Announcements

Assignment 4 due today.

Assignment 5 uploaded to website and Piazza. Will be due in two weeks, May 12th.

No class next week, May 5th.

Last lecture will be May 12th. Assignment 6 will be optional (see next slide).

We will meet again for Final Assessment on May 29.
HW Adjustments

Options for improving your grade:

1) On an assignment where you have done poorly, you can correct the problems, and receive half the points back.

   AND/OR

2) You can turn in the optional assignment 6 and its grade will replace your lowest grade.
Gravitational Wave
Data Analysis

Lecture 5: Gravitational Waves MSc Course
Crucial problem: How to extract a GW signal from the (typically) much larger detector noise?

Key components:
1. Characterizing detector noise
2. Understanding detectors’ angular sensitivity
3. Applying filtering techniques
Output of GW detector

**Signal Input** of detector has the form $h(t) = D^{ij} h_{ij}(t)$

$D^{ij}$ is the detector tensor; depends on detector geometry.

**Signal Output** of detector is related to input by

$$\tilde{h}_{out} = T(f) \tilde{h}(f)$$

$T(f)$ is the transfer function.

However, output of any real detector will have noise, so output will really be given by

$$s_{out}(t) = h_{out}(t) + n_{out}(t)$$

$$s(t) = h(t) + n(t)$$
Detector noise

To compare detector performances, we use the detectors’ noise $n(t)$.

If noise is stationary, then the different Fourier components are uncorrelated. Then the ensemble average of the Fourier components of the noise is of the form:

$$\langle \tilde{n}^*(f)\tilde{n}(f') \rangle = \delta(f - f') \frac{1}{2} S_n(f)$$

Ensemble average can be replaced by time average to give the variance of the noise:

$$\langle |\tilde{n}(f)|^2 \rangle = \frac{1}{2} S_n(f)T \quad \Delta f = \frac{1}{T}$$

$T$ is time of experiment.
Noise spectral density

$S_n(f)$ is the noise spectral density (aka noise spectral sensitivity or noise power spectrum):

$$\langle n^2(t) \rangle = \int_0^\infty df \ S_n(f)$$
Detector Pattern Functions

Polarization tensors where $\hat{u}, \hat{v}$ are orthogonal to propagation direction

\[
e_{ij}^+ (\hat{n}) = \hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j
\]
\[
e_{ij}^\times (\hat{n}) = \hat{u}_i \hat{v}_j + \hat{v}_i \hat{u}_j
\]

In the frame where $\hat{n}$ is along the $\hat{z}$ direction, we can choose

\[
\hat{u} = \hat{x} \quad \hat{v} = \hat{y}
\]

\[
e_{ab}^+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{ab} \quad e_{ab}^\times = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{ab}
\]

$a, b = 1, 2$ in the $(x,y)$ plane
Detector Pattern Functions

GW with given propagation direction $\hat{n}$ is given by

$$h_{ij}(t, x) = \sum_{A=+, \times} e_{ij}^A(\hat{n}) \int_{-\infty}^{\infty} df \tilde{h}_A(f) e^{-2\pi if(t - \hat{n} \cdot x / c)}$$

$$h_{ij}(t) = \sum_{A=+, \times} e_{ij}^A(\hat{n}) \int_{-\infty}^{\infty} df \tilde{h}_A(f) e^{-2\pi ift}$$

$$= \sum_{A=+, \times} e_{ij}^A(\hat{n}) \tilde{h}_A(t)$$

Fold in the detector tensor:

$$h(t) = D^{ij} h_{ij}(t)$$

$$h(t) = \sum_{A=+, \times} D^{ij} e_{ij}^A(\hat{n}) \tilde{h}_A(t)$$
Detector pattern functions depend on the direction of propagation $\hat{n} = (\theta, \phi)$ of the wave.

$$F_A(\hat{n}) = D^{ij} e^{A}_{ij}(\hat{n})$$

$$h(t) = h_+(t) F_+(\theta, \phi) + h_\times(t) F_\times(\theta, \phi)$$

Generic case: $F_{+,\times}(\theta, \phi; \psi)$
Detector Pattern Functions

\[ h(t) = h_+(t)F_+(\theta, \phi) + h_\times(t)F_\times(\theta, \phi) \]

GWs have two polarizations called + and \( \times \).

