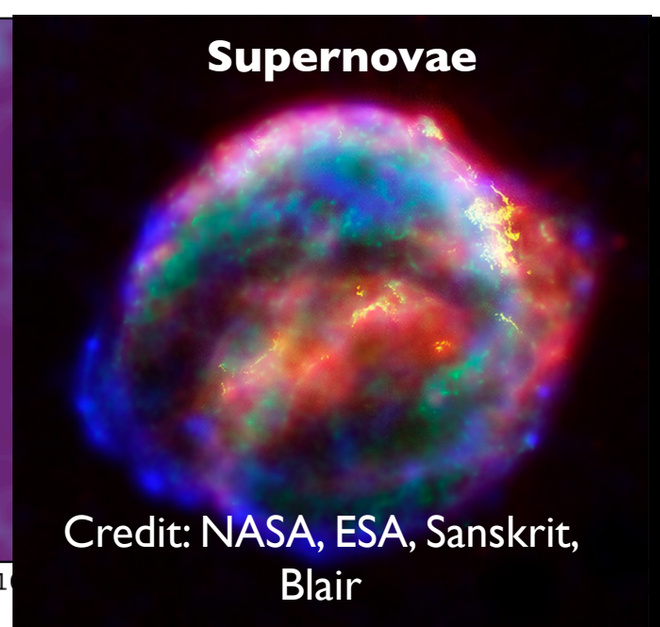
They should be parallel (i.e. proportional): $\boxed{\tilde{K}(f) = \mathrm{const.} \frac{\tilde{h}(f)}{S_n(f)}}$

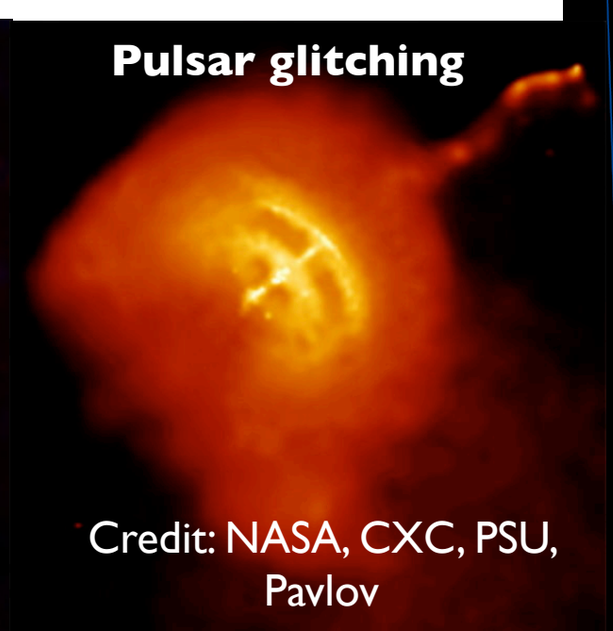
This is the **Wiener filter** (aka matched filter).

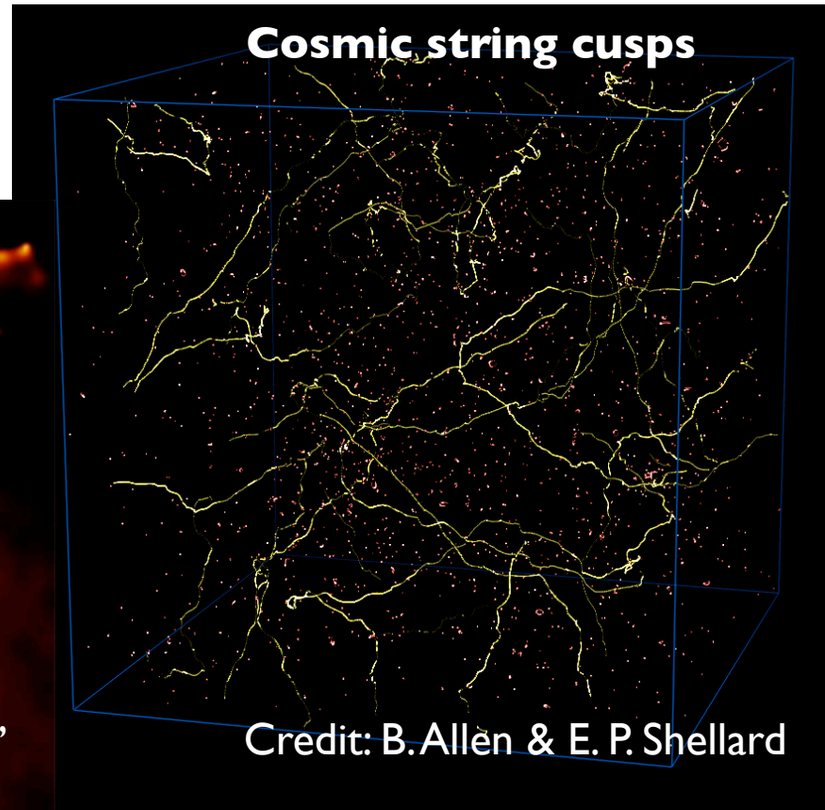
Burst Signals

- GW searches with minimal assumptions
  - Do not assume accurate source models - affected by noise
- Limited ability to measure waveforms, sky positions, polarizations
- Can detect the unexpected





