1. Indicate whether the following equations or terms are valid under Einstein Summation convention rules. Also indicate the free and dummy indices in each.

(a) \( s_{jk} = t_{kj} + t_{jk} \)
(b) \( A_i = B_j + C_i \)
(c) \( S_{jk} = A_j (B_k + C_k) \)
(d) \( \lambda \times \mu \) where \( \lambda = t_i t^i \) and \( \mu = q_i q^i \)
(e) \( x_i y_j z_i \)
(f) \( a_{ij} = t_j + b^k c_{kj} + d_{mnj} \)
(g) \( \Gamma^\alpha_{\beta\gamma} a^\alpha c^\beta c^\gamma = b^\alpha \)
(h) \( \partial x^\alpha / \partial x^\beta = \delta^\alpha_\beta \)
(i) \( \partial g_{\alpha\beta} / \partial x^\gamma = 0 \)

2. Once we have a metric, other geometric quantities can be computed. Express the following quantities in terms of metric and/or vector components:

(a) The cosine of the angle between two vectors, \( \cos \theta \)
(b) An element of area \( dA \) for a diagonal metric
(c) A four-volume \( dv \) element for a diagonal metric

3. Express the metric given the following infinitesimal length elements:

(a) spherical surface in polar coordinates: \( ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2 \)
(b) spherical surface in cylindrical coordinates: \( ds^2 = \frac{R^2 d\phi^2}{R^2 - \rho^2} + \rho^2 d\phi^2 \)

4. (a) In class, we saw that a finite length \( s \) can be computed as the line integral of \( ds \) which depends on the metric. From this expression, extract the Lagrangian as

\[ L(x, \dot{x}) = g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \]  \hspace{1cm} (1)

and use the Euler-Lagrange equation to obtain the geodesic equation which determines the trajectory of the “shortest curve”. Note the fact that the metric function \( g_{ab} \) depends on \( x \) but not \( \dot{x} \).

(b) Use the geodesic equation to confirm the familiar results that the geodesic is a straight line on a flat plane.

5. In Cartesian coordinates, the line element of Euclidean, 3-dimensional space is

\[ ds^2 = dx^2 + dy^2 + dz^2 \]  \hspace{1cm} (2)
(a) Compute the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ in this coordinate system. What do you conclude about the Riemann tensor?

(b) Express the line element of Eq. 2 in the usual spherical coordinates $(r, \theta, \phi)$.

(c) Compute the Christoffel symbols in spherical coordinates.

(d) Compute the Riemann tensor in spherical coordinates. Had you expected this result?