DESY 05-138, SFB/CPP-05-34
DCPT/05/88, IPPP/05/44
NIKHEF 05-013
August 2005

hep-ph/0508055

Three-Loop Results for Quark and Gluon Form Factors

S. Moch[a], J.A.M. Vermaseren[b] and A. Vogt[c]

[a] *Deutsches Elektronensynchrotron DESY*
Platanenallee 6, D–15735 Zeuthen, Germany

[b] *NIKHEF Theory Group*
Kruislaan 409, 1098 SJ Amsterdam, The Netherlands

[c] *IPPP, Department of Physics, University of Durham*
South Road, Durham DH1 3LE, United Kingdom

Abstract

We study the photon-quark-quark and Higgs-gluon-gluon form factors for on-shell massless quarks and gluons in perturbative QCD. Previous third-order results for the quark case are extended by calculating the fermion-loop contributions up to the finite terms in dimensional regularization. For the gluon case the complete set of infrared poles at three loops is derived. Using the exponentiation of the form factor, the latter results are employed to extract a function entering the infrared factorization of general third-order amplitudes. We evaluate the infrared finite absolute ratio of the time-like and space-like gluon form factors up to the fourth order in the strong coupling constant. The result supports previous indications that the perturbative expansion of the Higgs boson production rate at the LHC is under control.

The form factors of quarks and gluons, i.e., the QCD corrections to the qqX and ggX vertices with a colour-neutral particle X, are quantities of considerable phenomenological and theoretical interest. These three-point amplitudes represent important gauge invariant (if infrared divergent) parts of the perturbative corrections to processes of utmost relevance, like the production of lepton pairs and the Higgs boson H at proton colliders. Due to the exponentiation of the form factors [1–3], already the dimensionally regulated pole terms at order α_s^n, combined with appropriate lower-order information, suffice for deriving the finite absolute ratios of the time-like and space-like form factors to order α_s^{n+1}, thus providing valuable information about the size of the higher-order corrections. The form factors are also the simplest amplitudes to which the infrared factorization formulae of Refs. [4, 5] can be applied. Consequently explicit calculations of these quantities lead, through the resulting determinations of otherwise unspecified functions in these formulae, to unambiguous predictions for the pole structure of higher-order amplitudes involving more partons.

In a recent article [6] we have extracted all three-loop pole terms of the electromagnetic form factor $\mathcal{F}^q(\alpha_s, Q^2)$ of on-shell massless quarks from the calculation of the third-order coefficient function in deep-inelastic scattering [7] and extended the resummation of this form factor to the next-to-next-to-leading contributions. In this letter we present the corresponding results for the Hgg gluon form factor $\mathcal{F}^g(\alpha_s, Q^2)$. As a first step towards the complete three-loop computations, we moreover extend the fermion-loop (n_f) parts of $\mathcal{F}^q$ to the order ε^0 in dimensional regularization with $D = 4 - 2\varepsilon$.

Since results on deep-inelastic scattering are our starting point, we first address the space-like form factors. Thus the relevant amplitude for the $\gamma^* qq$ quark case is

$$\Gamma_\mu \,=\, \mathrm{i} e_\mathrm{q}\, (\bar{u}\gamma_\mu u)\, \mathcal{F}^q(\alpha_s, Q^2)\,, \tag{1}$$

where e_q represents the quark charge and Q^2 the virtuality of the photon. The corresponding Hgg vertex is an effective interaction in the limit of a heavy top quark,

$$\mathcal{L}_\mathrm{eff} \,=\, -\frac{1}{4}\, C_H\, H\, G^a_{\mu\nu} G^{a,\mu\nu}\,. \tag{2}$$

Here $G^a_{\mu\nu}$ denotes the gluon field strength tensor, and the prefactor C_H is determined by the top-quark loop including all QCD corrections. Neither these corrections nor the renormalization of $G^a_{\mu\nu} G^{a,\mu\nu}$ are relevant to the present study, hence we refer the reader to Refs. [8, 9] for details.

Besides the electromagnetic case utilized in Ref. [6], we have also calculated the third-order corrections to deep-inelastic scattering by exchange of a scalar interacting with gluons according to Eq. (2). The inclusion of this (experimentally entirely irrelevant) process was required for obtaining the full set of splitting functions governing the evolution of the parton distributions at the next-to-next-to-leading order [10, 11]. All terms up to ε^0 were consistently kept also in this part of the calculation. Hence we can repeat the procedure discussed in Ref. [6] for $\mathcal{F}^g(\alpha_s, Q^2)$, and derive the complete set of pole terms, $\varepsilon^{-6} \ldots \varepsilon^{-1}$, at three loops.