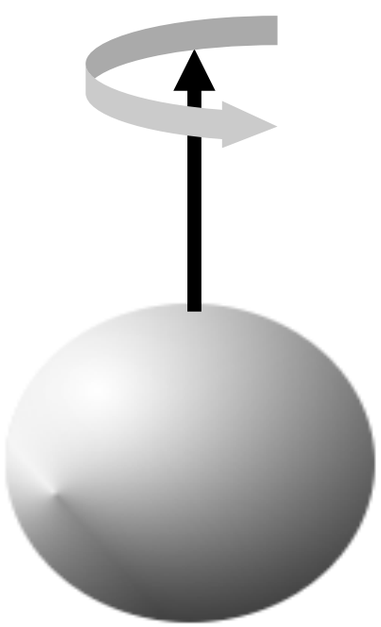


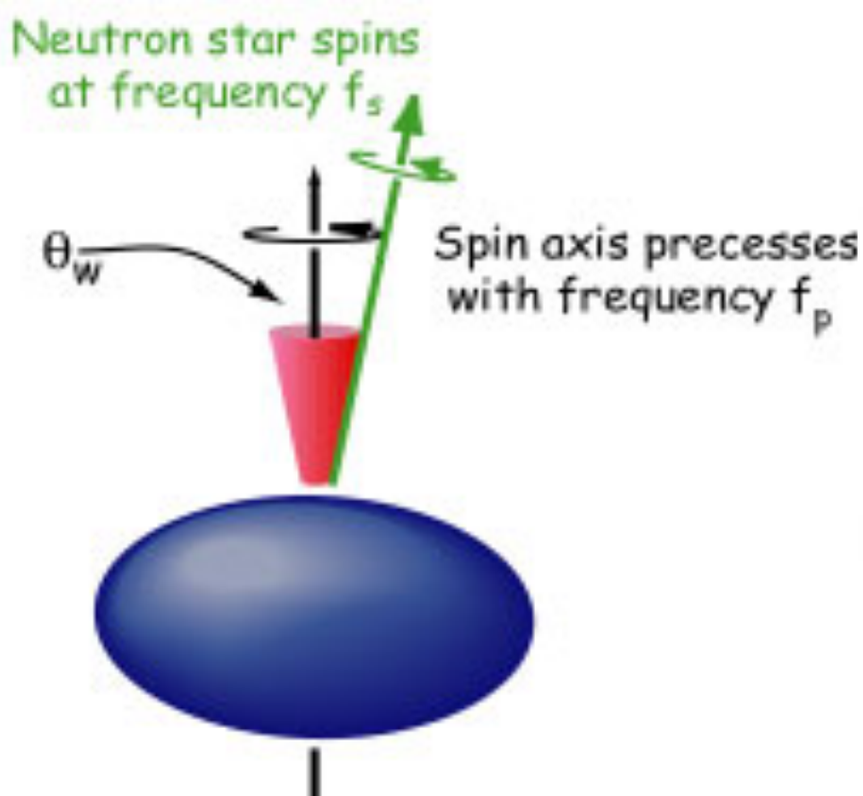


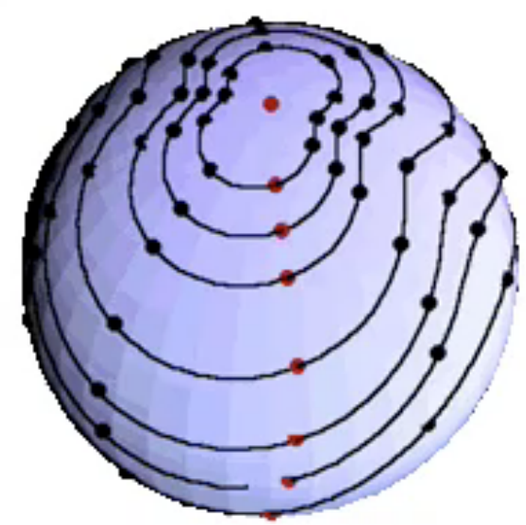
Continuous Gravitational Wave Sources

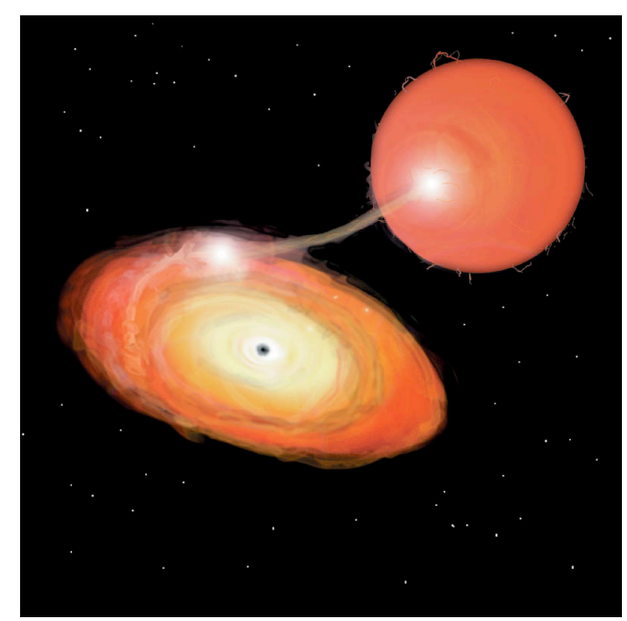
Non-axisymmetric rotating neutron stars;
asymmetry could arise from:

- equatorial ellipticity (mm-high mountain) $f_{\rm GW} = 2 f_{\rm rot}$
- free precession around rotation axis $f_{\rm GW} \sim f_{\rm rot} + f_{\rm prec}$
- excitation of long-lasting oscillations $f_{\rm GW} \sim 4/3 f_{\rm rot}$
- deformation due to matter accretion $f_{\rm GW} = 2 f_{\rm rot}$









What is a stochastic background?

