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The Three-Loop Splitting Functions in QCD: The Singlet Case

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Abstract

We compute the next-to-next-to-leading order (NNLO) contributions to the splitting functions governing the evolution of the unpolarized flavour-singlet parton densities in perturbative QCD. The exact expressions are presented in both Mellin-N and Bjorken-x space. We also provide accurate parametrizations for practical applications. Our results agree with all partial results available in the literature. As in the non-singlet case, the correct leading logarithmic predictions for small momentum fractions x do not provide good estimates of the respective complete splitting functions. We investigate the size of the corrections and the stability of the NNLO evolution under variation of the renormalization scale. The perturbative expansion appears to converge rapidly at $x \gtrsim 10^{-3}$. Relatively large third-order corrections are found at smaller values of x.

1 Introduction

Parton distributions form indispensable ingredients for the analysis of all hard-scattering processes involving initial-state hadrons. The scale-dependence (evolution) of these distributions can be derived from first principles in terms of an expansion in powers of the strong coupling constant α_s. The corresponding nth-order coefficients governing the evolution are referred to as the n-loop anomalous dimensions or splitting functions. Parton distributions evolved by including the terms up to order α_s^{n+1} in this expansion constitute, together with the corresponding results for the partonic cross sections for the observable under consideration, the N$^{\mathrm{n}}$LO (leading-order, next-to-leading-order, next-to-next-to-leading-order, etc.) approximation of perturbative QCD.

Presently the next-to-leading order is the standard approximation for most important processes. The corresponding one- and two-loop splitting functions have been known for a long time [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The NNLO corrections need to be included, however, in order to arrive at quantitatively reliable predictions for hard processes at present and future high-energy colliders. These corrections are so far known only for structure functions in deep-inelastic scattering (DIS) [12, 13, 14, 15] and for Drell-Yan lepton-pair and gauge-boson production in proton–(anti-)proton collisions [16, 17, 18, 19] and the related cross sections for Higgs production in the heavy-top-quark approximation [17, 20, 21, 22]. Work on NNLO cross sections for jet production is under way and expected to yield results in the near future, see Ref. [23] and references therein.

For the three-loop splitting functions, on the other hand, only partial results had been computed until very recently, especially the lowest six/seven (even or odd) integer-N Mellin moments [24, 25, 26] and the leading $(\ln x)/x$ small-x terms of three of the four singlet splitting functions [27, 28]. The results of Refs. [24, 25, 26] have been employed – directly [29, 30, 31, 32] and indirectly [33, 34] via Bjorken x-space approximations constructed in Refs. [35, 36, 37] from them and the small-x constraints [27, 28] – to improve the analysis of DIS data and hadron-collider predictions. This information is however not sufficient for quantitative predictions at small values of x.

We have recently published the non-singlet part of the unpolarized three-loop splitting functions [38]. In the present article we compute the corresponding singlet quantities. The article is organized as follows: In section 2 we set up our notations and very briefly discuss the method of our calculation. The Mellin-N space results are written down in section 3. The $(\ln N)/N$ subleading large-N term of the three-loop gluon-gluon splitting function is found to be related to the leading $\ln N$ contribution at second order, in complete analogy to the relation found for the non-singlet quark-quark case. In section 4 we present the exact results as well as compact parametrizations for the x-space splitting functions and study their behaviour at small x. We demonstrate that neither do the $(\ln x)/x$ terms dominate the splitting functions at experimentally relevant values of x, nor do even all $1/x$ terms dominate the Mellin convolutions by which the splitting functions enter the evolution equations. The numerical implications of our results for the scale dependence of the singlet-quark and gluon distributions are illustrated in section 5. As in the non-singlet case the perturbation series converges rapidly for $x \gtrsim 10^{-3}$, while relatively large corrections occur for smaller momentum fractions. Finally we briefly summarize our findings in section 6.

2 Notations and method

We start by setting up our notations for the singlet parton densities and the splitting functions governing their evolution. The singlet quark distribution of a hadron is given by

$$q_{\rm s}(x,\mu_f^2) \,=\, \sum_{i=1}^{n_f} \left[q_i(x,\mu_f^2) + \bar{q}_i(x,\mu_f^2) \right] \; . \tag{2.1}$$

Here $q_i(x,\mu_f^2)$ and $\bar{q}_i(x,\mu_f^2)$ represent the respective number distributions of quarks and antiquarks in the fractional hadron momentum x. The corresponding gluon distribution is denoted by $g(x,\mu_f^2)$. The subscript i indicates the flavour of the (anti-) quarks, and n_f stands for the number of effectively massless flavours. Finally μ_f represents the factorization scale. For the time being we do not need to introduce a renormalization scale μ_r different from μ_f.

Suppressing the functional dependences, the evolution equations for the singlet parton distributions read

$$\frac{d}{d\ln\mu_f^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \,=\, \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \; , \tag{2.2}$$

where $\otimes$ stands for the Mellin convolution in the momentum variable,

$$[a \otimes b](x) \,\equiv\, \int_x^1 \frac{dy}{y}\, a(y)\, b\Big(\frac{x}{y}\Big) \; . \tag{2.3}$$

The quark-quark splitting function $P_{\rm qq}$ in Eq. (2.2) can be expressed as

$$P_{\rm qq} \,=\, P_{\rm ns}^{+} + n_f (P_{\rm qq}^{\rm s} + P_{\rm \bar{q}q}^{\rm s}) \,\equiv\, P_{\rm ns}^{+} + P_{\rm ps} \; . \tag{2.4}$$

Here $P_{\rm ns}^{+}$ is the non-singlet splitting function which we have recently computed up to the third order in Ref. [38]. The $O(\alpha_{\rm s}^2)$ quantities $P_{\rm qq}^{\rm s}$ and $P_{\rm \bar{q}q}^{\rm s}$ are the flavour independent ('sea') contributions to the quark-quark and quark-antiquark splitting functions $P_{{\rm q}_i{\rm q}_k}$ and $P_{\bar{\rm q}_i{\rm q}_k}$, respectively. The non-singlet contribution dominates Eq. (2.4) at large x, where the 'pure singlet' term $P_{\rm ps}$ is very small. At small x, on the other hand, the latter contribution takes over as $xP_{\rm ps}$ does not vanish for $x \rightarrow 0$, unlike $xP_{\rm ns}^{+}$. The gluon-quark and quark-gluon entries in Eq. (2.2) are given by

$$P_{\rm qg} \,=\, n_f\, P_{{\rm q}_i{\rm g}} \; , \quad P_{\rm gq} \,=\, P_{{\rm gq}_i} \tag{2.5}$$

in terms of the flavour-independent splitting functions $P_{{\rm q}_i{\rm g}} = P_{\bar{\rm q}_i{\rm g}}$ and $P_{{\rm gq}_i} = P_{{\rm g}\bar{\rm q}_i}$. With the exception of the $\alpha_{\rm s}^1$ part of $P_{\rm qg}$, neither of the quantities $xP_{\rm qg}$, $xP_{\rm gq}$ and $xP_{\rm gg}$ vanishes for $x \rightarrow 0$.

Our calculation is performed in Mellin-N space, i.e., we compute the singlet anomalous dimensions $\gamma_{\rm ab}(N,\alpha_{\rm s})$ which are related to the splitting functions by a Mellin transformation,

$$\gamma_{\rm ab}(N,\alpha_{\rm s}) \,=\, -\int_0^1 dx\, x^{N-1}\, P_{\rm ab}(x,\alpha_{\rm s}) \; . \tag{2.6}$$

The additional relative sign is the standard convention. Note that in the older literature an additional factor of two is often included in Eq. (2.6).

The calculation follows the approach of Refs. [25, 26, 39, 40]. The optical theorem and the operator product expansion are employed to compute the Mellin moments of (partly fictitious, see below) deep-inelastic structure functions. Since the moment variable N is now an analytical parameter, we cannot apply the techniques of Refs. [25, 26], where the integrals were solved using the MINCER program [41, 42]. The introduction of new techniques was therefore necessary, and various aspects of those have already been discussed in Refs. [40, 43, 44, 45, 38]. A salient feature of our method is, however, that we can check our extensive manipulations at almost any stage by falling back on a MINCER evaluation of fixed low-integer moments. Note also that we will obtain the three-loop coefficient functions in DIS as well, once the present calculation is supplemented by a second Lorentz projection required to disentangle the structure functions F_2 and F_L [46].

The complete set of NNLO singlet anomalous dimensions can be extracted from the third-order amplitudes of the forward Compton processes

$$\text{parton}\,(P) + \text{probe}\,(Q) \;\longrightarrow\; \text{parton}\,(P) + \text{probe}\,(Q)\;, \tag{2.7}$$

where the probes are the photon (γ) and a fictitious classical scalar ϕ coupling directly only to the gluon field via $\phi\, G^a_{\mu\nu} G_a^{\mu\nu}$. The inclusion of the latter, required for obtaining also the anomalous dimensions $\gamma_{\rm gq}$ and $\gamma_{\rm gg}$ to the desired accuracy, leads to a substantial increase of the number of diagrams as shown in Table 1. Among the partons in Eq. (2.7) we also include an external ghost h. This is done in order to allow us to take the sum over external gluon spins by contracting with $-g_{\mu\nu}$ instead of the full physical expression which would, due to the presence of extra powers of P, lead to a complication of our task. For similar reasons we do not keep the gauge dependence in our all-N computations, but check its cancellation only for a few fixed values of N.

process	tree	1-loop	2-loop	3-loop
$q\gamma \to q\gamma$	1	3	25	359
$g\gamma \to g\gamma$		2	17	345
$h\gamma \to g\gamma$			2	56
$q\phi \to q\phi$		1	23	696
$g\phi \to g\phi$	1	8	218	6378
$h\phi \to h\phi$		1	33	1184
sum	2	15	318	9018

Table 1: The number of diagrams for the amplitudes employed for the calculation of the three-loop singlet anomalous dimensions. The roles of the ghost h and the scalar ϕ are discussed in the text.

The diagrams are generated automatically with the diagram generator QGRAF [47]. For all symbolic manipulations we use the latest version of FORM [48, 49]. The calculation is performed in dimensional regularization [50, 51, 52, 53]. The renormalization is carried out in the $\overline{\rm MS}$-scheme [54, 55] as described in detail in Ref. [25], using the result of Refs. [56, 57] for the renormalization of the operator $G^a_{\mu\nu} G_a^{\mu\nu}$ entering the scalar case.

3 Results in Mellin space

Here we present the anomalous dimensions $\gamma_{\rm ab}(N,\alpha_{\rm s})$ in the $\overline{\rm MS}$-scheme up to the third order in the running coupling constant $\alpha_{\rm s}$. The $\rm N^nLO$ expansion coefficients $\gamma_{\rm ab}^{(n)}(N)$ are normalized as

$$\gamma_{\rm ab}(\alpha_{\rm s},N) \;=\; \sum_{n=0} \left(\frac{\alpha_{\rm s}}{4\pi}\right)^{n+1} \gamma_{\rm ab}^{(n)}(N)\; . \tag{3.1}$$

The anomalous dimensions can be expressed in terms of harmonic sums [6, 7, 58, 59, 60]. Recall that, following the notation of Ref. [58], these sums are recursively defined by

$$S_{\pm m}(M) \;=\; \sum_{i=1}^{M} \frac{(\pm 1)^i}{i^m} \tag{3.2}$$

and

$$S_{\pm m_1,m_2,\ldots,m_k}(M) \;=\; \sum_{i=1}^{M} \frac{(\pm 1)^i}{i^{m_1}}\, S_{m_2,\ldots,m_k}(i)\; . \tag{3.3}$$

The sum of the absolute values of the indices m_k defines the weight of the harmonic sum. Sums up to weight $2l-1$ occur in the l-loop results written down below.

In order to arrive at a reasonably compact representation of our results, we employ the abbreviation $S_{\vec{m}} \equiv S_{\vec{m}}(N)$ in what follows, together with the notation

$$\mathbf{N_\pm}\, S_{\vec{m}} \;=\; S_{\vec{m}}(N\pm 1)\;, \qquad \mathbf{N_{\pm i}}\, S_{\vec{m}} \;=\; S_{\vec{m}}(N\pm i) \tag{3.4}$$

for arguments shifted by ± 1 or a larger integer i. In this notation the well-known one-loop (LO) singlet anomalous dimensions [1, 2] read

$$\begin{aligned}
\gamma_{\rm ps}^{(0)}(N) &= 0\\
\gamma_{\rm qg}^{(0)}(N) &= 2\,n_f\,(\mathbf{N_-} + 4\mathbf{N_+} - 2\mathbf{N_{+2}} - 3)\,S_1\\
\gamma_{\rm gq}^{(0)}(N) &= 2\,C_F\,(2\mathbf{N_{-2}} - 4\mathbf{N_-} - \mathbf{N_+} + 3)\,S_1\\
\gamma_{\rm gg}^{(0)}(N) &= C_A\left(4(\mathbf{N_{-2}} - 2\mathbf{N_-} - 2\mathbf{N_+} + \mathbf{N_{+2}} + 3)\,S_1 - \frac{11}{3}\right) + \frac{2}{3}\,n_f\; .
\end{aligned} \tag{3.5}$$

The corresponding second-order (NLO) quantities [5, 6, 10, 11] are given by

$$\begin{aligned}
\gamma_{\rm ps}^{(1)}(N) \;=\; & 4C_F n_f\Big(\frac{20}{9}(\mathbf{N_{-2}} - \mathbf{N_-})S_1 - (\mathbf{N_+} - \mathbf{N_{+2}})\Big[\frac{56}{9}S_1 + \frac{8}{3}S_2\Big] + (1-\mathbf{N_+})\Big[8S_1 - 4S_2\Big]\\
& - (\mathbf{N_-} - \mathbf{N_+})\Big[2S_1 + S_2 + 2S_3\Big]\Big)
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\gamma_{\rm qg}^{(1)}(N) \;=\; & 4C_A n_f\Big(\frac{20}{9}(\mathbf{N_{-2}} - \mathbf{N_-})S_1 - (\mathbf{N_-} - \mathbf{N_+})\Big[2S_1 + S_2 + 2S_3\Big] - (\mathbf{N_+} - \mathbf{N_{+2}})\Big[\frac{218}{9}S_1\\
& + 4S_{1,1} + \frac{44}{3}S_2\Big] + (1-\mathbf{N_+})\Big[27S_1 + 4S_{1,1} - 7S_2 - 2S_3\Big] - 2(\mathbf{N_-} + 4\mathbf{N_+} - 2\mathbf{N_{+2}} - 3)\Big[S_{1,-2}
\end{aligned}$$

$$+S_{1,1,1}\Big]\Big)+4C_Fn_f\Big(2(\mathbf{N_+}-\mathbf{N_{+2}})\Big[5S_1+2S_{1,1}-2S_2+S_3\Big]-(1-\mathbf{N_+})\Big[\frac{43}{2}S_1+4S_{1,1}-\frac{7}{2}S_2\Big]$$
$$+(\mathbf{N_-}-\mathbf{N_+})\Big[7S_1-\frac{3}{2}S_2\Big]+2(\mathbf{N_-}+4\mathbf{N_+}-2\mathbf{N_{+2}}-3)\Big[S_{1,1,1}-S_{1,2}-S_{2,1}+\frac{1}{2}S_3\Big]\Big) \tag{3.7}$$

$$\gamma^{(1)}_{\rm gq}(N) \;=\; 4C_AC_F\Big(2(2\mathbf{N_{-2}}-4\mathbf{N_-}-\mathbf{N_+}+3)\Big[S_{1,1,1}-S_{1,-2}-S_{1,2}-S_{2,1}\Big]+(1-\mathbf{N_+})\Big[2S_1$$
$$-13S_{1,1}-7S_2-2S_3\Big]+(\mathbf{N_{-2}}-2\mathbf{N_-}+\mathbf{N_+})\Big[S_1-\frac{22}{3}S_{1,1}\Big]+4(\mathbf{N_-}-\mathbf{N_+})\Big[\frac{7}{9}S_1+3S_2+S_3\Big]$$
$$+(\mathbf{N_+}-\mathbf{N_{+2}})\Big[\frac{44}{9}S_1+\frac{8}{3}S_2\Big]\Big)+4C_Fn_f\Big((\mathbf{N_{-2}}-2\mathbf{N_-}+\mathbf{N_+})\Big[\frac{4}{3}S_{1,1}-\frac{20}{9}S_1\Big]-(1-\mathbf{N_+})\Big[4S_1$$
$$-2S_{1,1}\Big]\Big)+4{C_F}^2\Big((2\mathbf{N_{-2}}-4\mathbf{N_-}-\mathbf{N_+}+3)\Big[3S_{1,1}-2S_{1,1,1}\Big]-(1-\mathbf{N_+})\Big[S_1-2S_{1,1}+\frac{3}{2}S_2$$
$$-3S_3\Big]-(\mathbf{N_-}-\mathbf{N_+})\Big[\frac{5}{2}S_1+2S_2+2S_3\Big]\Big) \tag{3.8}$$

$$\gamma^{(1)}_{\rm gg}(N) \;=\; 4C_An_f\Big(\frac{2}{3}-\frac{16}{3}S_1-\frac{23}{9}(\mathbf{N_{-2}}+\mathbf{N_{+2}})S_1+\frac{14}{3}(\mathbf{N_-}+\mathbf{N_+})S_1+\frac{2}{3}(\mathbf{N_-}-\mathbf{N_+})S_2\Big)$$
$$+4{C_A}^2\Big(2S_{-3}-\frac{8}{3}-\frac{14}{3}S_1+2S_3-(\mathbf{N_{-2}}-2\mathbf{N_-}-2\mathbf{N_+}+\mathbf{N_{+2}}+3)\Big[4S_{1,-2}+4S_{1,2}+4S_{2,1}\Big]$$
$$+\frac{8}{3}(\mathbf{N_+}-\mathbf{N_{+2}})S_2-4(\mathbf{N_-}-3\mathbf{N_+}+\mathbf{N_{+2}}+1)\Big[3S_2-S_3\Big]+\frac{109}{18}(\mathbf{N_-}+\mathbf{N_+})S_1+\frac{61}{3}(\mathbf{N_-}$$
$$-\mathbf{N_+})S_2\Big)+4C_Fn_f\Big(\frac{1}{2}+\frac{2}{3}(\mathbf{N_{-2}}-13\mathbf{N_-}-\mathbf{N_+}-5\mathbf{N_{+2}}+18)S_1+(3\mathbf{N_-}-5\mathbf{N_+}+2)S_2$$
$$-2(\mathbf{N_-}-\mathbf{N_+})S_3\Big)\,. \tag{3.9}$$

The pure-singlet contribution (2.4) to the three-loop (NNLO) anomalous dimension $\gamma^{(2)}_{\rm qq}(N)$ is

$$\gamma^{(2)}_{\rm ps}(N) \;=\; 16C_AC_Fn_f\Big(\frac{1}{3}(4\mathbf{N_{-2}}-\mathbf{N_-}-\mathbf{N_+}+4\mathbf{N_{+2}}-6)\Big[3S_1\zeta_3+S_{1,-2,1}-S_{1,1,-2}+S_{1,1,1,1}$$
$$-S_{1,1,2}\Big]+(\mathbf{N_{-2}}-\mathbf{N_-})\Big[\frac{571}{108}S_{1,1}-\frac{6761}{324}S_1-\frac{3}{2}S_{1,2}-\frac{52}{9}S_{1,-2}+\frac{56}{27}S_2-\frac{20}{9}S_{2,1}\Big]$$
$$-(\mathbf{N_{-2}}-\mathbf{N_-}-\mathbf{N_+}+\mathbf{N_{+2}})\Big[\frac{8}{3}S_{1,-3}+2S_{1,3}+\frac{1}{9}S_{1,1,1}+\frac{2}{3}S_{2,1,1}\Big]+(\mathbf{N_+}-\mathbf{N_{+2}})\Big[\frac{10279}{162}S_1$$
$$+\frac{106}{9}S_{1,-2}+\frac{151}{54}S_{1,1}+\frac{9}{2}S_{1,2}+4S_{2,-2}+\frac{2299}{54}S_2+\frac{28}{9}S_{2,1}+\frac{2}{3}S_{2,2}+\frac{83}{6}S_3+\frac{2}{3}S_{3,1}\Big]$$
$$+(1-\mathbf{N_+})\Big[\frac{4}{3}S_{1,2}-\frac{251}{4}S_1-\frac{50}{3}S_{1,-2}-\frac{29}{12}S_2-\frac{1165}{36}S_{1,1}+5S_{2,-2}+\frac{33}{4}S_{2,1}+S_{2,1,1}+\frac{3}{2}S_{2,2}$$
$$-\frac{37}{2}S_3-4S_{3,-2}+S_{3,1}-10S_4-7S_5\Big]-(\mathbf{N_-}+\mathbf{N_+}-2)\Big[\frac{1}{2}S_{1,-3}+3S_{1,-2,1}+\frac{3}{4}S_{1,1,1}+\frac{9}{4}S_{1,3}\Big]$$
$$+(\mathbf{N_-}-\mathbf{N_+})\Big[\frac{121}{12}S_1+\frac{16}{3}S_{1,-2}+\frac{437}{36}S_{1,1}-\frac{13}{6}S_{1,2}+\frac{3565}{108}S_2-6S_2\zeta_3+3S_{2,-3}+\frac{3}{2}S_{2,-2}$$
$$-\frac{479}{36}S_{2,1}+2S_{2,1,-2}+\frac{11}{6}S_{2,1,1}-2S_{2,1,1,1}+2S_{2,1,2}+S_{2,2}+\frac{7}{2}S_{2,3}+\frac{269}{36}S_3+5S_{3,-2}+\frac{29}{6}S_4$$
$$+\frac{59}{12}S_{3,1}+S_{3,1,1}+\frac{1}{2}S_{4,1}+4S_5\Big]\Big)+16C_F{n_f}^2\Big(\frac{2}{9}(\mathbf{N_{-2}}-\mathbf{N_-}-\mathbf{N_+}+\mathbf{N_{+2}})\Big[S_{1,1,1}+\frac{5}{3}S_{1,1}$$

$$+\frac{2}{3}S_1\Big] + (\mathbf{N}_+ - \mathbf{N}_{+2})\Big[2S_1 - S_{1,1} + \frac{19}{9}S_2 - \frac{2}{3}S_{2,1}\Big] - (1-\mathbf{N}_+)\Big[2S_1 - S_{1,1} + S_{2,1} + \frac{2}{27}S_2\Big]$$
$$+ (\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[\frac{77}{54}S_1 - \frac{63}{54}S_{1,1} - \frac{37}{27}S_2 + \frac{1}{6}S_{1,1,1}\Big] + \frac{1}{3}(\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{5}{3}S_{2,1} - S_{2,1,1} + \frac{29}{6}S_3$$
$$- 2S_{3,1} - S_4\Big]\Big) + 16C_F{}^2n_f\Big((\mathbf{N}_+ - \mathbf{N}_{+2})\Big[\frac{16}{3}S_{3,1} + \frac{13}{54}S_2 - \frac{163}{12}S_1 - \frac{85}{12}S_{1,1} + \frac{28}{9}S_{2,1} - \frac{22}{3}S_3$$
$$+ 4S_{2,2} - \frac{4}{3}S_{2,1,1} + 3S_{1,2} - \frac{22}{3}S_4\Big] - \frac{1}{3}(4\mathbf{N}_{-2} - \mathbf{N}_- - \mathbf{N}_+ + 4\mathbf{N}_{+2} - 6)\Big[3S_1\zeta_3 - S_{1,1,1} - S_{1,1,2}$$
$$+ S_{1,1,1,1} - \frac{1}{2}S_{1,3}\Big] + (\mathbf{N}_{-2} + \mathbf{N}_{+2})\Big[\frac{55}{12}S_1 - \frac{523}{108}S_{1,1} - \frac{23}{9}S_{1,2}\Big] - \frac{55}{6}S_1 + \frac{46}{9}S_{1,2} + \frac{523}{54}S_{1,1}$$
$$+ (1-\mathbf{N}_+)\Big[\frac{298}{27}S_1 - \frac{121}{9}S_{1,2} + \frac{2707}{108}S_{1,1} - \frac{497}{18}S_2 - \frac{63}{4}S_{2,1} + \frac{5}{6}S_{1,1,1} + 5S_{2,2} + \frac{181}{12}S_3 - S_4$$
$$- S_{2,1,1} + 5S_{3,1}\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{47}{9}S_{1,2} - \frac{971}{108}S_{1,1} + \frac{275}{216}S_1 - \frac{755}{72}S_2 - \frac{5}{12}S_{1,1,1} + 6S_2\zeta_3 - S_{2,3}$$
$$+ 17S_{2,1} + 2S_{2,1,1,1} - 2S_{2,1,2} - 3S_{2,1,1} + 2S_{2,2} - \frac{32}{3}S_3 - 2S_{3,1} - S_{3,1,1} + 4S_{3,2} - \frac{3}{2}S_4 + 6S_{4,1}$$
$$- 4S_5\Big]\Big)\,. \tag{3.10}$$

The non-singlet part of $\gamma_{\rm qq}^{(2)}(N)$ can be found in Eq. (3.7) of Ref. [38]. The third-order results for the off-diagonal anomalous dimensions $\gamma_{\rm qg}(N)$ and $\gamma_{\rm gq}(N)$ in Eq. (2.2) are given by

$$\gamma_{\rm qg}^{(2)}(N) = 16C_AC_Fn_f\Big((\mathbf{N}_- + 4\mathbf{N}_+ - 2\mathbf{N}_{+2} - 3)\Big[\frac{31}{2}S_1\zeta_3 - \frac{3997}{96}S_1 - \frac{11}{2}S_{1,-4} + 6S_{1,-3,1}$$
$$- \frac{3}{2}S_{1,-3} - \frac{9}{2}S_{1,-2} - 3S_{1,-2,-2} - \frac{5}{2}S_{1,-2,1} - 2S_{1,-2,1,1} + 2S_{1,-2,2} - \frac{2405}{216}S_{1,1} + 6S_{1,1,-3}$$
$$+ 3S_{1,1}\zeta_3 + \frac{5}{2}S_{1,1,-2} - 6S_{1,1,-2,1} - \frac{128}{9}S_{1,1,1} - 6S_{1,1,1,-2} - \frac{13}{3}S_{1,1,1,1} - 4S_{1,1,1,1,1} - 3S_{1,1,1,2}$$
$$- \frac{35}{12}S_{1,1,2} + 3S_{1,1,2,1} + S_{1,1,3} + \frac{53}{8}S_{1,2} + 3S_{1,2,-2} + \frac{15}{4}S_{1,2,1} + 6S_{1,2,1,1} - 6S_{1,3,1} - \frac{2833}{216}S_2$$
$$+ \frac{3}{2}S_{1,4} + 3S_2\zeta_3 - 6S_{2,-3} - \frac{5}{2}S_{2,-2} + 6S_{2,-2,1} + \frac{49}{4}S_{2,1} + 6S_{2,1,-2} - 6S_{2,1,1} + 3S_{2,1,2} - S_{2,2,1}$$
$$+ 2S_{2,1,1,1} + \frac{49}{4}S_{2,2} - 3S_{2,3} - \frac{551}{72}S_3 + \frac{173}{12}S_{3,1} - 2S_{3,1,1} - \frac{79}{6}S_4 + 2S_{4,1}\Big] + (\mathbf{N}_{-2} - 1)\Big[\frac{55}{12}S_1$$
$$- 4S_1\zeta_3 - \frac{371}{108}S_{1,1} + \frac{23}{9}S_{1,1,1} - \frac{2}{3}S_{1,1,1,1} + \frac{4}{3}S_{1,1,2} - \frac{23}{9}S_{1,2} + \frac{2}{3}S_{1,3}\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{8543}{192}S_1$$
$$- \frac{71}{2}S_1\zeta_3 - S_{1,-3} + 23S_{1,-2} - \frac{9}{2}S_{1,-2,1} + \frac{1301}{216}S_{1,1} + \frac{13}{2}S_{1,1,-2} + \frac{109}{18}S_{1,1,1} - \frac{5}{2}S_{1,2,1} + 4S_{3,2}$$
$$+ \frac{55}{6}S_{1,3} + \frac{23}{6}S_{1,1,1,1} + \frac{4}{3}S_{1,1,2} - \frac{235}{72}S_{1,2} + \frac{55}{8}S_2 + 9S_2\zeta_3 - \frac{21}{2}S_{2,-2} - \frac{269}{36}S_{2,1} - 4S_{2,1,-2}$$
$$+ 2S_{2,-3} + \frac{83}{12}S_{2,1,1} + \frac{3}{2}S_{2,1,1,1} - 3S_{2,1,2} - \frac{41}{4}S_{2,2} + S_{2,2,1} - \frac{5}{2}S_{2,3} - \frac{55}{48}S_3 + 3S_{3,-2} - \frac{143}{12}S_{3,1}$$
$$- 2S_{3,1,1} + \frac{49}{4}S_4 + 4S_{4,1} - 2S_5\Big] + (1-\mathbf{N}_+)\Big[\frac{145}{2}S_1\zeta_3 - \frac{3571}{64}S_1 + 2S_{1,-3} - \frac{58}{3}S_{1,3} - \frac{25}{9}S_{1,1,1}$$
$$+ \frac{23}{2}S_{1,-2,1} + \frac{335}{216}S_{1,1} - \frac{31}{2}S_{1,1,-2} - \frac{11}{3}S_{1,1,1,1} - \frac{5}{3}S_{1,1,2} + \frac{245}{72}S_{1,2} + \frac{3}{2}S_{2,1,1,1} + 8S_{4,1} - 2S_5$$