In order to access the finite (ε^0) parts of the form factors in this approach, the three-loop computations of deep-inelastic scattering need to be extended to order ε. This is not an option for the complete calculation with its more than 100 000 tabulated integrals. However, only very few genuine three-loop diagrams remain if we confine ourselves to the n_f contributions to the quark form factor (these diagrams can be found in Ref. [12]). As neither of those diagrams is of a particularly difficult type, we were able to carry out the required extensions for this subset.

We present our results in terms of the expansion coefficients $\mathcal{F}_l^p$ of the bare (unrenormalized) form factors ($p = q, g$)

$$\mathcal{F}_{\mathrm{b}}^{p}(\alpha_{\mathrm{s}}^{\mathrm{b}}, Q^2) = 1 + \sum_{l=1}^{\infty} \left(a_{\mathrm{s}}^{\mathrm{b}}\right)^l \left(\frac{Q^2}{\mu^2}\right)^{-l\varepsilon} \mathcal{F}_l^p\,, \quad a_{\mathrm{s}} \equiv \frac{\alpha_{\mathrm{s}}}{4\pi}\,. \tag{3}$$

The fermionic contributions to the quark form factor are given by

$$\mathcal{F}^q_{1,n_f} = 0\,, \tag{4}$$

$$\begin{aligned}
\mathcal{F}^q_{2,n_f} = {}& C_F n_f \bigg\{ \frac{1}{3\varepsilon^3} + \frac{14}{9\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{353}{54} + \frac{1}{3}\zeta_2\right) + \frac{7541}{324} + \frac{14}{9}\zeta_2 - \frac{26}{9}\zeta_3 \\
&+ \varepsilon\left(\frac{150125}{1944} + \frac{353}{54}\zeta_2 - \frac{364}{27}\zeta_3 - \frac{41}{30}\zeta_2^2\right) \\
&+ \varepsilon^2\left(\frac{2877653}{11664} - \frac{26}{9}\zeta_2\zeta_3 + \frac{7541}{324}\zeta_2 - \frac{4589}{81}\zeta_3 - \frac{287}{45}\zeta_2^2 - \frac{242}{15}\zeta_5\right)\bigg\}\,,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathcal{F}^q_{3,n_f} = {}& C_F^2 n_f \bigg\{ -\frac{2}{3\varepsilon^5} - \frac{37}{9\varepsilon^4} + \frac{1}{\varepsilon^3}\left(-\frac{545}{27} - \frac{1}{3}\zeta_2\right) + \frac{1}{\varepsilon^2}\bigg(-\frac{6499}{81} - \frac{133}{18}\zeta_2 \\
&+ \frac{146}{9}\zeta_3\bigg) + \frac{1}{\varepsilon}\left(-\frac{138865}{486} - \frac{2849}{54}\zeta_2 + \frac{2557}{27}\zeta_3 + \frac{337}{36}\zeta_2^2\right) \\
&- \frac{2732173}{2916} - \frac{45235}{162}\zeta_2 + \frac{51005}{81}\zeta_3 + \frac{8149}{216}\zeta_2^2 - \frac{343}{9}\zeta_2\zeta_3 + \frac{278}{45}\zeta_5\bigg\} \\
&+ C_F C_A n_f \bigg\{ \frac{88}{81\varepsilon^4} + \frac{1}{\varepsilon^3}\left(\frac{2254}{243} - \frac{16}{27}\zeta_2\right) + \frac{1}{\varepsilon^2}\bigg(\frac{13679}{243} + \frac{316}{81}\zeta_2 \\
&- \frac{256}{27}\zeta_3\bigg) + \frac{1}{\varepsilon}\left(\frac{623987}{2187} + \frac{11027}{243}\zeta_2 - \frac{6436}{81}\zeta_3 - \frac{44}{5}\zeta_2^2\right) + \frac{8560052}{6561} \\
&+ \frac{442961}{1458}\zeta_2 - \frac{45074}{81}\zeta_3 - \frac{1093}{27}\zeta_2^2 + \frac{368}{9}\zeta_2\zeta_3 - \frac{208}{3}\zeta_5\bigg\} \\
&+ C_F n_f^2 \bigg\{ -\frac{8}{81\varepsilon^4} - \frac{188}{243\varepsilon^3} + \frac{1}{\varepsilon^2}\left(-\frac{124}{27} - \frac{4}{9}\zeta_2\right) + \frac{1}{\varepsilon}\bigg(-\frac{49900}{2187} - \frac{94}{27}\zeta_2 \\
&+ \frac{136}{81}\zeta_3\bigg) - \frac{677716}{6561} - \frac{62}{3}\zeta_2 + \frac{3196}{243}\zeta_3 - \frac{83}{135}\zeta_2^2\bigg\}\,.
\end{aligned} \tag{6}$$

Here n_f stands for the number of effectively massless quark flavours, C_F and C_A are the usual QCD colour factors, $C_F = 4/3$ and $C_A = 3$, and the values of Riemann's zeta function are denoted by ζ_n. The ε^1 and ε^2 terms in Eq. (5) and the corresponding non-fermionic contributions [6] have recently been confirmed in Ref. [13], where exact expressions for both two-loop form factors were derived. The finite (ε^0) contributions in Eq. (6) represent a new result of the present letter.