- Stochastic (random) background of gravitational radiation

- Can arise from superposition of large number of unresolved GW sources
  1. Cosmological origin
  2. Astrophysical origin

- Strength of background measured as gravitational wave energy density ρ_{GW}

Detecting Stochastic Backgrounds

The filter function has the form:

$$\tilde{Q}(f) = N \frac{\gamma(f)\Omega_{\mathrm{GW}}(f)H_0^2}{f^3 P_1(f) P_2(f)}$$

overlap reduction function: $\gamma(f)$

power law template for GW spectrum: $\Omega_{\mathrm{GW}}(f) = \Omega_\alpha \left(f/100\,\mathrm{Hz}\right)^\alpha$

present value of Hubble parameter: H_0

noise in detector 1: $P_1(f)$

noise in detector 2: $P_2(f)$

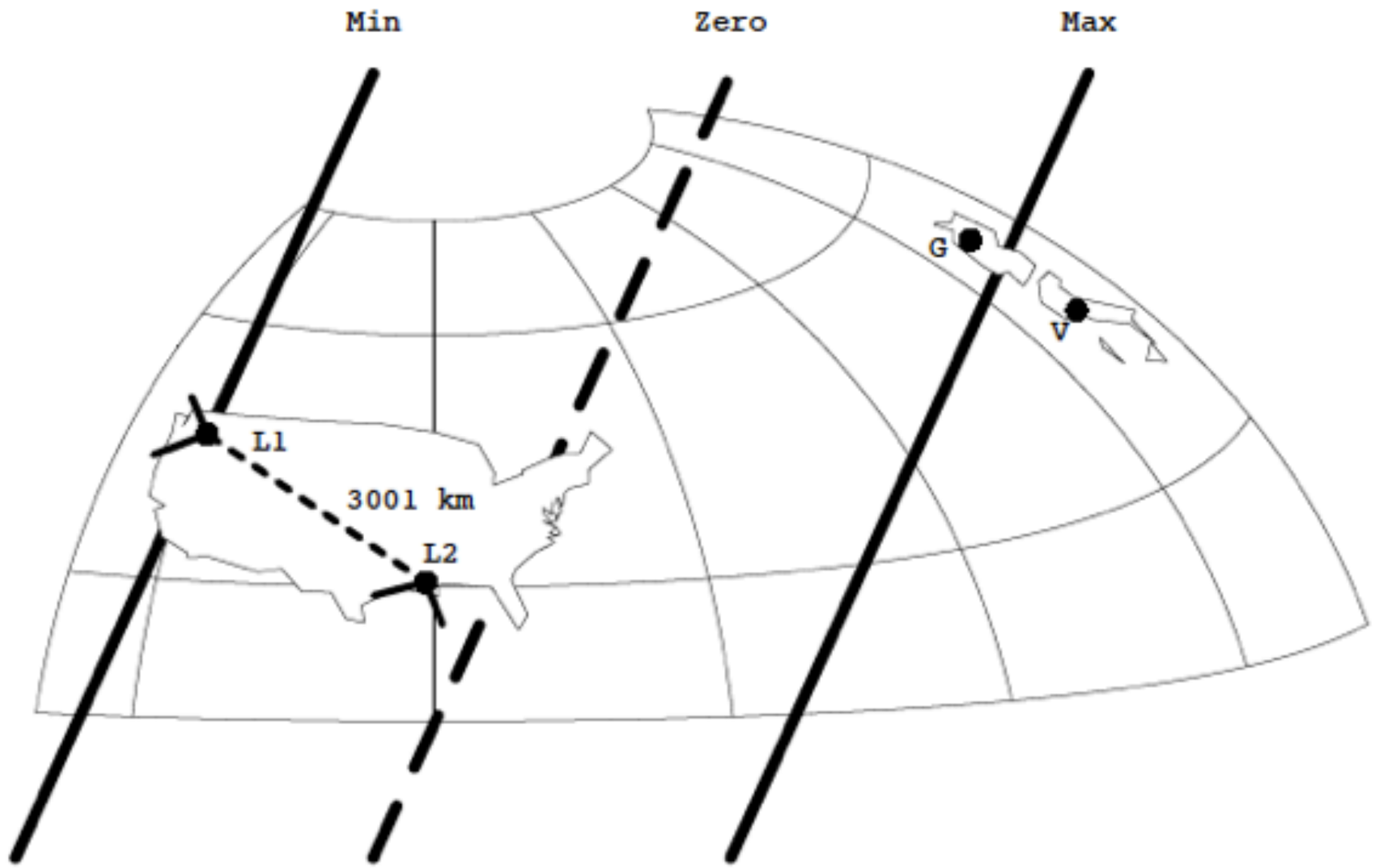
Purpose: Enhance SNR at frequencies where signal is strong and suppress SNR at frequencies where detector noise is large.

Overlap Reduction Function

Signal in two detectors will not be exactly the same because:

i) time delay between detectors

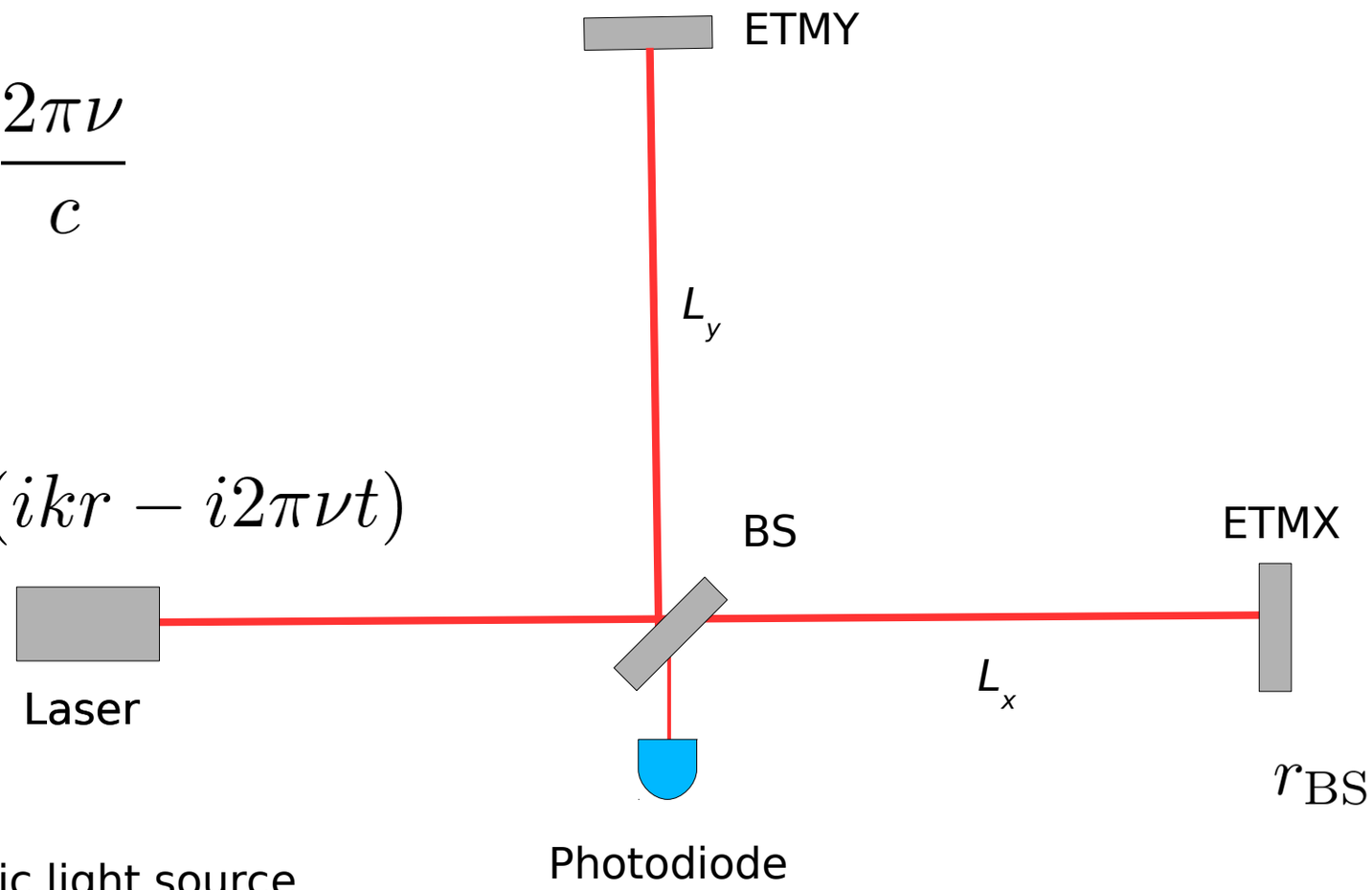
ii) non-alignment of detector



Michelson interferometer

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$$

$$E = E_0 \exp(ikr - i2\pi\nu t)$$



$$r_{\mathrm{BS}} = \pm\frac{1}{\sqrt{2}}$$

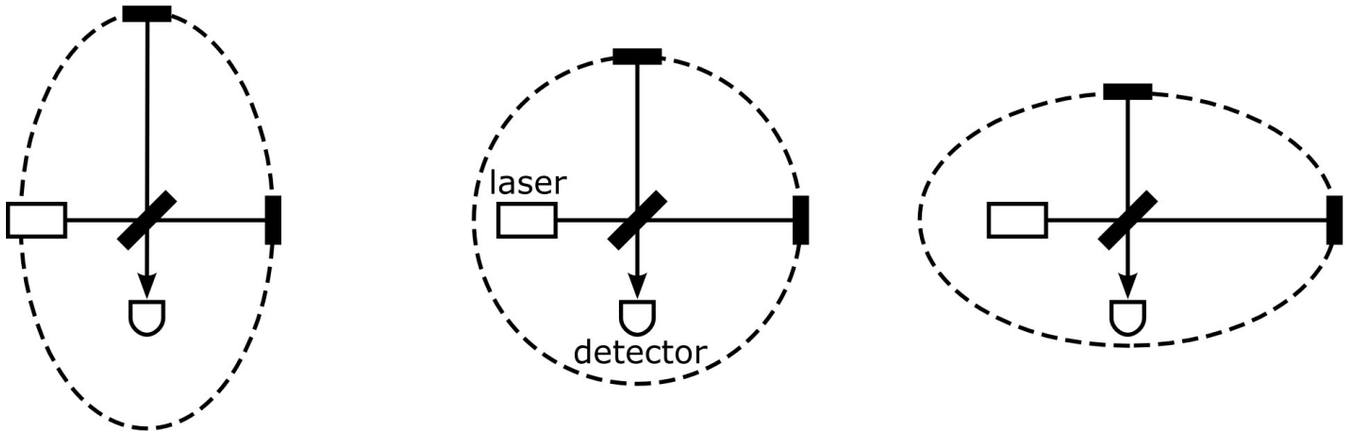
$$t_{\mathrm{BS}} = \frac{1}{\sqrt{2}}$$

$$r_{\mathrm{ETM}} = 1$$

$$P = |E|^2$$

- Monochromatic light source
- 50/50 beam-splitter, note sign flip in reflection coefficient to conserve energy (see Stokes relations)
- Perfectly reflecting end-mirrors (End Test Mass):
- Light of arms interferes on photodiode, which measures power

Interferometric GW detection



- Michelson interferometer is a natural fit for measuring gravitational waves: GW cause a differential change of arm length:

$$L_x = (1 + h/2)L$$
$$L_y = (1 - h/2)L$$
$$\Delta\phi = 2k(L_x - L_y) = 2khL$$

- Idea first proposed by Braginsky, first technical feasibility study by R. Weiss (1972)
- Note: interferometers measure the **amplitude** of the GW and not the power, so dependency on source distance is 1/R instead of 1/R^2
- A simple Michelson is not sensitive enough to detect GW, need several extra tricks ...