$$+\frac{1}{2}S_{1,2,1}-\frac{83}{2}S_{1,-2}+27S_2\zeta_3-8S_{2,-3}+\frac{3}{2}S_{2,-2}+8S_{2,-2,1}-\frac{183}{4}S_4+8S_{2,1,-2}-\frac{117}{4}S_{2,1,1}$$
$$-3S_{2,1,2}+\frac{157}{4}S_{2,2}-3S_{2,2,1}-\frac{9}{2}S_{2,3}-\frac{581}{16}S_3-S_{3,-2}+\frac{237}{4}S_{3,1}-8S_{3,1,1}+8S_{3,2}+\frac{73}{3}S_{2,1}$$
$$-\frac{4319}{48}S_2\Big]\Big)+16C_An_f^2\Big(\frac{1}{6}(\mathbf{N}_-+4\mathbf{N}_+-2\mathbf{N}_{+\mathbf{2}}-3)\Big[\frac{175}{27}S_1-2S_{1,-3}+\frac{7}{3}S_{1,-2}-\frac{7}{9}S_{1,1}+\frac{4}{3}S_3$$
$$+\frac{7}{3}S_{1,1,1}-S_{1,1,1,1}+S_{1,1,2}-S_{1,2,1}-S_{1,3}+\frac{229}{18}S_2\Big]+\frac{1}{6}(\mathbf{N}_--1)\Big[S_{1,-2}-\frac{4}{3}S_{1,1}+S_{1,1,1}\Big]$$
$$-\frac{53}{162}(\mathbf{N}_{-\mathbf{2}}-1)S_1-(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{149}{648}S_1+\frac{7}{4}S_2-\frac{2}{9}S_3-\frac{1}{3}S_4\Big]-(1-\mathbf{N}_+)\Big[\frac{473}{648}S_1-\frac{169}{36}S_2$$
$$+\frac{1}{6}S_{2,1}-\frac{43}{18}S_3+\frac{5}{3}S_4\Big]\Big)+16C_A{}^2n_f\Big((\mathbf{N}_-+4\mathbf{N}_+-2\mathbf{N}_{+\mathbf{2}}-3)\Big[\frac{3220}{27}S_1-\frac{3}{2}S_{1,-4}+\frac{277}{12}S_{1,-2}$$
$$-\frac{31}{2}S_1\zeta_3+\frac{61}{6}S_{1,-3}+2S_{1,-3,1}+3S_{1,-2,-2}-\frac{8}{3}S_{1,-2,1}+2S_{1,-2,1,1}-2S_{1,1,-2,1}+6S_{1,1,1,-2}$$
$$-\frac{95}{54}S_{1,1}-3S_{1,1}\zeta_3+2S_{1,1,-3}+\frac{20}{3}S_{1,1,-2}+\frac{47}{8}S_{1,1,1}+\frac{4}{3}S_{1,1,1,1}+2S_{1,1,1,1,1}-S_{1,1,3}+\frac{37}{6}S_{1,3}$$
$$+4S_{1,1,1,2}+\frac{21}{4}S_{1,1,2}+2S_{1,1,2,1}+\frac{69}{8}S_{1,2}-S_{1,2,-2}+\frac{23}{12}S_{1,2,1}-3S_{4,1}+2S_{2,3}-\frac{5}{2}S_{1,4}+95S_2$$
$$-3S_2\zeta_3-S_{2,-3}+\frac{25}{2}S_{2,-2}+2S_{2,-2,1}-\frac{155}{72}S_{2,1}+\frac{53}{6}S_{2,1,1}+3S_{1,3,1}-\frac{5}{12}S_{2,2}+\frac{31}{12}S_{3,1}-3S_4$$
$$+\frac{2561}{72}S_3-2S_{1,2,2}\Big]+(\mathbf{N}_{-\mathbf{2}}-1)\Big[4S_1\zeta_3-\frac{2351}{108}S_1-\frac{8}{3}S_{1,-3}-\frac{4}{3}S_{1,1,2}-\frac{52}{9}S_{1,-2}+\frac{4}{3}S_{1,-2,1}$$
$$+\frac{161}{36}S_{1,1}-\frac{4}{3}S_{1,1,-2}-\frac{10}{9}S_{1,1,1}+\frac{2}{3}S_{1,1,1,1}-\frac{3}{2}S_{1,2}+\frac{56}{27}S_2-\frac{20}{9}S_{2,1}-2S_{1,3}-\frac{2}{3}S_{2,1,1}\Big]$$
$$-(\mathbf{N}_--1)S_{1,2,1}+(\mathbf{N}_--\mathbf{N}_+)\Big[22S_1\zeta_3-\frac{1759}{24}S_1-\frac{13}{6}S_{1,-3}-\frac{799}{36}S_{1,-2}-\frac{8}{3}S_{1,-2,1}-\frac{21}{2}S_{1,3}$$
$$-\frac{37}{3}S_{1,1,-2}-\frac{425}{72}S_{1,1,1}-\frac{7}{12}S_{1,1,1,1}-\frac{35}{6}S_{1,1,2}-\frac{217}{24}S_{1,2}-\frac{1385}{18}S_2+\frac{593}{36}S_{1,1}-\frac{49}{6}S_{2,1,1}$$
$$+\frac{5}{2}S_{2,-3}-8S_{2,-2}-\frac{209}{24}S_{2,1}+3S_{2,1,-2}-S_{2,1,1,1}+2S_{2,1,2}+\frac{17}{12}S_{2,2}-6S_2\zeta_3+\frac{13}{4}S_{2,3}+\frac{9}{4}S_{4,1}$$
$$-\frac{1363}{72}S_3+\frac{9}{2}S_{3,-2}+\frac{1}{6}S_{3,1}+3S_{3,1,1}+\frac{25}{6}S_4+4S_5\Big]+(1-\mathbf{N}_+)\Big[\frac{15}{4}S_{2,2}+\frac{1783}{24}S_1-41S_1\zeta_3$$
$$+\frac{4}{3}S_{1,-3}+\frac{995}{36}S_{1,-2}+\frac{16}{3}S_{1,-2,1}-\frac{2731}{72}S_{1,1}+\frac{62}{3}S_{1,1,-2}+\frac{319}{72}S_{1,1,1}-\frac{7}{12}S_{1,1,1,1}+\frac{49}{6}S_{1,1,2}$$
$$+\frac{287}{24}S_{1,2}+\frac{79}{4}S_{1,3}+\frac{73141}{216}S_2-24S_2\zeta_3+\frac{17}{2}S_{2,-3}+\frac{93}{2}S_{2,-2}-\frac{1567}{72}S_{2,1}-\frac{34}{3}S_4-\frac{15}{4}S_{4,1}$$
$$+7S_{2,1,-2}+\frac{167}{6}S_{2,1,1}-3S_{2,1,1,1}+6S_{2,1,2}+\frac{53}{4}S_{2,3}+\frac{7385}{72}S_3-\frac{7}{2}S_{3,-2}+\frac{47}{4}S_{3,1}+5S_{3,1,1}$$
$$-19S_5\Big]\Big)+16C_Fn_f^2\Big((\mathbf{N}_-+4\mathbf{N}_+-2\mathbf{N}_{+\mathbf{2}}-3)\Big[\frac{2303}{324}S_1+\frac{7}{54}S_{1,1}-\frac{7}{18}S_{1,1,1}-\frac{1}{6}S_{2,1,1}-S_4$$
$$+\frac{4}{9}S_{1,2}+\frac{1}{6}S_{1,1,1,1}-\frac{1}{3}S_{1,3}+\frac{35}{18}S_2+\frac{7}{18}S_{2,1}-\frac{11}{9}S_3\Big]-\frac{1}{6}(\mathbf{N}_--1)\Big[S_{1,1,1}+S_{1,2}-S_{2,1}\Big]$$
$$-(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{59963}{2592}S_1-\frac{7}{18}S_{1,1}-\frac{251}{27}S_2+\frac{199}{24}S_3-\frac{25}{6}S_4+2S_5\Big]+(1-\mathbf{N}_+)\Big[\frac{163}{24}S_2+6S_5$$
$$+\frac{96277}{2592}S_1-\frac{17}{36}S_{1,1}-\frac{7}{24}S_3-\frac{19}{2}S_4\Big]+\frac{77}{81}(\mathbf{N}_{-\mathbf{2}}-1)S_1\Big)+16C_F{}^2n_f\Big((\mathbf{N}_--1)\Big[4S_{2,1,-2}$$

$$
\begin{aligned}
&+\frac{1}{2}S_{2,2}\Big]+(\mathbf{N}_-+4\mathbf{N}_+-2\mathbf{N}_{+2}-3)\Big[\frac{81}{32}S_1-S_{1,-4}+5S_{1,-3}-\frac{5}{2}S_{1,-2}+2S_{1,-2,-2}+4S_{1,1,-3}\\
&+\frac{87}{8}S_{1,1}-4S_{1,1,-2}+\frac{61}{8}S_{1,1,1}+3S_{1,1,1,1}+2S_{1,1,1,1,1}-S_{1,1,1,2}-\frac{5}{2}S_{1,1,2}+7S_{1,3,1}-3S_{1,4}\\
&-5S_{1,1,2,1}+4S_{1,1,3}-\frac{17}{2}S_{1,2}+2S_{1,2,-2}-\frac{11}{2}S_{1,2,1}-6S_{1,2,1,1}+6S_{1,2,2}+\frac{5}{2}S_{1,3}-\frac{87}{8}S_2+4S_5\\
&-4S_{2,-3}+4S_{2,-2}-\frac{61}{8}S_{2,1}-3S_{2,1,1}-2S_{2,1,1,1}+S_{2,1,2}+\frac{5}{2}S_{2,2}+5S_{2,2,1}-4S_{2,3}+6S_{3,1,1}\\
&+11S_3-4S_{3,-2}+\frac{11}{2}S_{3,1}-6S_{3,2}-\frac{15}{2}S_4-7S_{4,1}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{801}{64}S_1+\frac{27}{2}S_1\zeta_3-\frac{3}{2}S_{1,2}\\
&+3S_{1,-3}-\frac{35}{2}S_{1,-2}-\frac{103}{8}S_{1,1}-4S_{1,1,-2}-\frac{7}{8}S_{1,1,1}-\frac{13}{4}S_{1,1,1,1}+\frac{9}{2}S_{1,1,2}+\frac{7}{2}S_{1,2,1}-\frac{1}{2}S_{2,1,1,1}\\
&-\frac{9}{2}S_{1,3}+\frac{1}{4}S_2-3S_2\zeta_3+7S_{2,-2}+\frac{27}{8}S_{2,1}+\frac{3}{4}S_{2,1,1}+S_{2,1,2}-S_{2,2,1}+3S_{2,3}-\frac{87}{16}S_3-S_{3,1,1}\\
&-\frac{13}{4}S_{3,1}+2S_{3,2}+\frac{27}{4}S_4+\frac{7}{2}S_{4,1}-3S_5\Big]+(1-\mathbf{N}_+)\Big[17S_{1,1}-\frac{1759}{64}S_1-\frac{63}{2}S_1\zeta_3+\frac{17}{4}S_{1,1,1,1}\\
&-11S_{1,-3}+\frac{71}{2}S_{1,-2}+12S_{1,1,-2}-\frac{19}{8}S_{1,1,1}-\frac{13}{2}S_{1,1,2}+\frac{13}{2}S_{1,2}-\frac{3}{2}S_{1,2,1}+\frac{13}{2}S_{1,3}-3S_2\zeta_3\\
&-\frac{409}{16}S_2-4S_{2,-3}-S_{2,-2}+\frac{59}{8}S_{2,1}-\frac{1}{4}S_{2,1,1}+\frac{3}{2}S_{2,1,1,1}-3S_{2,1,2}+3S_{2,2,1}-5S_{2,3}+\frac{565}{16}S_3\\
&-8S_{3,-2}+\frac{17}{4}S_{3,1}+3S_{3,1,1}-6S_{3,2}-\frac{103}{4}S_4-\frac{21}{2}S_{4,1}+11S_5\Big]\Big) \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_{\rm gq}^{(2)}(N) \;=\; &16C_AC_Fn_f\Big((2\mathbf{N}_{-2}-4\mathbf{N}_--\mathbf{N}_++3)\Big[\frac{967}{144}S_1-2S_1\zeta_3+\frac{2}{3}S_{1,-3}+\frac{41}{18}S_{1,-2}-\frac{1}{3}S_{1,3}\\
&-\frac{2}{3}S_{1,-2,1}+\frac{251}{108}S_{1,1}-\frac{4}{3}S_{1,1,-2}-\frac{13}{4}S_{1,1,1}+\frac{5}{6}S_{1,1,1,1}-\frac{5}{6}S_{1,1,2}+\frac{10}{9}S_{1,2}-\frac{5}{6}S_{1,2,1}-\frac{151}{108}S_2\\
&-\frac{1}{3}S_{2,-2}+\frac{10}{9}S_{2,1}-\frac{5}{6}S_{2,1,1}+\frac{1}{3}S_{2,2}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{331}{72}S_1-4S_{2,-2}+\frac{28}{9}S_{1,-2}-\frac{11}{18}S_{1,1,1}\\
&+\frac{4}{3}S_{3,1}-\frac{2}{9}S_{2,1}+\frac{53}{54}S_{1,1}-\frac{733}{54}S_2+\frac{4}{3}S_{2,1,1}-\frac{22}{3}S_3\Big]+(1-\mathbf{N}_+)\Big[\frac{10}{3}S_{2,-2}+\frac{1}{12}S_{2,1}-\frac{1}{4}S_{1,1}\\
&-\frac{17}{3}S_{1,-2}-\frac{137}{144}S_1+\frac{5}{6}S_{1,2}+\frac{1}{4}S_{1,1,1}+\frac{565}{36}S_2-S_{2,1,1}+\frac{35}{12}S_3-\frac{2}{3}S_{3,1}\Big]-\frac{2}{9}(\mathbf{N}_--\mathbf{N}_{+2})\Big[S_3\\
&-3S_{2,1}+\frac{131}{4}S_1+S_{1,-2}-\frac{25}{6}S_{1,1}-S_{1,1,1}+\frac{125}{6}S_2\Big]-\frac{2}{3}(\mathbf{N}_--1)S_4\Big)+16C_AC_F{}^2\Big((2\mathbf{N}_{-2}\\
&-4\mathbf{N}_--\mathbf{N}_++3)\Big[\frac{163}{32}S_1-\frac{3}{2}S_{1,-4}-\frac{3}{2}S_{1,-3}+\frac{6503}{432}S_{1,1}-5S_{1,-2,-2}-3S_{1,-2,1}-4S_{1,1,1,1,1}\\
&+S_{1,-2}+2S_{1,-2,1,1}-9S_{1,1}\zeta_3-4S_{1,1,-3}+3S_{1,1,-2}+2S_{1,1,-2,1}+5S_{1,1,3}+6S_{1,1,1,-2}+S_{1,1,2,1}\\
&+3S_{1,1,1,2}+\frac{35}{3}S_{1,1,1,1}+\frac{2}{9}S_{1,1,1}-\frac{1}{12}S_{1,1,2}-\frac{191}{24}S_{1,2}-3S_{1,2,-2}-\frac{41}{12}S_{1,2,1}+4S_{1,3}-4S_{2,1}\\
&+2S_{1,2,1,1}-\frac{5}{2}S_{1,4}-\frac{9}{2}S_{2,1,1}+2S_{2,1,1,1}+S_{2,1,2}+3S_{2,2}+S_{2,2,1}-2S_{2,3}\Big]+(\mathbf{N}_--\mathbf{N}_{+2})\Big[6S_{2,1}\\
&+\frac{173}{54}S_{1,1}-\frac{26}{9}S_{1,1,1}-\frac{2}{3}S_{1,1,1,1}-\frac{335}{54}S_2+\frac{7}{2}S_1-2S_{2,1,1}-\frac{28}{9}S_3+\frac{8}{3}S_4\Big]-6(\mathbf{N}_--1)\Big[S_{2,-3}
\end{aligned}
$$

$$- 2S_{2,1,-2} + 3S_2\zeta_3\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[36S_1\zeta_3 - \frac{9703}{288}S_1 + 12S_{1,-3} - 36S_{1,-2} - \frac{2263}{216}S_{1,1} + 4S_{3,2}$$

$$-16S_{1,3} - 24S_{1,1,-2} - \frac{101}{36}S_{1,1,1} + \frac{5}{6}S_{1,1,1,1} - \frac{23}{12}S_{1,2} + 2S_{1,2,1} + \frac{12605}{432}S_2 + 36S_{2,-2} + \frac{79}{6}S_4$$

$$+\frac{55}{18}S_{2,1} - \frac{10}{3}S_{2,1,1} - 3S_{2,1,1,1} + \frac{17}{3}S_{2,2} - 2S_{2,2,1} - \frac{119}{8}S_3 - 14S_{3,-2} + \frac{47}{3}S_{3,1} - 7S_{3,1,1} + 4S_5$$

$$+10S_{2,3}\Big] + (1-\mathbf{N}_+)\Big[\frac{2005}{64}S_1 - \frac{117}{2}S_1\zeta_3 - \frac{39}{2}S_{1,-3} + \frac{315}{4}S_{1,-2} - S_{1,-2,1} + 3S_{1,1,1,1} - 2S_{4,1}$$

$$+\frac{2525}{144}S_{1,1} + 40S_{1,1,-2} - \frac{55}{12}S_{1,1,1} - 3S_{1,1,2} + \frac{197}{24}S_{1,2} - \frac{11}{2}S_{1,2,1} + \frac{53}{2}S_{1,3} + \frac{13}{2}S_{3,1,1} - 4S_{2,2}$$

$$-\frac{2831}{72}S_2 - 37S_{2,-2} + 13S_{3,-2} + \frac{1}{2}S_{2,1,1} + \frac{3}{2}S_{2,1,1,1} - \frac{15}{2}S_{3,1} + 3S_{2,2,1} - 12S_{2,3} + \frac{2407}{48}S_3$$

$$+\frac{3}{2}S_{2,1} - 6S_{3,2} - \frac{57}{2}S_4\Big]\Big) + 16C_A{}^2C_F\Big((2\mathbf{N}_{-2} - 4\mathbf{N}_- - \mathbf{N}_+ + 3)\Big[\frac{138305}{2592}S_1 - 2S_{1,-2,1,1}$$

$$-\frac{11}{2}S_{1,-4} + \frac{49}{6}S_{1,-3} + S_{1,-2,-2} - 10S_{1,1,-2,1} + \frac{109}{12}S_{1,-2} - \frac{3}{2}S_{1,-2,1} + 2S_{1,-2,2} - \frac{3379}{216}S_{1,1}$$

$$+8S_{1,-3,1} + 3S_{1,1}\zeta_3 + 12S_{1,1,-3} + \frac{19}{2}S_{1,1,-2} + 2S_{1,1,1,1,1} + \frac{65}{24}S_{1,1,1} - 6S_{1,1,1,-2} - \frac{43}{6}S_{1,1,1,1}$$

$$-4S_{1,1,1,2} + \frac{55}{12}S_{1,1,2} - 4S_{1,1,2,1} + 2S_{1,1,3} + \frac{71}{24}S_{1,2} + 5S_{1,2,-2} + \frac{55}{12}S_{1,2,1} - 4S_{1,2,1,1} + 6S_{1,2,2}$$

$$+\frac{11}{2}S_{1,3} + 4S_{1,3,1} - \frac{3}{2}S_{1,4} - \frac{395}{54}S_2 - 7S_{2,-3} - \frac{11}{6}S_{2,-2} + 4S_{2,-2,1} + 2S_{2,1,-2} - 2S_{2,1,1,1}$$

$$+\frac{17}{3}S_{2,1,1} + 3S_{2,1,2} - \frac{1}{3}S_{2,2} + 3S_{2,2,1} - 3S_{2,3} + 4S_{3,1,1} - 4S_{3,2}\Big] + (\mathbf{N}_- - 1)\Big[6S_2\zeta_3 - 8S_{2,-2,1}\Big]$$

$$+(\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{57595}{1296}S_1 - 12S_1\zeta_3 - \frac{31}{6}S_{1,-3} - \frac{143}{6}S_{2,-2} + \frac{25}{3}S_{1,-2,1} - \frac{689}{54}S_{1,1} + \frac{50}{3}S_{1,1,-2}$$

$$+\frac{11}{18}S_{1,1,1} - \frac{11}{6}S_{1,1,1,1} + \frac{229}{36}S_{1,2} + \frac{113}{12}S_{1,3} - \frac{2200}{27}S_2 - 3S_{2,-3} - 12S_{3,2} + 9S_{1,-2} + \frac{31}{2}S_{2,1}$$

$$-18S_{2,1,-2} + \frac{13}{6}S_{2,1,1} + 4S_{2,1,1,1} - \frac{37}{3}S_{2,2} - \frac{25}{2}S_{2,3} - 31S_3 - 9S_{3,-2} - \frac{463}{12}S_{3,1} + 4S_{3,1,1} + S_4$$

$$-\frac{13}{2}S_{4,1} - 8S_5\Big] + (\mathbf{N}_- - \mathbf{N}_{+2})\Big[\frac{4}{3}S_{1,-2,1} - \frac{2105}{81}S_1 - \frac{8}{3}S_{1,-3} - 10S_{1,-2} - \frac{109}{27}S_{1,1} - \frac{4}{3}S_{1,1,-2}$$

$$+\frac{37}{9}S_{1,1,1} + \frac{2}{3}S_{1,1,1,1} - \frac{145}{18}S_{1,2} - \frac{4}{3}S_{1,3} - \frac{584}{27}S_2 - 4S_{2,-2} - \frac{104}{9}S_{2,1} + \frac{8}{3}S_{2,1,1} - \frac{14}{3}S_{2,2}$$

$$-\frac{77}{18}S_3 - 6S_{3,1} + \frac{14}{3}S_4\Big] + (1-\mathbf{N}_+)\Big[\frac{39}{2}S_1\zeta_3 - \frac{29843}{864}S_1 + \frac{17}{2}S_{3,-2} + \frac{145}{6}S_{3,1} - \frac{29}{2}S_{1,-2,1}$$

$$-\frac{25}{2}S_{1,-2} - \frac{57}{2}S_{1,1,-2} - \frac{13}{12}S_{1,1,1} + \frac{5}{4}S_{1,1,1,1} + 4S_{1,1,2} - \frac{97}{24}S_{1,2} + 4S_{1,2,1} - \frac{41}{2}S_{1,3} + \frac{7417}{72}S_2$$

$$+\frac{1}{2}S_{2,-3} + \frac{92}{3}S_{2,-2} - \frac{53}{12}S_{2,1} + 15S_{2,1,-2} - \frac{9}{4}S_{2,1,1} - 3S_{2,1,1,1} + 5S_{2,2} + \frac{1}{4}S_{4,1} + 38S_3 + 8S_{3,2}$$

$$+\frac{41}{4}S_{2,3} + \frac{9}{2}S_{1,-3} + \frac{92}{3}S_{1,1} - 2S_{3,1,1} + \frac{25}{3}S_4 + \frac{31}{2}S_5\Big]\Big) + 16C_Fn_f{}^2\Big(\frac{1}{6}(1-\mathbf{N}_+)\Big[\frac{5}{3}S_1 - S_{1,1}\Big]$$

$$-\frac{1}{6}(2\mathbf{N}_{-2} - 4\mathbf{N}_- - \mathbf{N}_+ + 3)\Big[\frac{1}{3}S_1 + \frac{5}{3}S_{1,1} - S_{1,1,1}\Big]\Big) + 16C_F{}^2n_f\Big((\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{2}{3}S_{1,2} - \frac{371}{432}S_1$$

$$-\frac{35}{9}S_{1,-2} - \frac{1}{9}S_{1,1} - \frac{1}{3}S_{1,1,1} + \frac{1057}{72}S_2 + \frac{16}{3}S_{2,-2} - \frac{8}{9}S_{2,1} + \frac{1}{3}S_{2,1,1} - \frac{2}{3}S_{2,2} + \frac{181}{12}S_3 - \frac{2}{3}S_{3,1}$$

$$
\begin{aligned}
&-\frac{1}{3}S_4+4S_5\Big]+(2\mathbf{N}_{-2}-4\mathbf{N}_{-}-\mathbf{N}_{+}+3)\Big[2S_1\zeta_3-\frac{1}{3}S_{1,2,1}-\frac{31}{18}S_{1,-2}+\frac{95}{54}S_2+\frac{1}{2}S_{1,3}+\frac{1}{3}S_{1,2}\\
&-\frac{1625}{144}S_1-\frac{5}{6}S_{1,1,1,1}-\frac{2}{3}S_{1,1,2}-\frac{7}{108}S_{1,1}+\frac{83}{36}S_{1,1,1}+\frac{2}{3}S_{2,-2}\Big]-\frac{4}{9}(\mathbf{N}_{-}-\mathbf{N}_{+2})\Big[\frac{7}{2}S_1-\frac{11}{6}S_2\\
&-S_{1,-2}-S_3\Big]+(1-\mathbf{N}_{+})\Big[\frac{15137}{864}S_1+\frac{49}{6}S_{1,-2}-\frac{107}{36}S_{1,1}+\frac{19}{12}S_{1,1,1}-\frac{5}{6}S_{1,2}-10S_2-4S_{2,-2}\\
&-\frac{1}{2}S_{2,1,1}+S_{2,2}-\frac{155}{24}S_3+S_{3,1}+S_4-6S_5\Big]\Big)+16{C_F}^3\Big((2\mathbf{N}_{-2}-4\mathbf{N}_{-}-\mathbf{N}_{+}+3)\Big[6S_{1,-2,-2}\\
&-\frac{47}{16}S_1-S_{1,-4}-\frac{7}{2}S_{1,-2}+6S_{1,-3}-\frac{47}{16}S_{1,1}+6S_{1,1}\zeta_3+4S_{1,1,-3}-6S_{1,1,-2}-3S_{1,1,2}-3S_{1,1,3}\\
&-\frac{23}{8}S_{1,1,1}-\frac{9}{2}S_{1,1,1,1}+2S_{1,1,1,1,1}+S_{1,1,1,2}+3S_{1,1,2,1}+\frac{7}{4}S_{1,2}+2S_{1,2,-2}+2S_{1,2,1,1}-2S_{1,2,2}\\
&-\frac{3}{2}S_{1,3}\Big]+2(\mathbf{N}_{-}-1)\Big[6S_2\zeta_3-4S_{2,1,-2}+8S_{3,-2}\Big]+(\mathbf{N}_{-}-\mathbf{N}_{+})\Big[\frac{287}{32}S_1-24S_1\zeta_3+S_{1,1,1,1}\\
&-12S_{1,-3}+36S_{1,-2}+\frac{111}{8}S_{1,1}+16S_{1,1,-2}+\frac{1}{4}S_{1,1,1}+\frac{9}{2}S_{1,2}-2S_{1,2,1}+9S_{1,3}-4S_{3,1}-5S_{2,3}\\
&+3S_{3,1,1}-\frac{91}{16}S_2+8S_{2,-3}-30S_{2,-2}-\frac{41}{4}S_{2,1}+S_{2,1,1}-S_{2,1,1,1}+2S_{2,2,1}-\frac{35}{8}S_3-S_4+3S_{4,1}\\
&-2S_5\Big]+(1-\mathbf{N}_{+})\Big[39S_1\zeta_3-\frac{749}{64}S_1+20S_{1,-3}-\frac{141}{2}S_{1,-2}-\frac{433}{16}S_{1,1}+6S_{1,1,1}-\frac{17}{4}S_{1,1,1,1}\\
&-30S_{1,1,-2}-S_{1,1,2}-\frac{19}{4}S_{1,2}+\frac{3}{2}S_{1,2,1}-\frac{57}{4}S_{1,3}+21S_2-10S_{2,-3}+35S_{2,-2}-\frac{9}{2}S_{3,1,1}+\frac{37}{4}S_4\\
&+\frac{19}{4}S_{2,1}+\frac{9}{4}S_{2,1,1}+\frac{3}{2}S_{2,1,1,1}+3S_{2,2}-3S_{2,2,1}+\frac{11}{2}S_{2,3}-\frac{485}{16}S_3+\frac{27}{4}S_{3,1}-\frac{9}{2}S_{4,1}\Big]\Big)\ . \qquad (3.12)
\end{aligned}
$$