Detectors are not omnidirectional but exhibit an antenna pattern.

Credit: T. Fricke
Interferometric GW Detector Pattern Functions

\[
F_+(\theta, \phi; \psi = 0) = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi
\]

\[
F_\times(\theta, \phi; \psi = 0) = \cos \theta \sin 2\phi
\]

Thus GW interferometers have blind directions. For instance, for a GW with plus polarization,

\[
\phi = \pi/4 \quad \text{and} \quad F_+ = 0
\]

This wave produces the same displacement in the \( y \) and \( x \) arm.

Differential phase shift vanishes!
Finding signals

The detector output will be of the general form
\[ s(t) = h(t) + n(t) \]

Naively, we can detect a GW signal only when \(|h(t)|\) is larger than \(|n(t)|\)

But we will rather be in the situation
\[ |h(t)| \ll |n(t)| \]
Fundamental question:

How can we dig out the GW signal from a much larger noise?

Answer: We can detect values of $h(t)$ much smaller than the floor of the noise if we know, at least to some level of accuracy, the form of $h(t)$.

$$s(t) = h(t) + n(t)$$

$$\frac{1}{T} \int_{0}^{T} dt \ s(t) \ h(t) = \frac{1}{T} \int_{0}^{T} dt \ h^2(t) + \frac{1}{T} \int_{0}^{T} dt \ n(t)h(t)$$

$$\frac{1}{T} \int_{0}^{T} dt \ h^2(t) \sim h_0^2$$

$$\frac{1}{T} \int_{0}^{T} dt \ n(t)h(t) \sim \left( \frac{T_0}{T} \right)^{1/2} n_0 h_0$$
Fundamental question: How can we dig out the GW signal from a much larger noise?

\[
\frac{1}{T} \int_0^T dt \, n(t)h(t) \sim \left( \frac{\tau_0}{T} \right)^{1/2} n_0 h_0
\]

Goes to zero for large \( T \)!

Thus, it is not necessary to have \( h_0 > n_0 \).

It is sufficient to have \( h_0 > (\tau_0/T)^{1/2} n_0 \).
Let’s turn the previous procedure into a method we can use...

**Goal**: Obtain the highest possible signal-to-noise ratio (SNR).

\[
\hat{s} = \int_{-\infty}^{\infty} dt \, s(t)K(t)
\]

Consider \(K(t)\) as a generic filter function.

If we know the form of \(h(t)\), what is the filter function \(K(t)\) that maximizes the SNR? Aka matched filtering
Define the signal-to-noise ratio...

\[ \langle s(t) \rangle = \langle n(t) \rangle + \langle h(t) \rangle \]

\[ \langle n(t) \rangle = 0 \text{ (if noise is stationary)} \]

**Signal:** expected value if signal is present

\[ S = \int_{-\infty}^{\infty} dt \langle s(t) \rangle K(t) = \int_{-\infty}^{\infty} dt h(t)K(t) = \int_{-\infty}^{\infty} df \tilde{h}(f)\tilde{K}^*(f) \]

**Noise:** root-mean-square value if no signal present

\[ N^2 = \left[ \langle \hat{s}^2(t) \rangle - \langle \hat{s}(t) \rangle^2 \right]_{h=0} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' K(t)K(t') \langle n(t)n(t') \rangle \]

\[ = \int_{-\infty}^{\infty} df \left( \frac{1}{2} \right) S_n(f) |\tilde{K}(f)|^2 \]
Define the signal-to-noise ratio...

\[
\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f)}{\left[ \int_{-\infty}^{\infty} df \left(\frac{1}{2}\right) S_n(f) \left| \tilde{K}(f) \right|^2 \right]^{1/2}}
\]

Convenient definition: noise-weighted scalar product between two real functions \( A(t) \) and \( B(t) \)

\[
(A|B) = \text{Re} \int_{-\infty}^{\infty} df \frac{\tilde{A}^*(f) \tilde{B}(f)}{(1/2) S_n(f)}
\]

\[
= 4 \text{Re} \int_{0}^{\infty} df \frac{\tilde{A}^*(f) \tilde{B}(f)}{S_n(f)}
\]
Define the signal-to-noise ratio...