Bayes' Theorem

Given:

$$P(A \cap B) = P(A|B)P(B)$$

$$P(B \cap A) = P(B|A)P(A)$$

$$A \cap B = B \cap A$$

We can derive **Bayes' Theorem**:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

A = hypothesis (or parameters or theory)
B = data

P(hypothesis|data) $\propto$ P(data|hypothesis) P(hypothesis)

More on Bayes' Theorem

Initial Understanding + New Observation = Updated Understanding

Likelihood function

Prior probability

Posterior probability

$$p\left(h'|d\right) = \frac{p\left(d|h'\right)p\left(h'\right)}{p\left(d\right)}$$

Evidence

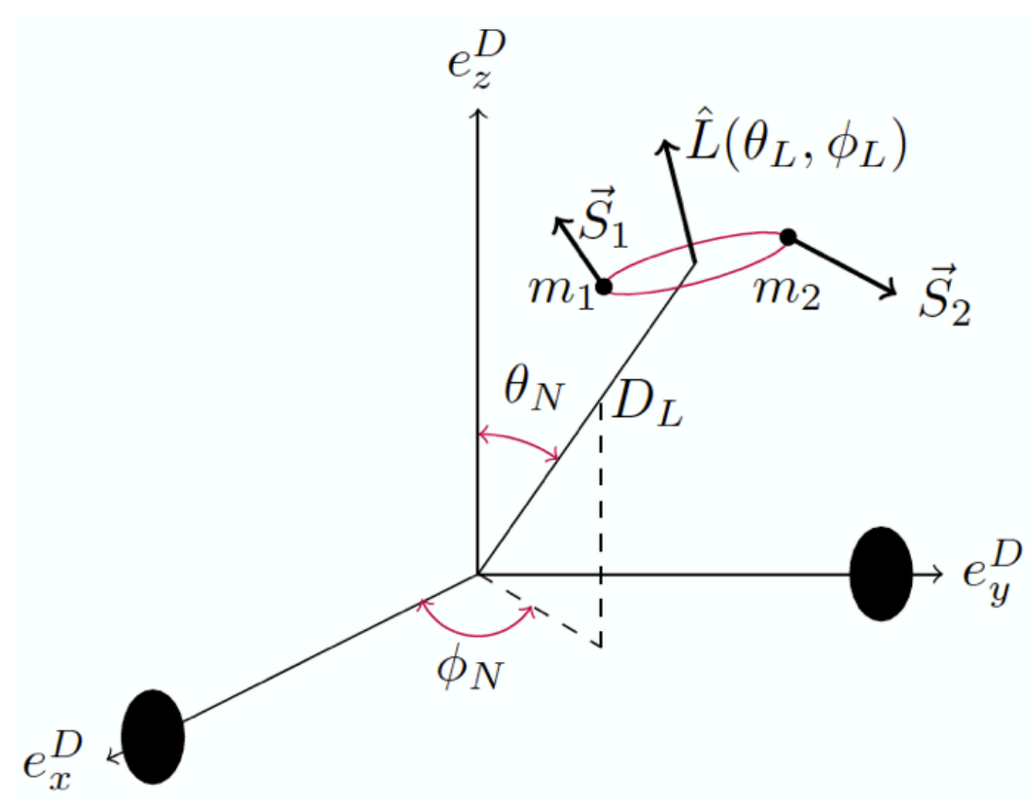
The likelihood function: the data

$$h_t \equiv h(\theta_t)$$

$$\Lambda(s|\theta_t) = \mathcal{N}\exp\left\{(h_t|s) - \frac{1}{2}(h_t|h_t) - \frac{1}{2}(s|s)\right\}$$

In this form, information might not be very manageable.

For binary coalescence there could be more than 15 parameters θ^i



The evidence: model selection

$$p(h'|d, M) = \frac{p(d|h', M)p(h'|M)}{p(d|M)}$$

M: any overall assumption or model (e.g. the signal is a GW, the binary black hole is spin-precessing, the binary components are neutron stars)

Odds Ratio: Compare competing models, for example "GW170817 was a BNS" vs "GW170817 was a BBH":

$$\mathcal{O}_{ij} = \frac{p(M_i|d)}{p(M_j|d)}$$

$$= \frac{p(M_i)p(d|M_i)}{p(M_j)p(d|M_j)}$$

What is the most probable value of the parameters, θ_t ?

