Name (please print legibly!) ________________________________

Gravitational Waves: Assignment 3
Derivation of GWs and Sources of GWs

1. Practice with raising, lowering, and contracting indices: the metric tensor plays the role of allowing one to raise/lower indices on tensors, i.e. switching between contravariant and covariant forms.

   (a) $g_{\mu\nu}A^{\nu} =$

   (b) If ∂_{ν} is the contravariant gradient operator, $\eta^{\mu\nu}\partial_{\nu} =$

   (c) $g^{ij}g_{jk} =$

   (d) $g^{\alpha\gamma}g^{\beta\delta}A_{\gamma\delta} =$

   (e) If $\hat{n}$ is a unit vector, then $n^i n_i =$

   (f) In spatial coordinates, $\delta^i_i =$

2. Start with a metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. From the definition of the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ and the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$, show that to linear order in $h_{\mu\nu}$, the Riemann tensor becomes

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_{\nu}\partial_{\rho}h_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}h_{\nu\rho} - \partial_{\mu}\partial_{\rho}h_{\nu\sigma} - \partial_{\nu}\partial_{\sigma}h_{\mu\rho}\right). \tag{1}$$

   Note that for the linear approximation, we can ignore all products of h with h or its derivatives and that we use the convention that indices are raised and lowered with the Minkowski flat metric $\eta_{\mu\nu}$.

3. Define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{2}$$

   Calculate how $\bar{h}_{\mu\nu}$ changes under gauge transformations. Then show that for any field configuration $h_{\mu\nu}(x)$, one can find a gauge transformation such that

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0. \tag{3}$$

   (Hint: first calculate how $\partial^{\nu}\bar{h}_{\mu\nu}$ changes under gauge transformations. Then using the Green's function of the d'Alembertian, construct the gauge transformation which makes it zero. Note that a Green's function $G(x)$ of the d'Alembertian satisfies $\Box_x G(x-y) = \delta^4(x-y)$.)

4. Let's show that $h^{\mathrm{TT}}_{ij} = \Lambda_{ij,kl}\bar{h}^{kl}$ really is transverse and traceless. Recall that the lambda tensor is

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{4}$$

   where $P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j$ is a symmetric, transverse projection operator and $\hat{n}$ is the direction of propagation of the GW.

   (a) Using results from Problem 1, show that $P^i_i = 2$.

   (b) Write out $\Lambda_{ij,kl}$ in terms of the Kroneker delta δ_{ij} and $\hat{n}$ and apply the operator to $\bar{h}^{kl}$.

(c) The condition for h_{ij}^{TT} to be transverse is that its component in the $\hat{n}$ direction vanishes, i.e., $n^i h_{ij}^{\mathrm{TT}} = 0$. Show that this is true.

(d) The condition for h_{ij}^{TT} to be traceless is $\delta^{ij} h_{ij}^{\mathrm{TT}} = 0$. Show that this is true.

5. When the direction of propagation $\hat{n}$ is equal to $\hat{z}$, P_{ij} is the diagonal matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

This means that P is a projector on the (x, y) plane. Then the quadrupole approximation reduces to

$$\begin{aligned} \left[h_{ij}^{\mathrm{TT}}\right]_{\mathrm{quad}} &= \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}(t - r/c) \\ &\to \left(P\ddot{M}P\right)_{ij} - \frac{1}{2}P_{ij}\mathrm{Tr}(P\ddot{M}). \end{aligned} \tag{6}$$

Show that this gives the following two polarization amplitudes for a gravitational wave propagating in the $\hat{z}$ direction:

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right), \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}. \end{aligned} \tag{7}$$

6. (a) Consider a binary on a fixed, circular Keplerian orbit. In the center-of-mass point of view for the system, suppose we have a single effective object of mass $\mu = (m_1 m_2)/(m_1 + m_2)$ described by coordinates:

$$\begin{aligned} x_0(t) &= R\cos(\omega_s t + \pi/2), \\ y_0(t) &= R\sin(\omega_s t + \pi/2), \\ z_0(t) &= 0. \end{aligned} \tag{8}$$

Determine the second derivatives of the second mass moments and plug these into the $h_+(t; \theta, \phi)$ and $h_\times(t; \theta, \phi)$ expressions for generic gravitational wave propagation that we saw in the lecture slides.

(b) Consider the cases where the binary orbit is edge-on ($\theta = \iota = \pi/2$) and face-on ($\theta = \iota = 0$). What happens to the contributions from h_+ and $h_\times$ in these cases?

7. Now consider a binary that loses energy due to the emission of gravitational waves. We saw in the lecture slides that the polarization amplitudes evolve as:

$$\begin{aligned} h_+(t) &= \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\frac{1 + \cos^2\iota}{2}\cos\left[\Phi(\tau)\right], \\ h_\times(t) &= \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\cos\iota\sin\left[\Phi(\tau)\right] \end{aligned} \tag{9}$$

where $\Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\tau^{5/8} + \Phi_0$, $\Phi_0 = \Phi(\tau = 0)$, and $\tau \equiv t_{\rm coal} - t$. We will generate the plots of a few chirp waveforms using a starting frequency of $f_{\rm gw}(\tau = t_{\rm coal}) = 40$ Hz. You will need to invert the following expression to determine the value of $\tau = t_{\rm coal}$ when $f_{\rm gw} = 40$ Hz:

$$f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}. \tag{10}$$

Assuming $\Phi_0 = 0$, $\iota = 0$ and $r = 1$ Mpc, create plots of h_+ and $h_\times$ versus t for the following masses:

(a) A binary neutron star: $m_1 = 1.4M_\odot$ and $m_2 = 1.4M_\odot$

(b) A neutron-star-black-hole system: $m_1 = 1.4M_\odot$ and $m_2 = 10M_\odot$

(c) A binary black hole: $m_1 = 10M_\odot$ and $m_2 = 10M_\odot$