Finally the three-loop gluon-gluon anomalous dimension reads

$$
\begin{aligned}
\gamma_{\rm gg}^{(2)}(N) \;=\; & 16C_AC_Fn_f\Big(\frac{241}{288}+(\mathbf{N}_{-2}-2\mathbf{N}_{-}-2\mathbf{N}_{+}+\mathbf{N}_{+2}+3)\Big[4S_1\zeta_3-\frac{15331}{648}S_1-\frac{44}{9}S_{1,-2}\\
&-\frac{2}{3}S_{1,-3}+\frac{4}{3}S_{1,-2,1}-\frac{521}{108}S_{1,1}-\frac{16}{3}S_{1,1,-2}+\frac{1}{9}S_{1,1,1}-\frac{4}{3}S_{1,1,1,1}+\frac{4}{3}S_{1,1,2}-\frac{17}{18}S_{1,2}-\frac{8}{3}S_{1,3}\\
&+\frac{86}{27}S_2+\frac{4}{3}S_{2,-2}-\frac{2}{3}S_{2,1}+\frac{2}{3}S_{2,1,1}-\frac{4}{3}S_{2,2}\Big]+(\mathbf{N}_{-}+\mathbf{N}_{+}-2)\Big[17S_1\zeta_3+\frac{25}{3}S_{1,-3}-\frac{8}{3}S_{1,-2,1}\\
&-\frac{70}{3}S_{1,1,-2}+\frac{31}{36}S_{1,1,1}-\frac{7}{3}S_{1,1,1,1}+\frac{7}{3}S_{1,1,2}-\frac{55}{6}S_{1,3}\Big]+(\mathbf{N}_{-}-\mathbf{N}_{+})\Big[\frac{133}{18}S_{1,2}-\frac{221}{9}S_{1,-2}\\
&-\frac{673}{54}S_{1,1}+\frac{4948}{81}S_1-\frac{49}{108}S_2-12S_2\zeta_3-4S_{2,-3}+17S_{2,-2}+\frac{119}{6}S_{2,1}+16S_{2,1,-2}+6S_{4,1}\\
&-\frac{7}{6}S_{2,1,1}+2S_{2,1,1,1}-2S_{2,1,2}-S_{2,2}+7S_{2,3}+\frac{251}{12}S_3-\frac{10}{3}S_{3,1}-S_{3,1,1}+4S_{3,2}-\frac{29}{6}S_4+8S_5\Big]\\
&-8(\mathbf{N}_{-}-1)S_{3,-2}+(\mathbf{N}_{-}-\mathbf{N}_{+2})\Big[\frac{127}{18}S_3-\frac{511}{12}S_1-6S_{1,-2}-\frac{97}{12}S_{1,1}-3S_{1,2}+2S_{3,1}-\frac{103}{27}S_2\\
&-\frac{8}{3}S_{2,-2}-\frac{16}{9}S_{2,1}-\frac{2}{3}S_{2,2}\Big]+(1-\mathbf{N}_{+})\Big[\frac{1807}{324}S_1+\frac{604}{9}S_{1,-2}+\frac{5311}{108}S_{1,1}-\frac{52}{9}S_{1,2}-\frac{1667}{54}S_2\\
&-\frac{68}{3}S_{2,-2}-\frac{53}{4}S_{2,1}-\frac{7}{3}S_{2,1,1}+\frac{19}{6}S_{2,2}+\frac{67}{12}S_3+\frac{9}{2}S_{3,1}+\frac{33}{2}S_4-20S_5\Big]+\frac{6923}{324}S_1-2S_1\zeta_3\\
&+\frac{2}{3}S_{1,-3}+\frac{44}{9}S_{1,-2}-\frac{4}{3}S_{1,-2,1}+\frac{521}{108}S_{1,1}+\frac{16}{3}S_{1,1,-2}-\frac{1}{9}S_{1,1,1}+\frac{4}{3}S_{1,1,1,1}-\frac{4}{3}S_{1,1,2}+\frac{8}{3}S_{1,3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{17}{18}S_{1,2}-\frac{86}{27}S_2-\frac{4}{3}S_{2,-2}+\frac{2}{3}S_{2,1}-\frac{2}{3}S_{2,1,1}+\frac{4}{3}S_{2,2}\Big)+16C_A n_f^2\Big(\frac{11}{72}(1-\mathbf{N}_+)S_2-\frac{65}{162}S_1\\
&+\frac{13}{54}S_{1,1}-\frac{29}{288}+(\mathbf{N}_{-\mathbf{2}}-2\mathbf{N}_--2\mathbf{N}_++\mathbf{N}_{+\mathbf{2}}+3)\Big[\frac{59}{162}S_1-\frac{13}{54}S_{1,1}\Big]-\frac{1}{9}(\mathbf{N}_--\mathbf{N}_+)\Big[S_2\\
&-2S_{2,1}+S_3\Big]+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[\frac{47}{648}S_1-\frac{19}{216}S_{1,1}\Big]-\frac{13}{54}(\mathbf{N}_--\mathbf{N}_{+\mathbf{2}})S_2\Big)+16C_A{}^2n_f\Big(\frac{233}{288}\\
&+(\mathbf{N}_{-\mathbf{2}}-2\mathbf{N}_--2\mathbf{N}_++\mathbf{N}_{+\mathbf{2}}+3)\Big[\frac{1204}{81}S_1-4S_1\zeta_3-\frac{2}{3}S_{1,-3}+\frac{19}{3}S_{1,-2}+2S_{1,1,-2}+\frac{11}{3}S_{1,2}\\
&-\frac{2}{3}S_{1,-2,1}+\frac{205}{108}S_{1,1}-\frac{71}{27}S_2-\frac{2}{3}S_{2,-2}+\frac{11}{3}S_{2,1}\Big]+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[\frac{305}{18}S_{1,-2}-\frac{1405}{648}S_1\\
&-8S_1\zeta_3-\frac{31}{6}S_{1,-3}+\frac{4}{3}S_{1,-2,1}+\frac{2441}{216}S_{1,1}+9S_{1,1,-2}+\frac{4}{9}S_{1,2}+\frac{25}{12}S_{1,3}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{109}{108}S_2\\
&+6S_2\zeta_3+3S_{2,-3}-\frac{59}{6}S_{2,-2}-\frac{71}{12}S_{2,1}-6S_{2,1,-2}-\frac{2}{3}S_{2,2}-\frac{3}{2}S_{2,3}-\frac{64}{9}S_3+5S_{3,-2}+\frac{5}{12}S_{3,1}\\
&-2S_4-\frac{3}{2}S_{4,1}\Big]+(\mathbf{N}_--\mathbf{N}_{+\mathbf{2}})\Big[\frac{2}{3}S_{2,-2}-\frac{2243}{108}S_2+\frac{31}{9}S_3-\frac{2}{3}S_{3,1}\Big]+(1-\mathbf{N}_+)\Big[\frac{6815}{216}S_2+S_5\\
&+\frac{25}{3}S_{2,-2}-\frac{8}{9}S_{2,1}-\frac{473}{36}S_3-4S_{3,-2}-\frac{25}{6}S_{3,1}+\frac{31}{6}S_4\Big]-\frac{10}{9}S_{-3}-\frac{1}{3}S_{1,3}-\frac{5443}{324}S_1+2S_1\zeta_3\\
&+\frac{2}{3}S_{1,-3}-\frac{37}{9}S_{1,-2}+\frac{2}{3}S_{1,-2,1}-\frac{205}{108}S_{1,1}-2S_{1,1,-2}-\frac{13}{9}S_{1,2}+\frac{2}{3}S_{-2,-2}+\frac{151}{54}S_2+\frac{2}{3}S_{2,-2}\\
&-\frac{13}{9}S_{2,1}-\frac{10}{9}S_3-\frac{1}{3}S_{3,1}\Big)+16C_A{}^3\Big((\mathbf{N}_{-\mathbf{2}}-2\mathbf{N}_--2\mathbf{N}_++\mathbf{N}_{+\mathbf{2}}+3)\Big[\frac{73091}{648}S_1-16S_{1,-4}\\
&+\frac{88}{3}S_{1,-3}+16S_{1,-3,1}+\frac{85}{6}S_{1,-2}+4S_{1,-2,-2}-11S_{1,-2,1}+4S_{1,-2,2}-\frac{413}{108}S_{1,1}+24S_{1,1,-3}\\
&+11S_{1,1,-2}-16S_{1,1,-2,1}+8S_{1,1,3}-\frac{67}{9}S_{1,2}+8S_{1,2,-2}+8S_{1,2,2}+\frac{55}{3}S_{1,3}+8S_{1,3,1}-8S_{1,4}\\
&-\frac{395}{27}S_2-14S_{2,-3}-\frac{11}{3}S_{2,-2}+8S_{2,-2,1}-\frac{67}{9}S_{2,1}+4S_{2,1,-2}+8S_{2,1,2}+\frac{22}{3}S_{2,2}+8S_{2,2,1}\\
&-10S_{2,3}+8S_{3,1,1}-8S_{3,2}\Big]+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[14S_{1,-2,1}-\frac{713}{324}S_1-\frac{26}{3}S_{1,-3}-\frac{61}{9}S_{1,-2}\\
&-\frac{80}{27}S_{1,1}+14S_{1,1,-2}-\frac{109}{18}S_{1,2}+4S_{1,3}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[\frac{473}{216}S_2-12S_{2,-3}+5S_{2,-2}-2S_{2,1}\\
&-8S_{2,1,-2}+\frac{23}{3}S_{2,2}-10S_{2,3}+\frac{665}{36}S_3-20S_{3,-2}+\frac{34}{3}S_{3,1}-16S_{3,2}-21S_4-26S_{4,1}\Big]\\
&+(\mathbf{N}_--\mathbf{N}_{+\mathbf{2}})\Big[8S_{2,-3}-\frac{9533}{108}S_2-\frac{77}{3}S_{2,-2}-8S_{2,-2,1}-8S_{2,1,-2}-\frac{44}{3}S_{2,2}-\frac{1517}{18}S_3-8S_5\\
&+8S_{3,-2}-\frac{121}{3}S_{3,1}+4S_{3,2}+44S_4+16S_{4,1}\Big]+(1-\mathbf{N}_+)\Big[\frac{8533}{108}S_2+\frac{103}{3}S_{2,-2}+\frac{1579}{18}S_3\\
&-8S_{2,-3}+8S_{2,-2,1}+\frac{109}{9}S_{2,1}+8S_{2,1,-2}+\frac{28}{3}S_{2,2}-4S_{3,2}+8S_{3,-2}+\frac{71}{3}S_{3,1}-16S_{4,1}+36S_5\\
&-\frac{98}{3}S_4\Big]-\frac{79}{32}+4S_{-5}-8S_{-4,1}+\frac{67}{9}S_{-3}-4S_{-3,-2}-2S_{-3,2}-4S_{-2,-3}-\frac{67}{9}S_{1,2}+\frac{413}{108}S_{1,1}\\
&-\frac{11}{3}S_{-2,-2}+4S_{-2,-2,1}+4S_{-2,1,-2}-\frac{16619}{162}S_1-\frac{88}{3}S_{1,-3}-\frac{523}{18}S_{1,-2}+11S_{1,-2,1}-\frac{22}{3}S_{2,2}\\
&-11S_{1,1,-2}-\frac{33}{2}S_{1,3}+\frac{781}{54}S_2-4S_{2,-3}+\frac{11}{3}S_{2,-2}+4S_{2,-2,1}-\frac{67}{9}S_{2,1}+4S_{2,1,-2}+\frac{11}{6}S_{3,1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{67}{9}S_3-4S_{3,-2}-2S_{3,2}-8S_{4,1}+4S_5\Big)+16C_F{n_f}^2\Big((\mathbf{N_{-2}}-2\mathbf{N_-}-2\mathbf{N_+}+\mathbf{N_{+2}}+3)\Big[\frac{4}{9}S_{1,2}\\
&-\frac{77}{81}S_1+\frac{16}{27}S_{1,1}-\frac{2}{9}S_{1,1,1}\Big]+\frac{7}{9}(\mathbf{N_-}+\mathbf{N_+}-2)\Big[S_{1,2}-\frac{1}{2}S_{1,1,1}\Big]-\frac{11}{144}+\frac{2}{9}S_{1,1,1}-\frac{16}{27}S_{1,1}\\
&+\frac{77}{81}S_1-\frac{4}{9}S_{1,2}+\frac{1}{3}(\mathbf{N_-}-\mathbf{N_+})\Big[\frac{211}{27}S_1-\frac{139}{18}S_{1,1}+\frac{11}{3}S_2+S_{2,1}+S_{2,1,1}-2S_{2,2}-2S_{3,1}+S_4\\
&+\frac{5}{2}S_3\Big]-(\mathbf{N_-}-\mathbf{N_{+2}})\Big[2S_1-S_{1,1}+\frac{11}{27}S_2+\frac{2}{9}S_{2,1}-\frac{4}{9}S_3\Big]+(1-\mathbf{N_+})\Big[\frac{64}{81}S_1+\frac{58}{27}S_{1,1}+\frac{1}{3}S_3\\
&-\frac{10}{3}S_2+\frac{1}{3}S_{2,1}\Big]\Big)+16{C_F}^2n_f\Big(\frac{4}{3}(\mathbf{N_{-2}}-2\mathbf{N_-}-2\mathbf{N_+}+\mathbf{N_{+2}}+3)\Big[\frac{5}{4}S_{1,2}+\frac{1}{2}S_{1,3}-S_{1,1,1}\\
&-S_{1,-3}+2S_{1,1,-2}+\frac{31}{16}S_{1,1}+S_{1,1,1,1}-\frac{11}{16}S_1-S_{1,1,2}\Big]+(\mathbf{N_-}+\mathbf{N_+}-2)\Big[\frac{25}{6}S_{1,3}-9S_1\zeta_3\\
&-\frac{16}{3}S_{1,-3}+\frac{67}{3}S_{1,-2}-\frac{23}{12}S_{1,1,1}+\frac{7}{3}S_{1,1,1,1}-\frac{7}{3}S_{1,1,2}+\frac{32}{3}S_{1,1,-2}\Big]+(\mathbf{N_-}-\mathbf{N_+})\Big[2S_{4,1}-2S_5\\
&-\frac{773}{24}S_1-\frac{8}{3}S_{1,1}+\frac{163}{8}S_2+6S_2\zeta_3+4S_{2,-3}-\frac{32}{3}S_{2,-2}-\frac{8}{3}S_{2,1}-8S_{2,1,-2}+\frac{5}{3}S_{2,1,1}+2S_{2,1,2}\\
&-2S_{2,1,1,1}-\frac{11}{3}S_{2,2}-3S_{2,3}-\frac{23}{2}S_3-4S_{3,1}+S_{3,1,1}+\frac{13}{6}S_4+\frac{17}{2}S_{1,2}\Big]+(\mathbf{N_-}-\mathbf{N_{+2}})\Big[\frac{85}{12}S_{1,1}\\
&+\frac{163}{12}S_1-3S_{1,2}-\frac{9}{2}S_2+\frac{8}{3}S_{2,-2}-\frac{4}{3}S_{2,1}+\frac{4}{3}S_{2,1,1}-\frac{4}{3}S_{2,2}+\frac{14}{3}S_3-\frac{2}{3}S_4\Big]+(1-\mathbf{N_+})\Big[4S_4\\
&-\frac{191}{12}S_{1,1}-8S_{1,2}+\frac{20}{3}S_2+8S_{2,-2}+\frac{11}{4}S_{2,1}+S_{2,1,1}-3S_{2,2}-\frac{215}{12}S_3-S_{3,1}+\frac{71}{3}S_1\Big]\\
&+8(\mathbf{N_-}-1)S_{3,-2}-\frac{1}{16}+\frac{11}{12}S_1+\frac{4}{3}S_{1,-3}-\frac{31}{12}S_{1,1}-\frac{8}{3}S_{1,1,-2}+\frac{4}{3}S_{1,1,1}-\frac{4}{3}S_{1,1,1,1}+\frac{4}{3}S_{1,1,2}\\
&-\frac{5}{3}S_{1,2}-\frac{2}{3}S_{1,3}\Big)\,. \qquad (3.13)
\end{aligned}
$$

Eqs. (3.10) – (3.13) represent new results of this article, with the only exception of the $C_A n_f^2$ part of Eq. (3.13) which has been obtained by Bennett and Gracey in Ref. [61]. Our results agree with the even moments $N = 2, \ldots, 12$ computed before [25, 26] using the MINCER program [41, 42].

The results (3.5) – (3.13) are assembled, after inserting the QCD values $C_F = 4/3$ and $C_A = 3$ for the colour factors, in Figs. 1 and 2 for four active flavours and a typical value $\alpha_s = 0.2$ for the strong coupling constant. The NNLO corrections are markedly smaller than the NLO contributions under these circumstances. At $N > 2$ they amount to less than 2% and 1% for the large diagonal quantities γ_{qq} and γ_{gg}, respectively, while for the much smaller off-diagonal anomalous dimensions γ_{qg} and γ_{gq} values of up to 6% and 4% are reached. The relative NNLO corrections are very large at $N > 2$ for γ_{ps}, which is however completely negligible in this region of N.

For $N \to \infty$ the off-diagonal n-loop anomalous dimensions vanish like $\frac{1}{N}\ln^{2n-2}N$, while the diagonal quantities behave as [62]

$$
\gamma_{aa}^{(n-1)}(N) \;=\; A_n^{a}\,(\ln N + \gamma_e) - B_n^{a} - C_n^{a}\,\frac{\ln N}{N} + O\left(\frac{1}{N}\right)\,, \qquad (3.14)
$$

where γ_e is the Euler-Mascheroni constant. The leading large-N coefficients A_n^{q} of γ_{qq} have been

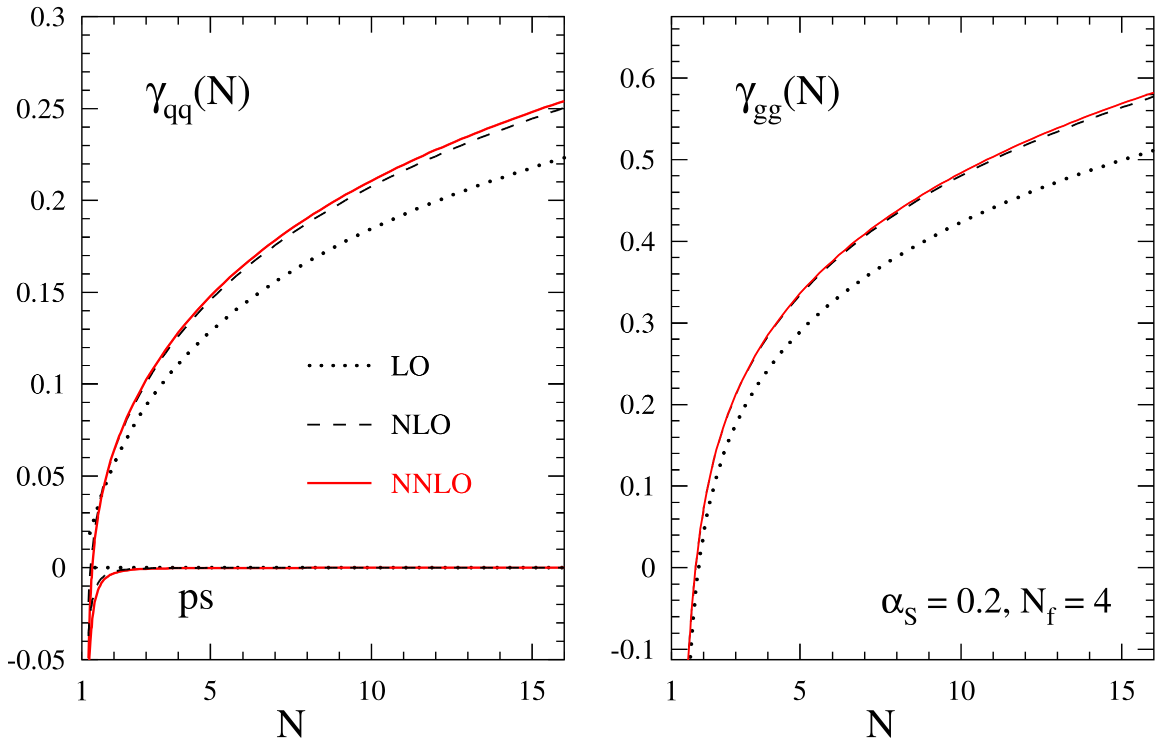

Figure 1: The perturbative expansion of the diagonal anomalous dimensions $\gamma_{qq}(N)$ and $\gamma_{gg}(N)$ for four flavours at $\alpha_s = 0.2$. The pure-singlet (ps) contribution to γ_{qq} is shown separately.

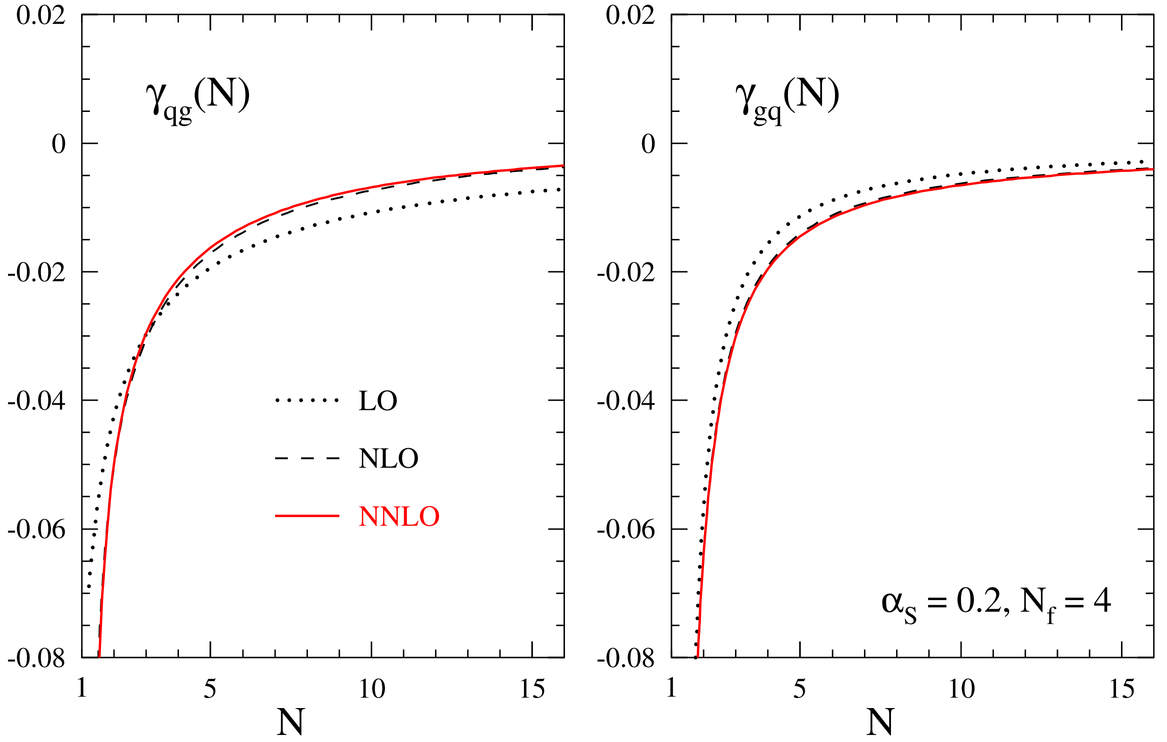

Figure 2: As Fig. 1, but for the off-diagonal anomalous dimensions $\gamma_{qg}(N)$ and $\gamma_{gq}(N)$.

specified up to $n = 3$ in Eq. (3.11) of Ref. [38]. As expected, the constants $A_n^{\rm g}$ are related to those results by

$$A_n^{\rm g} \;=\; \frac{C_A}{C_F} A_n^{\rm q} \; . \tag{3.15}$$

The coefficients $C_n^{\rm g}$ in Eq. (3.14) can be expressed in terms of the $A_n^{\rm g}$ by

$$C_1^{\rm g} \;=\; 0 \; , \quad C_2^{\rm g} \;=\; 4 C_A A_1^{\rm g} = (A_1^{\rm g})^2 \; , \quad C_3^{\rm g} \;=\; 8 C_A A_2^{\rm g} = 2 A_1^{\rm g} A_2^{\rm g} \; . \tag{3.16}$$

This result is completely analogous to the corresponding relation for $C_n^{\rm q}$ in Eq. (3.12) of Ref. [38]. Finally the N-independent contributions $B_n^{\rm g}$ can be read off directly from the $\delta(1-x)$ terms in Eqs. (4.6), (4.10) and (4.15) below.

4 Results in x-space

The N$^{\rm n}$LO singlet splitting functions $P_{\rm ab}^{(n)}(x)$ in

$$P_{\rm ab}\,(\alpha_{\rm s},x) \;=\; \sum_{n=0} \left(\frac{\alpha_{\rm s}}{4\pi}\right)^{n+1} P_{\rm ab}^{(n)}(x) \tag{4.1}$$

are obtained from the N-space results of the previous section by an inverse Mellin transformation which expresses these functions in terms of harmonic polylogarithms [63, 64, 65]. This transformation can be performed by a completely algebraic procedure [40, 65] based on the fact that harmonic sums occur as coefficients of the Taylor expansion of harmonic polylogarithms.

Our notation for the harmonic polylogarithms $H_{m_1,\ldots,m_w}(x)$, $m_j = 0, \pm 1$ follows Ref. [65] to which the reader is referred for a detailed discussion. For completeness, we recall the basic definitions: The lowest-weight ($w = 1$) functions $H_m(x)$ are given by

$$H_0(x) \;=\; \ln x \; , \qquad H_{\pm 1}(x) \;=\; \mp \ln(1 \mp x) \; . \tag{4.2}$$

The higher-weight ($w \geq 2$) functions are recursively defined as

$$H_{m_1,\ldots,m_w}(x) \;=\; \begin{cases} \displaystyle\frac{1}{w!} \ln^w x \; , & \text{if } m_1,\ldots,m_w = 0,\ldots,0 \\[2ex] \displaystyle\int_0^x dz\, f_{m_1}(z)\, H_{m_2,\ldots,m_w}(z) \; , & \text{else} \end{cases} \tag{4.3}$$

with

$$f_0(x) \;=\; \frac{1}{x} \; , \qquad f_{\pm 1}(x) \;=\; \frac{1}{1 \mp x} \; . \tag{4.4}$$

For chains of indices zero we again employ the abbreviated notation

$$H_{\underbrace{0,\ldots,0}_{m},\pm 1,\underbrace{0,\ldots,0}_{n},\pm 1,\ldots}(x) \;=\; H_{\pm(m+1),\pm(n+1),\ldots}(x) \; . \tag{4.5}$$

Corresponding to the maximal weight $2l-1$ of the harmonic sums in section 3, the l-loop splitting functions involve harmonic polylogarithms up to weight $2l-2$. Hence our three-loop results cannot be expressed in terms of standard polylogarithms which are sufficiently general only for $w \leq 3$.

For completeness we recall the one- and two-loop non-singlet splitting functions [3, 8]

$$\begin{aligned}
P^{(0)}_{\rm ps}(x) &= 0 \\
P^{(0)}_{\rm qg}(x) &= 2\,n_f\,p_{\rm qg}(x) \\
P^{(0)}_{\rm gq}(x) &= 2\,C_F\,p_{\rm gq}(x) \\
P^{(0)}_{\rm gg}(x) &= C_A\Big(4p_{\rm gg}(x)+\frac{11}{3}\delta(1-x)\Big)-\frac{2}{3}n_f\,\delta(1-x)
\end{aligned} \tag{4.6}$$

and

$$P^{(1)}_{\rm ps}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x}-2+6x-4{\rm H}_0+x^2\Big[\frac{8}{3}{\rm H}_0-\frac{56}{9}\Big]+(1+x)\Big[5{\rm H}_0-2{\rm H}_{0,0}\Big]\Big) \tag{4.7}$$

$$\begin{aligned}
P^{(1)}_{\rm qg}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x}-2+25x-2p_{\rm qg}(-x){\rm H}_{-1,0}-2p_{\rm qg}(x){\rm H}_{1,1}+x^2\Big[\frac{44}{3}{\rm H}_0-\frac{218}{9}\Big] \\
&+4(1-x)\Big[{\rm H}_{0,0}-2{\rm H}_0+x{\rm H}_1\Big]-4\zeta_2x-6{\rm H}_{0,0}+9{\rm H}_0\Big)+4C_Fn_f\Big(2p_{\rm qg}(x)\Big[{\rm H}_{1,0}+{\rm H}_{1,1}+{\rm H}_2 \\
&-\zeta_2\Big]+4x^2\Big[{\rm H}_0+{\rm H}_{0,0}+\frac{5}{2}\Big]+2(1-x)\Big[{\rm H}_0+{\rm H}_{0,0}-2x{\rm H}_1+\frac{29}{4}\Big]-\frac{15}{2}-{\rm H}_{0,0}-\frac{1}{2}{\rm H}_0\Big)
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
P^{(1)}_{\rm gq}(x) &= 4C_AC_F\Big(\frac{1}{x}+2p_{\rm gq}(x)\Big[{\rm H}_{1,0}+{\rm H}_{1,1}+{\rm H}_2-\frac{11}{6}{\rm H}_1\Big]-x^2\Big[\frac{8}{3}{\rm H}_0-\frac{44}{9}\Big]+4\zeta_2-2 \\
&-7{\rm H}_0+2{\rm H}_{0,0}-2{\rm H}_1x+(1+x)\Big[2{\rm H}_{0,0}-5{\rm H}_0+\frac{37}{9}\Big]-2p_{\rm gq}(-x){\rm H}_{-1,0}\Big)-4C_Fn_f\Big(\frac{2}{3}x \\
&-p_{\rm gq}(x)\Big[\frac{2}{3}{\rm H}_1-\frac{10}{9}\Big]\Big)+4{C_F}^2\Big(p_{\rm gq}(x)\Big[3{\rm H}_1-2{\rm H}_{1,1}\Big]+(1+x)\Big[{\rm H}_{0,0}-\frac{7}{2}+\frac{7}{2}{\rm H}_0\Big]-3{\rm H}_{0,0} \\
&+1-\frac{3}{2}{\rm H}_0+2{\rm H}_1x\Big)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
P^{(1)}_{\rm gg}(x) &= 4C_An_f\Big(1-x-\frac{10}{9}p_{\rm gg}(x)-\frac{13}{9}\Big(\frac{1}{x}-x^2\Big)-\frac{2}{3}(1+x){\rm H}_0-\frac{2}{3}\delta(1-x)\Big)+4{C_A}^2\Big(27 \\
&+(1+x)\Big[\frac{11}{3}{\rm H}_0+8{\rm H}_{0,0}-\frac{27}{2}\Big]+2p_{\rm gg}(-x)\Big[{\rm H}_{0,0}-2{\rm H}_{-1,0}-\zeta_2\Big]-\frac{67}{9}\Big(\frac{1}{x}-x^2\Big)-12{\rm H}_0 \\
&-\frac{44}{3}x^2{\rm H}_0+2p_{\rm gg}(x)\Big[\frac{67}{18}-\zeta_2+{\rm H}_{0,0}+2{\rm H}_{1,0}+2{\rm H}_2\Big]+\delta(1-x)\Big[\frac{8}{3}+3\zeta_3\Big]\Big)+4C_Fn_f\Big(2{\rm H}_0 \\
&+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\Big[4-5{\rm H}_0-2{\rm H}_{0,0}\Big]-\frac{1}{2}\delta(1-x)\Big)\,.
\end{aligned} \tag{4.10}$$

Here and in Eqs. (4.12) – (4.15) we suppress the argument x of the polylogarithms and use

$$\begin{aligned}
p_{\rm qg}(x) &= 1-2x+2x^2 \\
p_{\rm gq}(x) &= 2x^{-1}-2+x \\
p_{\rm gg}(x) &= (1-x)^{-1}+x^{-1}-2+x-x^2\,.
\end{aligned} \tag{4.11}$$

Divergences for $x \to 1$ are understood in the sense of $+$-distributions.