The first three expansion coefficients (3) of the unrenormalized gluon form factor read

$$\begin{aligned}
\mathcal{F}_1^g &= C_A\Big\{-\frac{2}{\varepsilon^2}+\zeta_2+\varepsilon\left(-2+\frac{14}{3}\zeta_3\right)+\varepsilon^2\left(-6+\frac{47}{20}\zeta_2^2\right)+\varepsilon^3\Big(-14+\zeta_2 \\
&\quad -\frac{7}{3}\zeta_2\zeta_3+\frac{62}{5}\zeta_5\Big)+\varepsilon^4\Big(-30+3\zeta_2+\frac{14}{3}\zeta_3+\frac{949}{280}\zeta_2^3-\frac{49}{9}\zeta_3^2\Big)\Big\}, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_2^g &= C_A^2\left\{\frac{2}{\varepsilon^4}-\frac{11}{6\varepsilon^3}+\frac{1}{\varepsilon^2}\left(-\frac{67}{18}-\zeta_2\right)+\frac{1}{\varepsilon}\left(\frac{68}{27}+\frac{11}{2}\zeta_2-\frac{25}{3}\zeta_3\right)+\frac{5861}{162}+\frac{67}{6}\zeta_2\right. \\
&\quad +\frac{11}{9}\zeta_3-\frac{21}{5}\zeta_2^2+\varepsilon\left(\frac{158201}{972}+\frac{106}{9}\zeta_2-\frac{1139}{27}\zeta_3-\frac{77}{60}\zeta_2^2+\frac{23}{3}\zeta_2\zeta_3+\frac{71}{5}\zeta_5\right) \\
&\quad +\varepsilon^2\left(\frac{3484193}{5832}+\frac{481}{54}\zeta_2-\frac{26218}{81}\zeta_3-\frac{1943}{60}\zeta_2^2-\frac{55}{3}\zeta_2\zeta_3+\frac{341}{15}\zeta_5+\frac{2313}{70}\zeta_2^3\right. \\
&\quad \left.\left.+\frac{901}{9}\zeta_3^2\right)\right\} + C_A n_f\left\{\frac{1}{3\varepsilon^3}+\frac{5}{9\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{26}{27}-\zeta_2\right)-\frac{808}{81}-\frac{5}{3}\zeta_2-\frac{74}{9}\zeta_3\right. \\
&\quad +\varepsilon\left(-\frac{23131}{486}-\frac{16}{9}\zeta_2-\frac{604}{27}\zeta_3-\frac{51}{10}\zeta_2^2\right)+\varepsilon^2\left(-\frac{540805}{2916}+\frac{28}{27}\zeta_2-\frac{3962}{81}\zeta_3\right. \\
&\quad \left.\left.-\frac{257}{18}\zeta_2^2+\frac{50}{3}\zeta_2\zeta_3-\frac{542}{15}\zeta_5\right)\right\} + C_F n_f\left\{-\frac{1}{\varepsilon}-\frac{67}{6}+8\zeta_3+\varepsilon\left(-\frac{2027}{36}\right.\right. \\
&\quad \left.+\frac{7}{3}\zeta_2+\frac{92}{3}\zeta_3+\frac{16}{3}\zeta_2^2\right)+\varepsilon^2\left(-\frac{47491}{216}+\frac{209}{18}\zeta_2+\frac{1124}{9}\zeta_3+\frac{184}{9}\zeta_2^2\right. \\
&\quad \left.\left.-\frac{40}{3}\zeta_2\zeta_3+32\zeta_5\right)\right\}, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_3^g &= C_A^3\left\{-\frac{4}{3\varepsilon^6}+\frac{11}{3\varepsilon^5}+\frac{361}{81\varepsilon^4}+\frac{1}{\varepsilon^3}\left(-\frac{3506}{243}-\frac{517}{54}\zeta_2+\frac{22}{3}\zeta_3\right)+\frac{1}{\varepsilon^2}\left(-\frac{17741}{243}\right.\right. \\
&\quad \left.+\frac{481}{162}\zeta_2-\frac{209}{27}\zeta_3+\frac{247}{90}\zeta_2^2\right)+\frac{1}{\varepsilon}\left(-\frac{145219}{2187}+\frac{20329}{243}\zeta_2+\frac{241}{9}\zeta_3-\frac{3751}{360}\zeta_2^2\right. \\
&\quad \left.\left.-\frac{85}{9}\zeta_2\zeta_3-\frac{878}{15}\zeta_5\right)\right\} + C_A^2 n_f\left\{-\frac{2}{3\varepsilon^5}-\frac{2}{81\varepsilon^4}+\frac{1}{\varepsilon^3}\left(\frac{1534}{243}+\frac{47}{27}\zeta_2\right)\right. \\
&\quad \left.+\frac{1}{\varepsilon^2}\left(\frac{4280}{243}-\frac{425}{81}\zeta_2+\frac{518}{27}\zeta_3\right)+\frac{1}{\varepsilon}\left(-\frac{92449}{2187}-\frac{7561}{243}\zeta_2+\frac{1022}{81}\zeta_3+\frac{2453}{180}\zeta_2^2\right)\right\} \\
&\quad +C_A C_F n_f\left\{\frac{20}{9\varepsilon^3}+\frac{1}{\varepsilon^2}\left(\frac{526}{27}-\frac{160}{9}\zeta_3\right)+\frac{1}{\varepsilon}\left(\frac{2783}{81}-\frac{22}{3}\zeta_2-\frac{224}{27}\zeta_3-\frac{176}{15}\zeta_2^2\right)\right\} \\
&\quad +C_F^2 n_f\left\{\frac{2}{3\varepsilon}\right\} + C_A n_f^2\left\{-\frac{8}{81\varepsilon^4}-\frac{80}{243\varepsilon^3}+\frac{1}{\varepsilon^2}\left(\frac{8}{9}+\frac{20}{27}\zeta_2\right)+\frac{1}{\varepsilon}\left(\frac{34097}{2187}+\frac{200}{81}\zeta_2\right.\right. \\
&\quad \left.\left.+\frac{664}{81}\zeta_3\right)\right\} + C_F n_f^2\left\{\frac{8}{9\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{424}{27}-\frac{32}{3}\zeta_3\right)\right\}. \qquad (9)
\end{aligned}$$

