4.4 Updated reference results for the evolution of parton distributions [1]

In this contribution we update and extend our benchmark tables, first presented in the report of the QCD/SM working group at the 2001 Les Houches workshop [1], for the evolution of parton distributions of hadrons in perturbative QCD. Since then the complete next-to-next-to-leading order (NNLO) splitting functions have been computed [2,3], see also section 4.2. Thus we can now replace the NNLO results of 2001 which were based on the approximate splitting functions of Ref. [4]. Furthermore we now include reference tables for the polarized case treated in neither Ref. [1] nor the earlier study during the 1995/6 HERA workshop [5]. Since the spin-dependent NNLO splitting functions are still unknown, we have to restrict ourselves to the polarized leading-order (LO) and next-to-leading-order (NLO) evolution.

As in Ref. [1], we employ two entirely independent and conceptually different FORTRAN programs. At this point, the x-space code of G.S. is available from the author upon request, while the Mellin-space program of A.V. has been published in Ref. [6]. The results presented below correspond to a direct iterative solution of the N$^{\rm m}$LO evolution equations for the parton distributions $f_p(x,\mu_{\rm f}^2) \equiv p(x,\mu_{\rm f}^2)$, where $p = q_i,\, \bar{q}_i\,, g$ with $i = 1,\, \ldots, N_{\rm f}$,

$$\frac{d\, f_p(x,\mu_{\rm f}^2)}{d \ln \mu_{\rm f}^2} \;=\; \sum_{l=0}^{m} a_{\rm s}^{\,l+1}(\mu_{\rm r}^2) \int_x^1 \frac{dy}{y} \sum_{p'} P_{pp'}^{(l)}\left(\frac{x}{y}, \frac{\mu_{\rm f}^2}{\mu_{\rm r}^2}\right) f_{p'}(y,\mu_{\rm f}^2) \tag{4.51}$$

with the strong coupling, normalized as $a_{\rm s} \equiv \alpha_{\rm s}/(4\pi)$, given in terms of

$$\frac{d\, a_{\rm s}}{d \ln \mu_r^2} \;=\; \beta_{\rm N^mLO}(a_{\rm s}) \;=\; -\sum_{l=0}^{m} a_{\rm s}^{\,l+2}\, \beta_l \tag{4.52}$$

with $\beta_0 = 11 - 2/3\, N_{\rm f}$ etc. $\mu_{\rm r}$ and $\mu_{\rm f}$ represent the renormalization and mass-factorization scales in the $\overline{\rm MS}$ scheme. The reader is referred to Refs. [1,6] for the scale dependence of the splitting functions $P^{(l)}$ and a further discussion of our solutions of Eqs. (4.51) and (4.52).

For the unpolarized case we retain the initial conditions as set up at the Les Houches meeting: The evolution is started at

$$\mu_{\rm f,0}^2 \;=\; 2\ {\rm GeV}^2 \;. \tag{4.53}$$

Roughly along the lines of the CTEQ5M parametrization [7], the input distributions are chosen as

$$\begin{aligned}
x u_{\rm v}(x,\mu_{\rm f,0}^2) &= 5.107200\, x^{0.8}\, (1-x)^3 \\
x d_{\rm v}(x,\mu_{\rm f,0}^2) &= 3.064320\, x^{0.8}\, (1-x)^4 \\
x g\,(x,\mu_{\rm f,0}^2) &= 1.700000\, x^{-0.1} (1-x)^5 \\
x \bar{d}\,(x,\mu_{\rm f,0}^2) &= .1939875\, x^{-0.1} (1-x)^6 \\
x \bar{u}\,(x,\mu_{\rm f,0}^2) &= (1-x)\, x\bar{d}\,(x,\mu_{\rm f,0}^2) \\
x s\,(x,\mu_{\rm f,0}^2) &= x\bar{s}\,(x,\mu_{\rm f,0}^2) \;=\; 0.2\, x(\bar{u}+\bar{d}\,)(x,\mu_{\rm f,0}^2)
\end{aligned} \tag{4.54}$$

where, as usual, $q_{i,\rm v} \equiv q_i - \bar{q}_i$. The running couplings are specified by Eq. (4.52) and

$$\alpha_{\rm s}(\mu_{\rm r}^2 = 2\ {\rm GeV}^2) \;=\; 0.35 \;. \tag{4.55}$$

For simplicity initial conditions (4.54) and (4.55) are employed regardless of the order of the evolution and the (fixed) ratio of the renormalization and factorization scales.

For the evolution with a fixed number $N_{\rm f} > 3$ of quark flavours the quark distributions not specified in Eq. (4.54) are assumed to vanish at $\mu_{\rm f,0}^2$, and Eq. (4.55) is understood to refer to the chosen

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value of $N_{\rm f}$. For the evolution with a variable $N_{\rm f} = 3 \ldots 6$, Eqs. (4.53) and (4.54) always refer to three flavours. $N_{\rm f}$ is then increased by one unit at the heavy-quark pole masses taken as

$$m_{\rm c} \,=\, \mu_{\rm f,0}\,, \quad m_{\rm b} \,=\, 4.5\,{\rm GeV}^2\,, \quad m_{\rm t} \,=\, 175\,{\rm GeV}^2\,, \tag{4.56}$$

i.e., Eqs. (4.51) and (4.52) are solved for a fixed number of flavours between these thresholds, and the respective matching conditions are invoked at $\mu_{\rm f}^2 = m_{\rm h}^2$, $h = c$, b, t. The matching conditions for the unpolarized parton distributions have been derived at NNLO in Ref. [8], and were first implemented in an evolution program in Ref. [9]. Note that, while the parton distributions are continuous up to NLO due to our choice of the matching scales, $\alpha_{\rm s}$ is discontinuous at these flavour thresholds already at this order for $\mu_{\rm r} \neq \mu_{\rm f}$, see Refs. [10, 11]. Again the reader is referred to Refs. [1, 6] for more details.

Since the exact NNLO splitting functions $P^{(2)}$ are rather lengthy and not directly suitable for use in a Mellin-space program (see, however, Ref. [12]), the reference tables shown below have been computed using the parametrizations (4.22) – (4.24) of Ref. [2] and (4.32) – (4.35) of Ref. [3]. Likewise, the operator matrix element $\widetilde{A}_{\rm hg}^{\rm S,2}$ entering the NNLO flavour matching is taken from Eq. (3.5) of Ref. [6]. The relative error made by using the parametrized splitting functions is illustrated in Fig. 1. It is generally well below 10^{-4}, except for the very small sea quark distributions at very large x.