The third-order pure-singlet contribution to the quark-quark splitting function (2.4), corresponding to the anomalous dimension (3.10), is given by

$$
\begin{aligned}
P^{(2)}_{\rm ps}(x) \; = \; & 16 C_A C_F n_f \Big(\frac{4}{3}(\frac{1}{x}+x^2)\Big[\frac{13}{3}\mathrm{H}_{-1,0} - \frac{14}{9}\mathrm{H}_0 + \frac{1}{2}\mathrm{H}_{-1}\zeta_2 - \mathrm{H}_{-1,-1,0} - 2\mathrm{H}_{-1,0,0} \\
& - \mathrm{H}_{-1,2}\Big] + \frac{2}{3}(\frac{1}{x}-x^2)\Big[\frac{16}{3}\zeta_2 + \mathrm{H}_{2,1} + 9\zeta_3 + \frac{9}{4}\mathrm{H}_{1,0} - \frac{6761}{216} + \frac{571}{72}\mathrm{H}_1 + \frac{10}{3}\mathrm{H}_2 + \mathrm{H}_1\zeta_2 - \frac{1}{6}\mathrm{H}_{1,1} \\
& - 3\mathrm{H}_{1,0,0} + 2\mathrm{H}_{1,1,0} + 2\mathrm{H}_{1,1,1}\Big] + (1-x)\Big[\frac{182}{9}\mathrm{H}_1 + \frac{158}{3} + \frac{397}{36}\mathrm{H}_{0,0} - \frac{13}{2}\mathrm{H}_{-2,0} + 3\mathrm{H}_{0,0,0,0} \\
& + \frac{13}{6}\mathrm{H}_{1,0} + 3x\mathrm{H}_{1,0} + \mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0,0} + \frac{1}{2}\mathrm{H}_{0,0}\zeta_2 + \frac{1}{2}\mathrm{H}_1\zeta_2 - \frac{9}{4}\mathrm{H}_{1,0,0} \\
& - \frac{3}{4}\mathrm{H}_{1,1} + \mathrm{H}_{1,1,0} + \mathrm{H}_{1,1,1}\Big] + (1+x)\Big[\frac{7}{12}\mathrm{H}_0\zeta_2 + \frac{31}{6}\zeta_3 + \frac{91}{18}\mathrm{H}_2 + \frac{71}{12}\mathrm{H}_3 + \frac{113}{18}\zeta_2 - \frac{826}{27}\mathrm{H}_0 \\
& + \frac{5}{2}\mathrm{H}_{2,0} + \frac{16}{3}\mathrm{H}_{-1,0} + 6x\mathrm{H}_{-1,0} + \frac{31}{6}\mathrm{H}_{0,0,0} - \frac{17}{6}\mathrm{H}_{2,1} + \frac{117}{20}\zeta_2^{\,2} + 9\mathrm{H}_0\zeta_3 + \frac{5}{2}\mathrm{H}_{-1}\zeta_2 + 2\mathrm{H}_{2,1,0} \\
& + \frac{1}{2}\mathrm{H}_{-1,0,0} - 2\mathrm{H}_{-1,2} + \mathrm{H}_2\zeta_2 - \frac{7}{2}\mathrm{H}_{2,0,0} + \mathrm{H}_{-1,-1,0} + 2\mathrm{H}_{2,1,1} + \mathrm{H}_{3,1} - \frac{1}{2}\mathrm{H}_4\Big] + 5\mathrm{H}_{-2,0} + \mathrm{H}_{2,1} \\
& + \mathrm{H}_{0,0,0,0} - \frac{1}{2}\zeta_2^{\,2} + 4\mathrm{H}_{-3,0} + 4\mathrm{H}_0\zeta_3 - \frac{32}{9}\mathrm{H}_{0,0} - \frac{29}{12}\mathrm{H}_0 - \frac{235}{12}\zeta_2 - \frac{511}{12} - \frac{97}{12}\mathrm{H}_1 + \frac{33}{4}\mathrm{H}_2 - \mathrm{H}_3 \\
& - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{11}{2}\zeta_3 - \frac{3}{2}\mathrm{H}_{2,0} - 10\mathrm{H}_{0,0,0} + \frac{2}{3}x^2\Big[\frac{83}{4}\mathrm{H}_{0,0} - \frac{243}{4}\mathrm{H}_0 + 10\zeta_2 + \frac{511}{8} + \frac{97}{8}\mathrm{H}_1 - \frac{4}{3}\mathrm{H}_2 \\
& - 4\zeta_3 - \mathrm{H}_0\zeta_2 + \mathrm{H}_3 + \mathrm{H}_{2,0} - 6\mathrm{H}_{-2,0}\Big]\Big) + 16 C_F {n_f}^2 \Big(\frac{2}{27}\mathrm{H}_0 - 2 - \mathrm{H}_2 + \zeta_2 + \frac{2}{3}x^2\Big[\mathrm{H}_2 - \zeta_2 + 3 \\
& - \frac{19}{6}\mathrm{H}_0\Big] + \frac{2}{9}(\frac{1}{x}-x^2)\Big[\mathrm{H}_{1,1} + \frac{5}{3}\mathrm{H}_1 + \frac{2}{3}\Big] + (1-x)\Big[\frac{1}{6}\mathrm{H}_{1,1} - \frac{7}{6}\mathrm{H}_1 + x\mathrm{H}_1 + \frac{35}{27}\mathrm{H}_0 + \frac{185}{54}\Big] \\
& + \frac{1}{3}(1+x)\Big[\frac{4}{3}\mathrm{H}_2 - \frac{4}{3}\zeta_2 + \zeta_3 + \mathrm{H}_{2,1} - 2\mathrm{H}_3 + 2\mathrm{H}_0\zeta_2 + \frac{29}{6}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0}\Big]\Big) + 16 {C_F}^2 n_f \Big(\frac{85}{12}\mathrm{H}_1 \\
& - \frac{25}{4}\mathrm{H}_{0,0} - \mathrm{H}_{0,0,0} + \frac{583}{12}\mathrm{H}_0 - \frac{101}{54} + \frac{73}{4}\zeta_2 - \frac{73}{4}\mathrm{H}_2 + \mathrm{H}_3 - 5\mathrm{H}_{2,0} - \mathrm{H}_{2,1} - \mathrm{H}_0\zeta_2 + x^2\Big[\frac{55}{12} \\
& - \frac{85}{12}\mathrm{H}_1 - \frac{22}{3}\mathrm{H}_{0,0} - \frac{109}{6} - \frac{13}{54}\mathrm{H}_0 + \frac{28}{9}\zeta_2 - \frac{28}{9}\mathrm{H}_2 - \frac{16}{3}\mathrm{H}_0\zeta_2 + \frac{16}{3}\mathrm{H}_3 + 4\mathrm{H}_{2,0} + \frac{4}{3}\mathrm{H}_{2,1} - \frac{26}{3}\zeta_3 \\
& + \frac{22}{3}\mathrm{H}_{0,0,0}\Big] + \frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{23}{12}\mathrm{H}_{1,0} - \frac{523}{144}\mathrm{H}_1 - 3\zeta_3 + \frac{55}{16} + \frac{1}{2}\mathrm{H}_{1,0,0} + \mathrm{H}_{1,1} - \mathrm{H}_{1,1,0} - \mathrm{H}_{1,1,1}\Big] \\
& + (1-x)\Big[\frac{1}{2}\mathrm{H}_{1,0,0} + \frac{7}{12}\mathrm{H}_{1,1} - \frac{2743}{72}\mathrm{H}_0 - \frac{53}{12}\mathrm{H}_{0,0} - \frac{251}{12}\mathrm{H}_1 - \frac{5}{4}\zeta_2 + \frac{5}{4}\mathrm{H}_2 - \frac{8}{3}\mathrm{H}_{1,0} + 3x\mathrm{H}_{1,0} \\
& + 3\mathrm{H}_0\zeta_2 - 3\mathrm{H}_3 - \mathrm{H}_{1,1,0} - \mathrm{H}_{1,1,1}\Big] + (1+x)\Big[\frac{1669}{216} + \frac{5}{2}\mathrm{H}_{0,0,0} + 4\mathrm{H}_{2,1} + 7\mathrm{H}_{2,0} + 10x\zeta_3 - \frac{37}{10}\zeta_2^{\,2} \\
& - 7\mathrm{H}_0\zeta_3 + 6\mathrm{H}_{0,0}\zeta_2 - 4\mathrm{H}_{0,0,0,0} + \mathrm{H}_{2,0,0} - 2\mathrm{H}_{2,1,0} - 2\mathrm{H}_{2,1,1} - 4\mathrm{H}_{3,0} - \mathrm{H}_{3,1} - 6\mathrm{H}_4\Big]\Big) \; . \qquad (4.12)
\end{aligned}
$$

Due to Eqs. (3.11) and (3.12) the three-loop gluon-quark and quark-gluon splitting functions read

$$
\begin{aligned}
P^{(2)}_{\rm qg}(x) \; = \; & 16 C_A C_F n_f \Big(p_{\rm qg}(x)\Big[\frac{39}{2}\mathrm{H}_1\zeta_3 - 4\mathrm{H}_{1,1,1} + 3\mathrm{H}_{2,0,0} - \frac{15}{4}\mathrm{H}_{1,2} + \frac{9}{4}\mathrm{H}_{1,1,0} + 3\mathrm{H}_{2,1,0} \\
& + \mathrm{H}_0\zeta_3 - 2\mathrm{H}_{2,1,1} + 4\mathrm{H}_2\zeta_2 - \frac{173}{12}\mathrm{H}_0\zeta_2 - \frac{551}{72}\mathrm{H}_{0,0} + \frac{64}{3}\zeta_3 - \zeta_2^{\,2} - \frac{49}{4}\mathrm{H}_2 - \frac{3}{2}\mathrm{H}_{1,0,0,0} - \frac{1}{3}\mathrm{H}_{1,0,0}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{385}{72}\mathrm{H}_{1,0}-\frac{31}{2}\mathrm{H}_{1,1}-\frac{113}{12}\mathrm{H}_{1}+\frac{49}{4}\mathrm{H}_{2,0}+\frac{5}{2}\mathrm{H}_{1}\zeta_2+\frac{79}{6}\mathrm{H}_{0,0,0}+\frac{173}{12}\mathrm{H}_{3}-\frac{1259}{32}+\frac{2833}{216}\mathrm{H}_{0}\\
&+6\mathrm{H}_{2,1}+3\mathrm{H}_{1,-2,0}+9\mathrm{H}_{1,0}\zeta_2+6\mathrm{H}_{1,1}\zeta_2+\mathrm{H}_{1,1,0,0}+3\mathrm{H}_{1,1,1,0}-4\mathrm{H}_{1,1,1,1}-3\mathrm{H}_{1,1,2}-6\mathrm{H}_{1,2,1}\\
&-6\mathrm{H}_{1,3}+\frac{49}{4}\zeta_2\Big]+p_{\mathrm{qg}}(-x)\Big[\frac{17}{2}\mathrm{H}_{-1}\zeta_3-\frac{5}{2}\mathrm{H}_{-1,-1,0}-\frac{5}{2}\mathrm{H}_{-1,2}-\frac{9}{2}\mathrm{H}_{-1,0}+\frac{5}{2}\mathrm{H}_{-2,0}+\frac{3}{2}\mathrm{H}_{-1,0,0}\\
&-2\mathrm{H}_{3,1}-2\mathrm{H}_{4}-6\mathrm{H}_{-2,2}+6\mathrm{H}_{-2,-1,0}-6\mathrm{H}_{-2,0,0}+2\mathrm{H}_{0,0}\zeta_2+9\mathrm{H}_{-2}\zeta_2+3\mathrm{H}_{-1,-2,0}-2\mathrm{H}_{-1,2,1}\\
&-6\mathrm{H}_{-1,-1,-1,0}+6\mathrm{H}_{-1,-1,0,0}+6\mathrm{H}_{-1,-1,2}+9\mathrm{H}_{-1,0}\zeta_2-9\mathrm{H}_{-1,-1}\zeta_2-2\mathrm{H}_{-1,2,0}-\frac{11}{2}\mathrm{H}_{-1,0,0,0}\\
&-6\mathrm{H}_{-1,3}\Big]+(\frac{1}{x}-x^2)\Big[\frac{55}{12}-4\zeta_3+\frac{23}{9}\mathrm{H}_{1,0}-\frac{4}{3}\mathrm{H}_{1,1,0}\Big]+(\frac{1}{x}+x^2)\Big[\frac{2}{3}\mathrm{H}_{1,0,0}-\frac{371}{108}\mathrm{H}_{1}+\frac{23}{9}\mathrm{H}_{1,1}\\
&-\frac{2}{3}\mathrm{H}_{1,1,1}\Big]+(1-x)\Big[6\mathrm{H}_{2,1,0}+3\mathrm{H}_{2,1,1}-\frac{5}{6}\mathrm{H}_{1,1,1}-7\mathrm{H}_{2,0,0}-2\mathrm{H}_{1,2}+39\mathrm{H}_{0}\zeta_3-4\mathrm{H}_{2}\zeta_2-\frac{16}{3}\zeta_3\\
&+\mathrm{H}_{1,1,0}+\frac{154}{3}\mathrm{H}_{0}\zeta_2+\frac{899}{24}\mathrm{H}_{0,0}+\frac{121}{10}{\zeta_2}^2+\frac{607}{36}\mathrm{H}_{2}-\frac{5}{2}\mathrm{H}_{1}\zeta_2+\frac{65}{6}\mathrm{H}_{1,0,0}-\frac{29}{12}\mathrm{H}_{1,0}-\frac{13}{18}\mathrm{H}_{1,1}\\
&-\frac{1189}{108}\mathrm{H}_{1}-\frac{67}{3}\mathrm{H}_{2,1}-29\mathrm{H}_{2,0}-\frac{949}{36}\zeta_2-\frac{67}{2}\mathrm{H}_{0,0,0}-\frac{142}{3}\mathrm{H}_{3}+\frac{215}{32}-\frac{3989}{48}\mathrm{H}_{0}+2\mathrm{H}_{-3,0}\Big]\\
&+(1+x)\Big[\mathrm{H}_{-1,0,0}-10\mathrm{H}_{-2}\zeta_2+6\mathrm{H}_{-2,0,0}+2\mathrm{H}_{0,0}\zeta_2-9\mathrm{H}_{-1,-1,0}-7\mathrm{H}_{-1,2}-9\mathrm{H}_{-2,0}-2\mathrm{H}_{3,1}\\
&-4\mathrm{H}_{-2,-1,0}-4\mathrm{H}_{4}-4\mathrm{H}_{3,0}-4\mathrm{H}_{0,0,0,0}+\frac{37}{2}\mathrm{H}_{-1,0}+\frac{5}{2}(1+x)\mathrm{H}_{-1}\zeta_2\Big]-4\mathrm{H}_{-2,0,0}+2\mathrm{H}_{0,0}\zeta_2\\
&+\mathrm{H}_{2}\zeta_2-3\mathrm{H}_{1,1,0}+2\mathrm{H}_{0,0,0,0}+\mathrm{H}_{-3,0}-9\mathrm{H}_{2,1,0}-\frac{9}{2}\mathrm{H}_{2,1,1}+\frac{11}{3}\mathrm{H}_{1,1,1}+\frac{19}{2}\mathrm{H}_{2,0,0}+\frac{9}{2}\mathrm{H}_{1,2}\\
&-\frac{91}{2}\mathrm{H}_{0}\zeta_3+8\mathrm{H}_{-2}\zeta_2+\frac{5}{2}\mathrm{H}_{-1,-1,0}+\frac{5}{2}\mathrm{H}_{-1,2}+\frac{9}{2}\mathrm{H}_{-1,0}+\frac{39}{2}\mathrm{H}_{-2,0}-\frac{473}{12}\mathrm{H}_{0}\zeta_2-\frac{1853}{48}\mathrm{H}_{0,0}\\
&-\frac{217}{12}\zeta_3-\frac{59}{4}{\zeta_2}^2-\frac{169}{18}\mathrm{H}_{2}-\frac{13}{4}\mathrm{H}_{1}\zeta_2-\frac{2}{3}\mathrm{H}_{1,0,0}+\frac{167}{24}\mathrm{H}_{1,0}+\frac{191}{18}\mathrm{H}_{1,1}+\frac{1283}{108}\mathrm{H}_{1}+\frac{185}{12}\mathrm{H}_{2,1}\\
&+\frac{75}{4}\mathrm{H}_{2,0}+\frac{170}{9}\zeta_2+\frac{85}{4}\mathrm{H}_{0,0,0}+\frac{425}{12}\mathrm{H}_{3}+\frac{7693}{192}+\frac{3659}{48}\mathrm{H}_{0}-2x\Big[x\mathrm{H}_{2,2}+4\mathrm{H}_{3,0}-4\mathrm{H}_{-2,2}\Big]\Big)\\
&+16C_A{n_f}^2\Big(\frac{1}{6}p_{\mathrm{qg}}(x)\Big[\mathrm{H}_{1,2}-\mathrm{H}_{1}\zeta_2-\mathrm{H}_{1,0,0}-\mathrm{H}_{1,1,0}-\mathrm{H}_{1,1,1}-\frac{229}{18}\mathrm{H}_{0}+\frac{4}{3}\mathrm{H}_{0,0}+\frac{11}{2}\Big]+x\Big[\frac{1}{6}\mathrm{H}_{2}\\
&-\frac{53}{18}\mathrm{H}_{0}+\frac{17}{6}\mathrm{H}_{0,0}-\zeta_3+\frac{11}{18}\zeta_2-\frac{139}{108}\Big]+\frac{1}{3}p_{\mathrm{qg}}(-x)\mathrm{H}_{-1,0,0}-\frac{53}{162}(\frac{1}{x}-x^2)-\frac{2}{9}(1-x)\Big[6\mathrm{H}_{0,0,0}\\
&-\frac{7}{6}x\mathrm{H}_{1}-\mathrm{H}_{0,0}+\frac{7}{2}x\mathrm{H}_{1,1}\Big]+\frac{7}{9}x(1+x)\mathrm{H}_{-1,0}+\frac{7}{4}\mathrm{H}_{0}-\frac{19}{54}\mathrm{H}_{1}+\mathrm{H}_{0,0,0}+\frac{5}{9}\mathrm{H}_{1,1}+\frac{5}{9}\mathrm{H}_{-1,0}\\
&-\frac{85}{216}\Big)+16{C_A}^2n_f\Big(p_{\mathrm{qg}}(x)\Big[3\mathrm{H}_{1,3}+\frac{31}{6}\mathrm{H}_{1,0,0}-\frac{17}{2}\mathrm{H}_{2,1}+\frac{7}{5}{\zeta_2}^2-\frac{55}{12}\mathrm{H}_{1,1,0}+\frac{31}{12}\mathrm{H}_{3}-\frac{31}{2}\mathrm{H}_{1}\zeta_3\\
&-\frac{5}{12}\mathrm{H}_{2,0}-\frac{63}{8}\mathrm{H}_{1,0}-\frac{23}{12}\mathrm{H}_{1,2}-\frac{155}{6}\zeta_3+\frac{25}{24}\mathrm{H}_{2}-\frac{2537}{27}\mathrm{H}_{0}+\frac{867}{8}-\frac{23}{2}\mathrm{H}_{-1,0,0}+3\mathrm{H}_{4}-\mathrm{H}_{1,1,1}\\
&+\frac{383}{72}\mathrm{H}_{1,1}-\frac{25}{2}\mathrm{H}_{-2,0}-\frac{3}{8}\zeta_2-\frac{7}{4}\mathrm{H}_{1}\zeta_2-3\mathrm{H}_{0,0}\zeta_2-\frac{31}{12}\mathrm{H}_{0}\zeta_2+\frac{103}{216}\mathrm{H}_{1}+\frac{5}{2}\mathrm{H}_{1,0,0,0}+\frac{2561}{72}\mathrm{H}_{0,0}\\
&+\mathrm{H}_{1,1,1}-2\mathrm{H}_{2,0,0}-3\mathrm{H}_{1,-2,0}-5\mathrm{H}_{1,0}\zeta_2+3\mathrm{H}_{0,0,0}-\mathrm{H}_{1,1}\zeta_2-\mathrm{H}_{1,1,0,0}-4\mathrm{H}_{1,1,1,0}+2\mathrm{H}_{1,1,1,1}\\
&-2\mathrm{H}_{1,1,2}-2\mathrm{H}_{1,2,0}\Big]+p_{\mathrm{qg}}(-x)\Big[\mathrm{H}_{-1,-1}\zeta_2-2\mathrm{H}_{-1,2}-6\mathrm{H}_{-1,-1,0}+\mathrm{H}_{1,1,1}+2\mathrm{H}_{-2}\zeta_2-\mathrm{H}_{-2,0,0}\\
&+\frac{727}{36}\mathrm{H}_{-1,0}-\mathrm{H}_{-1}\zeta_2-2\mathrm{H}_{-2,2}-\frac{5}{2}\mathrm{H}_{-1}\zeta_3-\mathrm{H}_{-1,-2,0}+2\mathrm{H}_{-1,-1,0,0}+2\mathrm{H}_{-1,-1,2}-\frac{3}{2}\mathrm{H}_{-1,0,0,0}
\end{aligned}
$$

$$+6\mathrm{H}_{-1,-1,-1,0}-2\mathrm{H}_{-1,3}+2\mathrm{H}_{-1,2,1}\Big]+(\frac{1}{x}-x^2)\Big[\frac{2}{3}\mathrm{H}_{2,1}+\frac{32}{9}\zeta_2-2\mathrm{H}_{1,0,0}+\frac{4}{3}\mathrm{H}_{1,1,0}-\frac{10}{9}\mathrm{H}_{1,1}$$
$$-\frac{8}{3}\mathrm{H}_{-1,0,0}+\frac{3}{2}\mathrm{H}_{1,0}+6\zeta_3+\frac{161}{36}\mathrm{H}_1-\frac{2351}{108}\Big]+\frac{2}{3}(\frac{1}{x}+x^2)\Big[\frac{26}{3}\mathrm{H}_{-1,0}-\frac{28}{9}\mathrm{H}_0-2\mathrm{H}_{-1,-1,0}$$
$$-2\mathrm{H}_{-1,2}+\mathrm{H}_1\zeta_2+\mathrm{H}_{-1}\zeta_2+\frac{10}{3}\mathrm{H}_2+\mathrm{H}_{1,1,1}\Big]+(1-x)\Big[15\mathrm{H}_{0,0,0,0}-5\mathrm{H}_2\zeta_2-\frac{65}{6}\zeta_3+\frac{11}{6}\mathrm{H}_{1,1,1}$$
$$-\frac{3}{2}\mathrm{H}_4+\frac{5}{2}\mathrm{H}_{0,0}\zeta_2+\mathrm{H}_{1,1,0}-\frac{31}{6}\mathrm{H}_{2,0}+\frac{17}{12}\mathrm{H}_{1,0}-\frac{551}{20}\zeta_2{}^2-\frac{29}{4}\mathrm{H}_{1,0,0}-\frac{113}{4}\mathrm{H}_2+\frac{18691}{72}\mathrm{H}_0$$
$$+\frac{2243}{108}+\frac{265}{6}\mathrm{H}_{-1,0,0}+\frac{33}{2}\mathrm{H}_{2,0,0}+19\mathrm{H}_{2,1}+\frac{31}{12}\mathrm{H}_{1,1}+\frac{23}{2}\mathrm{H}_{-2,0}-\frac{497}{36}\zeta_2+\frac{29}{6}\mathrm{H}_1\zeta_2-\frac{143}{12}\mathrm{H}_3$$
$$-\frac{11}{6}\mathrm{H}_{1,1,1}-\frac{19}{12}\mathrm{H}_0\zeta_2+\frac{1223}{72}\mathrm{H}_1-\frac{43}{6}\mathrm{H}_{0,0,0}-\frac{3011}{36}\mathrm{H}_{0,0}\Big]+(1+x)\Big[8\mathrm{H}_{2,1,0}-4\mathrm{H}_{-1,2}$$
$$+7\mathrm{H}_{-1,-1,0}-\frac{35}{6}\mathrm{H}_{1,1,1}-5\mathrm{H}_{-2}\zeta_2-11\mathrm{H}_{-2,0,0}+\frac{1}{3}\mathrm{H}_{-1,0}+\frac{15}{2}\mathrm{H}_{-1}\zeta_2+8\mathrm{H}_{3,1}-10\mathrm{H}_{-2,-1,0}$$
$$+5\mathrm{H}_2\zeta_2+4\mathrm{H}_{2,1,1}-\mathrm{H}_{-3,0}+36\mathrm{H}_0\zeta_3-5\mathrm{H}_2\zeta_2\Big]+2\mathrm{H}_{-1,2}+6\mathrm{H}_{-1,-1,0}-6\mathrm{H}_{2,1,0}-3\mathrm{H}_{2,1,1}$$
$$-11\mathrm{H}_{0,0,0,0}-5\mathrm{H}_{3,1}+\frac{25}{4}\mathrm{H}_{1,1,1}+\frac{13}{2}\mathrm{H}_{-2}\zeta_2+\frac{27}{2}\mathrm{H}_{-2,0,0}+\frac{11}{2}\mathrm{H}_{-3,0}+\frac{13}{2}\mathrm{H}_2\zeta_2-\frac{17}{4}\mathrm{H}_{1,0,0}$$
$$+13\mathrm{H}_{-2,-1,0}-\frac{17}{12}\mathrm{H}_{1,1,1}-\frac{3}{4}\mathrm{H}_4-\frac{1}{4}\mathrm{H}_{0,0}\zeta_2+\mathrm{H}_{1,2}+\frac{11}{2}\mathrm{H}_{1,1,0}+\frac{79}{12}\mathrm{H}_{2,0}+\frac{67}{8}\mathrm{H}_{1,0}+\frac{263}{8}\zeta_2{}^2$$
$$+\frac{119}{3}\zeta_3+\frac{967}{24}\mathrm{H}_2-\frac{305}{12}\mathrm{H}_{-1,0}-24\mathrm{H}_0\zeta_3+\mathrm{H}_{-1}\zeta_2-\frac{13375}{72}\mathrm{H}_0-\frac{1889}{18}-38\mathrm{H}_{-1,0,0}-\frac{21}{2}\mathrm{H}_{2,1}$$
$$-\frac{79}{4}\mathrm{H}_{2,0,0}-\frac{217}{24}\mathrm{H}_{1,1}-\frac{7}{2}\mathrm{H}_{-2,0}+\frac{79}{72}\zeta_2+\frac{4}{3}\mathrm{H}_1\zeta_2+\frac{17}{12}\mathrm{H}_{1,1,1}+\frac{17}{12}\mathrm{H}_0\zeta_2+\frac{31}{18}\mathrm{H}_1+3\mathrm{H}_{0,0,0}$$
$$+\frac{145}{12}\mathrm{H}_3+\frac{1553}{24}\mathrm{H}_{0,0}\Big)+16C_Fn_f{}^2\Big(\frac{7}{6}\mathrm{H}_{0,0,0}+\frac{11}{36}\mathrm{H}_1-\frac{739}{96}+\frac{163}{24}\mathrm{H}_0+\frac{7}{24}\mathrm{H}_{0,0}+2\mathrm{H}_{0,0,0,0}$$
$$-\frac{5}{9}\mathrm{H}_{1,1}-\frac{5}{9}\mathrm{H}_2-\frac{5}{18}\mathrm{H}_{1,0}+\frac{5}{9}\zeta_2+\frac{1}{6}p_{\mathrm{qg}}(x)\Big[\mathrm{H}_{2,1}+\frac{91}{2}-\frac{35}{3}\mathrm{H}_0-\frac{22}{3}\mathrm{H}_{0,0}+\mathrm{H}_{1,1,1}+6\mathrm{H}_{0,0,0}$$
$$-\zeta_3-2\mathrm{H}_{1,0,0}+\frac{7}{9}\mathrm{H}_1\Big]+\frac{77}{81}(\frac{1}{x}-x^2)+(1-x)\Big[\frac{1}{12}\mathrm{H}_1-\frac{6463}{432}-4\mathrm{H}_{0,0,0,0}-\frac{16}{3}\mathrm{H}_{0,0,0}+\frac{7}{9}x\mathrm{H}_{1,1}$$
$$+\frac{7}{9}x\mathrm{H}_2+\frac{8}{9}x\mathrm{H}_{1,0}-\frac{7}{9}x\zeta_2\Big]-(1+x)\Big[\frac{3475}{216}\mathrm{H}_0+\frac{103}{12}\mathrm{H}_{0,0}\Big]\Big)+16C_F{}^2n_f\Big(p_{\mathrm{qg}}(x)\Big[7\mathrm{H}_{1,3}+7\mathrm{H}_4$$
$$-2\mathrm{H}_{-3,0}-7\mathrm{H}_1\zeta_3+5\mathrm{H}_{2,2}+6\mathrm{H}_{3,0}+6\mathrm{H}_{3,1}+\mathrm{H}_{2,1,0}+4\mathrm{H}_{2,0,0}+3\mathrm{H}_{2,1}+2\mathrm{H}_{2,1,1}+\frac{5}{2}\mathrm{H}_{2,0}$$
$$+\frac{61}{8}\mathrm{H}_2-\frac{61}{8}\zeta_2+\frac{87}{8}\mathrm{H}_1+\frac{11}{2}\mathrm{H}_{1,2}+\frac{61}{8}\mathrm{H}_{1,1}+\frac{17}{2}\mathrm{H}_{1,0}-7\mathrm{H}_{0,0}\zeta_2+\frac{5}{2}\mathrm{H}_{1,0,0}+\frac{5}{2}\mathrm{H}_{1,1,0}-\frac{19}{2}\zeta_3$$
$$+\frac{81}{32}+\frac{11}{2}\mathrm{H}_3-\frac{11}{2}\mathrm{H}_0\zeta_2-\frac{7}{2}\mathrm{H}_1\zeta_2+\frac{15}{2}\mathrm{H}_{0,0,0}+\frac{87}{8}\mathrm{H}_0+\frac{11}{5}\zeta_2{}^2+3\mathrm{H}_{1,1,1}-5\mathrm{H}_2\zeta_2-7\mathrm{H}_0\zeta_3$$
$$+11\mathrm{H}_{0,0}-2\mathrm{H}_{1,-2,0}-7\mathrm{H}_{1,0}\zeta_2+3\mathrm{H}_{1,0,0,0}-5\mathrm{H}_{1,1}\zeta_2+4\mathrm{H}_{1,1,0,0}+\mathrm{H}_{1,1,1,0}+2\mathrm{H}_{1,1,1,1}+5\mathrm{H}_{1,1,2}$$
$$+6\mathrm{H}_{1,2,0}+6\mathrm{H}_{1,2,1}\Big]+4p_{\mathrm{qg}}(-x)\Big[\mathrm{H}_{0,0,0,0}-\mathrm{H}_{-2,0}+\mathrm{H}_{-1,-1,0}-\mathrm{H}_{-2,0,0}+\frac{1}{2}\mathrm{H}_{-1,-2,0}-\frac{5}{8}\mathrm{H}_{-1,0}$$
$$-\frac{5}{4}\mathrm{H}_{-1,0,0}-\frac{1}{2}\mathrm{H}_{-3,0}+\frac{1}{2}\mathrm{H}_{-1}\zeta_2+\mathrm{H}_{-1,-1,0,0}-\frac{1}{4}\mathrm{H}_{-1,0,0,0}\Big]+2(1-x)\Big[\mathrm{H}_{2,1,0}-\mathrm{H}_{2,0,0}-\mathrm{H}_{2,2}$$
$$-\mathrm{H}_{3,1}-2\mathrm{H}_{3,0}-2\mathrm{H}_{-1}\zeta_2+\mathrm{H}_{1,2}-\mathrm{H}_{1,0,0}-\mathrm{H}_{1,1,0}+\mathrm{H}_2\zeta_2-\zeta_2{}^2+\frac{43}{8}\mathrm{H}_2+\frac{49}{8}\zeta_2+\frac{13}{8}\mathrm{H}_{1,1}$$