Using this scalar product definition, we have:

\[
\frac{S}{N} = \frac{(u|h)}{(u|u)^{1/2}} \quad \text{where} \quad \tilde{u}(f) = \frac{1}{2}S_n(f)\tilde{K}(f)
\]

We are searching for vector \( \frac{u}{(u|u)^{1/2}} \) such that its scalar product with vector \( h \) is maximum.

They should be parallel (i.e. proportional):

\[
\tilde{K}(f) = \text{const.} \frac{\hat{h}(f)}{S_n(f)}
\]

This is the **Wiener filter** (aka matched filter).
Define the signal-to-noise ratio...

Optimal value of signal-to-noise ratio:

\[
\left( \frac{S}{N} \right) = (h|h)^{1/2}
\]

\[
\left( \frac{S}{N} \right)^2 = 4 \int_0^\infty df \frac{|\tilde{h}(f)|^2}{S_n(f)}
\]

Completely generic and independent of form of $\tilde{h}(f)$
Data Quality Vetoes

If a particularly noisy period of data is identified, and if it is correlated with a known instrument problem, then the data may be removed.

Smith, et al. CQG 28 235005 (2011)
Coincidence Window

$\sigma_1^2, \sigma_2^2$ - variances on arrival times in two detectors.

Then, variance on coincident search is $\sigma_{1,2} = \left(\sigma_1^2 + \sigma_2^2\right)^{1/2}$

Light travel time between detectors plus $k$ standard deviations.

$$|t_1 - t_2| \leq (\Delta t)_{\text{light}} + k \left(\sigma_1^2 + \sigma_2^2\right)^{1/2}$$

Coincidence must be within a few tens of milliseconds.
Gravitational-wave Sources

- Compact Binaries
- Continuous Waves
- Bursts
- Stochastic
Do we really need a matched filter search?

GW150914

GW151226
Formal Definition of Matched Filter

Cross-correlate data with template, weighted by detector noise:

\[ Z(t) = A \int_{-\infty}^{\infty} \frac{\tilde{s}(f)\tilde{h}^*(f)e^{2\pi i ft}}{S_n(f)} df \]

Cross-correlate template with template, weighted by detector noise:

\[ \sigma^2 = 2 \int_{0}^{\infty} \frac{\tilde{h}(f)\tilde{h}^*(f)}{S_n(f)} df \]

Normalize output of optimal filter:

\[ \rho(t) = \frac{|Z(t)|}{\sigma} \]
Matched Filter Search

\[ \chi_{1,2} \propto \vec{S}_{1,2} \cdot \hat{L} \]

Template Bank

Matched filter signal-to-noise ratio

- \(|\chi_1| < 0.9895, |\chi_2| < 0.05\)
- \(|\chi_{1,2}| < 0.05\)
- \(|\chi_{1,2}| < 0.9895\)

Data

Cross Correlation

Time (s)
The O1 UberBank

~200,000 templates, low frequency cutoff = 30 Hz
Signal-based Vetoes

Time-frequency Spectrograms:
Glitches versus Signals

Messick, et al, arXiv 1604.04324
Which glitch is not like the others?
Coincident Triggers

**Likelihood-based ranking statistic**

\[ L = \frac{P(d_H, d_L|\text{signal})}{P(d_H|\text{noise}) P(d_L|\text{noise})} \]

- **Numerator:** semi-analytic signal model
- **Denominator:** background noise model
A multivariate ranking statistic

\[ \mathcal{L} = \frac{P(d_H, d_L|\text{signal})}{P(d_H|\text{noise}) P(d_L|\text{noise})} \]
Signal and Background Models