Eqs. (4.53), (4.55) and (4.56) are used for the (longitudinally) polarized case as well, where Eq. (4.54) replaced by the sufficiently realistic toy input [6]

$$\begin{aligned}
x u_{\rm v} &= +1.3\, x^{0.7}\, (1-x)^3\, (1+3x) \\
x d_{\rm v} &= -0.5\, x^{0.7}\, (1-x)^4\, (1+4x) \\
x g &= +1.5\, x^{0.5}\, (1-x)^5 \\
x \bar{d} &= x \bar{u} \,=\, -0.05\, x^{0.3}\, (1-x)^7 \\
x s &= x \bar{s} \,=\, +0.5\, x \bar{d}\ .
\end{aligned} \tag{4.57}$$

As Eq. (4.54) in the unpolarized case, this input is employed regardless of the order of the evolution.

As in Ref. [1], we have compared the results of our two evolution programs, under the conditions specified above, at 500 x-$\mu_{\rm f}^2$ points covering the range $10^{-8} \leq x \leq 0.9$ and $2\ {\rm GeV}^2 \leq \mu_{\rm f}^2 \leq 10^6\ {\rm GeV}^2$. A representative subset of our results at $\mu_{\rm f}^2 = 10^4\ {\rm GeV}^4$, a scale relevant to high-E_T jets and close to $m_{\rm W}^2$, $m_{\rm Z}^2$ and, possibly, $m_{\rm Higgs}^2$, is presented in Tables 1 – 5. These results are given in terms of the valence distributions, defined below Eq. (4.54), $L_\pm \equiv \bar{d} \pm \bar{u}$, and the quark-antiquark sums $q_+ \equiv q - \bar{q}$ for $q = s$, c and, for the variable-$N_{\rm f}$ case, b.

For compactness an abbreviated notation is employed throughout the tables, i.e., all numbers $a \cdot 10^b$ are written as a^b. In the vast majority of the x-$\mu_{\rm f}^2$ points our results are found to agree to all five figures displayed, except for the tiny NLO and NNLO sea-quark distributions at $x = 0.9$, in the tables. Entries where the residual offsets between our programs lead to a different fifth digit after rounding are indicated by the subscript '$*$'. In these cases the number with the smaller modulus is given in the tables.

The approximate splitting functions [4], as mentioned above employed in the previous version [1] of our reference tables, have been used in (global) NNLO fits of the unpolarized parton distributions [13, 14], which in turn have been widely employed for obtaining NNLO cross sections, in particular for W and Higgs production. The effect of replacing the approximate results by the full splitting functions [2, 3] is illustrated in Figure 2. Especially at scales relevant to the above-mentioned processes, the previous approximations introduce an error of less than 0.2% for $x \gtrsim 10^{-3}$, and less than 1% even down to $x \simeq 10^{-5}$. Consequently the splitting-function approximations used for the evolution the parton distributions of Refs. [13, 14] are confirmed to a sufficient accuracy for high-scale processes at the LHC.

The unchanged unpolarized LO and NLO reference tables of Ref. [1] are not repeated here. Note that the one digit of the first (FFN) $\alpha_{\rm s}$ value was mistyped in the header of Table 1 in that report [2], the correct value can be found in Table 3 below.

[2] We thank H. Böttcher and J. Blümlein for pointing this out to us.

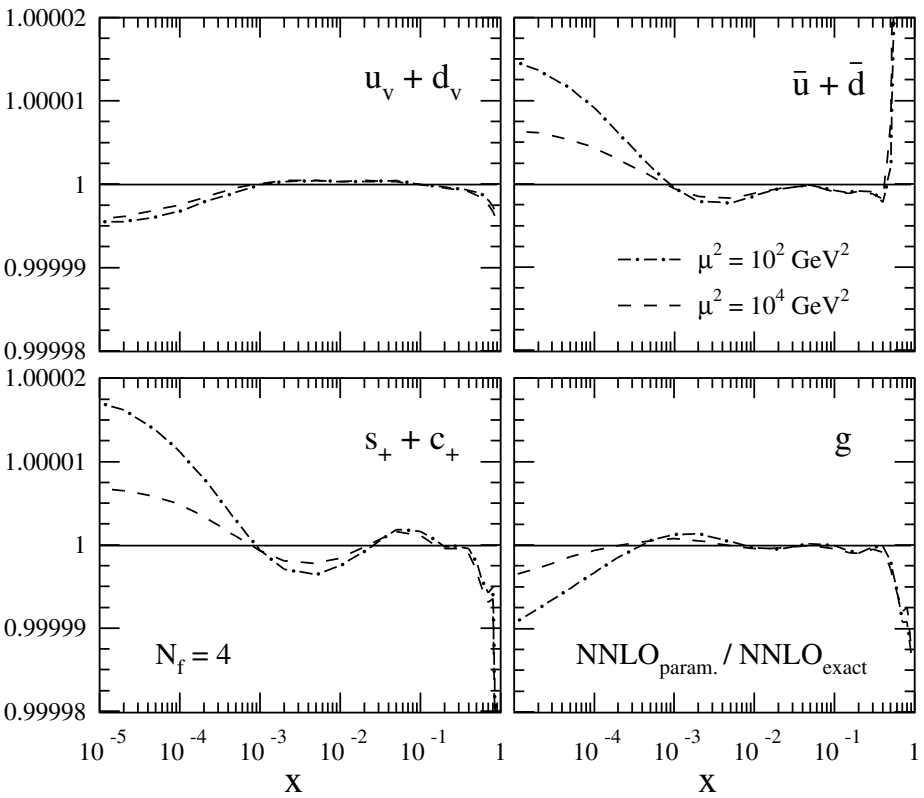

Fig. 1: Relative effects of using the parametrized three-loop splitting functions of Refs. [2, 3], instead of the exact expressions from the same source, on the NNLO evolution for the input (4.53) – (4.55) at two representative values of $\mu = \mu_r = \mu_f$.