A rule for assigning the most probable value is called an estimator. Choices of estimators include:

1. Maximum likelihood estimator

2. Maximum posterior probability

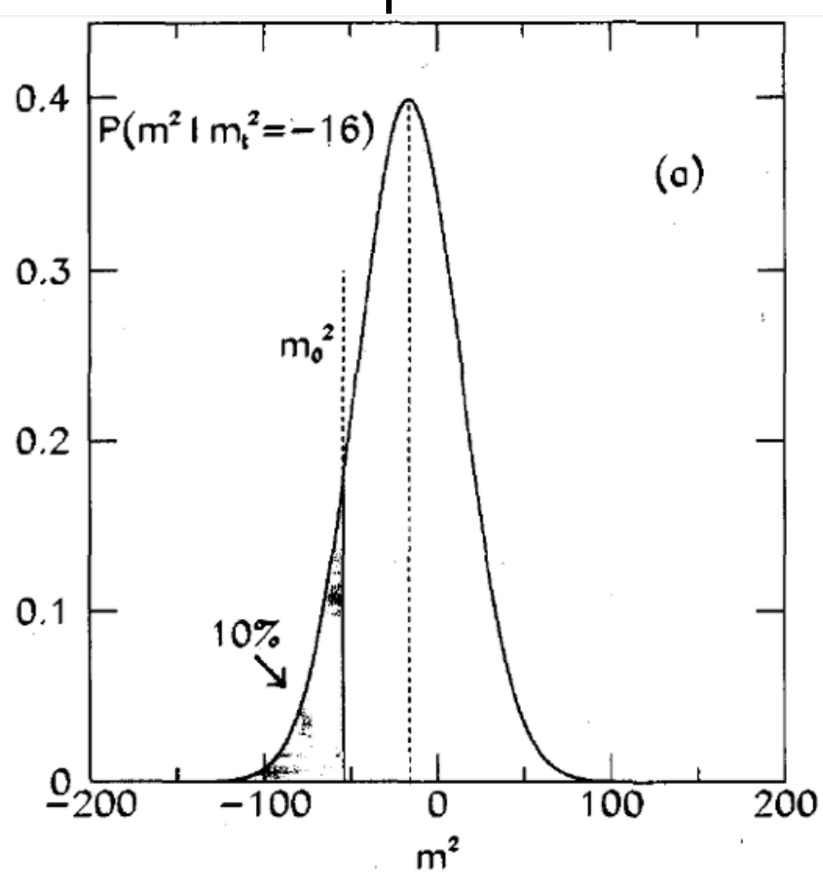
3. Bayes estimator

Confidence versus Credibility

Consider variable with bounded domain like a mass or rate. We can accommodate the physical constraint with a prior.

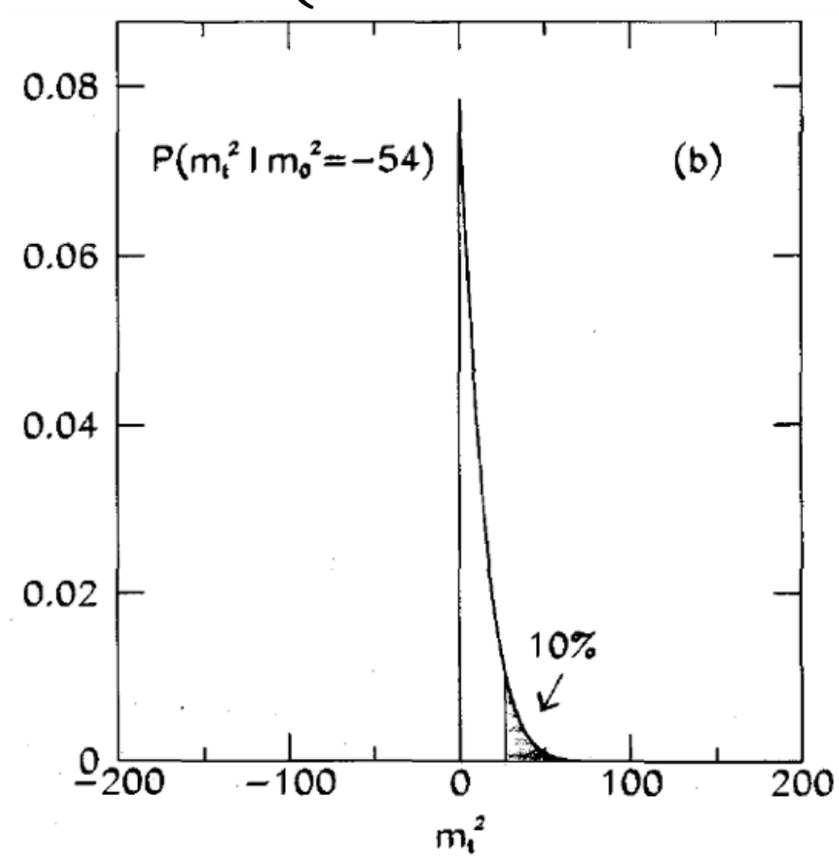
Example: square of mass of electron neutrino