As in Eq. (5), the one- and two-loop quantities (7) and (8) are written down to the accuracy in ε required for the extraction of the three-loop form factors to order ε^0. The terms up this order in Eq. (8) have been obtained before in Refs. [8,9]. Our corresponding coefficients of ε^1 and ε^2 agree with the recent all-order expression of Ref. [13]. Eq. (9), of which we will present an independent check below, is the main new result of this article.

In the context of Refs. [14,15] it is interesting to note that the terms of highest transcendentality, i.e, the coefficients of ζ_n and $\zeta_i\zeta_j$ with $i+j=n$, in the ε^{-2l+n} contributions to $\mathcal{F}_l$ agree between the quark and gluon form factors for the Super-Yang-Mills case $C_A = C_F = n_c$. In fact, up to two loops this can be readily shown to all orders in ε by expanding the prefactors of the master integrals in Eqs. (8) – (11) of Ref. [13].

The exponentiation of the form factors is performed for the (coupling-constant) renormalized quantities. These are obtained from Eqs. (3) – (9) by replacing the bare coupling a_s^b by its renormalized counterpart a_s according to

$$a_{\mathrm{s}}^{\mathrm{b}} = a_{\mathrm{s}}\left\{1-\frac{\beta_0}{\varepsilon}a_{\mathrm{s}}+\left(\frac{\beta_0^2}{\varepsilon^2}-\frac{1}{2}\frac{\beta_1}{\varepsilon}\right)a_{\mathrm{s}}^2-\left(\frac{\beta_0^3}{\varepsilon^3}-\frac{7}{6}\frac{\beta_1\beta_0}{\varepsilon^2}+\frac{1}{3}\frac{\beta_2}{\varepsilon}\right)a_{\mathrm{s}}^3\right\}, \tag{10}$$

where β_{i} are the usual expansion coefficients of the beta function of QCD, $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$ etc. Note that, unlike Refs. [9,13], we do not include the multiplicative renormalization of $G^a_{\mu\nu}G^{a,\mu\nu}$ in Eq. (2) into the definition of the (renormalized) gluon form factor.