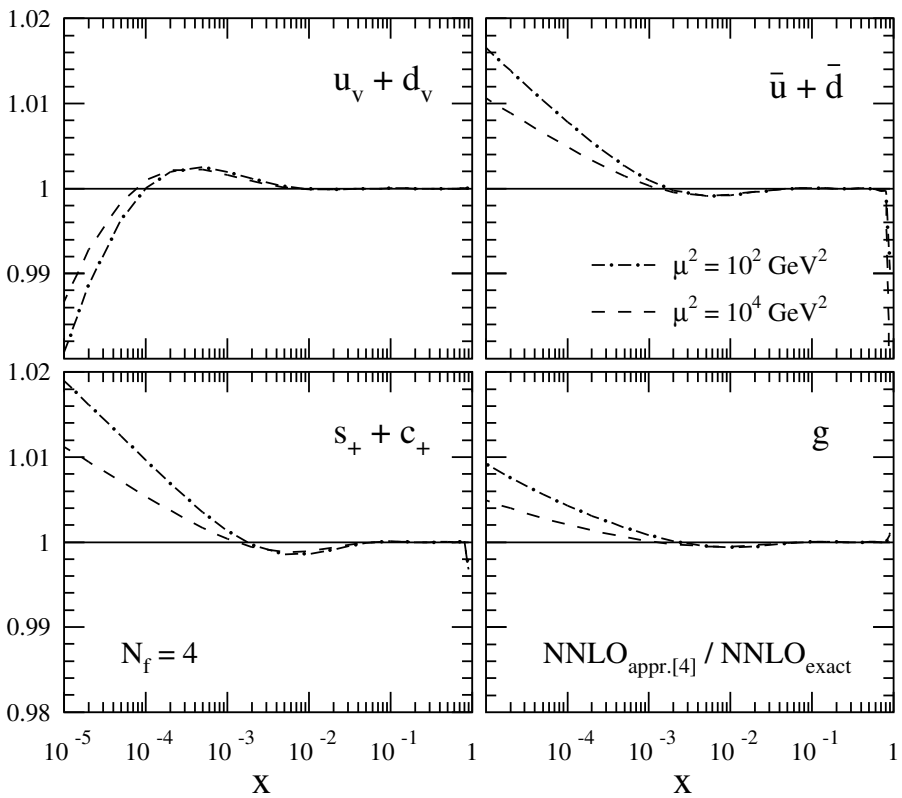

Fig. 2: Relative errors made by using the previous average approximations [4] for the three-loop splitting functions (used, e.g., in Refs. [13, 14]) instead of the full results [2, 3], on the NNLO evolution of the input (4.53) – (4.55) at $\mu_r = \mu_f$.

Table 1: Reference results for the $N_{\rm f} = 4$ next-next-to-leading-order evolution for the initial conditions (4.53) – (4.55). The corresponding value of the strong coupling is $\alpha_{\rm s}(\mu_{\rm r}^2 = 10^4\ {\rm GeV}^2) = 0.110141$. The valence distributions s_v and c_v are equal for the input (4.54). The notation is explained below Eq. (4.54) and in the paragraph below Eq. (4.57).

NNLO, $N_{\rm f} = 4$, $\mu_{\rm f}^2 = 10^4\ {\rm GeV}^2$								
x	xu_v	xd_v	xL_-	$2xL_+$	xs_v	xs_+	xc_+	xg
$\mu_{\rm r}^2 = \mu_{\rm f}^2$								
10^{-7}	1.5287^{-4}	1.0244^{-4}	5.7018^{-6}	1.3190^{+2}	3.1437^{-5}	6.4877^{+1}	6.4161^{+1}	9.9763^{+2}
10^{-6}	6.9176^{-4}	4.4284^{-4}	2.5410^{-5}	6.8499^{+1}	9.4279^{-5}	3.3397^{+1}	3.2828^{+1}	4.9124^{+2}
10^{-5}	3.0981^{-3}	1.8974^{-3}	1.0719^{-4}	3.3471^{+1}	2.2790^{-4}	1.6059^{+1}	1.5607^{+1}	2.2297^{+2}
10^{-4}	1.3722^{-2}	8.1019^{-3}	4.2558^{-4}	1.5204^{+1}	3.6644^{-4}	7.0670^{+0}	6.7097^{+0}	9.0668^{+1}
10^{-3}	5.9160^{-2}	3.4050^{-2}	1.6008^{-3}	6.3230^{+0}	1.4479^{-4}	2.7474^{+0}	2.4704^{+0}	3.1349^{+1}
10^{-2}	2.3078^{-1}	1.2919^{-1}	5.5688^{-3}	2.2752^{+0}	-5.7311^{-4}	8.5502^{-1}	6.6623^{-1}	8.1381^{+0}
0.1	5.5177^{-1}	2.7165^{-1}	1.0023^{-2}	3.9019^{-1}	-3.0627^{-4}	1.1386^{-1}	5.9773^{-2}	9.0563^{-1}
0.3	3.5071^{-1}	1.3025^{-1}	3.0098^{-3}	3.5358^{-2}	-3.1891^{-5}	9.0480^{-3}	3.3061^{-3}	8.4186^{-2}
0.5	1.2117^{-1}	3.1528^{-2}	3.7742^{-4}	2.3867^{-3}	-2.7215^{-6}	5.7965^{-4}	1.7170^{-4}	8.1126^{-3}
0.7	2.0077^{-2}	3.0886^{-3}	1.3434^{-5}	5.4244^{-5}	-1.0106^{-7}	1.2936^{-5}	3.5304^{-6}	3.8948^{-4}
0.9	3.5111^{-4}	1.7783^{-5}	8.651^{-9}	2.695^{-8}	-1.476^{-10}	7.132^{-9}	2.990^{-9}	1.2136^{-6}
$\mu_{\rm r}^2 = 2\,\mu_{\rm f}^2$								
10^{-7}	1.3416^{-4}	8.7497^{-5}	4.9751^{-6}	1.3020^{+2}	2.1524^{-5}	6.4025^{+1}	6.3308^{+1}	1.0210^{+3}
