Name (please print legibly!) ______________________________

Gravitational Waves: Assignment 2
Review of General Relativity

1. Indicate whether the following equations or terms are valid under Einstein Summation convention rules. Also indicate the free and dummy indices in each.

   (a) $s_{jk} = t_{kj} + t_{jk}$

   (b) $A_i = B_j + C_i$

   (c) $S_{jk} = A_j \left(B_k + C_k\right)$

   (d) $\lambda \times \mu$ where $\lambda = t_i t^i$ and $\mu = q_i q^i$

   (e) $x_i y^i z_i$

   (f) $a^i_{ij} = t_j + b^k c_{kj} + d^{mn}_{mnj}$

   (g) $\Gamma^\alpha_{\beta\gamma} a^\alpha c^\beta c^\gamma = b^\alpha$

   (h) $\partial x^\alpha / \partial x^\beta = \delta^\alpha_\beta$

   (i) $\partial g_{\alpha\beta} / \partial x^\gamma = 0$

2. Once we have a metric, other geometric quantities can be computed. Express the following quantities in terms of metric and/or vector components:

   (a) The cosine of the angle between two vectors, $\cos\theta$

   (b) An element of area dA for a diagonal metric

   (c) A four-volume dv element for a diagonal metric

3. Express the metric given the following infinitesimal length elements:

   (a) spherical surface in polar coordinates: $ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$

   (b) spherical surface in cylindrical coordinates: $ds^2 = \frac{R^2 d\rho^2}{R^2 - \rho^2} + \rho^2 d\phi^2$

4. (a) In class, we saw that a finite length s can be computed as the line integral of ds which depends on the metric. From this expression, extract the Lagrangian as

   $$L\left(x, \dot{x}\right) = g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \tag{1}$$

   and use the Euler-Lagrange equation to obtain the geodesic equation which determines the trajectory of the "shortest curve". Note the fact that the metric function g_{ab} depends on x but not $\dot{x}$.

   (b) Use the geodesic equation to confirm the familiar results that the geodesic is a straight line on a flat plane.

5. In Cartesian coordinates, the line element of Euclidean, 3-dimensional space is

   $$ds^2 = dx^2 + dy^2 + dz^2 \tag{2}$$

(a) Compute the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ in this coordinate system. What do you conclude about the Riemann tensor?

(b) Express the line element of Eq. 2 in the usual spherical coordinates (r, θ, ϕ).

(c) Compute the Christoffel symbols in spherical coordinates.

(d) Compute the Riemann tensor in spherical coordinates. Had you expected this result?

6. Define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{3}$$

Calculate how $\bar{h}_{\mu\nu}$ changes under gauge transformations. Then show that for any field configuration $h_{\mu\nu}(x)$, one can find a gauge transformation such that

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0. \tag{4}$$

(Hint: first calculate how $\partial^{\nu}\bar{h}_{\mu\nu}$ changes under gauge transformations. Then using the Green's function of the d'Alembertian, construct the gauge transformation which makes it zero.)