$$
\begin{aligned}
&-\frac{33}{16}H_1+\frac{5}{2}H_{1,0}+\frac{7}{2}H_{0,0}\zeta_2+\frac{21}{4}\zeta_3+\frac{479}{64}-\frac{1}{2}H_{1,1,1}-\frac{1}{2}H_3+\frac{1}{4}H_{2,1}+\frac{1}{2}H_{2,1,1}+\frac{3}{2}H_0\zeta_2\\
&+\frac{1}{2}H_0\zeta_3-\frac{7}{2}H_4+H_1\zeta_2-\frac{19}{2}H_{0,0,0}-\frac{239}{16}H_{0,0}-\frac{405}{32}H_0\Big]+8(1+x)\Big[H_{-1,-1,0}-H_{-1,0,0}\\
&-H_{0,0,0,0}+\frac{3}{4}H_{-2,0}-\frac{9}{4}H_{-1,0}\Big]-4H_{-1,-1,0}+5H_{0,0,0,0}+5H_{-1,0,0}-13H_{-2,0}+\frac{1}{2}H_{-1,0}\\
&+4xH_{-2,0,0}-\frac{113}{8}H_2-\frac{71}{8}\zeta_2-\frac{35}{4}H_1-\frac{11}{2}H_{1,2}-\frac{33}{8}H_{1,1}-\frac{7}{2}H_{1,0}-\frac{7}{2}H_{0,0}\zeta_2-\frac{5}{2}H_{1,0,0}\\
&-\frac{5}{2}H_{1,1,0}-\frac{5}{2}\zeta_3-\frac{157}{64}-\frac{9}{4}H_{1,1,1}-\frac{9}{4}H_3-\frac{5}{4}H_{2,1}-\frac{1}{2}H_{2,1,1}+\frac{1}{4}H_0\zeta_2+H_2\zeta_2+\frac{5}{2}H_0\zeta_3\\
&+\frac{9}{5}\zeta_2{}^2+\frac{7}{2}H_4+\frac{7}{2}H_1\zeta_2+\frac{49}{4}H_{0,0,0}+\frac{391}{16}H_{0,0}+\frac{401}{16}H_0-H_{2,0,0}-H_{2,1,0}+H_{2,2}+H_{3,1}\\
&+2H_{3,0}+6H_{-1}\zeta_2+\frac{1}{2}H_{2,0}+2H_{-2}\zeta_2+4H_{-2,-1,0}\Big) \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{(2)}_{\rm gq}(x) \;=\; & 16C_AC_Fn_f\Big(\frac{2}{9}x^2\Big[\frac{25}{6}H_1-\frac{131}{4}+3\zeta_2-H_{-1,0}-3H_2+H_{1,1}+\frac{125}{6}H_0-H_{0,0}\Big]\\
&+\frac{5}{6}p_{\rm gq}(x)\Big[H_{1,2}+H_{2,1}+\frac{967}{120}+\frac{251}{90}H_1-\frac{39}{10}H_{1,1}-3\zeta_3-\frac{2}{5}H_0\zeta_2-\frac{1}{5}H_1\zeta_2-\frac{4}{3}H_{1,0}+H_{1,1,0}\\
&-\frac{2}{5}H_{1,0,0}+H_{1,1,1}+\frac{2}{5}H_{2,0}\Big]+\frac{2}{3}p_{\rm gq}(-x)\Big[2H_{-1}\zeta_2+\frac{7}{4}\zeta_2+\frac{41}{12}H_{-1,0}-\frac{151}{72}H_0+\frac{1}{2}H_{-2,0}\\
&+\frac{5}{3}H_2+2H_{-1,-1,0}-H_{-1,0,0}-H_{-1,2}\Big]+\frac{2}{3}(1-x)\Big[H_{-2,0}+2\zeta_3-H_3\Big]+(1+x)\Big[\frac{179}{108}H_1\\
&+\frac{5}{9}\zeta_2+\frac{25}{9}H_{-1,0}-\frac{5}{36}H_{1,1}-\frac{167}{36}H_{0,0}-\frac{1}{3}H_{2,1}-\frac{4}{3}H_0\zeta_2\Big]-\frac{193}{72}+\frac{1}{4}H_1+\frac{1}{9}H_{-1,0}+4H_2\\
&-\frac{1}{4}H_{1,1}+\frac{227}{18}H_0-\frac{35}{12}H_{0,0}-H_{2,1}-\frac{2}{3}H_0\zeta_2+\frac{10}{3}H_{-2,0}+3\zeta_3+2H_3+\frac{2}{3}H_{0,0,0}+x\Big[\frac{11}{4}\zeta_2\\
&-\frac{523}{144}-\frac{19}{36}H_2+\frac{271}{108}H_0-\frac{5}{6}H_{1,0}\Big]\Big)+16C_AC_F{}^2\Big(x^2\Big[\frac{7}{2}+\frac{173}{54}H_1-2\zeta_3-\frac{2}{3}H_{1,1,1}-\frac{26}{9}H_{1,1}\\
&-6H_2+2H_{2,1}+6\zeta_2+\frac{335}{54}H_0-\frac{28}{9}H_{0,0}-\frac{8}{3}H_{0,0,0}\Big]+p_{\rm gq}(x)\Big[\frac{3}{2}H_1\zeta_3+\frac{163}{32}-5\zeta_2+\frac{27}{4}\zeta_3\\
&+\frac{6503}{432}H_1+\frac{2}{9}H_{1,1}+\frac{35}{3}H_{1,1,1}+4H_2+\frac{9}{2}H_{2,1}+4H_{1,0,0}+2H_{2,0,0}-H_2\zeta_2+\frac{41}{12}H_{1,2}+H_{2,2}\\
&+\frac{191}{24}H_{1,0}+3H_{2,0}-2H_{2,1,1}-\frac{3}{2}H_{-1}\zeta_2-\frac{59}{12}H_1\zeta_2+5H_{1,-2,0}+H_{1,0}\zeta_2+\frac{5}{2}H_{1,0,0,0}-2H_{1,1}\zeta_2\\
&+\frac{1}{12}H_{1,1,0}+5H_{1,1,0,0}-3H_{1,1,1,0}-4H_{1,1,1,1}-H_{1,1,2}-2H_{1,2,1}+H_{2,1,0}\Big]+p_{\rm gq}(-x)\Big[H_{-1,0}\\
&+H_{-1,0}\zeta_2+\frac{3}{2}H_{-1,0,0}+\frac{27}{10}\zeta_2{}^2-3H_{-1,-1,0}-\frac{11}{2}H_{-1}\zeta_3-3H_{-1,-2,0}-\frac{3}{2}H_{-1,0,0,0}-3H_{-1,2}\\
&+5H_{-1,-1}\zeta_2-4H_{-1,-1,0,0}-2H_{-1,-1,2}+6H_{-1,-1,-1,0}+2H_{-1,2,1}\Big]+(1-x)\Big[H_2\zeta_2-H_{2,2}\\
&+\frac{23}{12}H_{1,0}-\frac{7061}{432}H_0-\frac{4631}{144}H_{0,0}-\frac{38}{3}H_{0,0,0}-H_{-3,0}-2H_{3,0}-\frac{4433}{432}H_1-2H_{2,0,0}-\frac{21}{2}H_{1,0,0}\\
&-\frac{2}{5}\zeta_2{}^2-\frac{7}{2}H_{1,2}+\frac{23}{2}H_1\zeta_2-4H_0\zeta_3\Big]+(1+x)\Big[\frac{49}{6}H_3-H_{-2,0}-\frac{55}{6}H_0\zeta_2-\frac{1}{2}H_{3,1}-\frac{1159}{36}\zeta_2
\end{aligned}
$$

$$+\frac{655}{576}-\frac{151}{6}\zeta_3-\frac{185}{18}H_{1,1}+\frac{1}{6}H_{1,1,1}-\frac{95}{9}H_2+\frac{29}{6}H_{2,1}-\frac{171}{4}H_{-1,0}-12H_{-1,0,0}+7H_{-1}\zeta_2$$
$$+16H_{-1,-1,0}+\frac{5}{3}H_{2,0}+\frac{3}{2}H_{2,1,1}+4H_{0,0,0,0}\Big]-35H_{-2,0}-\frac{179}{27}H_0+\frac{2041}{144}H_{0,0}-\frac{19}{6}H_{0,0,0}$$
$$-2H_{3,0}-\frac{13}{2}H_0\zeta_2-13H_{-3,0}-\frac{13}{2}H_{3,1}+\frac{15}{2}H_3-\frac{2005}{64}+\frac{157}{4}\zeta_2+8\zeta_3+\frac{1291}{432}H_1+\frac{55}{12}H_{1,1}$$
$$+\frac{3}{2}H_2+\frac{1}{2}H_{2,1}+\frac{27}{4}H_{-1,0}-\frac{11}{2}H_{1,0,0}-8H_{2,0,0}-4\zeta_2^2+\frac{3}{2}H_{1,2}-H_{2,2}+\frac{5}{2}H_1\zeta_2+8H_{-1,-1,0}$$
$$+4H_{2,0}+\frac{3}{2}H_{2,1,1}-H_{-1}\zeta_2+7H_2\zeta_2+6H_{-2}\zeta_2+12H_{-2,-1,0}-6H_{-2,0,0}+x\Big[3H_{1,1,1}-H_{0,0}\zeta_2$$
$$+\frac{9}{2}H_{-1,0,0}-\frac{35}{8}H_{1,0}+2H_4+3H_{1,1,0}+H_{-1,2}\Big]\Big)+16C_A{}^2C_F\Big(x^2\Big[\frac{2}{3}H_1\zeta_2-\frac{2105}{81}-\frac{77}{18}H_{0,0}$$
$$-6H_3+\frac{16}{3}\zeta_3-10H_{-1,0}-\frac{14}{3}H_{2,0}-\frac{2}{3}H_{-1}\zeta_2-\frac{14}{3}H_{0,0,0}+\frac{104}{9}H_2-\frac{4}{3}H_{1,0,0}+\frac{37}{9}H_{1,1}$$
$$+\frac{4}{3}H_{-1,-1,0}-\frac{104}{9}\zeta_2-\frac{8}{3}H_{2,1}+\frac{145}{18}H_{1,0}+\frac{4}{3}H_{-1,2}+\frac{2}{3}H_{1,1,1}-\frac{109}{27}H_1+\frac{8}{3}H_{-1,0,0}+6H_0\zeta_2$$
$$+4H_{-2,0}+\frac{584}{27}H_0\Big]+p_{\rm gq}(x)\Big[\frac{7}{2}H_1\zeta_3+\frac{138305}{2592}-\frac{1}{3}H_{2,0}+\frac{13}{4}H_{-1}\zeta_2+2H_{2,1,1}+\frac{11}{2}H_{1,0,0}$$
$$+4H_{3,1}-\frac{43}{6}H_{1,1,1}-\frac{109}{12}\zeta_2-\frac{17}{3}H_{2,1}-\frac{71}{24}H_{1,0}-\frac{11}{6}H_{-2,0}-\frac{21}{2}\zeta_3+\frac{3}{2}H_{1,0,0,0}-H_{1,-2,0}$$
$$+\frac{395}{54}H_0-2H_{1,0}\zeta_2-H_{1,1}\zeta_2-\frac{55}{12}H_{1,1,0}+2H_{1,1,0,0}+4H_{1,1,1,0}+2H_{1,1,1,1}+4H_{1,1,2}-\frac{55}{12}H_{1,2}$$
$$+6H_{1,2,0}+4H_{1,2,1}+4H_{1,3}+3H_{2,1,0}+3H_{2,2}\Big]+p_{\rm gq}(-x)\Big[\frac{23}{2}H_{-1}\zeta_3+5H_{-2}\zeta_2+2H_{-2,-1,0}$$
$$+\frac{109}{12}H_{-1,0}+H_0\zeta_3+\frac{17}{5}\zeta_2^2+\frac{1}{6}H_1\zeta_2+2H_2\zeta_2-\frac{65}{24}H_{1,1}-\frac{19}{2}H_{-1,-1,0}-4H_{3,0}-3H_{2,0,0}$$
$$-7H_{-2,0,0}-\frac{3}{2}H_{-1,2}+\frac{3379}{216}H_1-4H_{-2,2}-\frac{49}{6}H_{-1,0,0}-\frac{11}{2}H_{-1,0,0,0}-13H_{-1,-1}\zeta_2-8H_{-1,3}$$
$$-6H_{-1,-1,-1,0}+12H_{-1,-1,0,0}+10H_{-1,-1,2}+10H_{-1,0}\zeta_2+5H_{-1,-2,0}-2H_{-1,2,0}-2H_{-1,2,1}$$
$$+\frac{11}{6}H_0\zeta_2\Big]+(1-x)\Big[\frac{41699}{2592}-3H_{-2,-1,0}-\frac{3}{2}H_{-2}\zeta_2-\frac{128}{9}\zeta_2-4H_{3,0}+\frac{26}{3}\zeta_3-\frac{5}{2}H_{-2,0,0}$$
$$-7H_1\zeta_2+\frac{97}{12}H_{1,0,0}+\frac{10}{3}H_{-1,0,0}+\frac{245}{12}H_3-8H_{0,0,0,0}\Big]+(1+x)\Big[4H_{3,1}-H_{2,1,1}+\frac{29}{6}H_{-1,2}$$
$$+\frac{17}{6}H_{-2,0}-12H_{2,0}-\frac{31}{12}H_{2,1}+\frac{1}{2}H_{2,0,0}-H_2\zeta_2+\frac{61}{36}H_{1,0}-4H_0\zeta_3-\frac{13}{3}H_{-1}\zeta_2-\frac{46}{3}H_{-1,-1,0}$$
$$+\frac{25}{4}H_4+\frac{93}{4}H_0\zeta_2-\frac{55}{9}H_{1,1}-\frac{71}{18}H_2+\frac{49}{18}H_{0,0}-\frac{13}{2}H_{0,0}\zeta_2-\frac{47}{40}\zeta_2^2\Big]+\frac{6131}{2592}-\frac{31}{2}H_{-2}\zeta_2$$
$$-15H_{-2,-1,0}+\frac{9}{2}H_{-1,0,0}-3H_{2,1,1}-\frac{9}{4}H_{2,1}+\frac{53}{3}H_{-2,0}-\frac{1}{2}H_{-2,0,0}-5H_{2,0}-\frac{7}{6}H_{1,1,1}-8H_0\zeta_3$$
$$-\frac{67}{40}\zeta_2^2+\frac{29}{6}H_{-1,2}-H_{-1,0}+8H_{-2,2}+25H_0\zeta_2+\frac{412}{9}H_1+\frac{928}{9}H_0+\frac{1}{4}H_4-65H_3-38H_{0,0}$$
$$-9H_{-3,0}-\frac{17}{3}H_{0,0,0}+x\Big[\frac{27}{2}H_{-1,0}-\frac{1}{2}H_{0,0,0,0}+\frac{3}{4}H_{0,0}\zeta_2+\frac{1}{2}H_{-3,0}-14H_{0,0,0}+\frac{1}{12}H_{1,1,1}$$
$$-\frac{43}{36}\zeta_2-\frac{1}{2}H_2\zeta_2+\frac{7}{72}H_0+\frac{749}{54}H_1+\frac{135}{4}\zeta_3+\frac{97}{24}H_{1,0}+\frac{43}{12}H_1\zeta_2-\frac{85}{12}H_{-1}\zeta_2-\frac{13}{3}H_{1,0,0}$$

$$
\begin{aligned}
&+\frac{53}{12}\mathrm{H}_2+\frac{39}{4}\mathrm{H}_{1,1}-2\mathrm{H}_{3,1}+\frac{13}{6}\mathrm{H}_{-1,-1,0}+\frac{7}{4}\mathrm{H}_{2,0,0}-4\mathrm{H}_{1,1,0}-4\mathrm{H}_{1,2}\Big]\Big)+16{C_F}{n_f}^2\Big(\frac{1}{9}-\frac{1}{9}\frac{1}{x}\\
&+\frac{2}{9}x-\frac{1}{6}x\mathrm{H}_1+\frac{1}{6}p_{\mathrm{gq}}(x)\Big[\mathrm{H}_{1,1}-\frac{5}{3}\mathrm{H}_1\Big]\Big)+16{C_F}^2{n_f}\Big(\frac{4}{9}x^2\Big[\mathrm{H}_{0,0}-\frac{11}{6}\mathrm{H}_0-\frac{7}{2}+\mathrm{H}_{-1,0}\Big]\\
&+\frac{1}{3}p_{\mathrm{gq}}(x)\Big[\mathrm{H}_{1,2}-\mathrm{H}_{1,0}-\mathrm{H}_1\zeta_2+9\zeta_3+\frac{83}{12}\mathrm{H}_{1,1}+2\mathrm{H}_{-2,0}-\frac{7}{36}\mathrm{H}_1+2\mathrm{H}_0\zeta_2-\frac{1625}{48}+\frac{3}{2}\mathrm{H}_{1,0,0}\\
&+2\mathrm{H}_{1,1,0}-\frac{5}{2}\mathrm{H}_{1,1,1}\Big]+\frac{31}{18}p_{\mathrm{gq}}(-x)\Big[\frac{95}{93}\mathrm{H}_0-\zeta_2-\mathrm{H}_{-1,0}\Big]+\frac{1}{3}(2-x)\Big[6\mathrm{H}_{0,0,0,0}-\mathrm{H}_3-\frac{13051}{288}\\
&-\frac{13}{2}\zeta_3-4\mathrm{H}_{-2,0}-\mathrm{H}_{2,0}-\frac{1}{2}\mathrm{H}_{1,0}-\frac{1}{2}\mathrm{H}_{2,1}+2\mathrm{H}_{0,0,0}-\frac{653}{24}\mathrm{H}_{0,0}\Big]+(1+x)\Big[\mathrm{H}_0\zeta_2-\frac{1187}{216}\mathrm{H}_0\\
&+\frac{8}{9}\mathrm{H}_2-\frac{85}{18}\mathrm{H}_{-1,0}-\frac{101}{18}\zeta_2\Big]-\frac{80}{27}\mathrm{H}_0+\frac{23}{18}\zeta_2-\frac{1}{3}\mathrm{H}_{1,1}+\frac{5}{4}x\mathrm{H}_{1,1}-\frac{1}{9}\mathrm{H}_1-\frac{37}{12}x\mathrm{H}_1+\frac{23}{18}\mathrm{H}_{-1,0}\\
&+\frac{1501}{54}+\mathrm{H}_0\zeta_2-\mathrm{H}_{0,0,0}+\frac{101}{3}\mathrm{H}_{0,0}-\frac{1}{3}\mathrm{H}_{1,0}\Big)+16{C_F}^3\Big(p_{\mathrm{gq}}(x)\Big[3\mathrm{H}_{1,1}\zeta_2+3\mathrm{H}_1\zeta_2+\frac{7}{2}\zeta_2\\
&-\frac{23}{8}\mathrm{H}_{1,1}-8\mathrm{H}_1\zeta_3-6\mathrm{H}_{1,-2,0}-2\mathrm{H}_{1,0}\zeta_2+3\mathrm{H}_{1,1,0}-3\mathrm{H}_{1,1,0,0}-\mathrm{H}_{1,1,1,0}+2\mathrm{H}_{1,1,1,1}-3\mathrm{H}_{1,1,2}\\
&-2\mathrm{H}_{1,2,0}-2\mathrm{H}_{1,2,1}-\frac{9}{2}\mathrm{H}_{1,1,1}-\frac{3}{2}\mathrm{H}_{1,0,0}-\frac{47}{16}-\frac{47}{16}\mathrm{H}_1-\frac{15}{2}\zeta_3\Big]+p_{\mathrm{gq}}(-x)\Big[2\mathrm{H}_{-1,-2,0}\\
&+6\mathrm{H}_{-1,-1,0}+3\mathrm{H}_{-1}\zeta_2+\frac{7}{4}\mathrm{H}_{1,0}-\frac{16}{5}{\zeta_2}^2-6\mathrm{H}_{-1,0,0}-\frac{7}{2}\mathrm{H}_{-1,0}+4\mathrm{H}_{-1,-1,0,0}-2\mathrm{H}_{-1,0}\zeta_2\\
&-\mathrm{H}_{-1,0,0,0}\Big]+(1-x)\Big[9\mathrm{H}_{1,0,0}+\mathrm{H}_{1,1,1}-10\mathrm{H}_1\zeta_2+3\mathrm{H}_0\zeta_3+\mathrm{H}_{2,2}-\mathrm{H}_2\zeta_2+\mathrm{H}_{0,0,0}+5\mathrm{H}_{2,0,0}\\
&-4\mathrm{H}_3+\mathrm{H}_{2,1,1}+3\mathrm{H}_{0,0}\zeta_2+3\mathrm{H}_{3,1}-3\mathrm{H}_4+\frac{211}{16}\mathrm{H}_1+\frac{49}{20}{\zeta_2}^2\Big]+(1+x)\Big[11\zeta_3+\frac{1}{4}\mathrm{H}_{1,1}+\frac{1}{4}\mathrm{H}_{1,0}\\
&+\frac{91}{16}\mathrm{H}_0+36\mathrm{H}_{-1,0}+8\mathrm{H}_{-1,0,0}-14\mathrm{H}_{-1,-1,0}-7\mathrm{H}_{-1}\zeta_2+2\mathrm{H}_{1,2}+4\mathrm{H}_0\zeta_2-\mathrm{H}_{2,1}+2\mathrm{H}_{-2,0,0}\\
&+5\mathrm{H}_{-2,0}+\frac{11}{2}\mathrm{H}_2-2\mathrm{H}_{0,0,0,0}\Big]-2\mathrm{H}_{-1,-1,0}-\mathrm{H}_{-1}\zeta_2-\frac{13}{4}\zeta_2+\frac{9}{4}\mathrm{H}_{1,0}+\frac{9}{20}{\zeta_2}^2+\frac{287}{32}+\frac{11}{16}\mathrm{H}_1\\
&+4\mathrm{H}_{-1,0,0}+16\mathrm{H}_{-3,0}-4\mathrm{H}_{-2}\zeta_2-8\mathrm{H}_{-2,-1,0}-5\mathrm{H}_2\zeta_2+\frac{19}{4}\mathrm{H}_2+\mathrm{H}_{2,2}-\frac{35}{8}\mathrm{H}_{0,0}+9\mathrm{H}_0\zeta_3\\
&+25\mathrm{H}_{-2,0}+6\mathrm{H}_{-2,0,0}+\frac{3}{2}x\Big[\frac{58}{3}\zeta_2-\frac{7}{3}\mathrm{H}_1\zeta_2+4\mathrm{H}_{1,1}-\frac{3}{2}\mathrm{H}_{1,1,1}+\frac{5}{2}\mathrm{H}_{1,0,0}-\frac{175}{96}+\mathrm{H}_{3,1}+\frac{19}{3}\zeta_3\\
&+2\mathrm{H}_{2,0}-14\mathrm{H}_0+\mathrm{H}_{0,0}\zeta_2-\mathrm{H}_{-1,0}-\mathrm{H}_4-\frac{3}{2}\mathrm{H}_{2,1}+\frac{1}{3}\mathrm{H}_{2,1,1}+3\mathrm{H}_{2,0,0}-\frac{5}{6}\mathrm{H}_3-\mathrm{H}_{1,2}-\frac{7}{6}\mathrm{H}_0\zeta_2\\
&+\frac{2}{3}\mathrm{H}_{1,1,0}-\frac{29}{6}\mathrm{H}_{0,0,0}-\frac{185}{8}\mathrm{H}_{0,0}\Big]\Big)\ . \qquad (4.14)
\end{aligned}
$$