10^{-6}	6.2804^{-4}	3.9406^{-4}	2.2443^{-5}	6.6914^{+1}	6.5149^{-5}	3.2602^{+1}	3.2032^{+1}	4.9626^{+2}
10^{-5}	2.9032^{-3}	1.7575^{-3}	9.6205^{-5}	3.2497^{+1}	1.5858^{-4}	1.5570^{+1}	1.5118^{+1}	2.2307^{+2}
10^{-4}	1.3206^{-2}	7.7673^{-3}	3.9093^{-4}	1.4751^{+1}	2.5665^{-4}	6.8388^{+0}	6.4807^{+0}	9.0162^{+1}
10^{-3}	5.8047^{-2}	3.3434^{-2}	1.5180^{-3}	6.1703^{+0}	1.0388^{-4}	2.6695^{+0}	2.3917^{+0}	3.1114^{+1}
10^{-2}	2.2930^{-1}	1.2857^{-1}	5.4626^{-3}	2.2492^{+0}	-3.9979^{-4}	8.4058^{-1}	6.5087^{-1}	8.0993^{+0}
0.1	5.5428^{-1}	2.7326^{-1}	1.0072^{-2}	3.9297^{-1}	-2.1594^{-4}	1.1439^{-1}	5.9713^{-2}	9.0851^{-1}
0.3	3.5501^{-1}	1.3205^{-1}	3.0557^{-3}	3.6008^{-2}	-2.2632^{-5}	9.2227^{-3}	3.3771^{-3}	8.5022^{-2}
0.5	1.2340^{-1}	3.2166^{-2}	3.8590^{-4}	2.4459^{-3}	-1.9420^{-6}	5.9487^{-4}	1.7699^{-4}	8.2293^{-3}
0.7	2.0597^{-2}	3.1751^{-3}	1.3849^{-5}	5.5722^{-5}	-7.2616^{-8}	1.3244^{-5}	3.5361^{-6}	3.9687^{-4}
0.9	3.6527^{-4}	1.8544^{-5}	9.050^{-9}	2.663^{-8}	-1.075^{-10}	6.713^{-9}	2.377^{-9}	1.2489^{-6}
$\mu_{\rm r}^2 = 1/2\,\mu_{\rm f}^2$								
10^{-7}	1.7912^{-4}	1.2521^{-4}	6.4933^{-6}_{*}	1.2714^{+2}	4.9649^{-5}	6.2498^{+1}	6.1784^{+1}	9.2473^{+2}
10^{-6}	7.7377^{-4}	5.1222^{-4}	2.8719^{-5}	6.7701^{+1}	1.4743^{-4}	3.2999^{+1}	3.2432^{+1}	4.6863^{+2}
10^{-5}	3.3184^{-3}	2.0760^{-3}	1.1977^{-4}	3.3644^{+1}	3.5445^{-4}	1.6147^{+1}	1.5696^{+1}	2.1747^{+2}
10^{-4}	1.4184^{-2}	8.4455^{-3}	4.6630^{-4}	1.5408^{+1}	5.6829^{-4}	7.1705^{+0}	6.8139^{+0}	8.9820^{+1}_{*}
10^{-3}	5.9793^{-2}	3.4418^{-2}	1.6996^{-3}	6.4042^{+0}	2.2278^{-4}	2.7892^{+0}	2.5128^{+0}	3.1336^{+1}
10^{-2}	2.3106^{-1}	1.2914^{-1}	5.7016^{-3}	2.2876^{+0}	-8.9125^{-4}	8.6205^{-1}	6.7377^{-1}	8.1589^{+0}
0.1	5.5039^{-1}	2.7075^{-1}	1.0031^{-2}	3.8850^{-1}	-4.7466^{-4}	1.1332^{-1}	5.9489^{-2}	9.0795^{-1}
0.3	3.4890^{-1}	1.2949^{-1}	2.9943^{-3}	3.5090^{-2}	-4.9304^{-5}	8.9667^{-3}	3.2670^{-3}	8.4309^{-2}
0.5	1.2026^{-1}	3.1269^{-2}	3.7428^{-4}	2.3729^{-3}	-4.1981^{-6}	5.7783^{-4}	1.7390^{-4}	8.1099^{-3}_{*}
0.7	1.9867^{-2}	3.0534^{-3}	1.3273^{-5}	5.4635^{-5}	-1.5541^{-7}	1.3275^{-5}	3.9930^{-6}	3.8824^{-4}
0.9	3.4524^{-4}	1.7466^{-5}	8.489^{-9}	3.030^{-8}	-2.255^{-10}	8.863^{-9}	4.803^{-9}	1.2026^{-6}