The exponentiation of the (coupling-constant) renormalized form factors can be written as [2,3]

$$\ln\mathcal{F}\left(\alpha_{\mathrm{s}},\frac{Q^2}{\mu^2},\varepsilon\right) = \frac{1}{2}\int_0^{Q^2/\mu^2}\frac{d\xi}{\xi}\left(K(\alpha_{\mathrm{s}},\varepsilon)+G(1,\bar{a}(\xi,a_s,\varepsilon),\varepsilon)+\int_\xi^1\frac{d\lambda}{\lambda}A(\bar{a}(\lambda,a_{\mathrm{s}},\varepsilon))\right). \tag{11}$$

Here $K^p(\alpha_{\mathrm{s}},\varepsilon)$ are scale-independent counter-term functions consisting of a series of poles in ε in the $\overline{\mathrm{MS}}$ scheme. These functions can be determined recursively from a renormalization group equation (see, e.g., Ref. [3]) in terms of the cusp anomalous dimensions A^p [16]. The functions G^p, on the other hand, can be expanded in non-negative powers of ε at all orders of α_{s}. Finally $\bar{a}$ is the running coupling in D dimensions, see Refs. [6, 17]. After expansion in α_{s}, the integrals in Eq. (11) can be solved using algorithms for the evaluation of nested sums [18, 19], see Ref. [6]. Transforming back to the unrenormalized expansion coefficients $\mathcal{F}_i^p$ in Eq. (3) the results read

$$\mathcal{F}_1 = -\frac{1}{2}\frac{1}{\varepsilon^2}A_1-\frac{1}{2}\frac{1}{\varepsilon}G_1, \tag{12}$$

$$\mathcal{F}_2 = \frac{1}{8}\frac{1}{\varepsilon^4}A_1^2+\frac{1}{8}\frac{1}{\varepsilon^3}A_1(2G_1-\beta_0)+\frac{1}{8}\frac{1}{\varepsilon^2}(G_1^2-A_2-2\beta_0G_1)-\frac{1}{4}\frac{1}{\varepsilon}G_2, \tag{13}$$

$$\begin{aligned}\mathcal{F}_3 = &-\frac{1}{48}\frac{1}{\varepsilon^6}A_1^3-\frac{1}{16}\frac{1}{\varepsilon^5}A_1^2(G_1-\beta_0)-\frac{1}{144}\frac{1}{\varepsilon^4}A_1(9G_1^2-9A_2-27\beta_0G_1+8\beta_0^2)\\ &-\frac{1}{144}\frac{1}{\varepsilon^3}(3G_1^3-9A_2G_1-18A_1G_2+4\beta_1A_1-18\beta_0G_1^2+16\beta_0A_2+24\beta_0^2G_1)\\ &+\frac{1}{72}\frac{1}{\varepsilon^2}(9G_1G_2-4A_3-6\beta_1G_1-24\beta_0G_2)-\frac{1}{6}\frac{1}{\varepsilon}G_3.\end{aligned} \tag{14}$$

The corresponding expression for the four-loop form factors can be found in Ref. [6].

From now on we confine ourselves to the gluon form factor $\mathcal{F}^g$. The corresponding anomalous dimensions A^g are related by a factor C_A/C_F to the quark quantities A^q [16] which are known to order α_s^3 [10,20]. This relation has been verified by the explicit calculation of Ref. [11]. As always using the expansion parameter $a_s = \alpha_s/(4\pi)$, the available coefficients read

$$A_1^g = 4C_A\,, \tag{15}$$

$$A_2^g = 8C_A^2\left(\frac{67}{18}-\zeta_2\right)+8C_An_f\left(-\frac{5}{9}\right), \tag{16}$$

$$\begin{aligned} A_3^g = {} & 16C_A^3\left(\frac{245}{24}-\frac{67}{9}\zeta_2+\frac{11}{6}\zeta_3+\frac{11}{5}\zeta_2^2\right)+16C_AC_Fn_f\left(-\frac{55}{24}+2\zeta_3\right) \\ & +16C_A^2n_f\left(-\frac{209}{108}+\frac{10}{9}\zeta_2-\frac{7}{3}\zeta_3\right)+16C_An_f^2\left(-\frac{1}{27}\right). \end{aligned} \tag{17}$$

Beyond the third order only the (small) leading-n_f contributions are known [21].

Using these coefficients, the l-loop pole terms $\varepsilon^{-2l}\ldots\varepsilon^{-2}$ in Eqs. (12) – (14) can be predicted from lower-order information, thus providing a strong check of either the exponentiation formula or the explicit l-th order calculation. Our results pass this check, thus Eqs. (7) – (9) can be employed to recursively derive the resummation functions G_i^g for $i = 1, 2,$ and 3 to the respective fifth, third and zeroth order in ε, yielding

$$\begin{aligned} G_1^g = {} & \varepsilon C_A(-2\zeta_2)+\varepsilon^2C_A\left(4-\frac{28}{3}\zeta_3\right)+\varepsilon^3C_A\left(12-\frac{47}{10}\zeta_2^2\right)+\varepsilon^4C_A\left(28-2\zeta_2\right. \\ & \left.+\frac{14}{3}\zeta_2\zeta_3-\frac{124}{5}\zeta_5\right)+\varepsilon^5C_A\left(60-6\zeta_2-\frac{28}{3}\zeta_3-\frac{949}{140}\zeta_2^3+\frac{98}{9}\zeta_3^2\right), \end{aligned} \tag{18}$$

