I. A GRAVITATIONAL WAVE IN A MICHELSON INTERFEROMETER

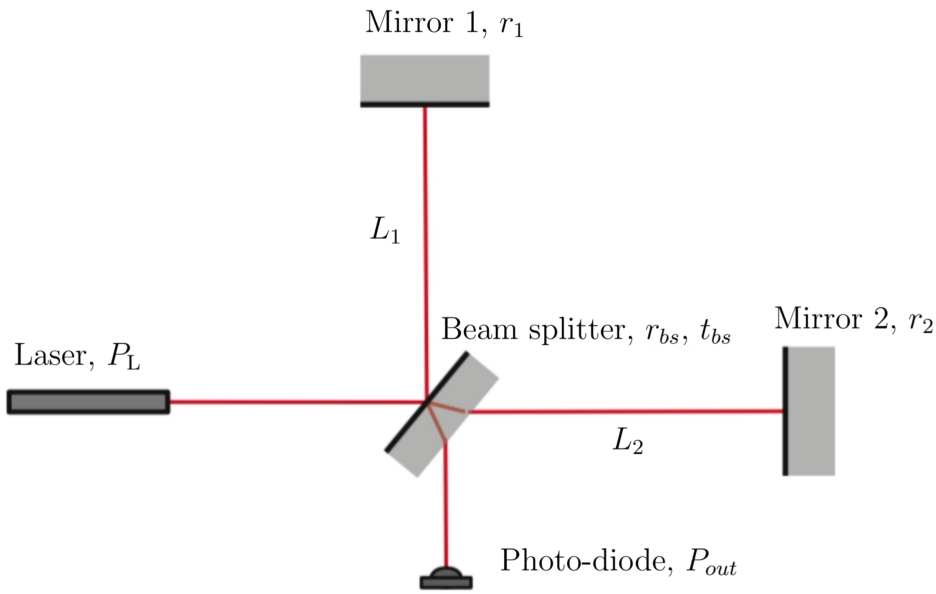

Figure 1: Simple Michelson interferometer: a laser with power P_L shoots light onto a beam splitter with reflection and transmission coeffiecients r_{bs} and t_{bs}.

A simple Michelson interferometer uses a laser producing planar monochromatic light waves having a total power P_L. The reflection of a 50-50 beam splitter can be modeled multiplying the incoming electric field component amplitude of the light wave by a factor $r_{bs} = 1/\sqrt{2}$ for reflection from one side and $r_{bs} = -1/\sqrt{2}$ from the other side, while the transmission multiplies it always with $t_{bs} = 1/\sqrt{2}$. Mirrors multiply that amplitude by r_1 and r_2 for mirror 1 and mirror 2, respectively.

(I.1) What is the power output $P_\mathrm{out,static}$ of a Michelson with a static arm length difference $\Delta L_\mathrm{static} = L_2 - L_1$? To simplify the expression, please fill in the given values for r_{bs} and t_{bs}.

(I.2) Define $\phi_0 = 4\pi/\lambda \times \Delta L_\mathrm{static}$ as the static phase difference between between the two returning beams at the beam splitter. What is the (time varying) phase difference between the two beams introduced by a passing gravitational wave with strain amplitude $h(t)$? Assume the wave is optimally oriented with respect to the interferometer such that the strain is aligned with the arms. Introduce this $\Delta\phi_\mathrm{GW}(t)$ somewhere in the expression found at (I.1) and explain why you chose to put it there.

(I.3) Use the answer found at (I.2) and the trigonomtic identity $cos(a+b) = cos(a)cos(b) - sin(a)sin(b)$ to extract the static part ($P_\mathrm{out,static}$, same as found at (I.1)) and a time varying part $\Delta P_\mathrm{out,GW}(t)$. At what value for ϕ_0 is $\Delta P_\mathrm{out,GW}(t)$ maximized?

II. MAXIMIZING SIGNAL-TO-NOISE RATIO (SNR)

In the absence of noise, the condition determined at (I.3) would be the most sensitive condition of the interferometer. However, the quantum nature of light results in a statistical fluctuation in the arrival times of the photons on the photo-diode. This is called *shot noise* and brings an uncertainty to the intensity measurement in which a GW signal could be hiding. To really maximize the sensitivity of the detector, the *signal-to-noise ratio* (SNR) needs to be maximized.

(II.1) If S_{GW} is the amplitude spectral density (ASD) of the phase induced by the gravitational wave, what is the ASD of the signal measured at the photo-diode? (Remember, a photo-diode measures power)

(II.2) Define the ASD of the shot noise is $S_{\text{shot}} = \sqrt{2P_{\text{out,static}}\hbar\omega}$, with ω the angular frequency of the laser light. Show that

$$SNR = \sqrt{\frac{P_L r_1^2 r_2^2}{2\hbar\omega} f(\phi)} S_{\text{GW}}, \tag{1}$$

where $f(\phi) = sin^2\phi_0/(r_1^2 + r_2^2 - 2r_1r_2cos\phi_0)$.

(II.3) At the *dark fringe*, i.e. $\phi_{0,df} = 2n\pi$, to what order is the signal in h? Is operating a gravitational wave detector in this condition the best idea? Could you also draw a similar conclusion from looking at eq. (1) when you assume the $r_1 \approx= r_2 \approx 1$?

(II.4) The SNR is optimised when f(ϕ) is maximal. Show that maximum is not exactly at the dark fringe, i.e. $cos(\phi_0') = r_1/r_2 \lesssim 1$, with ϕ_0' (a phase close to the dark fringe) if you assume the reflectivities to be slightly smaller than 1 and one slightly different from the other (choose $r_1 < r_2$). Then, assuming $r_1 \approx r_2 \approx 1$, show that ϕ_0' maximizes the SNR with $f(\phi_0') \approx 1$.

III. THE SENSITIVITY OF A SHOT NOISE LIMITED INTERFEROMETER

The sensitivity of an interferometer is defined as the ASD of the strain for which SNR=1. The goal of this exercise is to compute the sensitivity of the Michelson interferometer studied in the previous exercise parts.

(III.1) What is the ASD for the phase of a gravitational wave with SNR = 1?

(III.2) What is the ASD for the strain of the gravitational wave (with SNR=1)? (Hint: compute first the phase difference at the beam splitter assuming that the travel time of the photon in the interferometer arms is much smaller then the period of the gravitational wave. How do the ASD of the phase and the strain of the gravitational wave relate?)

(III.3) Assume an arm length of 3 km, an input laser power of 4 W, and a wavelength of 1064 nm. What is the value of the strain in the previous exercise?

(III.4) At (III.2) you found that the strain of a shot noise limited interferometer does not depend on the gravitational wave frequency. What does this imply for the bandwidth/sources? (Hint: the typical gravitational wave frequency of a source is source dependent: binary neutron stars emit typically gravitational waves of higher frequencies during merger than binary black holes)

(III.5) The sensitivity of the actual detectors is frequency dependent - what is unrealistic in the former computation?