Table 2: As Table 1, but for the variable-$N_{\rm f}$ evolution using the flavour matching conditions of Ref. [8,10,11]. The corresponding values for the strong coupling $\alpha_{\rm s}(\mu_{\rm r}^2 = 10^4\ {\rm GeV}^2)$ are given by 0.115818, 0.115605 and 0.115410 for $\mu_{\rm r}^2/\mu_{\rm f}^2 = 0.5$, 1 and 2, respectively. For brevity the small, but non-vanishing valence distributions s_v, c_v and b_v are not displayed.

NNLO, $N_{\rm f} = 3 \ldots 5$, $\mu_{\rm f}^2 = 10^4\ {\rm GeV}^2$								
x	xu_v	xd_v	xL_-	$2xL_+$	xs_+	xc_+	xb_+	xg
$\mu_{\rm r}^2 = \mu_{\rm f}^2$								
10^{-7}	1.5978^{-4}	1.0699^{-5}	6.0090^{-6}	1.3916^{+2}	6.8509^{+1}	6.6929^{+1}	5.7438^{+1}	9.9694^{+3}
10^{-6}	7.1787^{-4}	4.5929^{-4}	2.6569^{-5}	7.1710^{+1}	3.5003^{+1}	3.3849^{+1}	2.8332^{+1}	4.8817^{+2}
10^{-5}	3.1907^{-3}	1.9532^{-3}	1.1116^{-4}	3.4732^{+1}	1.6690^{+1}	1.5875^{+1}	1.2896^{+1}	2.2012^{+2}
10^{-4}	1.4023^{-2}	8.2749^{-3}	4.3744^{-4}	1.5617^{+1}	7.2747^{+0}	6.7244^{+0}	5.2597^{+0}	8.8804^{+1}
10^{-3}	6.0019^{-2}	3.4519^{-2}	1.6296^{-3}	6.4173^{+0}	2.7954^{+0}	2.4494^{+0}	1.8139^{+0}	3.0404^{+1}
10^{-2}	2.3244^{-1}	1.3000^{-1}	5.6100^{-3}	2.2778^{+0}	8.5749^{-1}	6.6746^{-1}	4.5073^{-1}	7.7912^{+0}
0.1	5.4993^{-1}	2.7035^{-1}	9.9596^{-3}	3.8526^{-1}	1.1230^{-1}	6.4466^{-2}	3.7280^{-2}	8.5266^{-1}
0.3	3.4622^{-1}	1.2833^{-1}	2.9572^{-3}	3.4600^{-2}	8.8410^{-3}	4.0134^{-3}	2.1047^{-3}	7.8898^{-2}
0.5	1.1868^{-1}	3.0811^{-2}	3.6760^{-4}	2.3198^{-3}	5.6309^{-4}	2.3752^{-4}	1.2004^{-4}	7.6398^{-3}
0.7	1.9486^{-2}	2.9901^{-3}	1.2957^{-5}	5.2352^{-5}	1.2504^{-5}	5.6038^{-6}	2.8888^{-6}	3.7080^{-4}
0.9	3.3522^{-4}	1.6933^{-5}	8.209^{-9}	2.574^{-8}	6.856^{-9}	4.337^{-9}	2.679^{-9}	1.1721^{-6}
$\mu_{\rm r}^2 = 2\,\mu_{\rm f}^2$								
10^{-7}	1.3950^{-4}	9.0954^{-5}	5.2113^{-6}	1.3549^{+2}	6.6672^{+1}	6.5348^{+1}	5.6851^{+1}	1.0084^{+3}
10^{-6}	6.4865^{-4}	4.0691^{-4}	2.3344^{-5}	6.9214^{+1}	3.3753^{+1}	3.2772^{+1}	2.7818^{+1}	4.8816^{+2}
10^{-5}	2.9777^{-3}	1.8020^{-3}	9.9329^{-5}	3.3385^{+1}	1.6015^{+1}	1.5306^{+1}	1.2601^{+1}	2.1838^{+2}
10^{-4}	1.3452^{-2}	7.9078^{-3}	4.0036^{-4}	1.5035^{+1}	6.9818^{+0}	6.4880^{+0}	5.1327^{+0}	8.7550^{+1}
10^{-3}	5.8746^{-2}	3.3815^{-2}	1.5411^{-3}	6.2321^{+0}	2.7012^{+0}	2.3747^{+0}	1.7742^{+0}	3.0060^{+1}
10^{-2}	2.3063^{-1}	1.2923^{-1}	5.4954^{-3}	2.2490^{+0}	8.4141^{-1}	6.5083^{-1}	4.4354^{-1}	7.7495^{+0}
0.1	5.5279^{-1}	2.7222^{-1}	1.0021^{-2}	3.8897^{-1}	1.1312^{-1}	6.2917^{-2}	3.7048^{-2}	8.5897^{-1}
0.3	3.5141^{-1}	1.3051^{-1}	3.0134^{-3}	3.5398^{-2}	9.0559^{-3}	3.8727^{-3}	2.0993^{-3}	8.0226^{-2}
0.5	1.2140^{-1}	3.1590^{-2}	3.7799^{-4}	2.3919^{-3}	5.8148^{-4}	2.2376^{-4}	1.1918^{-4}	7.8098^{-3}
0.7	2.0120^{-2}	3.0955^{-3}	1.3462^{-5}	5.4194^{-5}	1.2896^{-5}	5.0329^{-6}	2.8153^{-6}	3.8099^{-4}
0.9	3.5230^{-4}	1.7849^{-5}	8.687^{-9}	2.568^{-8}	6.513^{-9}	3.390^{-9}	2.407^{-9}	1.2188^{-6}
$\mu_{\rm r}^2 = 1/2\,\mu_{\rm f}^2$								
10^{-7}	1.8906^{-4}	1.3200^{-4}	6.9268^{-6}	1.3739^{+2}	6.7627^{+1}	6.5548^{+1}	5.5295^{+1}	9.4403^{+2}
10^{-6}	8.1001^{-4}	5.3574^{-4}	3.0345^{-5}	7.2374^{+1}	3.5337^{+1}	3.3846^{+1}	2.7870^{+1}	4.7444^{+2}
10^{-5}	3.4428^{-3}	2.1524^{-3}	1.2531^{-4}	3.5529^{+1}	1.7091^{+1}	1.6065^{+1}	1.2883^{+1}	2.1802^{+2}
10^{-4}	1.4580^{-2}	8.6744^{-3}	4.8276^{-4}	1.6042^{+1}	7.4886^{+0}	6.8276^{+0}	5.3044^{+0}	8.9013^{+1}
10^{-3}	6.0912^{-2}	3.5030^{-2}	1.7393^{-3}	6.5544^{+0}	2.8656^{+0}	2.4802^{+0}	1.8362^{+0}	3.0617^{+1}
10^{-2}	2.3327^{-1}	1.3022^{-1}	5.7588^{-3}	2.2949^{+0}	8.6723^{-1}	6.7688^{-1}	4.5597^{-1}	7.8243^{+0}_{*}
0.1	5.4798^{-1}	2.6905^{-1}	9.9470^{-3}	3.8192^{-1}	1.1124^{-1}	6.7091^{-2}	3.7698^{-2}	8.4908^{-1}
0.3	3.4291^{-1}	1.2693^{-1}	2.9239^{-3}	3.4069^{-2}	8.6867^{-3}	4.3924^{-3}	2.1435^{-3}	7.8109^{-2}
0.5	1.1694^{-1}	3.0310^{-2}	3.6112^{-4}	2.2828^{-3}	5.5537^{-4}	2.7744^{-4}	1.2416^{-4}	7.5371^{-3}
0.7	1.9076^{-2}	2.9217^{-3}	1.2635^{-5}	5.2061^{-5}	1.2677^{-5}	7.2083^{-6}	3.0908^{-6}	3.6441^{-4}
0.9	3.2404^{-4}	1.6333^{-5}	7.900^{-9}	2.850^{-8}	8.407^{-9}	6.795^{-9}	3.205^{-9}	1.1411^{-6}