$$\begin{aligned} G_2^g = {} & C_A^2\left(\frac{160}{27}-\frac{44}{3}\zeta_2-4\zeta_3\right)+C_An_f\left(\frac{104}{27}+\frac{8}{3}\zeta_2\right)+4C_Fn_f \\ & +\varepsilon C_A^2\left(-\frac{9022}{81}-\frac{134}{3}\zeta_2+\frac{88}{3}\zeta_3\right)+\varepsilon C_An_f\left(\frac{3448}{81}+\frac{20}{3}\zeta_2+\frac{80}{3}\zeta_3\right) \\ & +\varepsilon C_Fn_f\left(\frac{134}{3}-32\zeta_3\right)+\varepsilon^2C_A^2\left(-\frac{141677}{243}-\frac{568}{9}\zeta_2+\frac{4556}{27}\zeta_3+\frac{671}{30}\zeta_2^2\right. \\ & \left.+\frac{20}{3}\zeta_2\zeta_3-156\zeta_5\right)+\varepsilon^2C_An_f\left(\frac{48206}{243}+\frac{64}{9}\zeta_2+\frac{2416}{27}\zeta_3+\frac{259}{15}\zeta_2^2\right) \\ & +\varepsilon^2C_Fn_f\left(\frac{2027}{9}-\frac{28}{3}\zeta_2-\frac{368}{3}\zeta_3-\frac{64}{3}\zeta_2^2\right)+\varepsilon^3C_A^2\left(-\frac{3272297}{1458}\right. \\ & \left.-\frac{2060}{27}\zeta_2+\frac{98824}{81}\zeta_3+\frac{1943}{15}\zeta_2^2+\frac{506}{9}\zeta_2\zeta_3-\frac{5246}{35}\zeta_2^3-\frac{940}{3}\zeta_3^2\right) \\ & +\varepsilon^3C_An_f\left(\frac{554413}{729}-\frac{148}{27}\zeta_2+\frac{15848}{81}\zeta_3+\frac{514}{9}\zeta_2^2-\frac{572}{9}\zeta_2\zeta_3+128\zeta_5\right) \\ & +\varepsilon^3C_Fn_f\left(\frac{47491}{54}-\frac{418}{9}\zeta_2-\frac{4496}{9}\zeta_3-\frac{736}{9}\zeta_2^2+\frac{160}{3}\zeta_2\zeta_3-128\zeta_5\right), \end{aligned} \tag{19}$$

$$G_3^g = C_A^3\left(-\frac{373975}{729}-\frac{27320}{81}\zeta_2+\frac{4096}{27}\zeta_3+\frac{1276}{15}\zeta_2^2+\frac{80}{3}\zeta_2\zeta_3+32\zeta_5\right)$$

$$+C_A^2 n_f\left(\frac{266072}{729}+\frac{7328}{81}\zeta_2+\frac{56}{9}\zeta_3-\frac{328}{15}\zeta_2^2\right)+C_A C_F n_f\left(\frac{3833}{27}+8\zeta_2 -\frac{752}{9}\zeta_3+\frac{32}{5}\zeta_2^2\right)-4C_F^2 n_f+C_A n_f^2\left(-\frac{28114}{729}-\frac{160}{27}\zeta_2-\frac{256}{27}\zeta_3\right) +C_F n_f^2\left(-\frac{104}{3}+\frac{64}{3}\zeta_3\right). \tag{20}$$

Results analogous to the terms up to order ε in Eqs. (18) and (19) have been derived before, in a different notation, in Ref. [9]. The third-order coefficient (20) is a new result based on Eq. (9). Note that G_3^g is one of the quantities which determine the infrared structure of QCD amplitudes in the framework of Refs. [4,5].

We now turn to the check of our main results (9) and (20) announced above. Inspired by a key observation of Ref. [9], we decompose the resummation functions G_i^p ($p=q,\,g$) according to

$$\begin{aligned} G_1^p &= 2\left(B_1^p-\delta_{pg}\beta_0\right)\ +f_1^p+\varepsilon\widetilde{G_1^p}\ , \\ G_2^p &= 2\left(B_2^p-2\delta_{pg}\beta_1\right)+f_2^p+\beta_0\widetilde{G_1^p}(\varepsilon{=}0)+\varepsilon\widetilde{G_2^p}\ , \\ G_3^p &= 2\left(B_3^p-3\delta_{pg}\beta_2\right)+f_3^p+\beta_1\widetilde{G_1^p}(\varepsilon{=}0)+\beta_0\left[\widetilde{G_2^p}(\varepsilon{=}0)-\beta_0\widetilde{\widetilde{G_1^p}}(\varepsilon{=}0)\right]+\varepsilon\widetilde{G}_3^p \end{aligned} \tag{21}$$