Finally the Mellin inversion of Eq. (3.13) yields the NNLO gluon-gluon splitting function

$$
\begin{aligned}
P^{(2)}_{\mathrm{gg}}(x) \;=\; & 16{C_A}{C_F}{n_f}\Big(x^2\Big[\frac{4}{9}\mathrm{H}_2+3\mathrm{H}_{1,0}-\frac{97}{12}\mathrm{H}_1+\frac{8}{3}\mathrm{H}_{-2,0}-\frac{2}{3}\mathrm{H}_0\zeta_2+\frac{103}{27}\mathrm{H}_0-\frac{16}{3}\zeta_2+2\mathrm{H}_3\\
&-6\mathrm{H}_{-1,0}+2\mathrm{H}_{2,0}+\frac{127}{18}\mathrm{H}_{0,0}-\frac{511}{12}\Big]+p_{\mathrm{gg}}(x)\Big[2\zeta_3-\frac{55}{24}\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{17}{24}\mathrm{H}_{1,0}-\frac{43}{18}\mathrm{H}_0\\
&-\frac{521}{144}\mathrm{H}_1-\frac{6923}{432}-\frac{1}{2}\mathrm{H}_{2,1}+2\mathrm{H}_1\zeta_2+\mathrm{H}_0\zeta_2-2\mathrm{H}_{1,0,0}+\frac{1}{12}\mathrm{H}_{1,1}-\mathrm{H}_{1,1,0}-\mathrm{H}_{1,1,1}\Big]-\frac{175}{12}\mathrm{H}_2\\
&+6\mathrm{H}_{-1,0}+8\mathrm{H}_0\zeta_3-6\mathrm{H}_{-2,0}-\frac{53}{6}\mathrm{H}_0\zeta_2-\frac{49}{2}\mathrm{H}_0+\frac{185}{4}\zeta_2+\frac{511}{12}-\frac{1}{2}\mathrm{H}_{2,0}-3\mathrm{H}_{1,0}-4\mathrm{H}_{0,0,0,0}
\end{aligned}
$$

$$-\frac{67}{12}H_{0,0}+\frac{43}{2}\zeta_3-H_{2,1}+\frac{97}{12}H_1-4\zeta_2{}^2-\frac{9}{2}H_3-8H_{-3,0}+\frac{33}{2}H_{0,0,0}+\frac{4}{3}(\frac{1}{x}+x^2)\Big[\frac{1}{2}H_2-H_{2,0}$$
$$+\frac{11}{3}H_{-1,0}+H_{-2,0}+\frac{19}{6}\zeta_2+2\zeta_3-H_{-1}\zeta_2-4H_{-1,-1,0}-\frac{1}{2}H_{-1,0,0}-H_{-1,2}\Big]+(1-x)\Big[9H_1\zeta_2$$
$$+12H_{0,0,0,0}-\frac{293}{108}+\frac{61}{6}H_0\zeta_2-\frac{7}{3}H_{1,0}-\frac{857}{36}H_1-9H_0\zeta_3+16H_{-2,-1,0}-4H_{-2,0,0}+8H_{-2}\zeta_2$$
$$-\frac{13}{2}H_{1,0,0}+\frac{3}{4}H_{1,1}-H_{1,1,0}-H_{1,1,1}\Big]+(1+x)\Big[\frac{1}{6}H_{2,0}-\frac{95}{3}H_{-1,0}-\frac{149}{36}H_2+\frac{3451}{108}H_0$$
$$-7H_{-2,0}+\frac{302}{9}H_{0,0}+\frac{19}{6}H_3-\frac{991}{36}\zeta_2-\frac{163}{6}\zeta_3-\frac{35}{3}H_{0,0,0}+\frac{17}{6}H_{2,1}-\frac{43}{10}\zeta_2{}^2+13H_{-1}\zeta_2$$
$$+18H_{-1,-1,0}-H_{3,1}-6H_4-4H_{-1,2}+6H_{0,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}-2H_{2,1,1}-4H_{3,0}$$
$$-9H_{-1,0,0}\Big]-\frac{241}{288}\delta(1-x)\Big)+16C_An_f{}^2\Big(\frac{19}{54}H_0-\frac{1}{24}xH_0-\frac{1}{27}p_{gg}(x)+\frac{13}{54}(\frac{1}{x}-x^2)\Big[\frac{5}{3}-H_1\Big]$$
$$+(1-x)\Big[\frac{11}{72}H_1-\frac{71}{216}\Big]+\frac{2}{9}(1+x)\Big[\zeta_2+\frac{13}{12}xH_0-\frac{1}{2}H_{0,0}-H_2\Big]+\frac{29}{288}\delta(1-x)\Big)$$
$$+16C_A{}^2n_f\Big(x^2\Big[\zeta_3+\frac{11}{9}\zeta_2+\frac{11}{9}H_{0,0}-\frac{2}{3}H_3+\frac{2}{3}H_0\zeta_2+\frac{1639}{108}H_0-2H_{-2,0}\Big]+\frac{1}{3}p_{gg}(x)\Big[\frac{10}{3}\zeta_2$$
$$-\frac{209}{36}-8\zeta_3-2H_{-2,0}-\frac{1}{2}H_0-\frac{10}{3}H_{0,0}-\frac{20}{3}H_{1,0}-H_{1,0,0}-\frac{20}{3}H_2-H_3\Big]+\frac{10}{9}p_{gg}(-x)\Big[\zeta_2$$
$$+2H_{-1,0}+\frac{3}{10}H_0\zeta_2-H_{0,0}\Big]+\frac{1}{3}(\frac{1}{x}-x^2)\Big[H_3-H_0\zeta_2-\frac{13}{3}H_2+\frac{5443}{108}-3H_1\zeta_2+\frac{205}{36}H_1$$
$$-\frac{13}{3}H_{1,0}+H_{1,0,0}\Big]+(\frac{1}{x}+x^2)\Big[\frac{151}{54}H_0-\frac{8}{3}\zeta_2+\frac{1}{3}H_{-1}\zeta_2-\zeta_3+2H_{-1,-1,0}-\frac{2}{3}H_{-1,0,0}$$
$$-\frac{37}{9}H_{-1,0}+\frac{2}{3}H_{-1,2}\Big]+(1-x)\Big[\frac{5}{6}H_{-2,0}+H_{-3,0}+2H_{0,0,0}-\frac{269}{36}\zeta_2-\frac{4097}{216}-3H_{-2}\zeta_2$$
$$-6H_{-2,-1,0}+3H_{-2,0,0}-\frac{7}{2}H_1\zeta_2+\frac{677}{72}H_1+H_{1,0}+\frac{7}{4}H_{1,0,0}\Big]+(1+x)\Big[\frac{193}{36}H_2-\frac{11}{2}H_{-1}\zeta_2$$
$$+\frac{39}{20}\zeta_2{}^2-\frac{7}{12}H_3-\frac{53}{9}H_{0,0}+\frac{7}{12}H_0\zeta_2-\frac{5}{2}H_{0,0}\zeta_2+5\zeta_3-7H_{-1,-1,0}+\frac{77}{6}H_{-1,0}+\frac{9}{2}H_{-1,0,0}$$
$$+2H_{-1,2}-3H_2\zeta_2-\frac{2}{3}H_{2,0}+\frac{3}{2}H_{2,0,0}+\frac{3}{2}H_4\Big]+\frac{1}{9}\zeta_2+7H_{-2,0}+2H_2+\frac{458}{27}H_0+H_{0,0}\zeta_2$$
$$+\frac{3}{2}\zeta_2{}^2+4H_{-3,0}-x\Big[\frac{131}{12}H_{0,0}-\frac{8}{3}H_0\zeta_2+\frac{7}{2}H_3-H_{0,0,0,0}+\frac{7}{6}H_{0,0,0}+\frac{1943}{216}H_0+6H_0\zeta_3\Big]$$
$$-\delta(1-x)\Big[\frac{233}{288}+\frac{1}{6}\zeta_2+\frac{1}{12}\zeta_2{}^2+\frac{5}{3}\zeta_3\Big]\Big)+16C_A{}^3\Big(x^2\Big[33H_{-2,0}+33H_0\zeta_2-\frac{1249}{18}H_{0,0}$$
$$-44H_{0,0,0}-\frac{110}{3}H_3-\frac{44}{3}H_{2,0}+\frac{85}{6}\zeta_2+\frac{6409}{108}H_0\Big]+p_{gg}(x)\Big[\frac{245}{24}-\frac{67}{9}\zeta_2-\frac{3}{10}\zeta_2{}^2+\frac{11}{3}\zeta_3$$
$$-4H_{-3,0}+6H_{-2}\zeta_2+4H_{-2,-1,0}+\frac{11}{3}H_{-2,0}-4H_{-2,0,0}-4H_{-2,2}+\frac{1}{6}H_0-7H_0\zeta_3+\frac{67}{9}H_{0,0}$$
$$-8H_{0,0}\zeta_2+4H_{0,0,0,0}-6H_1\zeta_3-4H_{1,-2,0}+10H_{2,0,0}-6H_{1,0}\zeta_2+8H_{1,0,0,0}+8H_{1,1,0,0}+8H_4$$
$$+\frac{134}{9}H_{1,0}+\frac{11}{6}H_{1,0,0}+8H_{1,2,0}+8H_{1,3}+\frac{134}{9}H_2-4H_2\zeta_2+8H_{3,1}+8H_{2,2}+\frac{11}{6}H_3+10H_{3,0}$$
$$+8H_{2,1,0}\Big]+p_{gg}(-x)\Big[\frac{11}{2}\zeta_2{}^2-\frac{11}{6}H_0\zeta_2-4H_{-3,0}+16H_{-2}\zeta_2-12H_{-2,2}-\frac{134}{9}H_{-1,0}+2H_2\zeta_2$$
$$+8H_{-2,-1,0}+12H_{-1}\zeta_3-18H_{-2,0,0}+8H_{-1,-2,0}-16H_{-1,-1}\zeta_2+24H_{-1,-1,0,0}+16H_{-1,-1,2}$$

$$
\begin{aligned}
&+18\mathrm{H}_{-1,0}\zeta_2-16\mathrm{H}_{-1,0,0,0}-4\mathrm{H}_{-1,2,0}-16\mathrm{H}_{-1,3}-5\mathrm{H}_0\zeta_3-8\mathrm{H}_{0,0}\zeta_2+4\mathrm{H}_{0,0,0,0}+2\mathrm{H}_{3,0}\\
&-\frac{67}{9}\zeta_2+\frac{67}{9}\mathrm{H}_{0,0}+8\mathrm{H}_4\Big]+\Big(\frac{1}{x}-x^2\Big)\Big[\frac{16619}{162}+\frac{22}{3}\mathrm{H}_{2,0}-\frac{55}{2}\zeta_3-\frac{11}{2}\mathrm{H}_0\zeta_2-\frac{67}{9}\mathrm{H}_2-\frac{67}{9}\mathrm{H}_{1,0}\\
&-\frac{413}{108}\mathrm{H}_1-\frac{11}{2}\mathrm{H}_1\zeta_2+\frac{33}{2}\mathrm{H}_{1,0,0}\Big]+11(\frac{1}{x}+x^2)\Big[\frac{71}{54}\mathrm{H}_0-\frac{1}{6}\mathrm{H}_3-\frac{389}{198}\zeta_2-\frac{2}{3}\mathrm{H}_{-2,0}-\frac{1}{2}\mathrm{H}_{-1}\zeta_2\\
&+\mathrm{H}_{-1,-1,0}-\frac{523}{198}\mathrm{H}_{-1,0}+\frac{8}{3}\mathrm{H}_{-1,0,0}+\mathrm{H}_{-1,2}\Big]+(1-x)\Big[\frac{31}{36}\mathrm{H}_1+\frac{27}{2}\mathrm{H}_{1,0}-\frac{25}{2}\mathrm{H}_{1,0,0}-4\mathrm{H}_{-3,0}\\
&-\frac{263}{12}\mathrm{H}_{0,0}-\frac{29}{3}\mathrm{H}_{0,0,0}-\frac{19}{3}\mathrm{H}_{-2,0}-\frac{11317}{108}-4\mathrm{H}_{-2}\zeta_2-8\mathrm{H}_{-2,-1,0}-12\mathrm{H}_{-2,0,0}-\frac{3}{2}\mathrm{H}_1\zeta_2\Big]\\
&+(1+x)\Big[\frac{27}{2}\mathrm{H}_0\zeta_2-\frac{43}{6}\mathrm{H}_3+\frac{29}{3}\mathrm{H}_{2,0}+\frac{4651}{216}\mathrm{H}_0-\frac{329}{18}\zeta_2+\frac{11}{2}(1+x)\zeta_3-\frac{43}{5}{\zeta_2}^2-\frac{215}{6}\mathrm{H}_{-1,0}\\
&-22\mathrm{H}_{0,0}\zeta_2-8\mathrm{H}_0\zeta_3-3\mathrm{H}_{-1,-1,0}+38\mathrm{H}_{-1,0,0}+25\mathrm{H}_{-1,2}+10\mathrm{H}_{2,0,0}-4\mathrm{H}_2\zeta_2+16\mathrm{H}_{3,0}+26\mathrm{H}_4\\
&-\frac{158}{9}\mathrm{H}_2-\frac{53}{2}\mathrm{H}_{-1}\zeta_2\Big]-29\mathrm{H}_{0,0}-\frac{40}{3}\mathrm{H}_{0,0,0}+27\mathrm{H}_{-2,0}+\frac{41}{3}\mathrm{H}_0\zeta_2-20\mathrm{H}_3-24\mathrm{H}_{2,0}+\frac{53}{6}\zeta_2\\
&+\frac{601}{12}\mathrm{H}_0+24\zeta_3+2{\zeta_2}^2+27\mathrm{H}_2-4\mathrm{H}_{0,0}\zeta_2-16\mathrm{H}_0\zeta_3-16\mathrm{H}_{-3,0}+28x\mathrm{H}_{0,0,0,0}+\delta(1-x)\Big[\frac{79}{32}\\
&-\zeta_2\zeta_3+\frac{1}{6}\zeta_2+\frac{11}{24}{\zeta_2}^2+\frac{67}{6}\zeta_3-5\zeta_5\Big]\Big)+16C_F{n_f}^2\Big(\frac{2}{9}x^2\Big[\frac{11}{6}\mathrm{H}_0+\mathrm{H}_2-\zeta_2+2\mathrm{H}_{0,0}-9\Big]+\frac{1}{3}\mathrm{H}_2\\
&-\frac{1}{3}\zeta_2-\frac{10}{3}\mathrm{H}_0-\frac{1}{3}\mathrm{H}_{0,0}+2+\frac{2}{9}(\frac{1}{x}-x^2)\Big[\frac{8}{3}\mathrm{H}_1-2\mathrm{H}_{1,0}-\mathrm{H}_{1,1}-\frac{77}{18}\Big]-(1-x)\Big[\frac{1}{3}\mathrm{H}_{1,0}+\frac{1}{6}\mathrm{H}_{1,1}\\
&+\frac{4}{9}+\frac{13}{6}\mathrm{H}_1+x\mathrm{H}_1\Big]+\frac{1}{3}(1+x)\Big[\frac{68}{9}\mathrm{H}_0-\frac{4}{3}\mathrm{H}_2+\frac{4}{3}\zeta_2+\frac{29}{6}\mathrm{H}_{0,0}-\zeta_3+2\mathrm{H}_0\zeta_2-\mathrm{H}_{0,0,0}-2\mathrm{H}_3\\
&-\mathrm{H}_{2,1}-2\mathrm{H}_{2,0}\Big]+\frac{11}{144}\delta(1-x)\Big)+16{C_F}^2n_f\Big(\frac{4}{3}x^2\Big[\frac{163}{16}+\frac{27}{8}\mathrm{H}_0+\frac{7}{2}\mathrm{H}_{0,0}-\mathrm{H}_{2,0}-\zeta_2+\frac{9}{4}\mathrm{H}_{1,0}\\
&-\mathrm{H}_{2,1}+\frac{1}{2}\mathrm{H}_{0,0,0}+\frac{85}{16}\mathrm{H}_1+\mathrm{H}_2-2\mathrm{H}_{-2,0}-\frac{3}{2}\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{31}{16}\mathrm{H}_1-\frac{11}{16}-\frac{5}{4}\mathrm{H}_{1,0}+\frac{1}{2}\mathrm{H}_{1,0,0}\\
&-\mathrm{H}_1\zeta_2-\mathrm{H}_{1,1}+\mathrm{H}_{1,1,0}+\mathrm{H}_{1,1,1}+\zeta_3\Big]+\frac{4}{3}(\frac{1}{x}+x^2)\Big[\mathrm{H}_{-1}\zeta_2+2\mathrm{H}_{-1,-1,0}-\mathrm{H}_{-1,0,0}\Big]+\frac{215}{12}\mathrm{H}_{0,0}\\
&+\frac{20}{3}\mathrm{H}_0-\frac{131}{6}+3\mathrm{H}_{2,0}+\frac{205}{12}x\zeta_2-3\mathrm{H}_{1,0}+\mathrm{H}_{2,1}-\frac{85}{12}\mathrm{H}_1+\frac{11}{4}\mathrm{H}_2+8\mathrm{H}_{-2,0}+2{\zeta_2}^2-\mathrm{H}_0\zeta_2\\
&+\mathrm{H}_3+6\mathrm{H}_0\zeta_3+8\mathrm{H}_{-3,0}-4x\mathrm{H}_{0,0,0}+(1-x)\Big[\frac{107}{12}\mathrm{H}_1-\frac{5}{6}\mathrm{H}_{1,0}-4\zeta_2+\mathrm{H}_0\zeta_3-8\mathrm{H}_{-2,-1,0}\\
&-4\mathrm{H}_{-2}\zeta_2+4\mathrm{H}_{-2,0,0}-4\mathrm{H}_1\zeta_2+\frac{7}{2}\mathrm{H}_{1,0,0}-\frac{7}{12}\mathrm{H}_{1,1}+\mathrm{H}_{1,1,0}+\mathrm{H}_{1,1,1}\Big]+(1+x)\Big[\frac{5}{4}\mathrm{H}_2+\frac{33}{8}\\
&-\frac{99}{4}\mathrm{H}_{0,0}-8\mathrm{H}_{2,0}-\frac{541}{24}\mathrm{H}_0-4\mathrm{H}_{2,1}-\frac{3}{2}\mathrm{H}_{0,0,0}-2x\zeta_3+\frac{9}{2}{\zeta_2}^2+5\mathrm{H}_0\zeta_2-5\mathrm{H}_3-4\mathrm{H}_{-1}\zeta_2\\
&-8\mathrm{H}_{-1,-1,0}+\frac{67}{3}\mathrm{H}_{-1,0}+4\mathrm{H}_{-1,0,0}+2\mathrm{H}_{0,0}\zeta_2-2\mathrm{H}_{0,0,0,0}-4\mathrm{H}_2\zeta_2+3\mathrm{H}_{2,0,0}+2\mathrm{H}_{2,1,0}\\
&+2\mathrm{H}_{2,1,1}+\mathrm{H}_{3,1}-2\mathrm{H}_4\Big]+\frac{1}{16}\delta(1-x)\Big)\;. \qquad (4.15)
\end{aligned}
$$

The large-x behaviour of the gluon-gluon splitting function $P^{(2)}_{gg}(x)$ is given by

$$
P^{(2)}_{\mathrm{gg},x\to1}(x) \;=\; \frac{A^{\mathrm{g}}_3}{(1-x)_+} + B^{\mathrm{g}}_3\,\delta(1-x) + C^{\mathrm{g}}_3\,\ln(1-x) + O(1)\;. \qquad (4.16)
$$

The constants $A_3^{\rm g}$ and $C_3^{\rm g}$ have been specified in Eqs. (3.15) and (3.16), respectively, while the coefficients of $\delta(1-x)$ are explicit in Eq. (4.15). The corresponding limit of the gluon-quark and quark-gluon splitting functions is

$$P^{(2)}_{ab,x\to 1}(x) \;=\; \sum_{i=0}^{3} D_i^{\rm ab} \ln^{4-i}(1-x) \;+\; O(1) \tag{4.17}$$

with

$$\begin{aligned}
D_0^{\rm qg} &= \frac{4}{3}C_A^2 n_f - \frac{8}{3}C_A C_F n_f + \frac{4}{3}C_F^2 n_f \\
D_1^{\rm qg} &= -\frac{22}{9}C_A^2 n_f + \frac{40}{9}C_A C_F n_f - 2\,C_F^2 n_f + \frac{4}{9}C_A n_f^2 - \frac{4}{9}C_F n_f^2 \\
D_2^{\rm qg} &= \left[-\frac{268}{9} + 8\,\zeta_2\right] C_A^2 n_f + \frac{16}{9}C_A C_F n_f + [28 - 8\,\zeta_2]\, C_F^2 n_f + \frac{40}{9}C_A n_f^2 - \frac{40}{9}C_F n_f^2 \\
D_3^{\rm qg} &= \left[-\frac{950}{27} + \frac{44}{3}\zeta_2 + 80\,\zeta_3\right] C_A^2 n_f + \left[\frac{1904}{27} - 12\,\zeta_2 - 208\,\zeta_3\right] C_A C_F n_f \\
&\quad - [34 - 128\,\zeta_3]\, C_F^2 n_f + \left[\frac{152}{27} - \frac{8}{3}\zeta_2\right] C_A n_f^2 - \frac{188}{27}C_F n_f^2
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
D_0^{\rm gq} &= \frac{4}{3}C_A^2 C_F - \frac{8}{3}C_A C_F^2 + \frac{4}{3}C_F^3 \\
D_1^{\rm gq} &= \frac{182}{9}C_A^2 C_F - \frac{344}{9}C_A C_F^2 + 18\,C_F^3 - \frac{20}{9}C_A C_F n_f + \frac{20}{9}C_F^2 n_f \\
D_2^{\rm gq} &= \left[\frac{1093}{9} - 8\,\zeta_2\right] C_A^2 C_F - \frac{1342}{9}C_A C_F^2 + [29 + 8\,\zeta_2]\, C_F^3 - \frac{256}{9}C_A C_F n_f \\
&\quad + \frac{232}{9}C_F^2 n_f + \frac{4}{3}C_F n_f^2 \qquad\qquad (4.19)\\
D_3^{\rm gq} &= \left[\frac{9766}{27} - \frac{164}{3}\zeta_2 + 16\,\zeta_3\right] C_A^2 C_F - \left[\frac{9178}{27} - \frac{28}{3}\zeta_2 - 64\,\zeta_3\right] C_A C_F^2 + \frac{64}{9}C_F n_f^2 \\
&\quad + [36 + 32\,\zeta_2 - 80\,\zeta_3]\, C_F^3 - \left[\frac{2944}{27} - \frac{8}{3}\zeta_2\right] C_A C_F n_f + \left[\frac{1408}{27} + \frac{32}{3}\zeta_2\right] C_F^2 n_f \,.
\end{aligned}$$

It is worthwhile to notice that all the coefficients in Eqs. (4.18) and (4.19) except $D_3^{\rm gq}$ vanish for the choice

$$C_A \;\equiv\; n_c \;=\; C_F \;=\; n_f \tag{4.20}$$

of the colour factors leading to a $N\!=\!1$ supersymmetric theory. This is part of a general structure. The combination

$$\Delta_{\rm S}(x) \;\equiv\; P^{(n)}_{\rm qq}(x) + P^{(n)}_{\rm gq}(x) - P^{(n)}_{\rm qg}(x) - P^{(n)}_{\rm gg}(x) \tag{4.21}$$

of the $(n+1)$-loop $\overline{\rm MS}$ splitting functions is found to be much simpler than the functions $P^{(n)}_{\rm ab}(x)$ themselves. In fact, after transforming to the dimensional reduction (DR) scheme respecting the supersymmetry, $\Delta_{\rm S}(x)$ vanishes for both the unpolarized [9] and polarized (spin-dependent) [66, 67, 68] two-loop splitting functions. We are not (yet) in a position to present this scheme transformation at the third order. However, we do obtain the above-mentioned simplification within the $\overline{\rm MS}$ scheme; especially all harmonic polylogarithms of weight four cancel in the combination (4.21) for choice (4.20) of the colour factors. We plan to return to this issue in a later publication.

We now return to the end-point behaviour. At small x the three-loop splitting functions read

$$P^{(2)}_{\rm ab,x\to 0}(x) \;=\; E^{\rm ab}_1\,\frac{\ln x}{x} \;+\; E^{\rm ab}_2\,\frac{1}{x} \;+\; O(\ln^4 x)\;. \tag{4.22}$$

The coefficients of the $1/x$ terms of $P^{(2)}_{\rm qq}$ (which are, of course, entirely due the pure-singlet contribution given in Eq. (4.12)) are given by

$$\begin{aligned}
E^{\rm qq}_1 &= -\frac{896}{27}\,C_A C_F n_f\\
E^{\rm qq}_2 &= \left[-\frac{27044}{81}+\frac{512}{9}\zeta_2+96\zeta_3\right]C_A C_F n_f+\left[\frac{220}{3}-64\zeta_3\right]C_F^2 n_f+\frac{64}{27}C_F n_f^2\;,
\end{aligned}\tag{4.23}$$

or, after inserting $C_A=3$ and $C_F=4/3$ and the numerical values of ζ_2 and ζ_3,

$$\begin{aligned}
E^{\rm qq}_1 &\cong -132.741\,n_f\\
E^{\rm qq}_2 &\cong -505.999\,n_f+3.16049\,n_f^2\;.
\end{aligned}\tag{4.24}$$

The corresponding results for the gluon-quark splitting function (4.13) are

$$E^{\rm qg}_1 \;=\; -\frac{896}{27}\,C_A^2 n_f \;=\; \frac{C_A}{C_F}\,E^{\rm qq}_1 \tag{4.25}$$

$$E^{\rm qg}_2 \;=\; \left[-\frac{9404}{27}+\frac{512}{9}\zeta_2+96\zeta_3\right]C_A^2 n_f+\left[\frac{220}{3}-64\zeta_3\right]C_A C_F n_f-\frac{424}{81}C_A n_f^2+\frac{1232}{81}C_F n_f^2$$

and

$$\begin{aligned}
E^{\rm qg}_1 &\cong -298.667\,n_f\\
E^{\rm qg}_2 &\cong -1268.28\,n_f+4.57613\,n_f^2\;.
\end{aligned}\tag{4.26}$$

The coefficients E_1 in Eqs. (4.23) and (4.25) agree with those obtained by Catani and Hautmann in Ref. [27] from the small-x resummation.

The small-x coefficients of the quark-gluon splitting function (4.14) are given by

$$\begin{aligned}
E^{\rm gq}_1 &= \left[\frac{6320}{27}-\frac{176}{3}\zeta_2-32\zeta_3\right]C_A^2 C_F+\left[\frac{1208}{27}-\frac{32}{3}\zeta_2\right]C_A C_F n_f-\left[\frac{1520}{27}-\frac{64}{3}\zeta_2\right]C_F^2 n_f\\
E^{\rm gq}_2 &= \left[\frac{138305}{81}-\frac{872}{3}\zeta_2-336\zeta_3-\frac{544}{5}\zeta_2^2\right]C_A^2 C_F+\left[\frac{1934}{9}-\frac{112}{3}\zeta_2-80\zeta_3\right]C_A C_F n_f
\end{aligned}$$

$$
+\left[163-160\,\zeta_2+216\,\zeta_3-\frac{432}{5}\,\zeta_2^2\right]\,C_A C_F^2-\left[94-112\,\zeta_2+240\,\zeta_3-\frac{512}{5}\,\zeta_2^2\right]\,C_F^3
$$
$$
-\left[\frac{3250}{9}-\frac{496}{9}\,\zeta_2-96\,\zeta_3\right]\,C_F^2 n_f-\frac{16}{9}C_F n_f^2\;, \qquad (4.27)
$$

or

$$
\begin{aligned}
E_1^{\rm gq} &\cong 1189.27+71.0825\,n_f\\
E_2^{\rm gq} &\cong 6163.11-46.4075\,n_f-2.37037\,n_f^2\;.
\end{aligned} \qquad (4.28)
$$

Finally the corresponding coefficients of the three-loop gluon-gluon splitting function (4.15) read

$$
\begin{aligned}
E_1^{\rm gg} &= \left[\frac{6320}{27}-\frac{176}{3}\zeta_2-32\,\zeta_3\right]\,C_A^3+\left[\frac{1136}{27}-\frac{32}{3}\zeta_2\right]\,C_A^2 n_f-\left[\frac{1376}{27}-\frac{64}{3}\zeta_2\right]\,C_A C_F n_f\\
&= \frac{C_A}{C_F}\,E_1^{\rm gq}-\frac{8}{3}C_A n_f\,(C_A-2C_F)\;\rightarrow\;\frac{C_A}{C_F}\,E_1^{\rm gq}-\frac{8}{3}n_f \quad \text{for SU(N)}\\
E_2^{\rm gg} &= \left[\frac{146182}{81}-\frac{3112}{9}\zeta_2-\frac{1144}{3}\zeta_3-\frac{464}{5}\zeta_2^2\right]\,C_A^3+\left[\frac{19264}{81}-\frac{128}{3}\zeta_2-\frac{176}{3}\zeta_3\right]\,C_A^2 n_f\\
&\quad-\left[\frac{30662}{81}-\frac{608}{9}\zeta_2-\frac{224}{3}\zeta_3\right]\,C_A C_F n_f-\left[\frac{44}{3}-\frac{64}{3}\zeta_3\right]\,C_F^2 n_f+\frac{472}{81}C_A n_f^2\\
&\quad-\frac{1232}{81}C_F n_f^2
\end{aligned} \qquad (4.29)
$$

and

$$
\begin{aligned}
E_1^{\rm gg} &\cong 2675.85+157.269\,n_f\\
E_2^{\rm gg} &\cong 14214.2+182.958\,n_f-2.79835\,n_f^2\;.
\end{aligned} \qquad (4.30)
$$

The coefficient $E_1^{\rm gg}$ is identical to the result obtained from the next-to-leading logarithmic BFKL equation by Fadin and Lipatov in Ref. [28] after transformation to the $\overline{\rm MS}$ scheme (as given, e.g., in Eq. (4.7) of Ref. [36]). Numerically the simple relation $C_F E_1^{\rm gg} = C_A E_1^{\rm gq}$ is broken by less than 2% in the n_f part and less than 0.5% for the complete coefficients at $n_f = 3,\ldots,6$.

The three-loop splitting functions (4.12) – (4.15) are shown in Figs. 3 – 6 for $n_f = 4$ together with the approximate expressions inferred in Ref. [37] from the fixed-N results of Refs. [25, 26] and the small-x limits of Refs. [27, 28]. Also displayed are the respective leading small-x contributions $E_1^{\rm ab}x^{-1}\ln x$. Notice that all splitting functions have been multiplied by x for display purposes.

With the exception of $P_{\rm gq}^{(2)}$, where no small-x ‘anchor’ was available, our exact results comply with the error bands of Ref. [37] for the full range of x shown in the figures. Hence it is reasonable to expect that an extension of the results of Refs. [25, 26] to the next order, using a future four-loop generalization of the MINCER program [41, 42], would, together with small-x constraints, facilitate relevant estimates of $P_{\rm ab}^{(3)}(x)$. We expect that such an extension, while still a formidable task, will be performed much earlier than the fourth-order version of the present calculation.