Table 3: Reference results for the $N_{\mathrm{f}} = 4$ (FFN) and the variable-N_{f} (VFN) polarized leading-order evolution of the initial distributions (4.57), shown together with these boundary conditions. The respective values for $\alpha_{\mathrm{s}}(\mu_{\mathrm{r}}^2 = \mu_{\mathrm{f}}^2 = 10^4\ \mathrm{GeV}^2)$ read 0.117574 (FFN) and 0.122306 (VFN). The notation is the same as for the unpolarized case.

x	xu_v	$-xd_v$	$-xL_-$	$-2xL_+$	xs_+	xc_+	xb_+	xg
Pol. input, $\mu_{\mathrm{f}}^2 = 2\ \mathrm{GeV}^2$								
10^{-7}	1.6366^{-5}	6.2946^{-6}	7.9433^{-5}	1.5887^{-3}	-3.9716^{-4}	0.0^{+0}	0.0^{+0}	4.7434^{-4}
10^{-6}	8.2024^{-5}	3.1548^{-5}	1.5849^{-4}	3.1698^{-3}	-7.9244^{-4}	0.0^{+0}	0.0^{+0}	1.5000^{-3}
10^{-5}	4.1110^{-4}	1.5811^{-4}	3.1621^{-4}	6.3241^{-3}	-1.5810^{-3}	0.0^{+0}	0.0^{+0}	4.7432^{-3}
10^{-4}	2.0604^{-3}	7.9245^{-4}	6.3052^{-4}	1.2610^{-2}	-3.1526^{-3}	0.0^{+0}	0.0^{+0}	1.4993^{-2}
10^{-3}	1.0326^{-2}	3.9716^{-3}	1.2501^{-3}	2.5003^{-2}	-6.2507^{-3}	0.0^{+0}	0.0^{+0}	4.7197^{-2}
10^{-2}	5.1723^{-2}	1.9886^{-2}	2.3412^{-3}	4.6825^{-2}	-1.1706^{-2}	0.0^{+0}	0.0^{+0}	1.4265^{-1}
0.1	2.4582^{-1}	9.1636^{-2}	2.3972^{-3}	4.7943^{-2}	-1.1986^{-2}	0.0^{+0}	0.0^{+0}	2.8009^{-1}
0.3	3.6473^{-1}	1.1370^{-1}	5.7388^{-4}	1.1478^{-2}	-2.8694^{-3}	0.0^{+0}	0.0^{+0}	1.3808^{-1}
0.5	2.5008^{-1}	5.7710^{-2}	6.3457^{-5}	1.2691^{-3}	-3.1729^{-4}	0.0^{+0}	0.0^{+0}	3.3146^{-2}
0.7	8.4769^{-2}	1.1990^{-2}	1.9651^{-6}	3.9301^{-5}	-9.8254^{-6}	0.0^{+0}	0.0^{+0}	3.0496^{-3}
0.9	4.4680^{-3}	2.1365^{-4}	9.689^{-10}	1.9378^{-8}	-4.8444^{-9}	0.0^{+0}	0.0^{+0}	1.4230^{-5}
LO, $N_{\mathrm{f}} = 4$, $\mu_{\mathrm{f}}^2 = 10^4\ \mathrm{GeV}^2$								
10^{-7}	4.8350_{*}^{-5}	1.8556^{-5}	1.0385^{-4}	3.5124^{-3}	-1.2370^{-3}	-7.1774^{-4}	0.0^{+0}	1.4116^{-2}
10^{-6}	2.3504^{-4}	9.0090^{-5}	2.0700^{-4}	7.7716^{-3}	-2.8508^{-3}	-1.8158^{-3}	0.0^{+0}	4.2163^{-2}
10^{-5}	1.1220^{-3}	4.2916^{-4}	4.1147^{-4}	1.6007^{-2}	-5.9463^{-3}	-3.8889^{-3}	0.0^{+0}	1.0922^{-1}
10^{-4}	5.1990^{-3}	1.9818^{-3}	8.0948^{-4}	2.8757^{-2}	-1.0331^{-2}	-6.2836^{-3}	0.0^{+0}	2.4069^{-1}
10^{-3}	2.2900^{-2}	8.6763^{-3}	1.5309^{-3}	4.0166^{-2}	-1.2428^{-2}	-4.7739^{-3}	0.0^{+0}	4.2181^{-1}
10^{-2}	9.1489^{-2}	3.4200^{-2}	2.4502^{-3}	3.3928^{-2}	-4.7126^{-3}	7.5385^{-3}	0.0^{+0}	4.9485^{-1}
0.1	2.6494^{-1}	9.1898^{-2}	1.5309^{-3}	8.5427^{-3}	3.3830^{-3}	1.1037^{-2}	0.0^{+0}	2.0503^{-1}
0.3	2.2668^{-1}	6.2946^{-2}	2.1104^{-4}	6.6698^{-4}	7.2173^{-4}	1.7769^{-3}	0.0^{+0}	3.3980^{-2}
0.5	9.7647^{-2}	1.9652^{-2}	1.4789^{-5}	-1.8850^{-5}	8.3371^{-5}	1.5732^{-4}	0.0^{+0}	4.3802^{-3}
0.7	1.9545^{-2}	2.3809^{-3}	2.7279^{-7}	-4.1807^{-6}	3.4543^{-6}	4.8183^{-6}	0.0^{+0}	2.6355^{-4}
0.9	4.1768^{-4}	1.7059^{-5}	5.494^{-11}	-7.6712^{-9}	4.1103^{-9}	4.3850^{-9}	0.0^{+0}	9.8421^{-7}
LO, $N_{\mathrm{f}} = 3 \ldots 5$, $\mu_{\mathrm{f}}^2 = 10^4\ \mathrm{GeV}^2$								
10^{-7}	4.9026^{-5}	1.8815^{-5}	1.0422^{-4}	3.5315^{-3}	-1.2447^{-3}	-7.2356^{-4}	-6.2276^{-4}	1.3726^{-2}
10^{-6}	2.3818^{-4}	9.1286^{-5}	2.0774^{-4}	7.8108^{-3}	-2.8667^{-3}	-1.8280^{-3}	-1.5301^{-3}	4.1011^{-2}
10^{-5}	1.1359^{-3}	4.3445^{-4}	4.1289^{-4}	1.6070^{-2}	-5.9705^{-3}	-3.9060^{-3}	-3.1196^{-3}	1.0615^{-1}
10^{-4}	5.2567^{-3}	2.0035^{-3}	8.1206^{-4}	2.8811^{-2}	-1.0345^{-2}	-6.2849^{-3}	-4.5871^{-3}	2.3343^{-1}
10^{-3}	2.3109^{-2}	8.7537^{-3}	1.5345^{-3}	4.0125^{-2}	-1.2390^{-2}	-4.7174^{-3}	-2.4822^{-3}	4.0743^{-1}
10^{-2}	9.2035^{-2}	3.4391^{-2}	2.4501^{-3}	3.3804^{-2}	-4.6512^{-3}	7.5994^{-3}	6.4665^{-3}	4.7445^{-1}
0.1	2.6478^{-1}	9.1762^{-2}	1.5206^{-3}	8.5181^{-3}	3.3438^{-3}	1.0947^{-2}	6.5223^{-3}	1.9402^{-1}
0.3	2.2495^{-1}	6.2376^{-2}	2.0811^{-4}	6.6195^{-4}	7.0957^{-4}	1.7501^{-3}	9.2045^{-4}	3.1960^{-2}
0.5	9.6318^{-2}	1.9353^{-2}	1.4496^{-5}	-1.8549^{-5}	8.1756^{-5}	1.5424^{-4}	7.8577^{-5}	4.1226^{-3}
0.7	1.9147^{-2}	2.3281^{-3}	2.6556^{-7}	-4.0936^{-6}	3.3746^{-6}	4.7024^{-6}	2.4901^{-6}	2.4888^{-4}
0.9	4.0430^{-4}	1.6480^{-5}	5.285^{-11}	-7.4351^{-9}	3.9818^{-9}	4.2460^{-9}	2.6319^{-9}	9.2939^{-7}

Table 4: Reference results for the polarized next-to-leading-order polarized evolution of the initial distributions (4.57) with $N_{\rm f} = 4$ quark flavours. The corresponding value of the strong coupling is $\alpha_{\rm s}(\mu_{\rm r}^2 = 10^4\ {\rm GeV}^2) = 0.110902$. As in the leading-order case, the valence distributions s_v and c_v vanish for the input (4.57).