with

$$\widetilde{F} \ = \ \varepsilon^{-1}\left[F-F(\varepsilon{=}0)\right] \tag{22}$$

and B_n^p denoting the coefficients of $\delta(1-x)$ in the n-loop diagonal $\overline{\mathrm{MS}}$ splitting functions $P_{pp}^{(n-1)}(x)$ [10,11]. The crucial point of this decomposition is that the functions f_i^p are, like the cusp anomalous dimensions A_n^p, universal up to the factor C_A/C_F, i.e., $f_i^g=C_A/C_F\,f_i^q$. The functions f_i^p also exhibit the same maximally non-Abelian colour structure as the A_n^p [16], with

$$f_1^q \ = \ 0\ , \tag{23}$$

$$f_2^q \ = \ 2C_F\left\{-\beta_0\zeta_2-\frac{56}{27}n_f+C_A\left(\frac{404}{27}-14\zeta_3\right)\right\}\ , \tag{24}$$

$$\begin{aligned} f_3^q \ =& \ C_F C_A^2\left(\frac{136781}{729}-\frac{12650}{81}\zeta_2-\frac{1316}{3}\zeta_3+\frac{352}{5}\zeta_2^2+\frac{176}{3}\zeta_2\zeta_3+192\zeta_5\right) \\ &+C_A C_F n_f\left(-\frac{11842}{729}+\frac{2828}{81}\zeta_2+\frac{728}{27}\zeta_3-\frac{96}{5}\zeta_2^2\right)+C_F^2 n_f\left(-\frac{1711}{27}\right. \\ &\left.+4\zeta_2+\frac{304}{9}\zeta_3+\frac{32}{5}\zeta_2^2\right)+C_F n_f^2\left(-\frac{2080}{729}-\frac{40}{27}\zeta_2+\frac{112}{27}\zeta_3\right). \end{aligned} \tag{25}$$

Consequently, once f_3^q is known from the explicit calculation of the three-loop quark form factor [6], the pole terms of the gluon form factor can be derived from the results of Refs. [10,11] and lower-order quantities. This procedure confirms Eq. (20) and thus Eq. (9). For applications to other cases, like the coupling of a preudoscalar Higgs boson, it should be noted that the terms subtracted from B_i^g in Eqs. (21) are given by the renormalization of the operator in Eq. (2).

As a first concrete application of our results we consider the absolute ratio $|\mathcal{F}^g(q^2)/\mathcal{F}^g(-q^2)|$ of the renormalized time-like and space-like form factors. This quantity is infrared finite and directly enters the cross section for Higgs boson production in hadronic collisions. In terms of the coefficients A_i and $G_i(\varepsilon{=}0)$ this ratio is given by [6]

$$\begin{aligned}
\left|\frac{\mathcal{F}(q^2)}{\mathcal{F}(-q^2)}\right|^2 \;=\;& 1 + a_{\rm s}\{3\zeta_2 A_1\} + \alpha_{\rm s}^2\Big\{\frac{9}{2}\zeta_2^2 A_1^2 + 3\zeta_2(\beta_0 G_1 + A_2)\Big\} \\
&+ a_{\rm s}^3\Big\{\frac{9}{2}\zeta_2^3 A_1^3 + 3\zeta_2^2 A_1(3\beta_0 G_1 - \beta_0^2 + 3A_2) + 3\zeta_2(A_3 + \beta_1 G_1 + 2\beta_0 G_2)\Big\} \\
&+ a_{\rm s}^4\Big\{\frac{27}{8}\zeta_2^4 A_1^4 + \frac{9}{2}\zeta_2^3 A_1^2(3\beta_0 G_1 - 2\beta_0^2 + 3A_2) + \frac{3}{2}\zeta_2^2(-6\beta_0^2 A_2 + 3\beta_0^2 G_1^2 \\
&\quad + 3A_2^2 + 12\beta_0 A_1 G_2 + 6A_1 A_3 + 6\beta_1 A_1 G_1 - 5\beta_0\beta_1 A_1 + 6\beta_0 A_2 G_1 \\
&\quad - 6\beta_0^3 G_1) + 3\zeta_2(A_4 + \beta_2 G_1 + 3\beta_0 G_3 + 2\beta_1 G_2)\Big\} \;+\; O(a_{\rm s}^5)
\end{aligned} \tag{26}$$

for the couplings $a_{\rm s}(q^2) = a_{\rm s}(-q^2) = a_{\rm s}$. All terms contributing at the fourth order are now known, with the exception of the small four-loop cusp anomalous dimension A_4^g. For the latter we can employ the [1/1] Padé estimate of the quark case in Ref. [22], accordingly multiplied by C_A/C_F,

$$A_4^g \;\approx\; 17660\,,\; 9704\,,\; 3949 \quad \text{for} \quad n_f \;=\; 3\,,\, 4\,,\, 5\;. \tag{27}$$

Below a conservative (but numerically irrelevant) uncertainty of 50% is assigned to this estimate.