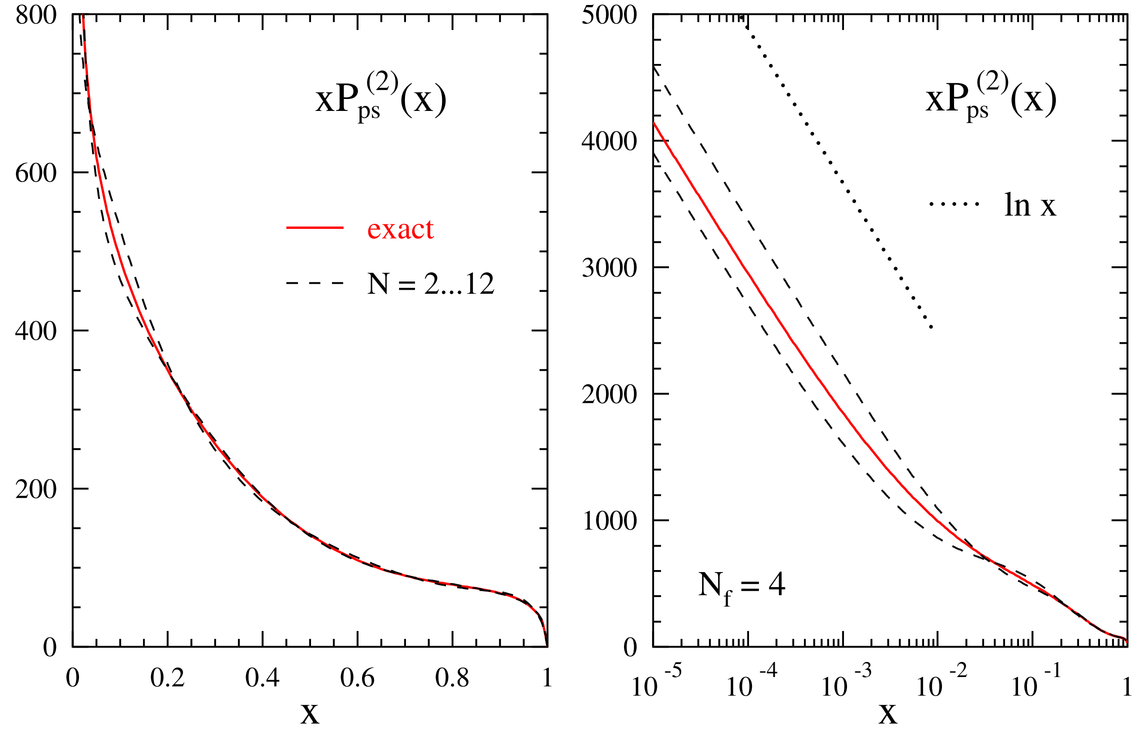

Figure 3: The three-loop pure-singlet splitting function (4.12) for four flavours, multiplied by x for display purposes. Also shown is the uncertainty band derived in Ref. [37] using the lowest six even-integer moments [25, 26] and the leading small-x term [27]. The latter contribution is shown separately on the right-hand-side (dotted line) for $x < 0.01$.

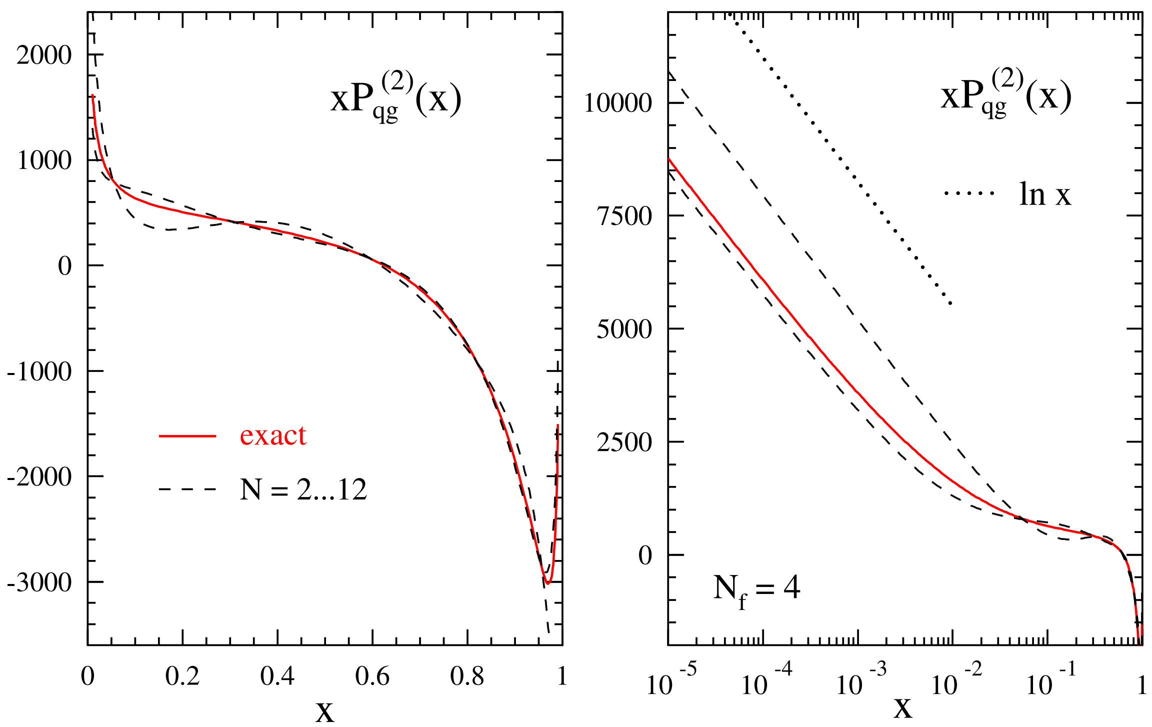

Figure 4: As Fig. 3, but for the third-order gluon-quark splitting function specified in Eq. (4.13).

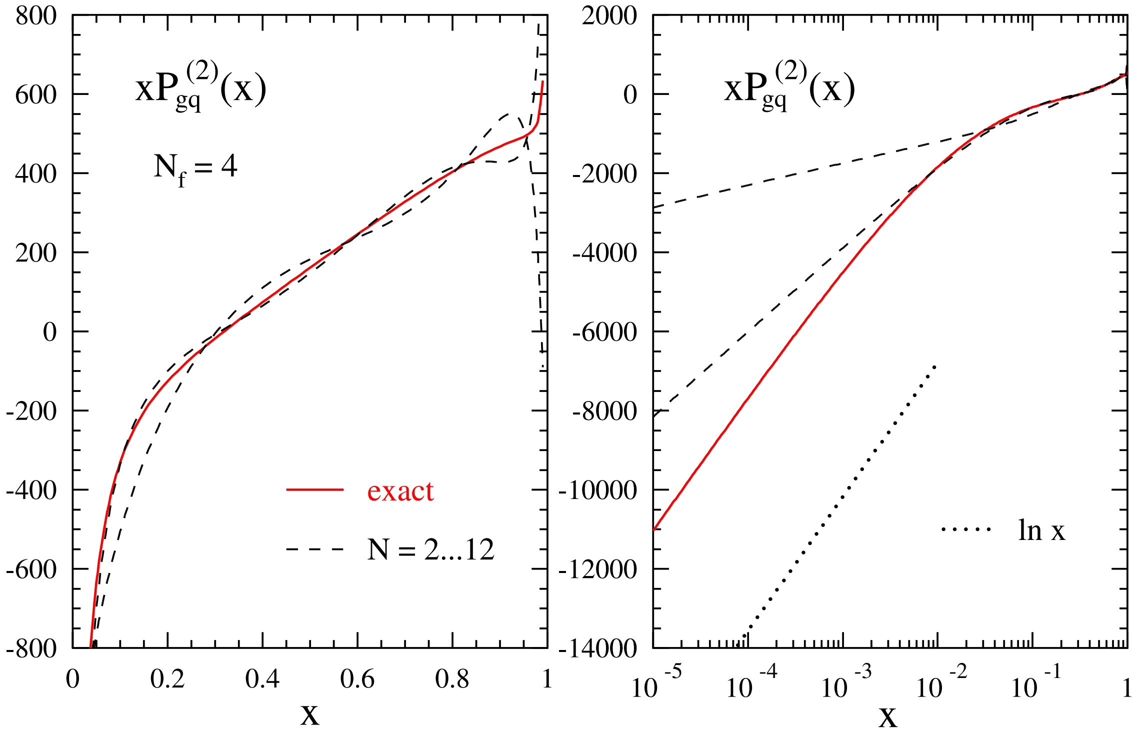

Figure 5: As Fig. 3, but for the three-loop quark-gluon splitting function (4.14). Note that in this case the leading small-x contribution was unknown before the present calculation.

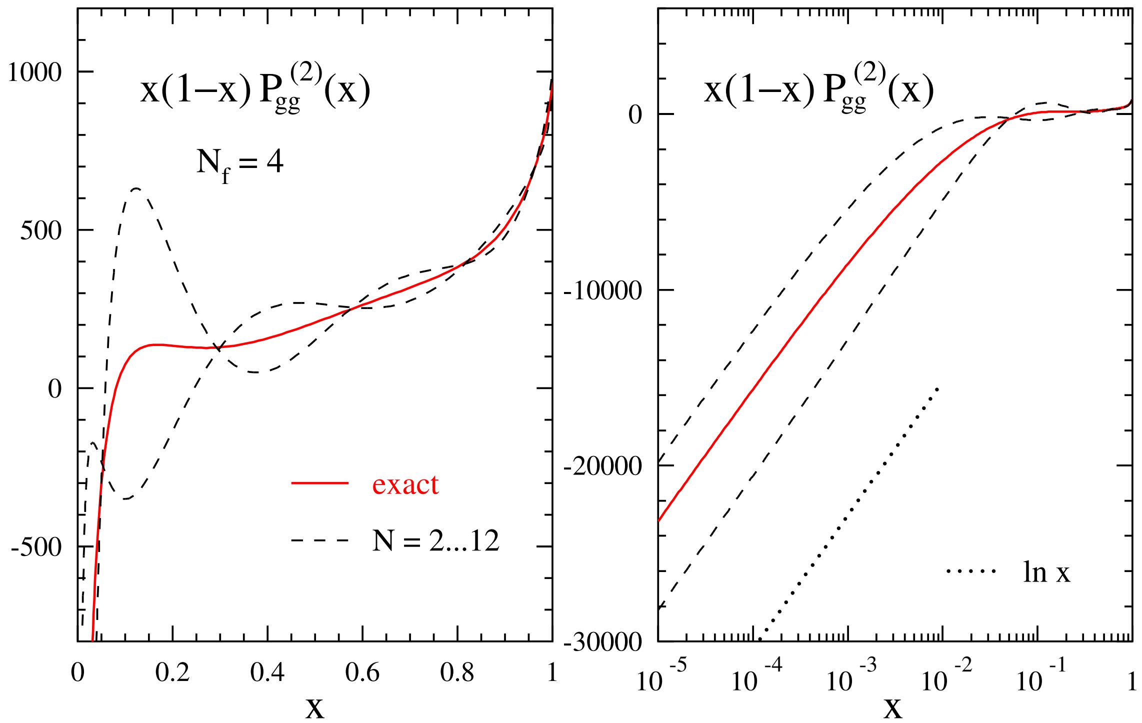

Figure 6: As Fig. 3, but for the third-order gluon-gluon splitting function specified in Eq. (4.15). This diagonal quantity has been additionally multiplied by $(1-x)$. The leading small-x term (again shown by the dotted line on the right-hand-side) has been first obtained in Ref. [28].

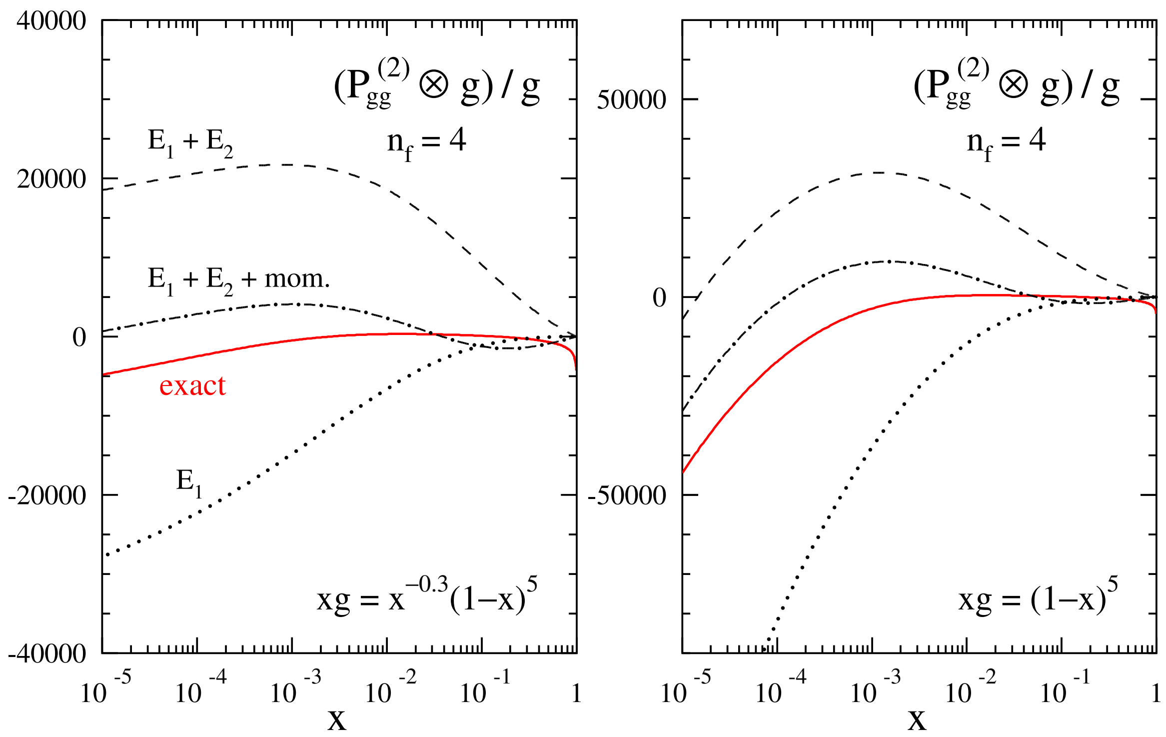

Figure 7: The convolution of the three-loop gluon-gluon splitting function (4.15) with schematic 'steep' (left) and 'flat' (right) gluon distributions. Also shown the results obtained by instead using only the leading (E_1) and the leading and next-to-leading ($E_1 + E_2$) small-x terms in Eq. (4.22), and by supplementing the latter by a constant term restoring the correct second moment.

As also illustrated in Figs. 3 – 6, the leading small-x terms $\sim x^{-1} \ln x$ alone do not provide good approximations of the full results (4.12)–(4.15) at experimentally relevant small values of x. At $x = 10^{-4}$, for example, they exceed the exact values of $P_{\rm ab}^{(2)}(x)$ by factors between 1.6 and 2.0 for $n_f = 4$. Good small-x approximations of these quantities are obtained by including all x^{-1} contributions as specified in Eq. (4.22) – (4.30). However this does not apply, as obvious from Fig. 7, to the convolution $[P_{\rm gg}^{(2)} \otimes g](x)$ by which $P_{\rm gg}^{(2)}$ enters the evolution equations (2.2). Even if the two terms explicit in Eq. (4.22) are (non-uniquely) supplemented by an x-independent contribution restoring the correct second moment, even the sign of the convolution remains wrong down to $x \simeq 10^{-5}$ for the simplified, but not unrealistic gluon distribution $xg \sim x^{-0.3}(1-x)^5$.

As our exact expressions (4.12) – (4.15) for the the functions $P_{\rm ab}^{(2)}(x)$ are neither particularly short nor especially simple, we also provide compact approximate representations built up, besides powers of x, only from the $+$-distribution (for $P_{\rm gg}^{(2)}(x)$) and the end-point logarithms

$$\mathcal{D}_0 \,=\, 1/(1-x)_+ \,, \quad L_1 \,=\, \ln(1-x) \,, \quad L_0 \,=\, \ln x \,. \tag{4.31}$$

Inserting the numerical values of the QCD colour factors, $P_{\rm ps}^{(2)}$ in Eq. (4.12) can be represented by

$$P_{\rm ps}^{(2)}(x) \;\cong\; \Big\{ n_f \Big(-5.926\, L_1^3 - 9.751\, L_1^2 - 72.11\, L_1 + 177.4 + 392.9\, x - 101.4\, x^2 - 57.04\, L_0 L_1$$

$$
\begin{aligned}
&\quad - 661.6\, L_0 + 131.4\, L_0^2 - 400/9\, L_0^3 + 160/27\, L_0^4 - 506.0\, x^{-1} - 3584/27\, x^{-1} L_0 \Big) \\
&+ \; n_f^2 \Big(1.778\, L_1^2 + 5.944\, L_1 + 100.1 - 125.2\, x + 49.26\, x^2 - 12.59\, x^3 - 1.889\, L_0 L_1 \\
&\quad + 61.75\, L_0 + 17.89\, L_0^2 + 32/27\, L_0^3 + 256/81\, x^{-1} \Big) \Big\} (1-x) \; . \qquad (4.32)
\end{aligned}
$$

Correspondingly the off-diagonal quantities (4.13) and (4.14) can be parametrized by

$$
\begin{aligned}
P^{(2)}_{\rm qg}(x) \;\cong\;& n_f \Big(100/27\, L_1^4 - 70/9\, L_1^3 - 120.5\, L_1^2 + 104.42\, L_1 + 2522 - 3316\, x + 2126\, x^2 \\
& + L_0 L_1 \,(1823 - 25.22\, L_0) - 252.5\, x L_0^3 + 424.9\, L_0 + 881.5\, L_0^2 - 44/3\, L_0^3 \\
& + 536/27\, L_0^4 - 1268.3\, x^{-1} - 896/3\, x^{-1} L_0 \Big) \\
+ \;& n_f^2 \Big(20/27\, L_1^3 + 200/27\, L_1^2 - 5.496\, L_1 - 252.0 + 158.0\, x + 145.4\, x^2 \\
& - 139.28\, x^3 - L_0 L_1 \,(53.09 + 80.616\, L_0) - 98.07\, x L_0^2 + 11.70\, x L_0^3 \\
& - 254.0\, L_0 - 90.80\, L_0^2 - 376/27\, L_0^3 - 16/9\, L_0^4 + 1112/243\, x^{-1} \Big) \qquad (4.33)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{(2)}_{\rm gq}(x) \;\cong\;& + 400/81\, L_1^4 + 2200/27\, L_1^3 + 606.3\, L_1^2 + 2193\, L_1 - 4307 + 489.3\, x + 1452\, x^2 \\
& + 146.0\, x^3 - 447.3\, L_0^2 L_1 - 972.9\, x L_0^2 + 4033\, L_0 - 1794\, L_0^2 + 1568/9\, L_0^3 \\
& - 4288/81\, L_0^4 + 6163.1\, x^{-1} + 1189.3\, x^{-1} L_0 \\
+ \;& n_f \Big(- 400/81\, L_1^3 - 68.069\, L_1^2 - 296.7\, L_1 - 183.8 + 33.35\, x - 277.9\, x^2 \\
& + 108.6\, x L_0^2 - 49.68\, L_0 L_1 + 174.8\, L_0 + 20.39\, L_0^2 + 704/81\, L_0^3 \\
& + 128/27\, L_0^4 - 46.41\, x^{-1} + 71.082\, x^{-1} L_0 \Big) \\
+ \;& n_f^2 \Big(96/27\, L_1^2 \,(x^{-1} - 1 + 1/2\, x) + 320/27\, L_1 \,(x^{-1} - 1 + 4/5\, x) \\
& - 64/27\, (x^{-1} - 1 - 2x) \Big) \; , \qquad (4.34)
\end{aligned}
$$

where the n_f^2 part is exact. Finally the gluon-gluon splitting function (4.15) can be approximated by

$$
\begin{aligned}
P^{(2)}_{\rm gg}(x) \;\cong\;& + 2643.521\, \mathcal{D}_0 + 4425.894\, \delta(1-x) + 3589\, L_1 - 20852 + 3968\, x - 3363\, x^2 \\
& + 4848\, x^3 + L_0 L_1 \,(7305 + 8757\, L_0) + 274.4\, L_0 - 7471\, L_0^2 + 72\, L_0^3 - 144\, L_0^4 \\
& + 14214\, x^{-1} + 2675.8\, x^{-1} L_0 \\
+ \;& n_f \Big(- 412.172\, \mathcal{D}_0 - 528.723\, \delta(1-x) - 320\, L_1 - 350.2 + 755.7\, x - 713.8\, x^2 \\
& + 559.3\, x^3 + L_0 L_1 \,(26.15 - 808.7\, L_0) + 1541\, L_0 + 491.3\, L_0^2 + 832/9\, L_0^3 \\
& + 512/27\, L_0^4 + 182.96\, x^{-1} + 157.27\, x^{-1} L_0 \Big) \\
+ \;& n_f^2 \Big(- 16/9\, \mathcal{D}_0 + 6.4630\, \delta(1-x) - 13.878 + 153.4\, x - 187.7\, x^2 + 52.75\, x^3 \\
& - L_0 L_1 \,(115.6 - 85.25\, x + 63.23\, L_0) - 3.422\, L_0 + 9.680\, L_0^2 - 32/27\, L_0^3 \\
& - 680/243\, x^{-1} \Big) \; . \qquad (4.35)
\end{aligned}
$$

The coefficients of $1/x$, $(\ln x)/x$, $\ln^3 x$ and $\ln^4 x$ are exact in Eqs. (4.32) – (4.35), up to a truncation of the irrational numbers. The same holds for the coefficients of $\ln^3(1-x)$ and $\ln^4(1-x)$ in Eqs. (4.33) and (4.34), and those of $\mathcal{D}_0$ and $\ln(1-x)$ in Eq. (4.35). The remaining terms (except, or course, for the $\delta(1-x)$ parts in Eq. (4.35)) have been obtained by fits to the exact results (4.12) – (4.15) at $10^{-6} \leq x \leq 1-10^{-6}$ which we evaluated using the FORTRAN code of Ref. [69]. Smaller values of x are not needed, as all $1/x$ terms are exact. Except for values of x very close to zeros of $P^{(2)}_{\rm ab}(x)$, the parametrizations (4.32) – (4.35) deviate from the exact expressions by less than one part in a thousand, which should be amply sufficient for foreseeable numerical applications.

Finally the coefficients of $\delta(1-x)$ in Eq. (4.35) have been slightly adjusted from their exact values using the lowest integer moments. This is a somewhat tricky point, so let us briefly elaborate on it. For $P_{\rm qq}$ and $P_{\rm gg}$ the low moments, and partly also the convolutions with the parton distributions, involve large cancellations between the integrals over the (fitted) regular parts and the $\delta(1-x)$ contributions. The second moment of the n_f-independent part of $P_{\rm gg}$, for example, vanishes due to the momentum sum rule (recall that $P_{\rm qg}$ has no n_f^0 contribution, cf. Eq. (2.5)), while the respective third-order coefficient of $\delta(1-x)$ is as large as $4 \cdot 10^3$. A maximal accuracy of the parametrization (4.35), and of the convolutions with the gluon distribution, is thus achieved by 'fitting' this coefficient to the second moment. For the case under consideration this actually leads to a very small adjustment of about 0.01%.

One important approach to implementing higher-order results into the numerical evolution of the parton distributions and the analysis of general hard processes is the moment-space technique [70, 71, 72, 73, 74], which requires the analytic continuation of the anomalous dimensions (2.6) to certain complex values of N. Also these complex-N moments can be readily obtained to a perfectly sufficient accuracy using Eqs. (4.32) – (4.35) together with the corresponding non-singlet results in Eqs. (4.22) – (4.24) of Ref. [38]. The Mellin transform of these parametrizations involve only simple harmonic sums $S_{m>0}(N)$ of which the analytic continuations in terms of logarithmic derivatives of Euler's Γ-function are well known. The reader is referred to Refs. [59, 75] for a more mathematical approach to the analytic continuations.

5 Numerical implications

We are now ready to illustrate the numerical effect of our new three-loop splitting functions $P^{(2)}_{\rm ab}(x)$ on the evolution (2.2) of the singlet-quark and gluon distributions $q_{\rm s}(x,\mu_f^2)$ and $g(x,\mu_f^2)$. For all figures we choose a reference scale $\mu_f^2 = \mu_0^2 \simeq 30\ {\rm GeV}^2$ – a scale relevant, for example, for deep-inelastic scattering both at fixed-target experiments and the *ep* collider HERA – and employ the sufficiently realistic model distributions

$$
\begin{aligned}
xq_{\rm s}(x,\mu_0^2) &= 0.6\,x^{-0.3}(1-x)^{3.5}\,(1+5.0\,x^{0.8}) \\
xg(x,\mu_0^2) &= 1.6\,x^{-0.3}(1-x)^{4.5}\,(1-0.6\,x^{0.3})
\end{aligned}
\tag{5.1}
$$

irrespective of the order of the expansion. This order-independence does not hold for actual data-fitted parton distributions like those in Refs. [33, 34], but here it facilitates direct comparisons of the various contributions to the scale derivatives $\dot{f} \equiv d\ln f/d\ln\mu_f^2$ for $f = q_s$, g. For the same reason we employ an order-independent value for the strong coupling constant,

$$\alpha_s(\mu_0^2) = 0.2 \;, \tag{5.2}$$

corresponding to a fairly standard value at the Z mass, $\alpha_s(M_Z^2) \simeq 0.116$, beyond the leading order. Finally our default for the number of effectively massless flavours is $n_f = 4$.

The respective scale derivatives of the singlet-quark and gluon distributions are graphically displayed in Figs. 8 and 9 over a wide range of x. Numerical values can be found for four characteristic x-values in Tables 2 and 3, where we also show the dependence on n_f and the break-up into the quark- and gluon-initiated contributions. As these two terms can occur with different signs, and since the LO and NLO results partly display a somewhat anomalous behaviour (see below), the picture is much less clear-cut here than in the non-singlet sector discussed in Ref. [38].

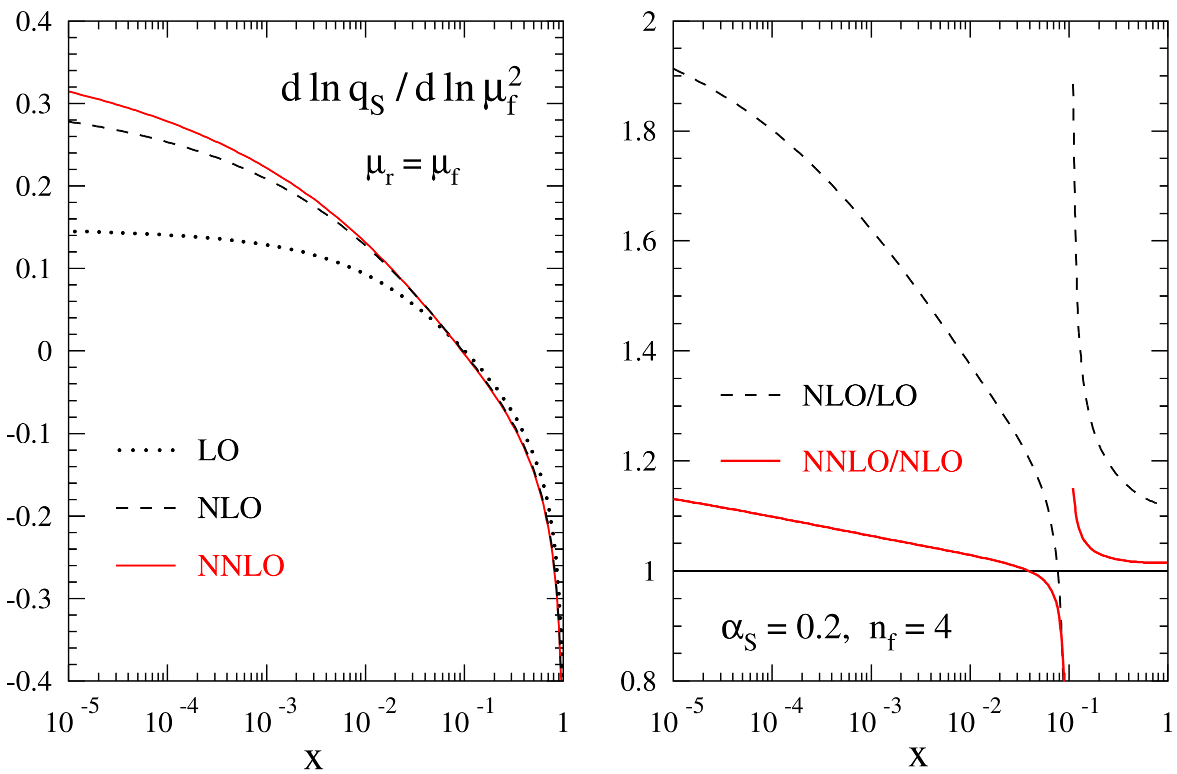

Figure 8: The perturbative expansion of the scale derivative $\dot{q}_s \equiv d\ln q_s/d\ln\mu_f^2$ of the singlet quark density at $\mu_f^2 = \mu_0^2$ for the initial conditions specified in Eqs. (5.1) and (5.2).

The scale derivative of the quark distribution (Fig. 8 and Table 2) is dominated at large x (small x) by the $P_{qq} \otimes q_s$ ($P_{qg} \otimes g$) contributions. The former (latter) is actually negligible for very small (large) values of x. The NNLO corrections are small at large x with respect to both the total derivative and the NLO contributions. At small-x all NLO contributions are very large (or the LO terms are abnormally small, recall that $xP^{(0)}_{\rm qq}$ and $xP^{(0)}_{\rm qg}$ vanish for $x \to 0$). Consequently the total NNLO corrections, while reaching 10% at $x = 10^{-4}$, remain smaller than the NLO results by a factor of eight or more over the full x-range.