Pol. NLO, $N_{\rm f} = 4$, $\mu_{\rm f}^2 = 10^4\ {\rm GeV}^2$							
x	xu_v	xd_v	xL_-	$2xL_+$	xs_+	xc_+	xg
$\mu_{\rm r}^2 = \mu_{\rm f}^2$							
10^{-7}	6.7336^{-5}	-2.5747^{-5}	-1.1434^{-4}	-5.2002^{-3}	-2.0528^{-3}	-1.5034^{-3}	2.6955^{-2}
10^{-6}	3.1280^{-4}	-1.1938^{-4}	-2.3497^{-4}	-1.0725^{-2}	-4.2774^{-3}	-3.1845^{-3}	6.5928^{-2}
10^{-5}	1.4180^{-3}	-5.3982^{-4}	-4.8579^{-4}	-1.9994^{-2}	-7.8594^{-3}	-5.6970^{-3}	1.4414^{-1}
10^{-4}	6.2085^{-3}	-2.3546^{-3}	-9.8473^{-4}	-3.1788^{-2}	-1.1749^{-2}	-7.5376^{-3}	2.7537^{-1}
10^{-3}	2.5741^{-2}	-9.7004^{-3}	-1.8276^{-3}	-3.8222^{-2}	-1.1427^{-2}	-3.6138^{-3}	4.3388^{-1}
10^{-2}	9.6288^{-2}	-3.5778^{-2}	-2.6427^{-3}	-2.6437^{-2}	-1.2328^{-3}	1.0869^{-2}	4.8281^{-1}
0.1	2.5843^{-1}	-8.9093^{-2}	-1.4593^{-3}	-7.5546^{-3}	3.4258^{-3}	1.0639^{-2}	2.0096^{-1}
0.3	2.1248^{-1}	-5.8641^{-2}	-1.9269^{-4}	-1.2210^{-3}	3.5155^{-4}	1.3138^{-3}	3.4126^{-2}
0.5	8.9180^{-2}	-1.7817^{-2}	-1.3125^{-5}	-9.1573^{-5}	1.9823^{-5}	8.5435^{-5}	4.5803^{-3}
0.7	1.7300^{-2}	-2.0885^{-3}	-2.3388^{-7}	-1.9691^{-6}	1.8480^{-7}	1.3541^{-6}	2.9526^{-4}
0.9	3.4726^{-4}	-1.4028^{-5}	-4.407^{-11}	-4.247^{-9}	-1.903^{-9}	-1.683^{-9}	1.2520^{-6}
$\mu_{\rm r}^2 = 2\,\mu_{\rm f}^2$							
10^{-7}	6.1781^{-5}	-2.3641^{-5}	-1.1137^{-4}	-4.6947^{-3}	-1.8092^{-3}	-1.2695^{-3}	2.2530^{-2}
10^{-6}	2.8974^{-4}	-1.1068^{-4}	-2.2755^{-4}	-9.8528^{-3}	-3.8580^{-3}	-2.7838^{-3}	5.7272_{*}^{-2}
10^{-5}	1.3281^{-3}	-5.0612^{-4}	-4.6740^{-4}	-1.8799^{-2}	-7.2908^{-3}	-5.1629^{-3}	1.2975^{-1}
10^{-4}	5.8891^{-3}	-2.2361^{-3}	-9.4412^{-4}	-3.0787^{-2}	-1.1292^{-2}	-7.1363^{-3}	2.5644^{-1}
10^{-3}	2.4777^{-2}	-9.3502^{-3}	-1.7632^{-3}	-3.8610^{-2}	-1.1658^{-2}	-3.9083^{-3}	4.1725^{-1}
10^{-2}	9.4371^{-2}	-3.5129^{-2}	-2.6087^{-3}	-2.8767^{-2}	-2.3430_{*}^{-3}	9.7922_{*}^{-3}	4.7804^{-1}
0.1	2.6008^{-1}	-8.9915^{-2}	-1.4923^{-3}	-8.3806^{-3}	3.1932^{-3}	1.0585^{-2}	2.0495^{-1}
0.3	2.1837^{-1}	-6.0497^{-2}	-2.0143^{-4}	-1.2157^{-3}	3.9810^{-4}	1.4042^{-3}	3.5366^{-2}
0.5	9.3169^{-2}	-1.8699^{-2}	-1.3954^{-5}	-7.9331^{-5}	3.0091^{-5}	9.9849^{-5}	4.7690^{-3}
0.7	1.8423^{-2}	-2.2357^{-3}	-2.5360^{-7}	-1.0062^{-6}	7.6483^{-7}	2.0328^{-6}	3.0796^{-4}
0.9	3.8293^{-4}	-1.5559^{-5}	-4.952^{-11}	-1.955^{-9}	-7.298^{-10}	-4.822^{-10}	1.3247^{-6}
$\mu_{\rm r}^2 = 1/2\,\mu_{\rm f}^2$							
10^{-7}	7.4443^{-5}	-2.8435^{-5}	-1.1815^{-4}	-5.7829^{-3}	-2.3341^{-3}	-1.7739^{-3}	3.2071^{-2}
10^{-6}	3.4143^{-4}	-1.3016^{-4}	-2.4482^{-4}	-1.1668^{-2}	-4.7305^{-3}	-3.6168^{-3}	7.5123^{-2}
10^{-5}	1.5256^{-3}	-5.8002^{-4}	-5.1085^{-4}	-2.1193^{-2}	-8.4295^{-3}	-6.2295_{*}^{-3}	1.5788^{-1}
10^{-4}	6.5726^{-3}	-2.4891^{-3}	-1.0409^{-3}	-3.2697^{-2}	-1.2166^{-2}	-7.8952^{-3}	2.9079^{-1}
10^{-3}	2.6766^{-2}	-1.0070^{-2}	-1.9171^{-3}	-3.7730^{-2}	-1.1160^{-2}	-3.2890_{*}^{-3}	4.4380^{-1}
10^{-2}	9.8073^{-2}	-3.6370^{-2}	-2.6942^{-3}	-2.4056^{-2}	-1.2354_{*}^{-4}	1.1929^{-2}	4.8272^{-1}
0.1	2.5628^{-1}	-8.8133^{-2}	-1.4304^{-3}	-6.9572^{-3}	3.5561^{-3}	1.0604^{-2}	1.9831^{-1}
0.3	2.0709^{-1}	-5.6988^{-2}	-1.8541^{-4}	-1.3308^{-3}	2.5993^{-4}	1.1855^{-3}	3.3524^{-2}
0.5	8.5835^{-2}	-1.7089^{-2}	-1.2463^{-5}	-1.1920^{-4}	2.6972_{*}^{-6}	6.4995^{-5}	4.5044^{-3}
0.7	1.6405^{-2}	-1.9723^{-3}	-2.1859_{*}^{-7}	-3.6817^{-6}	-7.4795_{*}^{-7}	3.4496^{-7}	2.9100^{-4}
0.9	3.2011^{-4}	-1.2870^{-5}	-4.000^{-11}	-8.173^{-9}	-3.886^{-9}	-3.686^{-9}	1.2230^{-6}

Table 5: As Table 4, but for the variable-$N_{\rm f}$ evolution using Eqs. (4.53), (4.54) and (4.57). The corresponding values for the strong coupling $\alpha_{\rm s}(\mu_{\rm r}^2 = 10^4\ {\rm GeV}^2)$ are given by 0.116461, 0.116032 and 0.115663 for $\mu_{\rm r}^2/\mu_{\rm f}^2 = 0.5$, 1 and 2, respectively.