Signal Model

Background Model

+ - GW150914
+ - GW151226
+ - LVT151012
Evolution of Noise Model with Mass

The search background is calculated separately for each mass and chi bin.
Only when we leave coincident triggers in our noise model, i.e. gravitational waves, do we form accidental coincidences at the same level of significance or higher.
Burst Signals

• GW searches with minimal assumptions
  • Do not assume accurate source models - affected by noise
• Limited ability to measure waveforms, sky positions, polarizations
• Can detect the unexpected
Burst Analysis with Wavelets

- Wavelets are waveforms of limited duration and bandwidth
- GW bursts can be described as superposition of wavelets

Credit: MathWorks
The Fourier Transform does not represent abrupt changes efficiently. Sum of Sine waves are not localized in time or space and oscillate forever.
For time series that have abrupt changes, we need functions that are well localized in time and frequency.
Time-frequency Pixel Maps

Example: (10, 10) M⊙ binary black hole merger with an eccentric orbit at 4 different time-frequency representations.

Credit: S. Klimenko
Cluster Selection

Identify time-frequency areas with excess energy above a predetermined threshold.

Cluster excesses together.
Combine wavelet layers to make a supercluster. These form burst events.
Coherent Burst Search

- Wavelet amplitudes need to be calculated and signal reconstructed for each time-frequency pixel and sky location
- Joint analysis from all detectors (H1, L1, V1,...)
- Synchronize detectors by searching over entire sky (~200,000 points)
Maximum Likelihood Statistic

\[ L = c_c E_s \]

- **Similarity of waveforms in detectors**
- **Total energy of reconstructed waveforms**

\[ c_c = \frac{E_c}{E_c + E_n} \]

- **\( E_c \) - coherent energy**
- **\( E_n \) - energy of residual noise after signal subtracted**
Example: 2-Detector Constraint

Magnitude of antenna pattern vectors in for L1-H1 network

Ratio of antenna pattern vectors

\[ |f_\times| \ll |f_+| \]

LIGO network nearly aligned; blind to second polarization in most sky locations.
3 Search Classes

**C1**
Time-frequency morphology of known populations of noise

**C2**
Frequency increasing with time

**C3**
All remaining events
A Burst Result

The first gravitational wave was found by a burst pipeline, within 3 minutes!

LVC, PRL 116, 061102 (2016)
Gravitational-wave Sources

- Compact Binaries
- Continuous Waves
- Bursts
- Stochastic
Types of CW Searches

- **Targeted searches** for known pulsars: known position, rotation frequency and spin down

- **Directed searches** in known position: SNRs, galactic center, accreting neutron stars in low-mass x-ray binaries

- **All-sky searches** for isolated, unknown sources
The Continuous Wave from an Isolated NS

• The source should emit a nearly monochromatic sinusoidal wave

• Limit on observation comes from total available observation time

• But the detector will see a modified signal

\[ \tilde{A} = \{ f, \dot{f}, \ddot{f}, \ldots, \alpha, \delta, h_0, \cos \nu, \psi, \phi_0 \} \]

• Four phase evolution parameters

• Four amplitude parameters
Doppler Effect Due to Earth Rotation and Orbit

CREDIT: Bill Saxton, NRAO/AUI/NSF

CREDIT: S. Walsh
The All-sky Search Goal

\[ \text{SNR} \propto \frac{h_0}{\sqrt{S_n}} \sqrt{T} \]

- Search over broad frequency, spin-down range over whole sky over many months of data
- Not really computationally feasible
- Solution: use semi-coherent methods
Semi-coherent Methods

Split data into short segments which are analyzed coherently

\[ \Delta T = \frac{T_{\text{obs}}}{N} \]

Short (~1000s): signal stays in single Fourier bin
Semi-coherent Methods

Split data into short segments which are analyzed coherently

Long (~hrs/days): need to account for signal modulation within segment

\[ \Delta T = \frac{T_{\text{obs}}}{N} \]
Under the assumption of white detector noise, F-statistic will look like

\[ F \approx \frac{2}{\sigma^2} \left( \frac{|F_a|^2}{\langle a^2 \rangle} + \frac{|F_b|^2}{\langle b^2 \rangle} \right) \]

Maximize with respect to four unknown amplitude parameters:
\[ \{ h_0, \cos \iota, \psi, \phi_0 \} \]

Dimension of parameter space: 8 to 4

Under the assumption of white detector noise, F-statistic will look like

\[ \Lambda(x) \equiv \max_\mathcal{L} (x; \mathcal{A}) = \exp^F(x) \]
Stack-Slide Method

CREDIT: S. Walsh
Stack-Slide Method

Add power after frequency bins are shifted.