No prior



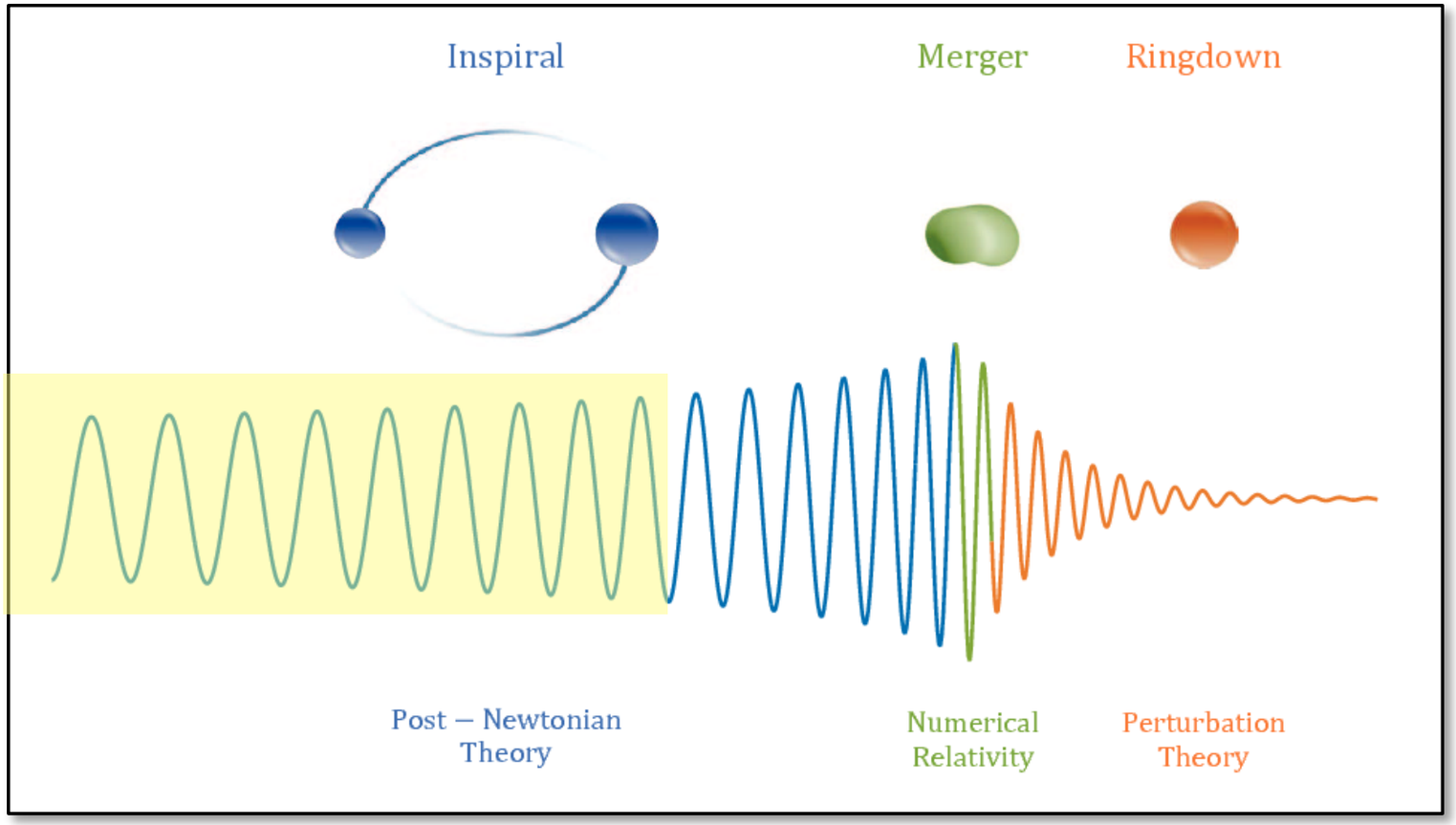
$$m^2 = (-54 \pm 30)\mathrm{eV}^2$$

$$P(m^2) = \begin{cases} 0 & m^2 < 0 \\ \text{uniform} & m^2 \geq 0 \end{cases}$$



$$m^2 < 26.6\mathrm{eV}^2$$

FD Cousins (1995)



During inspiral, phase evolution $\phi_{\rm GW}(t; m_{1,2}, \mathbf{S}_{1,2})$ can be computed with PN-theory in powers of v/c.

leading order

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{M^{1/5}}$$

$$\simeq \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3}\dot{f}\right]^{3/5}$$