Pol. NLO, $N_{\rm f} = 3 \ldots 5$, $\mu_{\rm f}^2 = 10^4\ {\rm GeV}^2$								
x	xu_v	$-xd_v$	$-xL_-$	$-2xL_+$	xs_+	xc_+	xb_+	xg
$\mu_{\rm r}^2 = \mu_{\rm f}^2$								
10^{-7}	6.8787^{-5}_{*}	2.6297^{-5}	1.1496^{-4}	5.2176^{-3}	-2.0592^{-3}	-1.5076^{-3}	-1.2411^{-3}	2.5681^{-2}
10^{-6}	3.1881^{-4}	1.2165^{-4}	2.3638^{-4}	1.0770^{-2}	-4.2953^{-3}	-3.1979^{-3}	-2.4951^{-3}	6.3021^{-2}
10^{-5}	1.4413^{-3}	5.4856^{-4}	4.8893^{-4}	2.0077^{-2}	-7.8934^{-3}	-5.7228^{-3}	-4.1488^{-3}	1.3809^{-1}
10^{-4}	6.2902^{-3}	2.3849^{-3}	9.9100^{-4}	3.1883^{-2}	-1.1785^{-2}	-7.5596^{-3}	-4.8420^{-3}	2.6411^{-1}
10^{-3}	2.5980^{-2}	9.7872^{-3}	1.8364^{-3}	3.8224^{-2}	-1.1416^{-2}	-3.5879^{-3}	-1.1723^{-3}	4.1601^{-1}
10^{-2}	9.6750^{-2}	3.5935^{-2}	2.6452^{-3}	2.6306^{-2}	-1.1774^{-3}	1.0917^{-2}	8.1196^{-3}	4.6178^{-1}
0.1	2.5807^{-1}	8.8905^{-2}	1.4509^{-3}	7.4778^{-3}	3.4207^{-3}	1.0591^{-2}	6.1480^{-3}	1.9143^{-1}
0.3	2.1104^{-1}	5.8186^{-2}	1.9054^{-4}	1.2026^{-3}	3.4999^{-4}	1.3015^{-3}	7.2795^{-4}	3.2621^{-2}
0.5	8.8199^{-2}	1.7601^{-2}	1.2924^{-5}	8.9668^{-5}	1.9771^{-5}	8.4378^{-5}	5.2125^{-5}	4.4207^{-3}
0.7	1.7027^{-2}	2.0531^{-3}	2.2921^{-7}	1.9243^{-6}	1.8384^{-7}	1.3298^{-6}	1.2157^{-6}	2.8887^{-4}
0.9	3.3898^{-4}	1.3676^{-5}	4.284^{-11}	4.260^{-9}	-1.916^{-9}	-1.701^{-9}	-7.492^{-11}	1.2435^{-6}
$\mu_{\rm r}^2 = 2\,\mu_{\rm f}^2$								
10^{-7}	6.2819^{-5}_{*}	2.4035^{-5}	1.1180^{-4}	4.6896^{-3}	-1.8050^{-3}	-1.2637^{-3}	-1.0544^{-3}	2.1305^{-2}
10^{-6}	2.9408^{-4}	1.1232^{-4}	2.2855^{-4}	9.8538^{-3}	-3.8554^{-3}	-2.7780^{-3}	-2.2077^{-3}	5.4411^{-2}
10^{-5}	1.3450^{-3}	5.1245^{-4}	4.6965^{-4}	1.8815^{-2}	-7.2936^{-3}	-5.1597^{-3}	-3.8359^{-3}	1.2368^{-1}
10^{-4}	5.9485^{-3}	2.2582^{-3}	9.4866^{-4}	3.0816^{-2}	-1.1297^{-2}	-7.1323^{-3}	-4.7404^{-3}	2.4503^{-1}
10^{-3}	2.4951^{-2}	9.4134^{-3}	1.7698^{-3}	3.8618^{-2}	-1.1654^{-2}	-3.8925^{-3}	-1.5608^{-3}	3.9912^{-1}
10^{-2}	9.4706^{-2}	3.5243^{-2}	2.6108^{-3}	2.8761^{-2}	-2.3471^{-3}	9.7827^{-3}	7.5188^{-3}	4.5698^{-1}
0.1	2.5982^{-1}	8.9780^{-2}	1.4862^{-3}	8.3807^{-3}	3.1615^{-3}	1.0522^{-2}	6.1973^{-3}	1.9561^{-1}
0.3	2.1732^{-1}	6.0165^{-2}	1.9984^{-4}	1.2086^{-3}	3.9371^{-4}	1.3919^{-3}	7.6929^{-4}	3.3906^{-2}
0.5	9.2445^{-2}	1.8539^{-2}	1.3804^{-5}	7.8411^{-5}	2.9799^{-5}	9.8805^{-5}	5.7333^{-5}	4.6166^{-3}
0.7	1.8219^{-2}	2.2090^{-3}	2.5004^{-7}_{*}	9.8927^{-7}_{*}	7.5552^{-7}	2.0057^{-6}	1.4438^{-6}	3.0231^{-4}
0.9	3.7653^{-4}	1.5285^{-5}	4.855^{-11}	2.005^{-9}	-7.599^{-10}	-5.171^{-10}	3.809^{-10}	1.3232^{-6}
$\mu_{\rm r}^2 = 1/2\,\mu_{\rm f}^2$								
10^{-7}	7.6699^{-5}	2.9289^{-5}	1.1912^{-4}	5.8548^{-3}	-2.3667^{-3}	-1.8030^{-3}	-1.4521^{-3}	3.1009^{-2}
10^{-6}	3.5067^{-4}	1.3364^{-4}	2.4707^{-4}	1.1806^{-2}	-4.7934^{-3}	-3.6731^{-3}	-2.7846^{-3}	7.2690^{-2}
10^{-5}	1.5611^{-3}	5.9329^{-4}	5.1593^{-4}	2.1406^{-2}	-8.5248^{-3}	-6.3125^{-3}	-4.4072^{-3}	1.5274^{-1}
10^{-4}	6.6957^{-3}	2.5346^{-3}	1.0509^{-3}	3.2903^{-2}	-1.2252^{-2}	-7.9608^{-3}	-4.8402^{-3}	2.8097^{-1}
10^{-3}	2.7125^{-2}	1.0200^{-2}	1.9310^{-3}	3.7698^{-2}	-1.1127^{-2}	-3.2334^{-3}	-7.5827^{-4}	4.2756^{-1}
10^{-2}	9.8758^{-2}	3.6602^{-2}	2.6980^{-3}	2.3675^{-2}	5.1386^{-5}	1.2092^{-2}	8.6053^{-3}	4.6241^{-1}
0.1	2.5572^{-1}	8.7847^{-2}	1.4179^{-3}	6.7523^{-3}	3.5944^{-3}	1.0578^{-2}	6.0904^{-3}	1.8838^{-1}
0.3	2.0497^{-1}	5.6318^{-2}	1.8228^{-4}	1.2965^{-3}	2.6142^{-4}	1.1713^{-3}	6.8941^{-4}	3.1884^{-2}
0.5	8.4404^{-2}	1.6775^{-2}	1.2174^{-5}	1.1604^{-4}	2.8309^{-6}	6.3682^{-5}	4.7009^{-5}	4.3221^{-3}
0.7	1.6013^{-2}	1.9215^{-3}	2.1196^{-7}_{*}	3.6047^{-6}	-7.4260^{-7}	3.1714^{-7}	9.6419^{-7}	2.8268^{-4}
0.9	3.0848^{-4}	1.2377^{-5}	3.829^{-11}	8.129^{-9}	-3.873^{-9}	-3.681^{-9}	-6.816^{-10}	1.2009^{-6}

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