Inserting the numerical values of Eqs. (15) – (20) for $n_f = 5$ quark flavours, we obtain

$$\left|\frac{\mathcal{F}^g(q^2)}{\mathcal{F}^g(-q^2)}\right|^2 = 1 + 4.712\,\alpha_{\rm s} + 13.69\,\alpha_{\rm s}^2 + 25.94\,\alpha_{\rm s}^3 + (36.65 \pm 0.35)\,\alpha_{\rm s}^4\,. \tag{28}$$

The contributions up to order $\alpha_{\rm s}^2$ have been derived before in Ref. [8], where the factor 153.2 in the penultimate line of Eq. (19) should read 135.2. The low-order contributions to Eq. (28) are larger than for the corresponding quark quantity in Ref. [6]. On the other hand the growth of the coefficients is slower in the present case, surprisingly suggesting a more benign higher-order behaviour. As for Higgs boson production couplings of the order $\alpha_{\rm s} \approx 0.1$ are relevant, Eq. (28) further supports indications from the next-to-next-to-leading logarithmic threshold resummation [23, 24] that the corrections beyond the next-to-next-to-leading order ($\alpha_{\rm s}^2$) of Refs. [25–28] are rather small.

Acknowledgments

The work of S.M. has been supported in part by the Helmholtz Gemeinschaft under contract VH-NG-105 and by the Deutsche Forschungsgemeinschaft in Sonderforschungsbereich/Transregio 9. The work of J.V. has been part of the research program of the Dutch Foundation for Fundamental Research of Matter (FOM).

References

[1] J.C. Collins, Adv. Ser. Direct. High Energy Phys. 5 (1989) 573, hep-ph/0312336

[2] L. Magnea and G. Sterman, Phys. Rev. D42 (1990) 4222

[3] L. Magnea, Nucl. Phys. B593 (2001) 269, hep-ph/0006255

[4] S. Catani, Phys. Lett. B427 (1998) 161, hep-ph/9802439

[5] G. Sterman and M.E. Tejeda-Yeomans, Phys. Lett. B552 (2003) 48, hep-ph/0210130

[6] S. Moch, J.A.M. Vermaseren and A. Vogt, hep-ph/0507039 (JHEP, in press)

[7] J.A.M. Vermaseren, A. Vogt, and S. Moch, hep-ph/0504242 (Nucl. Phys. B, in press)

[8] R. Harlander, Phys. Lett. B492 (2000) 74, hep-ph/0007289

[9] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B704 (2005) 332, hep-ph/0408315

[10] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192

[11] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111

[12] S. Moch, J.A.M. Vermaseren, and A. Vogt, Nucl. Phys. B646 (2002) 181, hep-ph/0209100

[13] T. Gehrmann, T. Huber and D. Maitre, hep-ph/0507061 (Phys. Lett. B, in press)

[14] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, and V.N. Velizhanin Phys. Lett. B595 (2004) 521, hep-th/0404092

[15] Z. Bern, L.J. Dixon, and V.A. Smirnov, hep-th/0505205

[16] G. P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257

[17] H. Contopanagos, E. Laenen, and G. Sterman, Nucl. Phys. B484 (1997) 303, hep-ph/9604313

[18] J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037, hep-ph/9806280

[19] S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. 43 (2002) 3363, hep-ph/0110083

[20] J. Kodaira and L. Trentadue, Phys. Lett. 112B (1982) 66

[21] J.A. Gracey, Phys. Lett. B322 (1994) 141, hep-ph/9401214

[22] S. Moch, J.A.M. Vermaseren, and A. Vogt, hep-ph/0506288 (Nucl. Phys. B, to appear)

[23] A. Vogt, Phys. Lett. B497 (2001) 228, hep-ph/0010146

[24] S. Catani, D. de Florian, M. Grazzini and P. Nason, JHEP 0307 (2003) 028, hep-ph/0306211

[25] R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. 88 (2002) 201801, hep-ph/0201206

[26] C. Anastasiou and K. Melnikov, Nucl. Phys. B646 (2002) 220, hep-ph/0207004

[27] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B665 (2003) 325, hep-ph/0302135

[28] C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0501130