higher order

$$q = \frac{m_2}{m_1} \leq 1$$

$$\mathbf{S}_{1,2} \parallel \mathbf{L}$$

even higher order

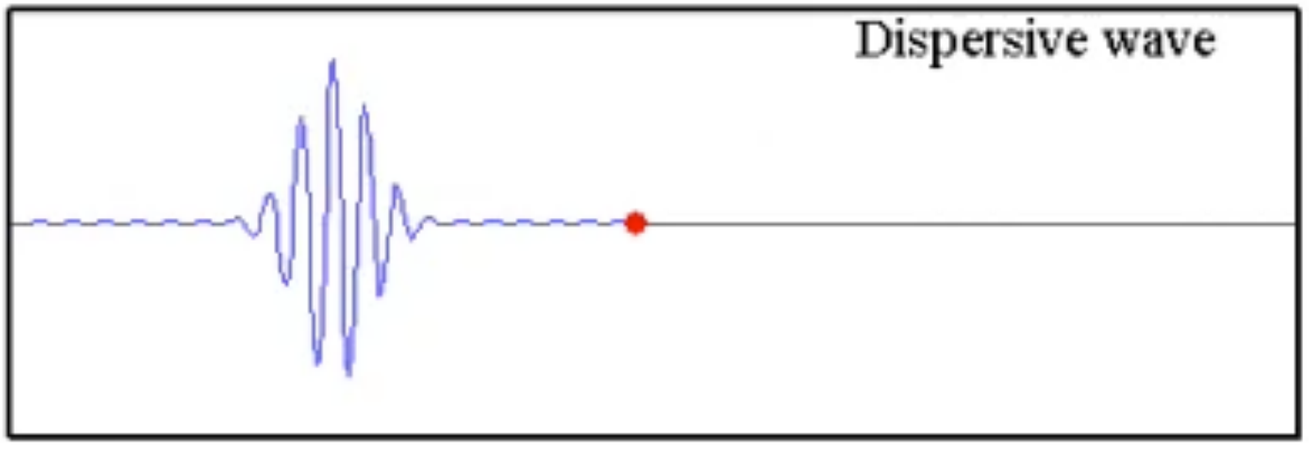
$$S_{1x}, S_{1y}, S_{1z}$$

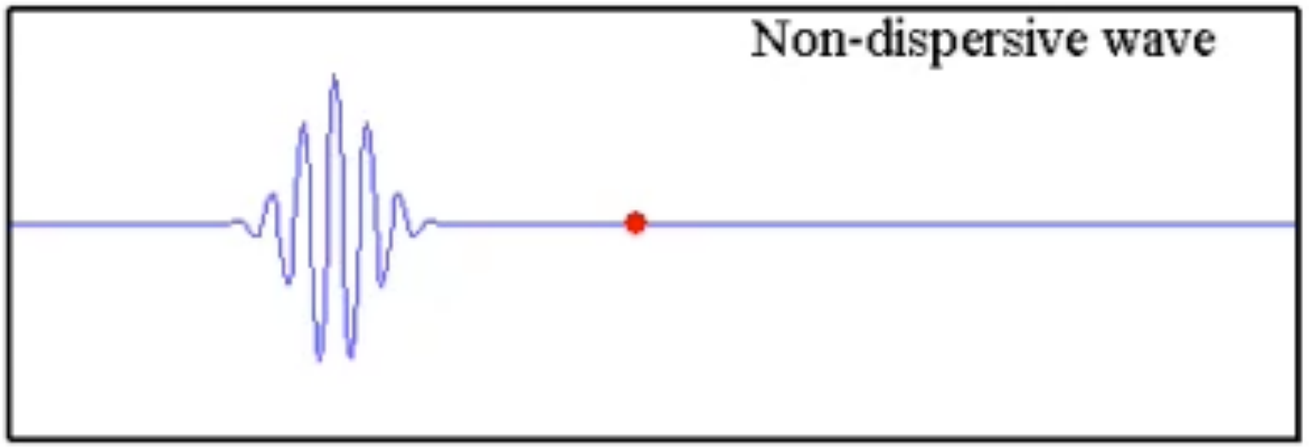
$$S_{2x}, S_{2y}, S_{2z}$$

Constraints on Lorentz violations

In GR, GWs are nondispersive.

But modifications to the dispersion relation can arise in theories that include violations of local Lorentz invariance.





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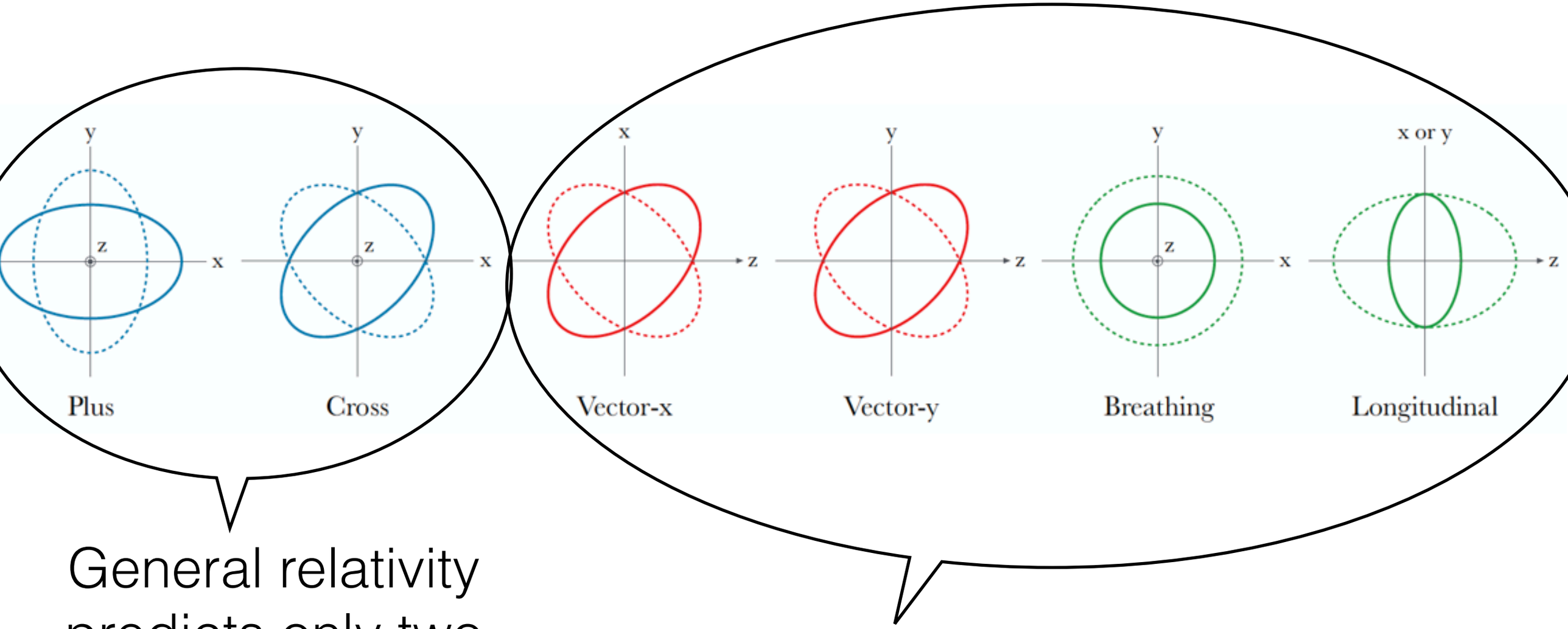
Thus, modified propagation of GWs can be mapped to Lorentz violation.

Gravitational-wave Polarizations



General relativity predicts only two tensor GW polarizations.

Alternate theories allow for up to four additional vector and scalar modes.

In principle, full generic metric theories predict any combination of tensor, vector or scalar polarizations.