CREDIT: S. Walsh
Hough Transform Method

\[ y = ax + b \]

\[ b = -x_i a + y_i \]

\[ b = -x_j a + y_j \]
Hough Transform Method

\[ y = ax + b \]

\[ b = -x_i a + y_i \]

\[ b = -x_j a + y_j \]

Credit: Cornelissen, et. al.
Identify Interesting Frequency Bands

Identify 50 mHz bands disturbed by detector artifacts. Maximum density and mean 2F value are metrics to identify disturbed bands.

Credit: S. Zhu, et. al., PRD 94 (2016)
Contamination in CW Searches

Two power spectra, one with an unusually high low-frequency noise contribution up to 300 Hz.
Contamination in CW Searches

Harmonics of single-frequency noise source create comb-like patterns in noise spectrum. If not flagged, searches can identify them as possible sources.

Credit: A. Neunzert
A CW Result

At 55Hz, the O1 all-sky search can exclude sources with ellipticity above $10^{-5}$ within 100pc.

LVC, arXiv: 1707.02669
What is a stochastic background?

• Stochastic (random) background of gravitational radiation

• Can arise from superposition of large number of unresolved GW sources
  1. Cosmological origin
  2. Astrophysical origin

• Strength of background measured as gravitational wave energy density $\rho_{GW}$
Cosmological Gravitational Wave Background

GW spectrum: \( \Omega_{GW}(f) = \frac{f}{\rho_c} \frac{d \rho_{GW}}{df} \)

Critical energy density of universe: \( \rho_c = \frac{3c^2 H_0^2}{8\pi G} \)
Astrophysical Gravitational Wave Backgrounds

- Produced by an extremely large number of weak, independent, and unresolved gravity-wave sources
- Binary black holes and/or neutron stars
- Supernovae
- White dwarf binaries

For example, the expected contribution from double white dwarfs for LISA
Potential background from binary black hole mergers

Astrophysical Gravitational Wave Backgrounds
Detecting Stochastic Backgrounds

Cross-correlation of two interferometers’ data streams multiplied by a filter function:

\[ Y = \int_{-\infty}^{+\infty} \, df \, \int_{-\infty}^{+\infty} \, df' \, \delta_T(f - f') \tilde{s}_1(f) \ast \tilde{s}_2(f') \tilde{Q}(f') \]
Detecting Stochastic Backgrounds

The filter function has the form:

$$\tilde{Q}(f) = N \frac{\gamma(f) \Omega_{GW}(f) H_0^2}{f^3 P_1(f) P_2(f)}$$

overlap reduction function: $\gamma(f)$

power law template for GW spectrum: $\Omega_{GW}(f) = \Omega_{\alpha} \left(\frac{f}{100\text{ Hz}}\right)^\alpha$

present value of Hubble parameter: $H_0$

noise in detector 1: $P_1(f)$

noise in detector 2: $P_2(f)$

Purpose: Enhance SNR at frequencies where signal is strong and suppress SNR at frequencies where detector noise is large.
Overlap Reduction Function

Signal in two detectors will not be exactly the same because:

i) time delay between detectors
ii) non-alignment of detector
Get involved...

Einstein@Home uses your computer's idle time to search for weak astrophysical signals from spinning neutron stars (often called pulsars) using data from the LIGO gravitational-wave detectors, the Arecibo radio telescope, and the Fermi gamma-ray satellite.

Gravity Spy - citizen science project for identifying and classifying glitches