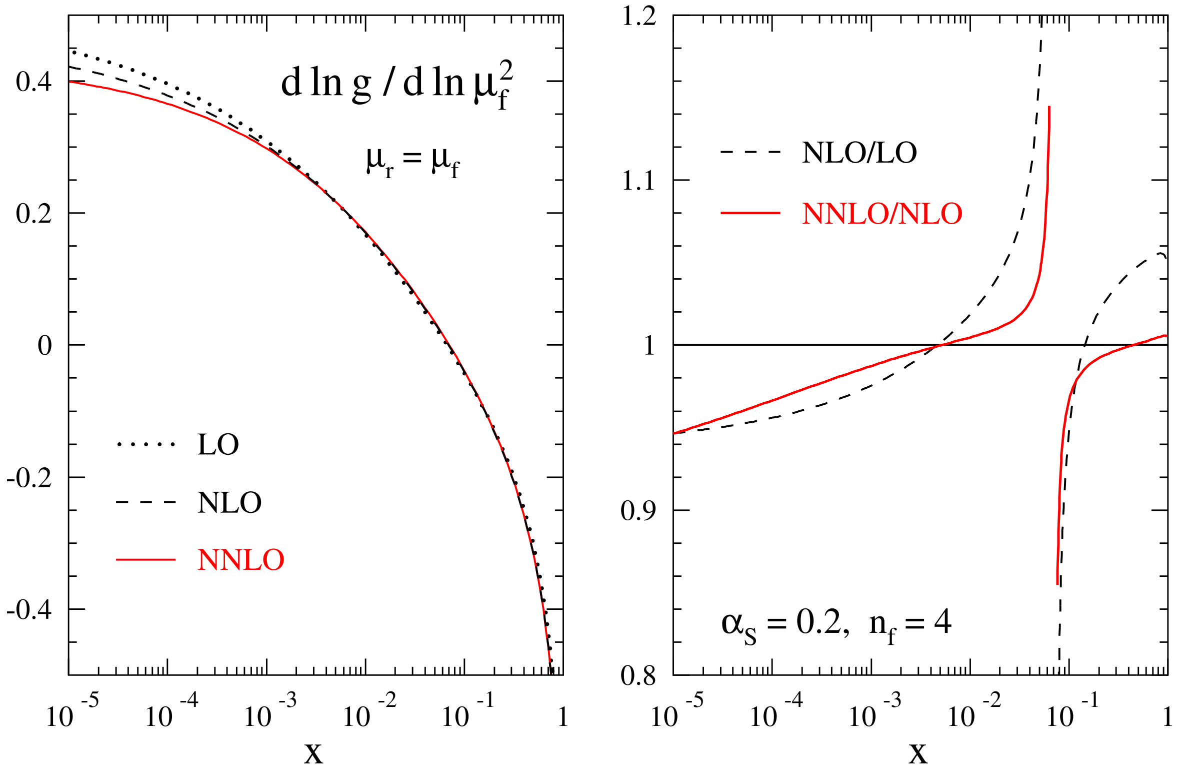

Figure 9: As Fig. 8, but for the gluon density. The spikes close to $x = 0.1$ in the right parts of both figures are due to zeros of the LO and NLO predictions and do not represent large corrections.

The situation is rather different for the evolution of the gluon density (Fig. 9 and Table 3). The contribution from $P_{\rm gg} \otimes g$ dominates for all x (except for extremely large values not considered here), but the $P_{\rm gq} \otimes q_{\rm s}$ part is nowhere negligible. Already the NLO corrections are small especially at small x and furthermore the g- and $q_{\rm s}$-initiated terms cancel each other to some extent. Thus the ratio r_2/r_1 of the relative NNLO and NLO corrections is rather large at small values of x, despite the NNLO contribution amounting to only 3% for x as low as 10^{-4}.

We now turn to the stability of the perturbative expansions in Figs. 8 and 9 under variations of the renormalization scale μ_r. For $\mu_r \neq \mu_f$ the expansion of the splitting functions in Eq. (4.1) is, using the abbreviation $a_{\rm s} \equiv \alpha_{\rm s}/(4\pi)$, replaced by

$$\begin{aligned} P_{\rm ab}(\mu_f,\mu_r) \;=\;& a_{\rm s}(\mu_r^2)\, P^{(0)}_{\rm ab} \;+\; a_{\rm s}^2(\mu_r^2) \left(P^{(1)}_{\rm ab} - \beta_0\, P^{(0)}_{\rm ab} \ln \frac{\mu_f^2}{\mu_r^2} \right) \\ & + a_{\rm s}^3(\mu_r^2) \left(P^{(2)}_{\rm ab} - \left\{ \beta_1 P^{(0)}_{\rm ab} + 2\beta_0\, P^{(1)}_{\rm ab} \right\} \ln \frac{\mu_f^2}{\mu_r^2} + \beta_0^2\, P^{(0)}_{\rm ab} \ln^2 \frac{\mu_f^2}{\mu_r^2} \right) + \ldots \, , \end{aligned} \tag{5.3}$$

where β_k represent the $\overline{\rm MS}$ expansion coefficients of the β-function of QCD [76, 77, 78, 79].

x	LO	NLO	NNLO	r_1	r_2	r_2/r_1
complete (Fig. 8)						
10^{-4}	$1.405\cdot10^{-1}$	$2.532\cdot10^{-1}$	$2.781\cdot10^{-1}$	0.802	0.099	0.12
0.002	$1.218\cdot10^{-1}$	$1.890\cdot10^{-1}$	$1.991\cdot10^{-1}$	0.552	0.053	0.10
0.25	$-5.783\cdot10^{-2}$	$-6.919\cdot10^{-2}$	$-7.093\cdot10^{-2}$	0.196	0.025	0.13
0.75	$-2.056\cdot10^{-1}$	$-2.311\cdot10^{-1}$	$-2.346\cdot10^{-1}$	0.124	0.015	0.12
$P_{\mathrm{qg}}\otimes g$ contribution						
10^{-4}	$1.624\cdot10^{-1}$	$2.610\cdot10^{-1}$	$2.787\cdot10^{-1}$	0.607	0.068	0.11
0.002	$1.421\cdot10^{-1}$	$1.996\cdot10^{-1}$	$2.053\cdot10^{-1}$	0.404	0.029	0.07
0.25	$1.146\cdot10^{-2}$	$1.087\cdot10^{-2}$	$1.037\cdot10^{-2}$	-0.051	-0.046	0.89
0.75	$3.773\cdot10^{-4}$	$1.682\cdot10^{-4}$	$1.578\cdot10^{-4}$	-0.554	-0.062	0.11
$P_{\mathrm{qq}}\otimes q$ contribution						
10^{-4}	$-2.185\cdot10^{-2}$	$-7.838\cdot10^{-3}$	$-5.308\cdot10^{-4}$	-0.641	-0.932	1.45
0.002	$-2.036\cdot10^{-2}$	$-1.056\cdot10^{-2}$	$-6.182\cdot10^{-3}$	-0.481	-0.415	0.86
0.25	$-6.929\cdot10^{-2}$	$-8.006\cdot10^{-2}$	$-8.130\cdot10^{-2}$	0.156	0.015	0.10
0.75	$-2.060\cdot10^{-1}$	$-2.313\cdot10^{-1}$	$-2.347\cdot10^{-1}$	0.123	0.015	0.12
complete, but $n_f = 3$						
10^{-4}	$9.993\cdot10^{-2}$	$1.831\cdot10^{-1}$	$2.018\cdot10^{-1}$	0.832	0.102	0.12
0.002	$8.625\cdot10^{-2}$	$1.354\cdot10^{-1}$	$1.429\cdot10^{-1}$	0.570	0.055	0.10
0.25	$-6.070\cdot10^{-2}$	$-7.293\cdot10^{-2}$	$-7.527\cdot10^{-2}$	0.202	0.032	0.16
0.75	$-2.057\cdot10^{-1}$	$-2.344\cdot10^{-1}$	$-2.397\cdot10^{-1}$	0.139	0.023	0.16
low scale: $n_f = 3\,,\ \alpha_{\mathrm{s}} = 0.4$ and modified input (see Table 3)						
10^{-4}	$2.132\cdot10^{-1}$	$9.317\cdot10^{-1}$	$1.416\cdot10^{-0}$	3.37	0.520	0.15
0.002	$2.047\cdot10^{-1}$	$6.047\cdot10^{-1}$	$7.762\cdot10^{-1}$	1.95	0.284	0.15
0.25	$-8.394\cdot10^{-2}$	$-1.227\cdot10^{-1}$	$-1.384\cdot10^{-1}$	0.462	0.128	0.28
0.735	$-3.870\cdot10^{-1}$	$-4.962\cdot10^{-1}$	$-5.362\cdot10^{-1}$	0.282	0.081	0.29

Table 2: The LO, NLO and NNLO logarithmic derivatives of the singlet quark distribution at four representative values of x, together with the ratios $r_n = \mathrm{N^nLO}/\mathrm{N^{n-1}LO} - 1$ for the default input parameters specified in the first paragraph of this section and some variations thereof.

x	LO	NLO	NNLO	r_1	r_2	r_2/r_1
complete (Fig. 9)						
10^{-4}	$3.956 \cdot 10^{-1}$	$3.782 \cdot 10^{-1}$	$3.655 \cdot 10^{-1}$	-0.044	-0.034	0.76
0.002	$2.730 \cdot 10^{-1}$	$2.689 \cdot 10^{-1}$	$2.670 \cdot 10^{-1}$	-0.015	-0.007	0.47
0.25	$-1.614 \cdot 10^{-1}$	$-1.661 \cdot 10^{-1}$	$-1.653 \cdot 10^{-1}$	0.029	-0.005	-0.18
0.75	$-4.721 \cdot 10^{-1}$	$-4.980 \cdot 10^{-1}$	$-5.000 \cdot 10^{-1}$	0.055	0.004	0.08
$P_{\mathrm{gg}} \otimes g$ contribution						
10^{-4}	$3.085 \cdot 10^{-1}$	$2.895 \cdot 10^{-1}$	$2.814 \cdot 10^{-1}$	-0.062	-0.028	0.45
0.002	$1.898 \cdot 10^{-1}$	$1.825 \cdot 10^{-1}$	$1.825 \cdot 10^{-1}$	-0.038	0.000	0.00
0.25	$-2.129 \cdot 10^{-1}$	$-2.287 \cdot 10^{-1}$	$-2.295 \cdot 10^{-1}$	0.074	0.003	0.04
0.75	$-5.226 \cdot 10^{-1}$	$-5.667 \cdot 10^{-1}$	$-5.717 \cdot 10^{-1}$	0.084	0.009	0.11
$P_{\mathrm{gq}} \otimes q$ contribution						
10^{-4}	$8.715 \cdot 10^{-2}$	$8.873 \cdot 10^{-2}$	$8.409 \cdot 10^{-2}$	0.018	-0.052	2.89
0.002	$8.323 \cdot 10^{-2}$	$8.642 \cdot 10^{-2}$	$8.454 \cdot 10^{-2}$	0.038	-0.022	0.57
0.25	$5.154 \cdot 10^{-2}$	$6.264 \cdot 10^{-2}$	$6.424 \cdot 10^{-2}$	0.215	0.026	0.12
0.75	$5.047 \cdot 10^{-2}$	$6.871 \cdot 10^{-2}$	$7.168 \cdot 10^{-2}$	0.361	0.043	0.12
complete, but $n_f = 3$						
10^{-4}	$4.062 \cdot 10^{-1}$	$4.053 \cdot 10^{-1}$	$3.973 \cdot 10^{-1}$	-0.002	-0.020	8.29
0.002	$2.836 \cdot 10^{-1}$	$2.915 \cdot 10^{-1}$	$2.927 \cdot 10^{-1}$	0.028	0.004	0.15
0.25	$-1.508 \cdot 10^{-1}$	$-1.578 \cdot 10^{-1}$	$-1.579 \cdot 10^{-1}$	0.047	0.001	0.01
0.75	$-4.615 \cdot 10^{-1}$	$-4.971 \cdot 10^{-1}$	$-5.028 \cdot 10^{-1}$	0.077	0.012	0.15
low scale: $n_f = 3\,,\ \alpha_{\mathrm{s}} = 0.4$ and $xq_{\mathrm{s}}(x,\mu_0^2) \;=\; 0.6\,x^{-0.1}(1-x)^3\,(1+10\,x^{0.8})$ $xg(x,\mu_0^2) \;=\; 1.2\,x^{-0.1}(1-x)^4\,(1+1.5\,x)$						
10^{-4}	$2.135 \cdot 10^{-0}$	$2.015 \cdot 10^{-0}$	$1.536 \cdot 10^{-0}$	-0.056	-0.238	4.23
0.002	$1.412 \cdot 10^{-0}$	$1.430 \cdot 10^{-0}$	$1.376 \cdot 10^{-0}$	0.013	-0.038	-3.00
0.25	$-1.663 \cdot 10^{-1}$	$-1.631 \cdot 10^{-1}$	$-1.523 \cdot 10^{-1}$	-0.019	-0.067	3.46
0.75	$-9.055 \cdot 10^{-1}$	$-1.064 \cdot 10^{-0}$	$-1.116 \cdot 10^{-0}$	0.175	0.049	0.28

Table 3: As Table 2, but for the scale derivative $d\ln g/d\ln\mu_f^2$ of the gluon distribution.

In Figs. 10 and 11 the respective consequences of varying μ_r over the rather wide range $\frac{1}{8}\mu_f^2 \leq \mu_r^2 \leq 8\mu_f^2$ are displayed for the logarithmic μ_f-derivatives of the singlet-quark and gluon distributions (5.1) at six representative values of x. In both cases the scale dependence is considerably reduced over the full x-range by including the third-order corrections. With the exception of the smallest x-value considered, $x = 10^{-5}$ (and of $x = 0.05$ in Fig. 11, where the derivative is very small anyway), the points of fastest apparent convergence and of minimal μ_r-sensitivity, $\partial \dot{f}/\partial\mu_r = 0$, are rather close to the 'natural' choice $\mu_r = \mu_f$ for the renormalization scale.

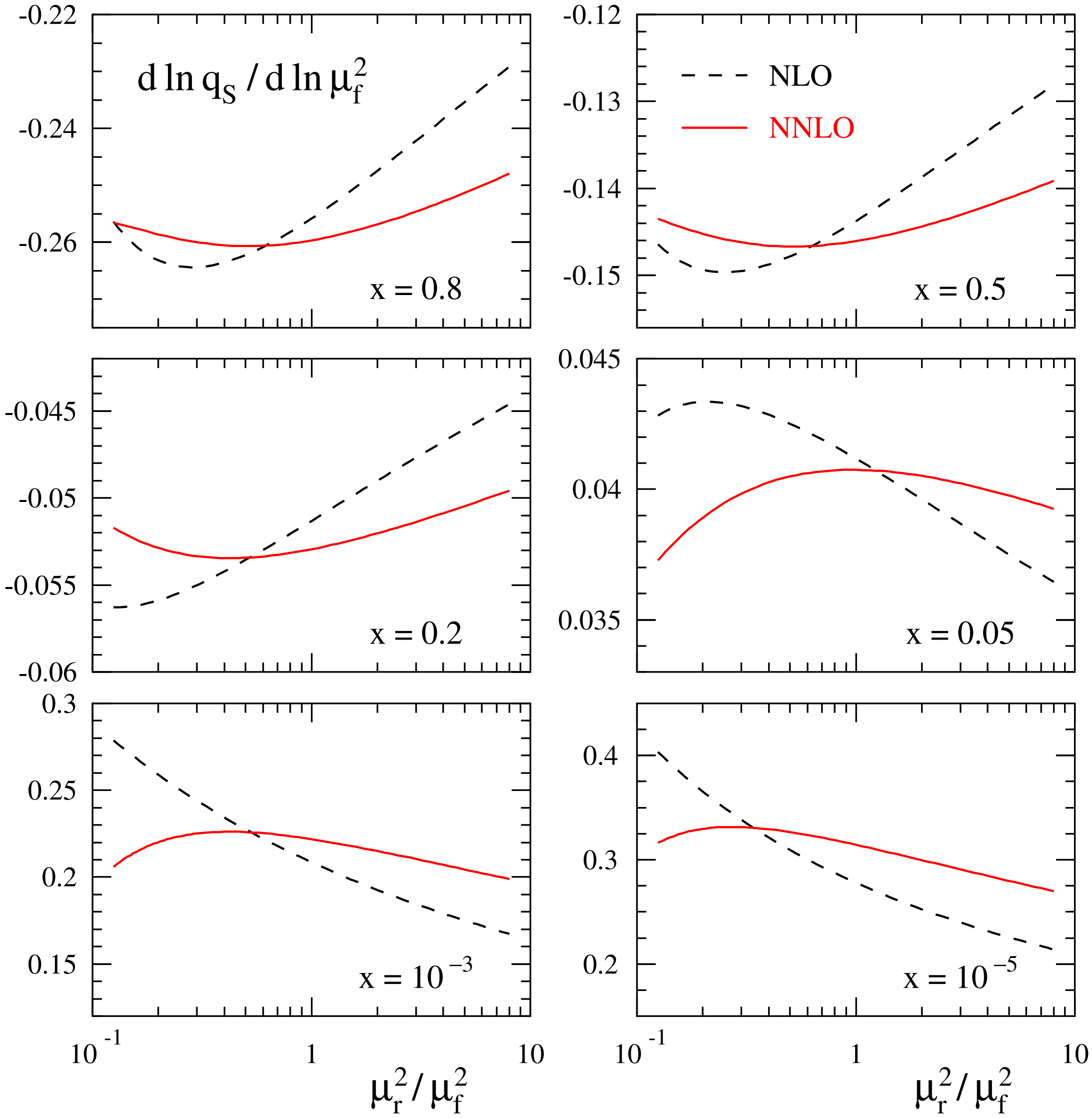

Figure 10: The dependence of the NLO and NNLO predictions for the derivative $d\ln q_{\rm s}/d\ln\mu_f^2$ of the singlet-quark distribution on the renormalization scale μ_r for six typical values of x. The initial conditions are given in Eqs. (5.1) and (5.2).

The relative scale uncertainties $\Delta\dot{q}_{\rm s}$ and $\Delta\dot{g}$ of the average derivatives, estimated using the

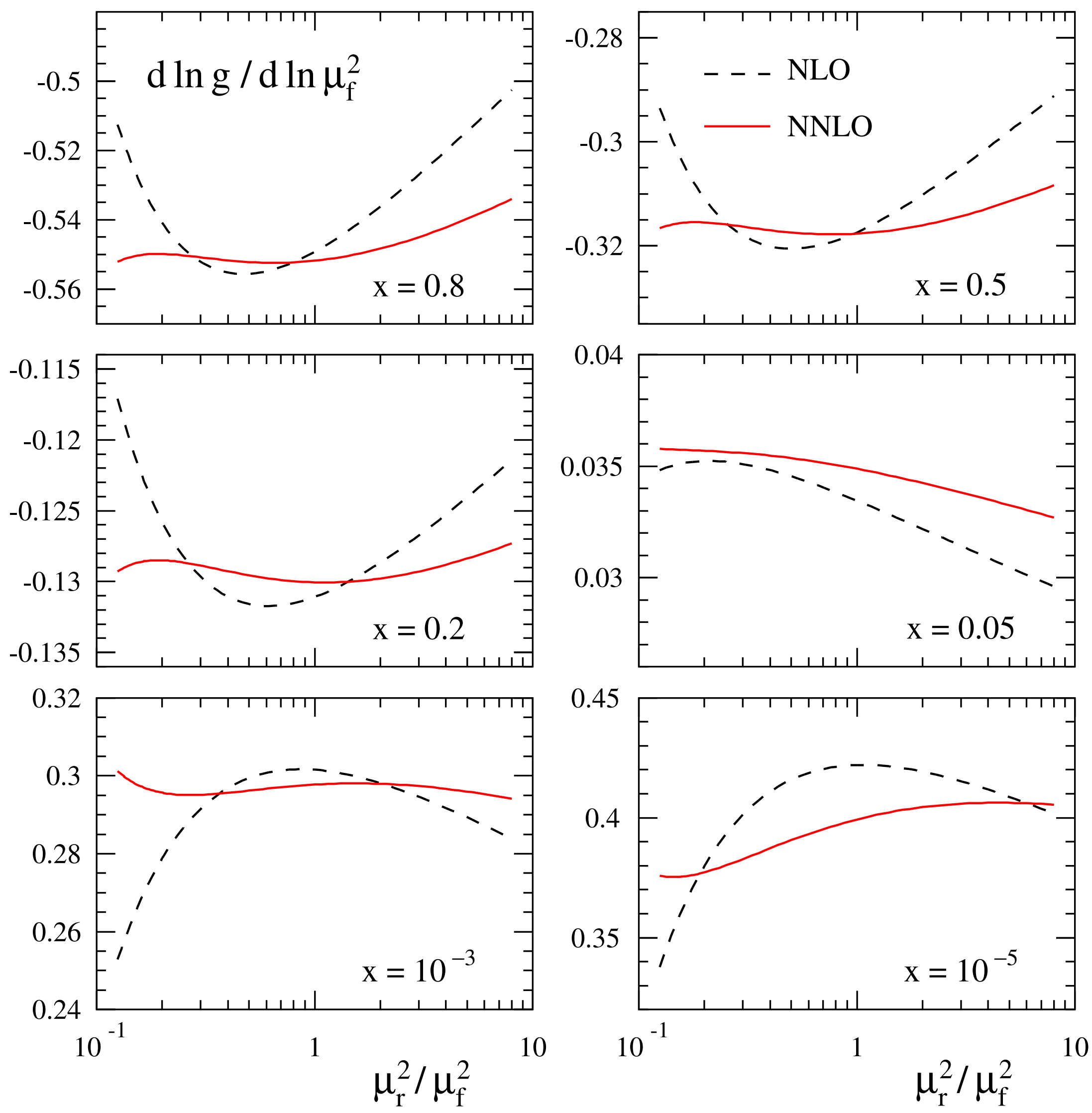

Figure 11: As Fig. 10, but for the derivative $\dot{g} \equiv d\ln g/d\ln\mu_f^2$ of the gluon distribution. Notice that the scales of the ordinates of the graphs differ within as well as between the two figures.

conventional interval $\frac{1}{2}\mu_f \leq \mu_r \leq 2\mu_f$,

$$\Delta\dot{f} \equiv \frac{\max\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \ldots 2\mu_f)\right] - \min\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \ldots 2\mu_f)\right]}{2\,|\,\mathrm{average}\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \ldots 2\mu_f)\right]|}\,, \tag{5.4}$$

are finally shown in Fig. 12. For the singlet-quark (gluon) distribution, these uncertainty estimates amount to 2% (1%) or less at $x > 10^{-2}$ ($4 \cdot 10^{-3}$), an improvement by more than a factor of three with respect to the corresponding NLO results. Taking into account also the apparent convergence of the series in Figs. 6 and 7, it is not unreasonable to expect that the effect of the higher-order singlet splitting functions will be about 1% or less for $x \gtrsim 10^{-3}$. Larger corrections have to be expected at small x. One should also keep in mind that at fourth order also terms with the colour structure $d^{abc}d_{abc}/n_c$ — which enter the non-singlet case already at three loops and have a large effect at $x < 10^{-3}$ [38] — will contribute to the singlet splitting functions.

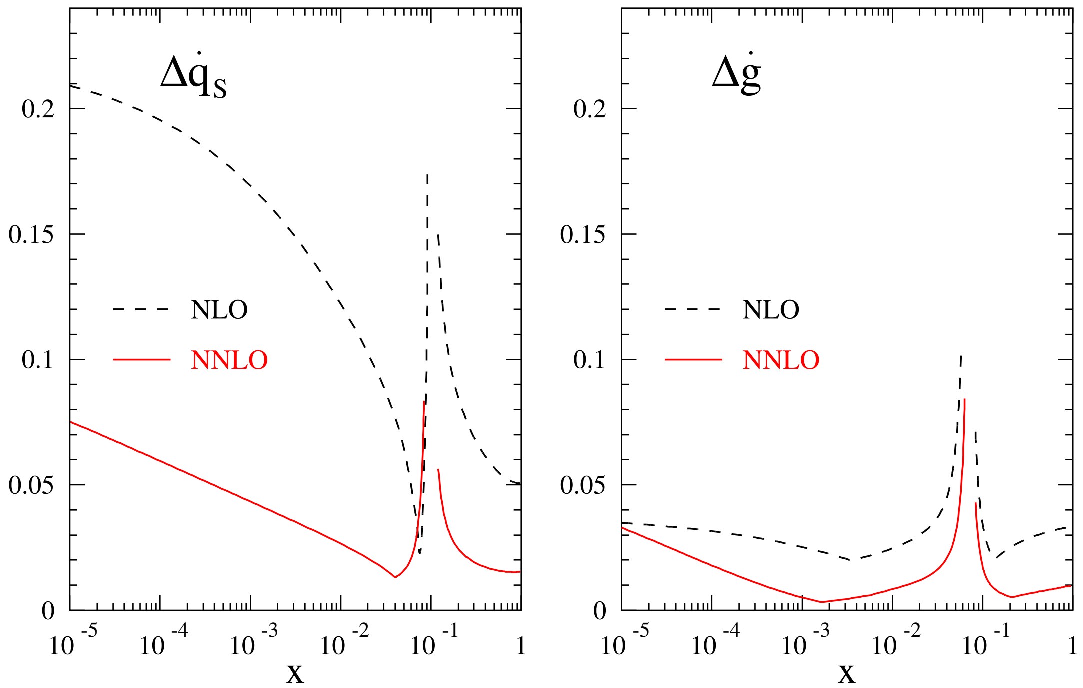

Figure 12: The renormalization scale uncertainty of the NLO and NNLO predictions for the scale derivatives of the singlet-quark density (right) and the gluon distribution (left) as estimated by the respective quantities $\Delta\dot{q}_s$ and $\Delta\dot{g}$ defined in Eq. (5.4).

6 Summary

We have calculated the complete third-order contributions to the splitting functions governing the evolution of unpolarized flavour-singlet parton distribution in perturbative QCD. Our calculation is performed in Mellin-N space and follows the previous fixed-N computations [25, 26] inasmuch as we compute the partonic structure functions in deep-inelastic scattering at even N using the optical theorem and a dispersion relation as discussed in [25]. Our calculation, however, is not restricted to low fixed values of N but provides the complete N-dependence from which the x-space splitting functions can be obtained by a (by now) standard Mellin inversion. This progress has been made possible by an improved understanding of the mathematics of harmonic sums, difference equations and harmonic polylogarithms [58, 65, 40], and the implementation of corresponding tools, together with other new features [49], in the symbolic manipulation program FORM [48] which we have employed to handle the almost prohibitively large intermediate expressions.

Our results have been presented in both Mellin-N and Bjorken-x space, in the latter case we have also provided easy-to-use accurate parametrizations. We agree with all partial results available in the literature, in particular we reproduce the lowest six even-integer moments computed before [25, 26]. We also agree with the resummation predictions of Refs. [27, 28] for the leading small-x logarithms $(\ln x)/x$ of the splitting functions P_{qq}, P_{qg} and P_{gg}, and with the large-n_f result [61] for the simple $C_A n_f^2$ part of P_{gg}. Our results respect the supersymmetric relation between all four splitting functions for $C_A = C_F = n_f$ to the extend expected for the $\overline{\rm MS}$ scheme. At large x we verify the expected simple relation between the leading $1/(1-x)_+$ terms of $P_{\rm qq}$ and $P_{\rm gg}$. We find that also for the gluon-gluon splitting function the coefficients of the leading integrable term $\ln(1-x)$ at order $n = 2, 3$ are proportional to the coefficient of the $+$-distribution $1/(1-x)_+$ at order $n-1$, in complete analogy with our surprising findings in the non-singlet case [38].

We have investigated the numerical impact of the three-loop (NNLO) contributions on the evolution of the singlet-quark and gluon densities. At $x \gtrsim 10^{-3}$ the perturbative expansion for the scale derivatives $\dot{f} \equiv d\ln f(x,\mu_f^2)/d\ln\mu_f^2$, $f = q_{\rm s}$, g appears to be very well convergent and suggests a residual higher-order uncertainty of about 1% or less at $\alpha_{\rm s} \lesssim 0.2$. Consequently the perturbative evolution can be safely extended to considerably larger values of $\alpha_{\rm s}$, hence lower scales, in this range of x. The situation is much less clear at smaller x. For $\alpha_{\rm s} = 0.2$ and realistic initial distributions with $xq_{\rm s}, xg \sim x^{-0.3}$ at small x, the NNLO corrections for $\dot{q}_{\rm s}$ and $\dot{g}$ rise towards $x \to 0$, respectively reaching 13% and -6% at $x = 10^{-5}$. We stress that the results of the small-x resummation alone cannot help here. For example, not even a qualitatively reliable prediction can be expected for the convolution $P_{\rm gg} \otimes g$, by which $P_{\rm gg}$ enters the evolution equations, even when all $1/x$ terms are included. Besides knowledge of as many of these terms as possible, further progress at small x would require at least a four-loop generalization of the fixed-N calculations [25, 26] and of the x-space approximations [37] linking them to the small-x limits.

FORM files of our results, and FORTRAN subroutines of our exact and approximate splitting functions can be obtained from the preprint server `http://arXiv.org` by downloading the source. Furthermore they are available from the authors upon request.

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